11. Optimization with Costs

11.1. **Single-period optimization with costs.** Let $h_0 \in \mathbb{R}^n$ denote our current portfolio holdings in dollars and let $h \in \mathbb{R}^n$ denote a hypothetical set of portfolio holdings that we could trade into. Let $\pi(h)$ denote the random variable which represents our P/L if we do these trades, and then liquidate the portfolio h so that we hold all cash. As discussed last time,

$$\pi(h) = h'R - [\mathrm{slip}(h_0, h) + \mathrm{liqslip}(h)]$$

We want to optimize $\mathbb{E}[u(w_T)]$ where T is the final time (eg. after the liquidation has been completed). We assume asset returns follow a multivariate elliptical distribution, and hence there exists some $\kappa > 0$ such that we can equivalently maximize

$$\mathbb{E}[w_T] - \frac{\kappa}{2}\mathbb{V}[w_T]$$

The final wealth is $w_T = w_0 + \pi(h)$ and at this point, dependence on the initial wealth w_0 drops out of the problem (it adds a constant term).

We are thus left with the problem

$$\max_h \left\{ \mathbb{E}[\pi(h)] - \frac{\kappa}{2}\mathbb{V}[\pi(h)] \right\}$$

Suppose we assume that $\mathbb{V}[\pi(h)]$ is well approximated by the variance of the term $h'R$ in $\pi(h)$, in other words

$$\mathbb{V}[\pi(h)] \approx h'\Sigma h \text{ where } \Sigma := \mathrm{cov}(R).$$

If we aren't planning to liquidate the portfolio h after the next period, and we are happy to identify unrealized (ie. paper, mark-to-market) P/L with other P/L realized as cash, then we can ignore the liquidation slippage term.

Then the mean-variance problem becomes

$$\max_h \left\{ h'\mathbb{E}[R] - \frac{\kappa}{2}h'\Sigma h - \mathbb{E}[\mathrm{slip}(h_0, h)] \right\} \tag{11.1}$$

Note that (11.1) can equivalently be considered as an optimization over the *trade list*

$$\delta := h - h_0 \in \mathbb{R}^n$$

since h_0 is fixed, and is not a parameter in the optimization. The trade list δ is the more natural variable for the largest sources of cost; slippage $\mathrm{slip}(h_0, h)$ is a function of δ, as are commissions.

Financing costs, on the other hand, are functions of h rather than δ. You pay to finance the portfolio you end up with. On the short side, financing costs are usually called *borrow costs*, and can be quite high for stocks that are hard to borrow. On the long side, you will pay to finance a portfolio that is larger (in notional terms) than your capital, usually at the rate of 25bps per year times financed notional value.

Note that there are (at least) two functional forms for the latter two terms in (11.1) which allow for easy solution of the mean-variance maximization problem: (1) purely quadratic, and (2) quadratic plus absolute-value type penalty terms. In the first case, the entire problem remains quadratic, while in the second case, the problem becomes mathematically equivalent to a Lasso regression. The Almgren et al. (2005) form does not lead to such a well-known procedure as Lasso, but the associated problem is convex and differentiable, hence standard optimization routines can be expected to perform well.

11.2. **General Multiperiod Problems.** Gârleanu and Pedersen (2013) studied the multiperiod quantitative-trading problem under the somewhat restrictive assumptions that the alpha models follow mean-reverting dynamics and that the only source of trading frictions are purely linear market impacts (leading to purely quadratic impact-related trading costs). We want to do something similar, but not so restrictive and general enough to apply to real trading scenarios.

We now place ourselves into the position of a rational agent planning a sequence of trades beginning presently and extending into the future. Specifically, a *trading plan* for the agent is modeled as a specific portfolio sequence

$$\boldsymbol{h} = (h_1, h_2, \ldots, h_T),$$

where h_t is the portfolio the agent plans to hold at time t in the future. If r_{t+1} is the vector of asset returns over $[t, t+1]$, then the trading profit (ie. difference between initial and final wealth) associated to the trading plan $\boldsymbol{h}$ is given by

$$\pi(\boldsymbol{h}) = \sum_t [h_t \cdot r_{t+1} - c_t(h_{t-1}, h_t)] \tag{11.2}$$

where $c_t(h_{t-1}, h_t)$ is the total cost (including but not limited to market impact, spread pay, borrow costs, ticket charges, financing, etc.) associated with holding portfolio h_{t-1} at time $t-1$ and ending up with h_t at time t.

