Reflection (DSkernel and CPS)

Juko Yamamoto

2022年2月25日

Definition 1. *Maps $\star$ and $\#$ form a Galois Connection from $\lambda^{\mathcal{S}'}_{c**}$ to λ_{cps} whenever*

$$M \twoheadrightarrow_{\lambda^{\mathcal{S}'}_c} N^{\#} \ if\ and\ only\ if\ M^{\star} \twoheadrightarrow_{\lambda_{cps}} N$$

Proposition 2. *Maps $\star$ and $\#$ form a Galois Connection from $\lambda^{\mathcal{S}'}_{c}$ to λ_{cps} if and only if the following four condition hold.*

$$\begin{array}{ll} (1) & M \twoheadrightarrow M^{\star\#} \\ (2) & N^{\#\star} \twoheadrightarrow N \\ (3) & M \twoheadrightarrow M' \ implies\ M^{\star} \twoheadrightarrow M'^{\star} \\ (4) & N \twoheadrightarrow N' \ implies\ N^{\#} \twoheadrightarrow N'^{\#} \end{array}$$

Proposition 2 is an alternative characterization of Definition 1.

DSkterm

The kernel computational calculus, $\lambda^{\mathcal{S}'}_{c**}$

$$\begin{array}{llll} terms & L_\Delta, M_\Delta, N_\Delta & ::= & K_\Delta[V] \mid K_\Delta[P] \\ values & V, W & ::= & n \mid x \mid \lambda x.\, \mathcal{S}(\lambda k.\, \langle M_k \rangle) \mid \mathcal{S} \\ nonvalues & P, Q & ::= & VW \mid \langle M_\bullet \rangle \\ contexts & K_\Delta & ::= & {}_{(\Delta=k)}k\,\texttt{[ ]} \mid {}_{(\Delta=\bullet)}\texttt{[ ]} \mid \mathsf{let}\ x = \texttt{[ ]}\ \mathsf{in}\ M_\Delta \end{array}$$

$$\begin{array}{llll} (\beta.v) & K_\Delta[(\lambda x.\, \mathcal{S}(\lambda k.\, \langle M_k \rangle))V] & \longrightarrow & M_k[x := V][k := K_\Delta] & \\ (\eta.v) & \lambda x.\, \mathcal{S}(\lambda k.\, \langle k(Vx) \rangle) & \longrightarrow & V & if\ x \notin fv(V) \\ (\beta.let) & \mathsf{let}\ x = V\ \mathsf{in}\ M_\Delta & \longrightarrow & M_\Delta[x := V] & \\ (\eta.let) & \mathsf{let}\ x = \texttt{[ ]}\ \mathsf{in}\ K_\Delta[x] & \longrightarrow & K_\Delta & if\ x \notin fv(K_\Delta) \\ (\beta.\mathcal{S}) & \langle J_\bullet[\mathcal{S}W] \rangle & \longrightarrow & \langle W(\lambda y.\, \mathcal{S}(\lambda k.\, \langle k \langle J_\bullet[y] \rangle \rangle)) \rangle & \\ (\beta.R) & \langle V \rangle & \longrightarrow & V & \end{array}$$

CPSterm

Continuation-passing style calculus, λ_{cps}

$$\begin{array}{llll} terms & L_\Delta, M_\Delta, N_\Delta & ::= & K_\Delta V \mid VWK_\Delta \mid K_\Delta M_\bullet \\ values & V, W & ::= & n \mid x \mid \lambda xk.\, M_k \mid S \\ shift & S & ::= & \lambda wj.\, w(\lambda yk.\, k(jy))(\lambda x.\, x) \\ continuations & K_\Delta & ::= & {}_{(\Delta=k)}k \mid {}_{(\Delta=\bullet)}\lambda x.\, x \mid \lambda x.\, M_\Delta \end{array}$$

$$\begin{array}{lllll} (\beta.v) & (\lambda xk'.\, M)VK_\Delta & \longrightarrow & M[x := V][k' := K_\Delta] & \\ (\eta.v) & \lambda xk.\, Vxk & \longrightarrow & V & \\ (\beta.let) & (\lambda x.\, M_\Delta)V & \longrightarrow & M[x := V] & \\ (\eta.let) & \lambda x.\, K_\Delta x & \longrightarrow & K_\Delta & if\ x \notin fv(K) \\ (\beta.\mathcal{S}) & SWJ_\bullet & \longrightarrow & W(\lambda yk.\, k(J_\bullet y))(\lambda x.\, x) & \\ (\beta.\mathcal{R}) & (\lambda x.\, x)V & \longrightarrow & V & \end{array}$$

