5. The Capital Asset Pricing Model

5.1. **The Capital Asset Pricing Model in its original form.** We begin by introducing the Capital Asset Pricing Model in its *original form* due to Sharpe (1964). We may, time permitting, return to discuss extensions due to Black and Lintner.

The Capital Asset Pricing Model is an extremely idealized model of an open market place. Everyone knows it isn't a true description of how our markets actually function, as all of the assumptions one must make to derive it are gross idealizations. Nonetheless, sometimes it's useful to work out what would be true in an ideal world. It's then instructive to understand the exact ways in which our world deviates from the predictions of the idealized model.

Here are all of the assumptions we need to make in order to derive the CAPM:

(a) an open market place, where all the risky assets are available to all
(b) there exists a risk-free asset (idealized treasury bond) for borrowing and/or lending in unlimited quantities with interest rate r_f
(c) all information is available to all such as covariances, variances, mean rates of return, etc.
(d) everyone is a risk-averse rational investor who uses Markowitz mean-variance portfolio theory
(e) there are no trading costs
(f) the market has reached equilibrium

Assumptions (c) and (d) together imply a unique, up to scalar multiplication, portfolio of risky assets. It must be proportional to $\Sigma^{-1}\mathbb{E}[r]$ where $\Sigma \in S^n_{++}$ is the common covariance matrix and r is an n-dimensional random vector denoting the next-period asset returns.

Let's spend a few moments discussing why all of these assumptions are false. For the record, I am not saying anything revolutionary here – even those responsible for the development of the CAPM and related theories understand very well that all of the assumptions are gross idealizations. This is probably only slightly worse than modeling a neuron as a sigmoid activation function applied to the output of a linear function. However, the latter approximation is extremely useful as well; without it, there would be no deep learning.

Sharpe (1990) says:

> *The initial version of the CAPM, developed over 25 years ago, was extremely parsimonious. It dealt with the central aspects of equilibrium in capital markets and assumed away many important aspects of such markets as they existed at the time.*

— Sharpe (1990)

Assumption (a) is false because there are many assets that aren't tradable by everyone (think A-shares in China, shares like Berkshire with very high prices so that retail investors can't even afford one share, etc.)

Assumption (b) is perhaps the closest to being semi-realistic among assumptions (a)-(f). Of course no asset is risk-free, as governments and corporations frequently default on even relatively short-term debt. Also, individual investors do not have the same credit rating as, say, the US government and hence cannot borrow at the "low-risk rate" of a sovereign bond. No investor or institution can borrow or lend in infinite quantities. However, for institutional investors, (b) may actually be a reasonable approximation.

Assumption (c) is probably the most egregiously false of all of these, with (d) coming in a close second place. No-one knows what the asset return covariance matrix is going to look like in the future, although with a lot of effort, some still-noisy-but-useful statistical forecasts can be made. There isn't universal agreement on the "best" forecasting method, although some forecasting methods can be ruled out as obviously worthless.

The situation for the *first* moment of asset returns $\mathbb{E}[r]$ is even worse – no-one has much of an idea here, and those who do have typically come by this information via extremely labor-intensive processes and aren't prepared to freely share the results of their labor. The more expertise anyone has in predicting returns, the less inclined they are to share the results of their analysis. To come up with a statistical forecasting method which realizes out-of-sample a cross-sectional R^2 of 0.01 when forecasting residual returns to a factor model is possible, but difficult.

There is *no consensus* about $\mathbb{E}[r]$ even among sophisticated, professional investors with the same information set. Two such investors with identical, and fairly complete, information concerning Apple, Inc in 2001 could still disagree their forecasts of how popular the iPod might become. Even assuming (d) holds, these two investors would put different $\mathbb{E}[r_i]$ for AAPL into their optimizers.

Assumption (d) is also quite egregiously false. Most investors don't know how to use modern portfolio theory effectively, and so they don't use it, falling back instead on simple heuristics ("cut your losers, let your winners run," etc.) This is probably why the HFRX type indexes, which are supposed to be broad indexes of the hedge fund industry, have correlations of 0.7 (or more; one recent report from one of my prime brokers said 0.9) to the S&P 500. Most large quantitative or multi-strategy hedge funds do use modern portfolio theory in some way or other. Those that do often either realize low correlation to the market, or else it is done deliberately as in a benchmark-aware strategy.