Trading profit $\pi(\boldsymbol{h})$ is a random variable, since many of its components are future quantities unknowable at time $t = 0$. Thus the problem we treat initially is that of maximizing $u(\boldsymbol{h})$, where

$$u(\boldsymbol{h}) := \mathbb{E}[\pi(\boldsymbol{h})] - (\kappa/2)\mathbb{V}[\pi(\boldsymbol{h})] \tag{11.3}$$

We will often refer to a planned portfolio sequence $\boldsymbol{h} = (h_1, h_2, \ldots, h_T)$ simply as a "path." Similarly we sometimes refer to (11.3) as the "utility of the path $\boldsymbol{h}$," while remembering the more complex link to utility theory noted above. Our task, in this simpler language, is to find the maximum-utility path $\boldsymbol{h}^* = \operatorname{argmax}_{\boldsymbol{h}} u(\boldsymbol{h})$.

Combining (11.2) with (11.3), and defining

$$\alpha_t := \mathbb{E}[r_{t+1}] \quad \text{and} \quad \Sigma_t := \boldsymbol{V}[r_{t+1}],$$

one has

$$u(\mathbf{x}) = \sum_t \left[h_t' \alpha_t - \frac{\kappa}{2} h_t' \Sigma_t h_t - c_t(h_{t-1}, h_t) \right] \tag{11.4}$$

Note that any symmetric, positive-definite matrix Q defines a norm

$$N_Q(x) = x^\top Q x$$

and an associated metric $d_Q(x, y) := N_Q(x - y)$, and bilinear form

$$b_Q(x, y) = N_Q(x - y).$$

The bilinear form associated to $\kappa\Sigma_t$ will be useful to us. It is

$$b_{\kappa\Sigma_t}(x_t, y_t) := \frac{1}{2}(y_t - x_t)^\top \kappa\Sigma_t (y_t - x_t) \tag{11.5}$$

Now, let us take a certain specific choice of y_t, namely the unconstrained Markowitz portfolio

$$y_t := (\kappa\Sigma_t)^{-1}\alpha_t. \tag{11.6}$$

and evaluate the quadratic form at the pont (11.6)

$$\begin{aligned} b_{\kappa\Sigma_t}(x_t, y_t) &= \frac{1}{2}(y_t - x_t)^\top \kappa\Sigma_t (y_t - x_t) \\ &= \frac{1}{2}\left[y_t^\top (\kappa\Sigma_t) y_t - 2x_t^\top (\kappa\Sigma_t) y_t + x_t^\top (\kappa\Sigma_t) x_t \right] \\ &= \frac{1}{2} y_t^\top (\kappa\Sigma_t) y_t - x_t^\top \alpha_t + \frac{1}{2} x_t^\top (\kappa\Sigma_t) x_t \end{aligned}$$

We denote by $O(y^2)$ any collection of terms which is a quadratic function of y_t and which doesn't contain x_t at all. Then we have shown the following:

$$-b_{\kappa\Sigma_t}(x_t, y_t) = O(y^2) + x_t^\top \alpha_t - \frac{1}{2} x_t^\top (\kappa\Sigma_t) x_t$$

Therefore the first two terms in the utility calculation (11.4) (ie. all the terms not dealing with costs) are given by

$$-b_{\kappa\Sigma_t}(h_t, y_t)$$

Note that $b_{\kappa\Sigma_t}$ measures variance of the tracking error to the unconstrained Markowitz portfolio. Then up to x-independent terms,

$$u(\boldsymbol{h}) = -\sum_t \left[b_{\kappa\Sigma_t}(h_t, y_t) + c_t(h_{t-1}, h_t) \right] \tag{11.7}$$

In any multiperiod optimization problem with transaction costs, one can always ask what the solution would be in an ideal world without transaction costs, or equivalently, in the limit as costs tend to zero. We call this solution the *ideal* *sequence*, and always denote it by y_t. By mean-variance equivalence, if asset returns follow an elliptical distribution and there are no constraints, then the ideal sequence is (11.6).

Intuition 11.1. Multiperiod portfolio optimization is mathematically equivalent to optimally tracking a sequence y_t, called the *ideal sequence*, which is the portfolio sequence that would be optimal in a transaction-cost free world.

The general guiding principle expressed as Intuition 11.1 extends beyond the case in which the ideal sequence is $(\kappa\Sigma_t)^{-1}\alpha_t$, and indeed, beyond the case in which y_t has a clean derivation from a utility function. For computing optimal liquidation paths in the spirit of Almgren and Chriss (1999), the ideal sequence is clearly $y_t = 0$ for all t. For hedging exposure to derivatives, y_t should be our expectation of the offsetting replicating portfolio at all future times until expiration.