Ismorphism

$\star : \lambda_{c**}^{\mathcal{S}'} \to \lambda_{cps}$

$$
\begin{array}{lcl}
x^{\dagger} & = & x \\
(\lambda x.\, \mathcal{S}(\lambda k.\, \langle M \rangle))^{\dagger} & = & \lambda xk.\, M_k^{\circ} \\
\mathcal{S}^{\dagger} & = & \lambda wj.\, w(\lambda yk.\, k(jy))(\lambda x.\, x) \\
\\
(k\texttt{[ ]})_k^{\ddagger} & = & k \\
\texttt{[ ]}_{\bullet}^{\ddagger} & = & \lambda x.\, x \\
(\mathsf{let}\ x = \texttt{[ ]}\ \mathsf{in}\ N)_{\Delta}^{\ddagger} & = & \lambda x.\, N_{\Delta}^{\circ} \\
\\
(K[V])_{\Delta}^{\circ} & = & K_{\Delta}^{\ddagger} V^{\dagger} \\
(K[VW])_{\Delta}^{\circ} & = & V^{\dagger} W^{\dagger} K_{\Delta}^{\ddagger} \\
(K[\langle M \rangle])_{\Delta}^{\circ} & = & K_{\Delta}^{\ddagger} M_{\bullet}^{\circ}
\end{array}
$$

DSTranslation

$\# : \lambda_{cps} \to \lambda_{c}^{\mathcal{S}'}$

$$
\begin{array}{lcl}
x^{\natural} & = & x \\
(\lambda xk.\, M_k)^{\natural} & = & \lambda x.\, \mathcal{S}(\lambda k.\, \langle M_k^{\sharp} \rangle) \\
(\lambda wj.\, w(\lambda yk.\, k(jy))(\lambda x.\, x))^{\natural} & = & \mathcal{S} \\
\\
k^{\flat} & = & k\texttt{[ ]} \\
(\lambda x.\, x)^{\flat} & = & \texttt{[ ]} \\
(\lambda x.\, N_{\Delta})^{\flat} & = & \mathsf{let}\ x = \texttt{[ ]}\ \mathsf{in}\ N_{\Delta}^{\sharp} \\
\\
(K_{\Delta} V)^{\sharp} & = & K_{\Delta}^{\flat}[V^{\natural}] \\
(VWK_{\Delta})^{\sharp} & = & K_{\Delta}^{\flat}[V^{\natural} W^{\natural}] \\
(K_{\Delta} M_{\bullet})^{\sharp} & = & K_{\Delta}^{\flat}[\langle M_{\bullet}{}^{\sharp} \rangle]
\end{array}
$$

Theorem 3 (Left near inverse of $\star$, $\dagger$, $\ddagger$ and $\circ$)**.** *The following implication holds.*

1. $M \equiv M^{\circ\sharp}$
2. $V \equiv V^{\dagger\natural}$
3. $K \equiv K^{\ddagger\flat}$

Proof. By mutual structural induction on R, M, V, and K.

1. $M \equiv M^{\circ\sharp}$

Case $M = K[V]$:

$$\begin{array}{rcl} K[V] & \equiv_{2,3} & K^{\ddagger\flat}[V^{\dagger\natural}] \\ & \equiv & (K^{\ddagger}V^{\dagger})^{\sharp} \\ & \equiv & (K[V])^{\circ\sharp} \end{array}$$

Case $M = K[VW]$:

$$\begin{array}{rcl} K[VW] & \equiv_{2,3} & K^{\ddagger\flat}[V^{\dagger\natural}W^{\dagger\natural}] \\ & \equiv & (V^{\dagger}W^{\dagger}K^{\ddagger})^{\sharp} \\ & \equiv & (K[VW])^{\circ\sharp} \end{array}$$

Case $M = K[\langle M\rangle]$:

$$\begin{array}{rcl} K[\langle M\rangle] & \equiv_{1,3} & K^{\ddagger\flat}[\langle M^{\circ\sharp}\rangle] \\ & \equiv & K^{\ddagger}M^{\circ\sharp} \\ & \equiv & (K[\langle M\rangle])^{\circ\sharp} \end{array}$$

2. $V \twoheadrightarrow V^{\dagger\natural}$

Case $V = x$:

$$x \equiv x^{\dagger\natural}$$

Case $V = \lambda x.\, \mathcal{S}(\lambda k.\, \langle M\rangle)$:

$$\begin{array}{rcl} \lambda x.\, \mathcal{S}(\lambda k.\, \langle M\rangle) & \equiv_1 & \lambda x.\, \mathcal{S}(\lambda k.\, \langle M^{\circ\sharp}\rangle) \\ & \equiv & \lambda x.\, (\lambda k.\, M^{\circ})^{\#} \\ & \equiv & (\lambda x.\, \mathcal{S}(\lambda k.\, \langle M\rangle))^{\dagger\natural} \end{array}$$