Assumption (e) is also false. The more actively-traded a strategy is, the more trading costs eat into any potential alpha. Depending on exactly how they're constructed, naive implementations of Markowitz portfolios can turn over quite a lot. Arrow-Pratt utility theory calls for optimization of utility of final wealth anyway, and final wealth includes any transaction costs that are paid, so utility theory doesn't actually say to trade the (cost-unaware) Markowitz portfolio – it says we should optimize with costs in the consideration.

Assumption (f) is unrealistic; stock prices routinely move by several percents over the time span of a few minutes even when there's no news or information that should affect the firm's intrinsic value in any way. To make an analogy with thermodynamic equilibrium, real markets much more closely resemble the surface of a bubbling, churning, boiling cauldron than a flat, placid, reflective mountain lake. The settling down into a thermal equilibrium never seems to occur. This isn't really a problem for the CAPM – the CAPM creators would probably say that they are trying to model what would happen if the system were in equilibrium, not to model the fluctuations around equilibrium.

Note that we have not assumed normality of returns anywhere. Sharpe confirms:

> *The CAPM makes no assumptions about the "return generating process." Hence, its results are completely consistent with any such process.*
>
> — Sharpe (1990)

With these caveats, let's continue with deriving the mathematical implications of the above assumptions. The efficient fund proportional to $\Sigma^{-1}\mathbb{E}[r]$ and used by all is called "the market portfolio" and will henceforth be denoted by M. Let V_i be the market capitalization of the i-th company. The total number of dollars in the entire equity market is then $V = \sum_{i=1}^{n} V_i$. If all investors hold the same portfolio (as the *risky* part of their allocation), then it doesn't matter whether there are a bunch of small investors or one big one. If there were one big one, then we know the total value of her investment in risk assets: it's V. The "weights" of the one big investor's portfolio are then $w_i = V_i/V$. Hence the vector $\mathbf{w}$ must be proportional to $\Sigma^{-1}\mathbb{E}[r]$.

Note that we found the (risky part of the) one big investor's portfolio without having to know Σ and $\mathbb{E}[r]$. This argument relied on all of the (tenuous) assumptions above. Note also that in this model, there are no assets with $\mathbb{E}[r] < r_f$ and hence no need for anyone to hold any short positions, although it isn't explicitly ruled out. One can see this as follows: if $\mathbb{E}[r_i] < r_f$ at the current price of asset i, then no investors will buy the asset at that price (they'd prefer to buy the risk-free bond). Hence no-one will buy it, period, unless the price is set low enough so that *at the new price* $\mathbb{E}[r_i] > r_f$.

> *When a riskless asset is available, the only negative holdings in equilibrium will involve borrowing by investors with above-average risk tolerance who wish to finance added investment in a portfolio representing the overall capital market.*
>
> — Sharpe (1990)

Also, a short position must be borrowed from an owner – someone who holds a long position. Hence shorting is at odds with the notion that there is only one efficient fund of risky assets.

Since there's only one risky portfolio in this model, all investors must hold some combination of this portfolio and the risk-free asset. Any portfolio with weights adding to 1 must be of the form $1-h$ units of the risk-free asset, and h units of M. The expected return and risk of this portfolio are:

$$\overline{r} = (1-h)r_f + h\overline{r}_M, \quad \sigma = h\sigma_M$$

where, for notational convenience, we will denote $\mathbb{E}[r]$ as $\overline{r}$, and continue this notation throughout the lecture.

We can then eliminate $h = \sigma/\sigma_M$ to find

$$\overline{r} = (1 - \frac{\sigma}{\sigma_M})r_f + \frac{\sigma}{\sigma_M}\overline{r}_M = r_f + \frac{\sigma}{\sigma_M}(\overline{r}_M - r_f) \tag{5.1}$$

Eq. (5.1) is called *the capital market line.* It tells us that the expected excess return of any *efficient* portfolio is a constant times its risk, where the constant is the so-called *price of risk*

$$\frac{\overline{r}_M - r_f}{\sigma_M} \tag{5.2}$$

Note that eq. (5.1) is for an efficient portfolio; it does not hold for an inefficient portfolio (such as a single stock). The analogous single-stock relation is as follows.