Tracking the portfolios of Black and Litterman (1992) is also a special case of our framework in which y_t is the solution to a mean-variance problem with a Bayesian posterior distribution for the expected returns. Since the posterior is Gaussian in the original Black-Litterman model, the two-moment approximation to utility is exact, and one simply replaces α_t and Σ_t with the appropriate quantities.

11.3. **Non-differentiable Optimization.** Given a convex, differentiable map $f : \mathbb{R}^n \to \mathbb{R}$, if we are at a point x such that $f(x)$ is minimized along each coordinate axis, have we found a global minimizer? In other words, does

$$f(x + d \cdot e_i) \geq f(x) \quad \text{for all} \quad d, i$$

imply that $f(x) = \min_z f(z)$? Here $e_i = (0, ..., 1, ...0) \in \mathbb{R}^n$, the i-th standard basis vector.

The answer is: Yes!

Now consider the same question, but without the differentiability assumption.

The answer changes to no, and Fig. 11.1 below gives a counterexample.

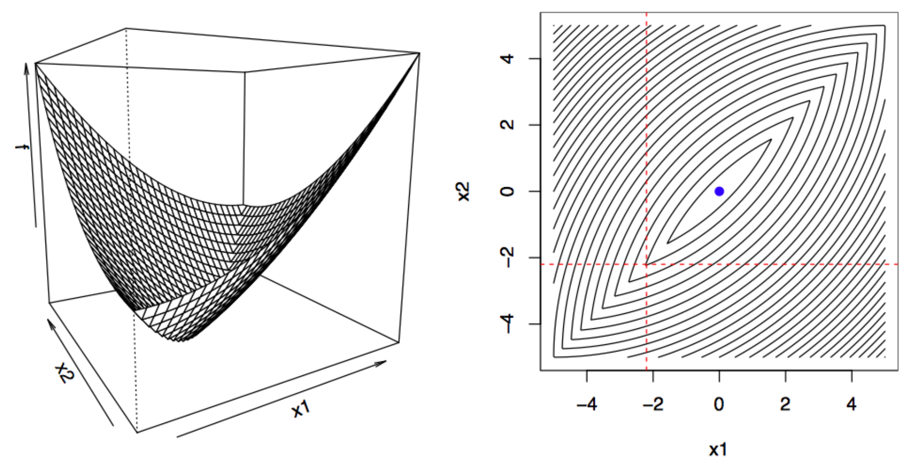

FIGURE 11.1. A convex function for which coordinate descent will get "stuck" before finding the global minimum.

Consider the same question again: "if we are at a point x such that $f(x)$ is minimized along each coordinate axis, have we found a global minimizer?" only now

$$f(x) = g(x) + \sum_{i=1}^{n} h_i(x_i)$$

with g convex, differentiable and each h_i convex? (In this case, we say the non-differentiable part is *separable*.)

If the non-differentiable term is separable, the answer is yes once again. This is a special case of a deep general result proved by Tseng (2001), which we will call "Tseng's theorem." The main take-away is: we can easily optimize

$$f(x) = g(x) + \sum_{i=1}^{n} h_i(x_i)$$

with g convex, differentiable and each h_i convex, by coordinate-wise optimization.

Tseng's results also suggest an algorithm, called *blockwise coordinate descent* (BCD).

Algorithm 11.1. Chose an initial guess for x. Repeatedly iterate cyclically through $i = 1, \ldots, N$, and perform the following optimization and update:

$$x_i = \underset{\omega}{\operatorname{argmin}}\, f(x_1, \ldots, x_{i-1},\ \omega,\ x_{i+1}, \ldots, x_N)$$

Tseng (2001) shows that for functions of the form above, any limit point of the BCD iteration is a minimizer of f. The order of cycling through coordinates is arbitrary; and we can use any scheme that visits each of $\{1, 2, \ldots, n\}$ every M

steps for fixed constant M. We can also everywhere replace individual coordinates with blocks of coordinates.

11.4. **Single-asset trading paths.** Now let us consider the multiperiod problem for a single asset, in which case the ideal sequence $\mathbf{y} = (y_t)$ and the holdings (or equivalently, hidden states) $\boldsymbol{h} = (h_t)$ are both univariate time series. Since the multiperiod many-asset problem can be reduced to iteratively solving a sequence of single-asset problems, the methods we develop in this section are important even if our main interest is in multi-asset portfolios.

A very important class of examples arises when there are no constraints, but the cost function is a convex and non-differentiable function of the difference

$$\delta_t := h_t - h_{t-1}.$$

This allows for non-quadratic terms as in Almgren et al. (2005) and non-differentiable terms such as linear proportional costs.