Case $V = \mathcal{S}$:

$$\mathcal{S} \equiv \mathcal{S}^{\dagger\natural}$$

3. $K \twoheadrightarrow K^{\ddagger\flat}$

Case $K = k\mathtt{[\ ]}$:

$$k\mathtt{[\ ]} \equiv (k\mathtt{[\ ]})^{\ddagger\flat}$$

Case $K = \mathtt{[\ ]}$:

$$\mathtt{[\ ]} \equiv \mathtt{[\ ]}^{\ddagger\flat}$$

Case $K = (\textsf{let } x = \mathtt{[\ ]} \textsf{ in } N)$:

$$\begin{array}{rcl} \mathsf{let}\ x = \mathtt{[\ ]}\ \mathsf{in}\ N & \equiv_1 & \mathsf{let}\ x = \mathtt{[\ ]}\ \mathsf{in}\ N^{\circ\sharp} \\ & \equiv & (\lambda x.\, N^{\circ})^{\flat} \\ & \equiv & (\mathsf{let}\ x = \mathtt{[\ ]}\ \mathsf{in}\ N)^{\ddagger\flat} \end{array}$$

□

Theorem 4 (Right inverse of $\star$)**.** *The following implication holds.*

1. $M_\Delta \equiv M_\Delta^{\sharp\circ}$
2. $V \equiv V^{\natural\dagger}$
3. $K_\Delta \equiv K_\Delta^{\flat\ddagger}$

Proof. By mutual structural induction on R, M_Δ, V, and K_Δ.

1. $M_\Delta \equiv M_\Delta^{\sharp\circ}$

Case $M_\Delta = K_\Delta V$:

$$\begin{array}{rcl} (K_\Delta V)^{\sharp\circ} & \equiv & (K_\Delta^{\flat}[V^{\natural}])^{\circ} \\ & \equiv & K_\Delta^{\flat\ddagger}V^{\natural\dagger} \\ & \equiv_{2,3} & K_\Delta V \end{array}$$

Case $M_\Delta = VWK_\Delta$:

$$\begin{array}{rcl} (VWK_\Delta)^{\sharp\circ} & \equiv & (K_\Delta^{\flat}[V^{\natural}W^{\natural}])^{\circ} \\ & \equiv & V^{\natural\dagger}W^{\natural\dagger}K_\Delta^{\flat\ddagger} \\ & \equiv_{2,3} & VWK_\Delta \end{array}$$

Case $M_\Delta = K_\Delta M_\bullet$:

$$\begin{array}{rcl} (K_\Delta M_\bullet)^{\sharp\circ} & \equiv & (K_\Delta^{\flat}[\langle M_\bullet^{\circ}\rangle])^{\circ} \\ & \equiv & K_\Delta^{\flat\ddagger}M_\bullet^{\sharp\circ} \\ & \equiv_{1,3} & K_\Delta M_\bullet \end{array}$$

2. $V \equiv V^{\natural\dagger}$

Case $V = x$:

$$x^{\natural\dagger} \equiv x$$

Case $V = S$:

$$S^{\natural\dagger} \equiv S$$

Case $V = \lambda xk.\, M_k$:

$$\begin{array}{rcl} (\lambda xk.\, M_k)^{\natural\dagger} & \equiv & \lambda xk.\, M_k^{\sharp\circ} \\ & \equiv_1 & \lambda xk.\, M_k \end{array}$$

3. $K_\Delta \equiv K_\Delta^{\flat\ddagger}$

Case $K = k$:

$$k^{\flat\ddagger} \equiv k$$

Case $K = \lambda x.\, x$:

$$(\lambda x.\, x)^{\flat\ddagger} \equiv \lambda x.\, x$$

Case $K = \lambda x.\, N_\Delta$:

$$\begin{array}{rcl} (\lambda x.\, N_\Delta)^{\flat\ddagger} & \equiv & (\mathsf{let}\ x = \mathtt{[\ ]}\ \mathsf{in}\ N_\Delta^{\sharp})^{\ddagger} \\ & \equiv & \lambda x.\, N_\Delta^{\sharp\circ} \\ & \equiv_2 & \lambda x.\, N_\Delta \end{array}$$

□

Lemma 5 (Substitution lemma)**.** *The following implications hold:*

1. ${V_1}^\dagger[x := V^\dagger] = (V_1[x := V])^\dagger$
2. ${K_1}^\ddagger[x := V^\dagger] = (K_1[x := V])^\ddagger$
3. $M^\circ[x := V^\dagger] = (M[x := V])^\circ$
4. ${K_1}^\ddagger[x := V^\dagger][k' := K^\ddagger] = (K_1[x := V][k' := K])^\ddagger$
5. $M^\circ[x := V^\dagger][k' := K^\ddagger] = (M[x := V][k' := K])^\circ$

Proof.