Theorem 5.1. Under the assumptions above, for any asset i we have

$$\overline{r}_i - r_f = \beta_i(\overline{r}_M - r_f) \tag{5.3}$$

where $\beta_i = \text{cov}(r_i, r_M)/\sigma_M^2$.

Under the strict assumptions of the CAPM, (5.3) is not a regression; rather it is a relation which holds exactly. The variances and covariances appearing in $\beta_i = \text{cov}(r_i, r_M)/\sigma_M^2$ are the *ex ante* values agreed upon by all investors in assumption (c).

Proof. Form a portfolio with holding λ in asset i and $1-\lambda$ in the market M. Trivially, this portfolio has expected return and variance

$$\overline{r}(\lambda) = \lambda(\overline{r}_i - \overline{r}_M) + \overline{r}_M \tag{5.4}$$

$$\begin{aligned}\sigma^2(\lambda) &= \lambda^2(\sigma_i^2 + \sigma_M^2 - 2\,\mathrm{cov}(r_i, r_M)) \\ &\quad + 2\lambda(\mathrm{cov}(r_i, r_M) - \sigma_M^2) + \sigma_M^2\end{aligned}$$

Think of (5.4) as parametrizing a curve in (σ, r) space, just as (5.1) is a line in (σ, r) space.

Then its trivial to see that the line (5.1) is tangent to the curve (5.4) at the point $\lambda = 0$. This follows by evaluating (5.4) at $\lambda = 0$ to show that it is the market point $(\overline{r}_M, \sigma_M)$, and other points on the curve aren't efficient portfolios, so they lie below the line (5.1). The slope of this tangent line is, of course, (5.2), so we can equate this to the slope calculated by taking the derivative:

$$\frac{\overline{r}_M - r_f}{\sigma_M} = \left.\frac{d\overline{r}(\lambda)}{d\sigma(\lambda)}\right|_{\lambda=0} = \frac{\overline{r}_i - \overline{r}_M}{(\mathrm{cov}(r_i, r_M) - \sigma_M^2)/\sigma_M}$$

Solving this equation for the unknown variable $\overline{r}_i$ leads directly to (5.3), completing the proof. □

Theorem 5.2. Consider a portfolio of risky assets with holdings vector $h \in \mathbb{R}^n$ where $h_i \geq 0$ and $\sum_i h_i = 1$. Then one has

$$\overline{r}_h - r_f = \beta_h(\overline{r}_M - r_f) \quad \text{where} \quad \beta_h := \sum_i h_i \beta_i = \boldsymbol{h}'\boldsymbol{\beta} \tag{5.5}$$

Proof. Note that

$$\overline{r}_h - r_f = -r_f + \sum_i h_i \overline{r}_i = \sum_i h_i(\overline{r}_i - r_f) \tag{5.6}$$

$$= \sum_i h_i \beta_i (\overline{r}_M - r_f) = \beta_h(\overline{r}_M - r_f) \tag{5.7}$$

To go from the first line to the second line, we used (5.3). This concludes the proof of Theorem 5.2.

5.2. **Estimation.** Eq. (5.3) suggests that we could attempt to estimate β_i from a linear regression of $r_i - r_f$ on $r_m - r_f$. Indeed, Sharpe seems to allow for this:

> *The value of β may be given an interpretation similar to that found in regression analysis utilizing historic data, although in the context of the CAPM it is to be interpreted strictly as an ex ante value based on probabilistic beliefs about future outcomes.*

— Sharpe (1990)

Of course, it may be in many cases that a historically estimated regression coefficient is a good *ex ante* forecast. This is likely to be true only in cases where the capital structure of the underlying company remains stationary. If the company has just undergone a major spinoff or restructuring, or a major debt issue, then the historical beta estimated via regression is likely less accurate as a forecast of ex ante beta. We now look at the empirical data for Apple over the three-year period from 2012-2014.

