Discrete-Time Markov Chains
21-344 Final Project

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Markov Chains Overview

- Stochastic Process: $\{X_t : t \in T\}$, state space S
- First-Order MC:

$$P(X_{n+1} = j | X_0 = i_0, X_1 = i_1, \cdots, X_n = i_n) = P(X_{n+1} = j | X_n = i_n)$$

future past current future current

- Transition Matrix P
- $P_{ij} = P(X_{n+1} = j | X_n = i)$
- $P_{ij} \geq 0$
- $\sum_{j \in S} P_{ij} = 1$

Application: Pittsburgh Weather

- Weather Underground, May 2019–May 2021
  - Mostly Sunny
  - Partly Cloudy
  - Mostly Cloudy
  - Cloudy
  - Foggy
  - Scattered Showers
  - Thunderstorms
  - Snow

Application: Pittsburgh Weather

$$P = \begin{pmatrix}
0.3204 & 0.1553 & 0.1748 & 0.3301 & 0 & 0 & 0.0097 & 0.0097 \\
0.1831 & 0.1408 & 0.2113 & 0.3521 & 0.0141 & 0.0986 & 0 & 0 \\
0.1513 & 0.1513 & 0.2773 & 0.3277 & 0 & 0.0504 & 0.0252 & 0.0168 \\
\boxed{0.0714} & 0.0652 & 0.1273 & 0.5373 & 0.0155 & 0.1056 & 0 & 0.0776 \\
0.25 & 0 & 0.25 & 0.375 & 0.125 & 0 & 0 & 0 \\
0.1481 & 0.0741 & 0.0926 & 0.4444 & 0.0185 & 0.1481 & 0 & 0.0741 \\
0.5 & 0 & 0.25 & 0.25 & 0 & 0 & 0 & 0 \\
0.0816 & 0.0408 & 0.0816 & 0.449 & 0 & 0 & 0 & 0.3469
\end{pmatrix}$$

N-Step Transition Probabilities

- $P_{ij}^n = P(X_{n+k} = j | X_k = i)$
- Chapman-Kolmogorov Equations: $P_{ij}^{n+m} = \sum_{k \in S} P_{kj}^m P_{ik}^n$
- Let $P^{(n)}$ be the matrix of P_{ij}^n,

$$P^{(m+n)} = P^{(m)} \cdot P^{(n)}$$

$$P^{(n)} = P^n$$

Application: Pittsburgh Weather

$$P^7 = \begin{pmatrix} 0.1415 & 0.0974 & 0.1632 & 0.4393 & 0.011 & 0.0754 & 0.0055 & 0.0667 \\ 0.1414 & 0.0974 & 0.1631 & 0.4394 & 0.011 & 0.0754 & 0.0055 & 0.0668 \\ 0.1414 & 0.0974 & 0.1631 & 0.4394 & 0.011 & 0.0754 & 0.0055 & 0.0668 \\ 0.1411 & 0.0972 & 0.1629 & 0.4397 & 0.011 & 0.0754 & 0.0055 & 0.0673 \\ 0.1415 & 0.0974 & 0.1632 & 0.4393 & 0.011 & 0.0754 & 0.0055 & 0.0667 \\ 0.1412 & 0.0972 & 0.1629 & 0.4396 & 0.011 & \boxed{0.0754} & 0.0055 & 0.0672 \\ 0.1416 & 0.0975 & 0.1632 & 0.4392 & 0.0109 & 0.0754 & 0.0055 & .0667 \\ 0.1408 & 0.097 & 0.1627 & 0.4399 & 0.011 & 0.0754 & 0.0055 & 0.0677 \end{pmatrix}$$

Limiting Distribution

- Estimates the long-term behavior of the Markov Chain
- $\pi = \begin{pmatrix} \pi_0 & \pi_1 & \cdots & \pi_s \end{pmatrix}$

  $\pi_j = \lim_{n \to \infty} P(X_n = j | X_0 = i)$
- Independent of initial state
- Not all MCs have a limiting distribution!
- To calculate:
  - See if there are identical rows in P^∞
  - Solve $\pi = \pi P$

Application: Pittsburgh Weather

$$\pi = \begin{pmatrix} 0.14124 & 0.09728 & 0.16297 & 0.43957 & 0.01096 & 0.07539 & 0.00548 & 0.0671 \end{pmatrix}$$

Thank You!