1. ${V_1}^\dagger[x := V^\dagger] = (V_1[x := V])^\dagger$

Case $V_1 = x$:

$$\begin{array}{rcl} x^\dagger[x := V^\dagger] &=& x[x := V^\dagger] \\ &=& V^\dagger \\ &=& (x[x := V])^\dagger \end{array}$$

Case $V_1 = y$:

$$\begin{array}{rcl} y^\dagger[x := V^\dagger] &=& y[x := V^\dagger] \\ &=& y \\ &=& (y[x := V])^\dagger \end{array}$$

Case $V_1 = \mathcal{S}$:

$$\begin{array}{rcl} \mathcal{S}^\dagger[x := V^\dagger] &=& S[x := V^\dagger] \\ &=& S \\ &=& (\mathcal{S}[x := V])^\dagger \end{array}$$

Case $V_1 = \lambda x.\, \mathcal{S}(\lambda k.\, \langle M\rangle)$:

$$\begin{array}{rl} & (\lambda x.\, \mathcal{S}(\lambda k.\, \langle M\rangle))^\dagger[x := V^\dagger] \\ = & (\lambda xk.\, M^\circ_k)[x := V^\dagger] \\ = & \lambda xk.\, M^\circ_k[x := V^\dagger] \\ =_3 & \lambda xk.\, (M[x := V])^\circ_k \\ = & ((\lambda x.\, \mathcal{S}(\lambda k.\, \langle M\rangle))[x := V])^\dagger \end{array}$$

2. ${K_1}^\ddagger[x := V^\dagger] = (K_1[x := V])^\ddagger$

Case $K_1 = k\texttt{[ ]}$:

$$\begin{array}{rcl} (k\texttt{[ ]})^\ddagger[x := V^\dagger] &=& k[x := V^\dagger] \\ &=& k \\ &=& ((k\texttt{[ ]})[x := V])^\ddagger \end{array}$$

Case $K_1 = \texttt{[ ]}$:

$$\begin{array}{rcl} \texttt{[ ]}^\ddagger[x := V^\dagger] &=& (\lambda x.\, x)[x := V^\dagger] \\ &=& \lambda x.\, x \\ &=& (\texttt{[ ]}[x := V])^\ddagger \end{array}$$

Case $K_1 = (\textsf{let } x = \texttt{[ ]} \textsf{ in } N)$:

$$\begin{array}{rl} & (\textsf{let } x = \texttt{[ ]} \textsf{ in } N)^\ddagger[x := V^\dagger] \\ = & (\lambda x.\, N^\circ)[x := V^\dagger] \\ = & \lambda x.\, N^\circ[x := V^\dagger] \\ =_3 & \lambda x.\, (N[x := V])^\circ \\ = & ((\textsf{let } x = \texttt{[ ]} \textsf{ in } N)[x := V])^\ddagger \end{array}$$

3. $M^\circ[x := V^\dagger] = (M[x := V])^\circ$

Case $M = K_1[V_1]$:

$$\begin{array}{rl} & (K_1[V_1])^\circ[x := V^\dagger] \\ = & ({K_1}^\ddagger {V_1}^\dagger)[x := V^\dagger] \\ = & ({K_1}^\ddagger[x := V^\dagger])({V_1}^\dagger[x := V^\dagger]) \\ =_{1,2} & (K_1[x := V])^\ddagger(V_1[x := V])^\dagger \\ = & ((K_1[V_1])[x := V])^\circ \end{array}$$

Case $M = K_1[V_1 W_1]$:

$$\begin{array}{rl} & (K_1[V_1W_1])^\circ[x := V^\dagger] \\ = & ({V_1}^\dagger {W_1}^\dagger {K_1}^\ddagger)[x := V^\dagger] \\ = & ({V_1}^\dagger[x := V^\dagger])({W_1}^\dagger[x := V^\dagger])({K_1}^\ddagger[x := V^\dagger]) \\ =_{1,2} & (V_1[x := V])^\dagger(W_1[x := V])^\dagger(K_1[x := V])^\ddagger \\ = & ((K_1[V_1W_1])[x := V])^\circ \end{array}$$