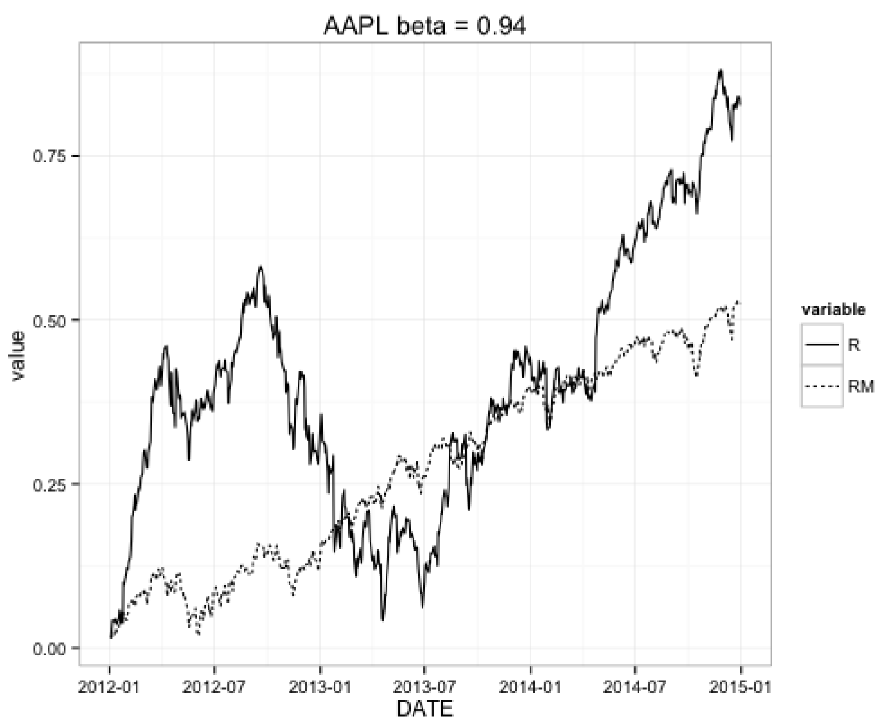

Figure 5.1. Cumulative returns to AAPL and the S&P 500 over a three-year period.

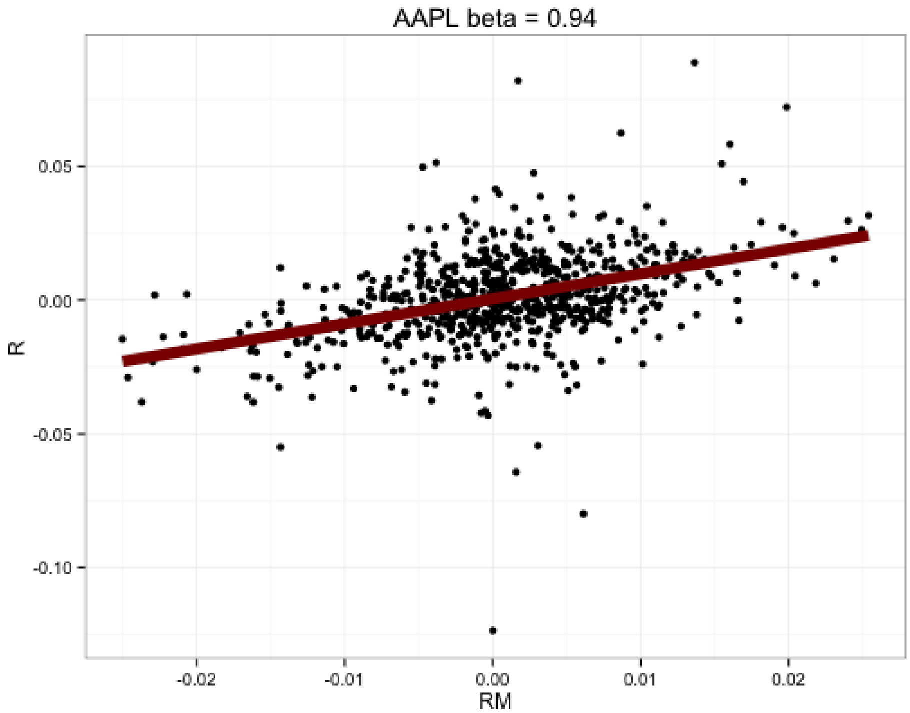

FIGURE 5.2. Scatterplot of returns to AAPL and the S&P 500 together with the line suggested by a CAPM regression.