In this case, we can use Tseng's theorem, applied to *trades* rather than *positions.* Writing

$$h_t = h_0 + \sum_{s=1}^{t} \delta_s,$$

the objective function becomes

$$u(\boldsymbol{h}) = -\sum_t \left[b\Big(h_0 + \sum_{s=1}^{t} \delta_s, y_t\Big) + c_t(\delta_t) \right] \tag{11.8}$$

Eq. (11.8) satisfies the convergence criteria of Tseng (2001) that the non-differentiable term is separable *across time*, while the non-separable term is differentiable. One then performs coordinate descent over the trades $\delta_1, \delta_2, \ldots, \delta_T$. Almost any reasonable starting point will do to initialize the iteration, but if a warm start from a previous optimization is available, that may speed things along. This approach was introduced in Kolm and Ritter (2015).

Problem 11.1. Consider optimally trading a single stock over $T = 30$ days. Each period is one day, and you can trade once per day. The stock's daily return volatility is σ. Suppose your forecast is 50 basis points for the first period, and decays exponentially with half-life 5 days. This means that

$$\alpha_t := \mathbb{E}[r_{t,t+1}] = 50 \times 10^{-4} \times 2^{-t/5}. \tag{11.9}$$

Let $c(\delta)$ be the cost, in dollars, of trading δ dollars of this stock. For selling, $\delta < 0$. Following Almgren, we assume that

$$c(\delta) = PX\left(\frac{\gamma\sigma}{2}\frac{X}{V}\left(\frac{\Theta}{V}\right)^{1/4} + \operatorname{sign}(X)\,\eta\sigma\left|\frac{X}{V}\right|^{\beta}\right), \quad X = \delta/P$$

where P is the current price in dollars, X is the signed trade size in shares, V is the daily volume in shares, Θ is the total number of shares outstanding, and finally $\gamma = 0.314$ and $\eta = 0.142$ and $\beta = 0.6$ are constants fit to market data. For concreteness, suppose the asset we are trading has

$$P = \$40,\ V = 2 \times 10^6,\ \Theta = 2 \times 10^8, \quad \sigma = 0.02.$$

For a trading path $\mathbf{x} = (x_0, x_1, \ldots, x_T)$ where x_t denotes dollar holdings of the stock at time t, define the profit (also in dollars) as

$$\pi(\mathbf{x}) = \sum_{t=1}^{t} [x_t r_{t,t+1} - c(x_t - x_{t-1})]$$

This is a random variable due to the presence of $r_{t,t+1}$ which you can assume is Gaussian with mean $\alpha_t := \mathbb{E}[r_{t,t+1}]$ and variance $\mathbb{V}[r_{t,t+1}] = \sigma^2$. In this problem, always assume $x_0 = 0$ is fixed.

(a) Find the sequence of positions $x_1, x_2, \ldots, x_T$ that maximizes

$$u(x_1, \ldots, x_T) = \sum_{t=1}^{T} \left[x_t \alpha_t - \frac{\kappa}{2} \sigma^2 x_t^2 - c(x_t - x_{t-1}) \right] \tag{11.10}$$

with risk-aversion $\kappa = 10^{-7}$. Set tolerance so that your algorithm does not terminate unless each $x_t \in \mathbb{R}$ is within a distance of one dollar to the true optimal path. Plot the optimal path $\mathbf{x}^* := (x_0 = 0, x_1^*, \ldots, x_T^*)$ and also report its values in a table. Also report the computation time.

Submit your code and a clear explanation of the algorithm you used, why your chose it over other possible algorithms, and how you know that it converges. For example, if you used a method that requires convexity, explain why the function you are optimizing is convex.

(b) Use the program you wrote in part (a) to plot expected profit of the optimal path, $\mathbb{E}[\pi(\mathbf{x}^*)]$ and *ex ante* Sharpe ratio of the optimal path, defined as

$$\text{Sharpe}(\mathbf{x}^*) = \sqrt{252}\ \frac{\mathbb{E}[\pi(\mathbf{x}^*)]}{\sqrt{\mathbb{V}[\pi(\mathbf{x}^*)]}}$$

as a function of κ, as a function of the half-life (which was taken to be 5 in equation (11.9) above), as a function of the initial strength (taken to be 50 in equation (11.9)), and as a function of σ. So you need to do eight plots in all: profit and Sharpe ratio, each as a function of one of four parameters (holding the others fixed). Choose appropriate intervals around the parameter values in part (a). Note that κ cannot be negative in reasonable models.

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