5.3. **Fully-Invested Portfolios with no riskless asset.** Consider first a universe of n risky assets, with no risk-free asset at all.

In the notation of Merton (1972), let E_i denote the expected return on the i-th security, and let Ω denote the covariance matrix of returns of a universe of n securities. The bold face $\mathbf{E} = (E_1, \dots, E_n)$ denotes the vector of expected returns for all securities.

A portfolio is said to be a *frontier portfolio* if it has the smallest variance for a prescribed level E of expected return. Merton (1972) writes this as a constrained optimization problem:

$$\min \frac{1}{2}\sigma^2 \quad \text{subject to}$$
$$\sigma^2 = \langle \mathbf{x}, \Omega \mathbf{x} \rangle, \quad E = \langle \mathbf{x}, \mathbf{E} \rangle, \quad 1 = \mathbf{1} \cdot \mathbf{x}$$

One may then see by applying the KKT conditions that all solutions are of the form

$$\mathbf{x} = \gamma_1 \Omega^{-1} \mathbf{E} + \gamma_2 \Omega^{-1} \mathbf{1}$$

Indeed, defining

$$A = \langle \mathbf{1}, \Omega^{-1}\mathbf{E} \rangle, \quad B = \langle \mathbf{E}, \Omega^{-1}\mathbf{E} \rangle, \quad C = \langle \mathbf{1}, \Omega^{-1}\mathbf{1} \rangle$$

Merton shows that

$$\gamma_1 = \frac{CE - A}{D}, \quad \gamma_2 = \frac{B - AE}{D}$$

where $D := BC - A^2 > 0$, and it follows using non-singularity of Ω that $B > 0$ and $C > 0$.

One of the Lagrangian first-order conditions is

$$\Omega\mathbf{x} = \gamma_1 \mathbf{E} + \gamma_2 \mathbf{1}.$$

If we take the inner product of both sides of this equation on the left by $\mathbf{x}$, we obtain

$$\sigma^2 = \langle \mathbf{x}, \Omega\mathbf{x} \rangle = \gamma_1 E + \gamma_2$$

where we use the fully-invested constraint that $\langle \mathbf{x}, \mathbf{1} \rangle = 1$. Substituting the expressions above for γ_1 and γ_2, we find

$$\sigma^2 = \frac{C}{D}E^2 - \frac{2A}{D}E + \frac{B}{D}.$$

Thus the frontier in mean-variance space is a parabola. The unique minimum point (the minimum-variance portfolio) is located at $\overline{E} = A/C$ with variance $\overline{\sigma}^2 = 1/C$. The weights of the minimum-variance portfolio are

$$\overline{\mathbf{x}} = \frac{1}{C}\Omega^{-1}\mathbf{1}.$$

Suppose one makes a change of coordinates, representing E as a function of standard deviation σ, instead of variance σ^2. Also, we switch the axes, representing E on the ordinate and σ on the abscissa. In this parameterization, the frontier is a hyperbola whose leftmost point $(\overline{\sigma}, \overline{E})$ corresponds to the minimum variance portfolio. If, bizarrely, you insisted upon even lower return than the minimum variance portfolio (while remaining fully invested and without being able to access a risk-free asset), then you would have to accept more risk than $\overline{\sigma}$. This contradicts rationality, so no-one would ever do it. Hence the lower half of the hyperbola is not relevant or important; only the upper half is important. The whole hyperbola is called the *frontier*; the upper half is called the *efficient frontier*.

This is nicely visualized by measuring the historical frontiers for stocks and bonds over a century, for various holding periods. In each case, we see that 100% bonds is in the bottom half of the hyperboloid – the silly region that one should never consider. It is also worth emphasizing that these are based on *historical* returns and variances, which should never be confused with predictive or *ex ante* versions of same.

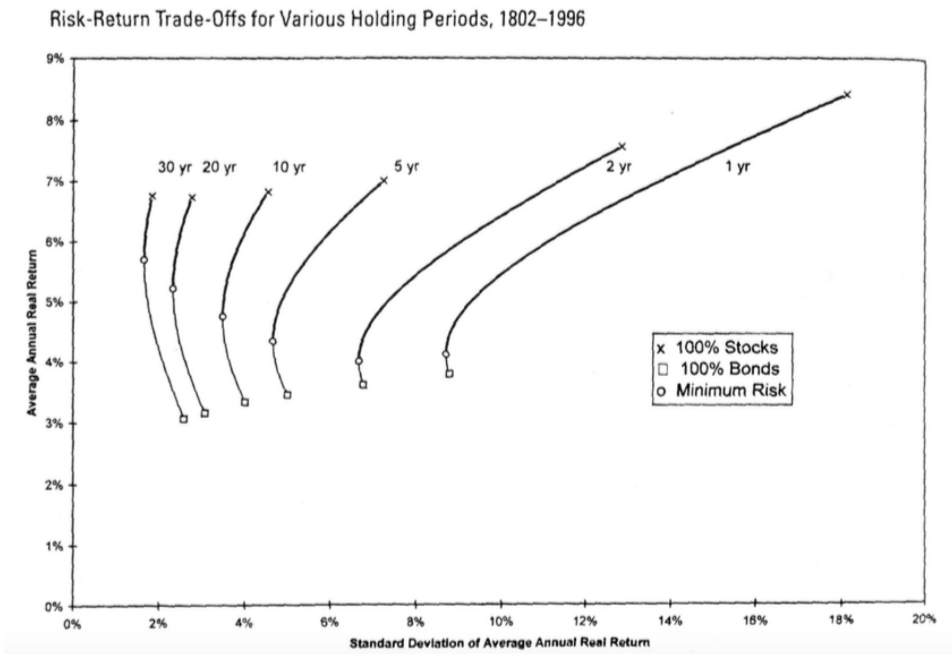

FIGURE 5.3.

In some sense, the slightly weird-looking shape of the hyperbolae in Fig. 5.3 is closely-related to the constraint $\sum_i x_i = 1$. This means, intuitively, that all of your money must be invested in one of the menu of risky assets; you cannot opt out, even with part of your account. This leads to various (perhaps counterintuitive) properties such as the fact that there is an absolute minimum variance, and you cannot achieve lower variance no matter what you do in this model.

The constraint $\sum_i x_i = 1$ is also responsible for the fact that if you insisted upon lower return than the minimum variance portfolio's return, you would have to accept more risk. After all, the capital has to go somewhere, and if it isn't going to the minimum variance weights, where's it going to go? You guessed it: something else, which must then display more than the minimum variance.

Clearly, different efficient portfolios have different Sharpe ratios. If this seems odd, it's also essentially due to the constraint $\sum_i x_i = 1$. That constraint basically means that in order to get risk that is very different from the minimum risk $\overline{\sigma}$ while still remaining fully invested, you might have to construct rather strange portfolios which have suboptimal risk-return tradeoffs.

The Sharpe ratio is

$$S = \frac{E - r_f}{\sigma}$$

where r_f is the risk-free rate. In a world with no riskless asset at all, I suppose $r_f = 0$, in which case the Sharpe ratio is simply E/σ. In the coordinates, (σ, E) the Sharpe ratio at a point on the efficient frontier is the slope of a line from the origin to that point. One can show with a little calculus (exercise) that the maximum such slope is realized when the line from the origin is tangent to the hyperbola.

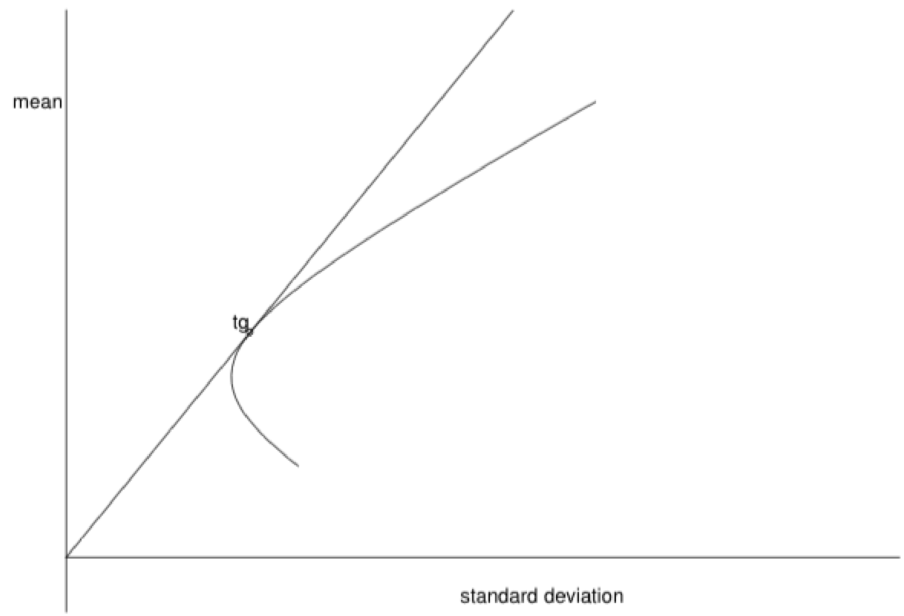

FIGURE 5.4. The maximum Sharpe ratio is when the line from the origin is tangent to the hyperbola.

I'd like to re-emphasize that if you apply the above using historical returns the tangency portfolio is the one that *was* the maximum Sharpe ratio *in the past*, and really all statements about return and risk should be suitably transformed into the past tense. The "minimum variance portfolio" is the one that was the minimum variance in the past, etc. This should not be confused with a portfolio that is forecasted to be the minimum variance portfolio in the future.

5.4. **Removing the fully-invested constraint.** Continue from the previous section considering a universe of n risky assets, with no risk-free asset at all. It makes sense to ask whether any of the foregoing technology could possibly apply to dollar-neutral, self-financing portfolios. Recall that a portfolio is said to be *self-financing* if $\sum_i x_i = 0$, which is achieved in equity markets when You borrow stock worth, say, 100 million dollars, sell it in the marketplace for precisely that amount, and then use the proceeds to purchase 100 million dollars' worth of other stocks. Such a portfolio is said to be "self-financing," but in fact, it requires stock borrow, which is a form of financing!

Without the constraint $\sum_i x_i = 1$, the original goal of finding the minimum risk for a desired level of expected return still makes sense:

$$\min \sigma^2(\mathbf{x}) \text{ s.t. } \mathbf{x} \cdot \mathbf{E} = E$$

Applying the method of Lagrange multipliers, this is equivalent to

$$\min \left\{ \frac{1}{2} \kappa \sigma^2(\mathbf{x}) - \mathbf{x} \cdot \mathbf{E} \right\} \tag{5.8}$$

The solution to this is $\kappa^{-1} \Omega^{-1} \mathbf{E}$, which has already been derived using the Arrow-Pratt theory of decision-making under uncertainty.

Indeed, as long as the asset returns follow an elliptical distribution, then for any concave increasing utility function, and for any initial wealth, there exists some $\kappa > 0$ such that the portfolio that optimizes expected utility of wealth is the same as the one optimizing (5.8).

Hence, as soon as we remove the constraint $\sum_i x_i = 1$, there is only one efficient portfolio of risky assets, up to scaling. The risk-aversion constant κ controls the size of the portfolio, trivially, because it scales with $1/\kappa$. All (unconstrained) efficient portfolios are proportional to $\Omega^{-1}\mathbf{E}$. An overall scaling does not change the Sharpe ratio, so all efficient portfolios have the same Sharpe ratio, and no other portfolio that can be constructed from the same set of assets has a higher Sharpe ratio.

If trading costs are included, then we are once again in a scenario where the Sharpe ratio depends on κ. As we have shown from utility theory, with trading costs, (5.8) becomes

$$\max \left\{ \mathbf{x} \cdot \mathbf{E} - \frac{1}{2} \kappa \sigma^2(\mathbf{x}) - \text{cost}(\mathbf{x}_0, \mathbf{x}) \right\} \tag{5.9}$$

where $\mathbf{x}_0$ is the initial portfolio that we are trading out of.

Now it is still true that κ controls the size of the portfolio, and it is still true that smaller κ means larger portfolios, but the relationship is no longer linear. Larger portfolios trade more in absolute dollar terms, and hence more of their pre-cost Sharpe ratio is eroded in the form of trading costs, with spread cost and market impact being some of the largest costs for large institutional investors.

Hence in the presence of trading costs, smaller κ translates into lower Sharpe ratio, and the maximum Sharpe ratio that could be achieved by any portfolio is the Sharpe ratio of the unconstrained Markowitz portfolio, but in reality that upper bound is never achieved. It should be noted that by "Markowitz portfolio" we are not implying that the mean and covariance estimates must be sample means or sample covariance. Rather, they should be your best and most reasonable *ex* *ante* (forward-looking) forecasts. The sample estimates for mean return and return covariance are typically not very good forecasts.

Problem 5.1. The data set accompanying this homework gives daily returns for three stocks: TSLA, AAPL, and IBM.

(a) Calculate the historical (regressed, no intercept) beta, for each of these assets as of Dec 31, 2014. In each case, calculate the appropriate t-statistic on the coefficient to test the null hypothesis $\beta = 0$ and state whether you reject the null hypothesis.
(b) Compute the holdings vector $h \in \mathbb{R}^3$ for the unique portfolio which is dollar-neutral (ie. self-financing) and which has unit exposure to AAPL and zero exposure to beta as of Dec 31, 2014.
(c) Compute the daily returns of the portfolio from (b) over the period Jan 1, 2015 to Dec 31, 2015. Assume that each day, the portfolio is rebalanced back to the initial holdings vector $h \in \mathbb{R}^3$. Plot the cumulative sum of the log returns.
(d) Compute the realized correlation of the returns in part (c) to the market's return. Construct a statistical test of the null hypothesis that the correlation is zero. Is the realized correlation significantly different from zero at the 95% level?

Problem 5.2. Suppose you are a fund-of-funds manager with investments in n different hedge funds for some $n \geq 2$. Let r_i denote the annualized return of the i-th fund. Suppose that

$$r_i = \beta r_M + \epsilon_i, \quad \mathrm{var}(\epsilon_i) = \sigma_i^2$$

where r_M denotes the return of the market portfolio (approximated by the S&P 500 in the US) with variance σ_M^2. Suppose that ϵ_i and ϵ_j are independent random variables if $i \neq j$, and that ϵ_i is independent from r_M for all $i = 1, \ldots, n$. Suppose that your fund-of-funds has invested $h_i > 0$ dollars in the i-th hedge fund, so their profit/loss is

$$\pi = h'r = \sum_i h_i r_i.$$

Throughout the following, assume $h = (1/n, 1/n, \ldots, 1/n) \in \mathbb{R}^n$ for simplicity, ie. the fund-of-funds has one unit of capital evenly distributed across its constituents.

(a) Calculate $\mathbb{E}[h'r]$ and $\mathbb{V}[h'r]$. Note that $\mathbb{V}[h'r]$ can be expressed as $\mathbb{V}(h'r) = f(\beta, \sigma_M^2) + g(\sigma_1^2, \ldots, \sigma_n^2)$; find functions $f()$ and $g()$ explicitly.
(b) Take $\beta = 0.5$ and $\sigma_M = 0.2$. Assume that each constituent fund has an annualized volatility target of 10% and all $\sigma_i \approx 0.03$. The "fraction of variance explained by the market" for the fund-of-funds is defined to be

$f/(f+g)$. Numerically compute and plot this fraction as a function of n for $n = 2 \ldots 30$.

(c) Take the same assumptions as (b). Further assume that each ϵ_i has a Sharpe ratio of 1.5, so that $\mathbb{E}[\epsilon_i] = 1.5 \cdot \sigma_i$, and the market's expected annual return is $\mathbb{E}[r_M] = 0.07$. The fund-of-funds charges a fee of 0.01 on capital. Numerically compute and plot the Sharpe ratio, $\mathbb{E}[h'r - 0.01]/\sqrt{\mathbb{V}[h'r]}$ as a function of n for $n = 2 \ldots 30$. How does this change if the Sharpe ratio of ϵ_i is 2.0 rather than 1.5?

(d) If the fund-of-funds could simply invest in a single fund with the same properties as the others except that this fund has $\beta = 0$ and $\sigma_i = 0.1$, would that be better or worse, in terms of Sharpe ratio, than the above scenario?

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