Case $M = K_1[\langle M_1\rangle]$:

$$\begin{array}{rl} & (K_1[\langle M_1\rangle])^\circ[x := V^\dagger] \\ = & ({K_1}^\ddagger {M_1}^\circ)[x := V^\dagger] \\ = & ({K_1}^\ddagger[x := V^\dagger])({M_1}^\circ[x := V^\dagger]) \\ =_{1,3} & (K_1[x := V])^\ddagger(M_1[x := V])^\circ \\ = & ((K_1[\langle M_1\rangle])[x := V])^\circ \end{array}$$

4. ${K_1}^\ddagger[x := V^\dagger][k' := K^\ddagger] = (K_1[x := V][k' := K])^\ddagger$

Case $K_1 = k\texttt{[ ]}$:

$$\begin{array}{rl} & (k\texttt{[ ]})^\ddagger[x := V^\dagger][k' := K^\ddagger] \\ = & k[x := V^\dagger][k' := K^\ddagger] \\ = & K^\ddagger \\ = & ((k\texttt{[ ]})[x := V][k' := K])^\ddagger \end{array}$$

Case $K_1 = (\textsf{let } x = \texttt{[ ]} \textsf{ in } N)$:

$$\begin{array}{rl} & (\textsf{let } x = \texttt{[ ]} \textsf{ in } N_\Delta)^\ddagger[x := V^\dagger][k' := K^\ddagger] \\ = & (\lambda x.\, N^\circ_\Delta)[x := V^\dagger][k' := K^\ddagger] \\ = & \lambda x.\, N^\circ_\Delta[x := V^\dagger][k' := K^\ddagger] \\ =_5 & \lambda x.\, (N_\Delta[x := V][k' := K])^\circ \\ = & ((\textsf{let } x = \texttt{[ ]} \textsf{ in } N_\Delta)[x := V][k' := K])^\ddagger \end{array}$$

Case $K_1 = \texttt{[ ]}$:

$$
\begin{array}{ll}
 & \texttt{[ ]}^{\ddagger}[x := V^{\dagger}][k' := K^{\ddagger}] \\
= & (\lambda x.\, x)[x := V^{\dagger}][k' := K^{\ddagger}] \\
= & \lambda x.\, x \\
= & (\texttt{[ ]}[x := V][k' := K])^{\ddagger}
\end{array}
$$

6. $M^{\circ}[x := V^{\dagger}][k' := K^{\ddagger}] = (M[x := V][k' := K])^{\circ}$

Case $M = K_1[V_1]$:

$$
\begin{array}{ll}
 & (K_1[V_1])^{\circ}[x := V^{\dagger}][k' := K^{\ddagger}] \\
= & ({K_1}^{\ddagger}{V_1}^{\dagger})[x := V^{\dagger}][k' := K^{\ddagger}] \\
= & ({K_1}^{\ddagger}[x := V^{\dagger}][k' := K^{\ddagger}])({V_1}^{\dagger}[x := V^{\dagger}]) \\
=_{1,3} & (K_1[x := V][k' := K])^{\ddagger}(V_1[x := V])^{\dagger} \\
= & ((K_1[V_1])[x := V][k' := K])^{\circ}
\end{array}
$$

Case $M = K_1[\langle M_1 \rangle]$:

$$
\begin{array}{ll}
 & (K_1[\langle M_1 \rangle])^{\circ}[x := V^{\dagger}][k' := K^{\ddagger}] \\
= & ({K_1}^{\ddagger}{M_1}^{\circ})[x := V^{\dagger}][k' := K^{\ddagger}] \\
= & ({K_1}^{\ddagger}[x := V^{\dagger}][k' := K^{\ddagger}])({M_1}^{\circ}[x := V^{\dagger}]) \\
=_{1,4} & (K_1[x := V][k' := K])^{\ddagger}(M_1[x := V])^{\circ} \\
= & ((K_1[\langle M_1 \rangle])[x := V][k' := K])^{\circ}
\end{array}
$$

Case $M = K_1[V_1 W_1]$:

$$
\begin{array}{ll}
 & (K_1[V_1 W_1])^{\circ}[x := V^{\dagger}][k' := K^{\ddagger}] \\
= & ({V_1}^{\dagger}{W_1}^{\dagger}{K_1}^{\ddagger})[x := V^{\dagger}][k' := K^{\ddagger}] \\
= & ({V_1}^{\dagger}[x := V^{\dagger}])({W_1}^{\dagger}[x := V^{\dagger}])({K_1}^{\ddagger}[x := V^{\dagger}][k' := K^{\ddagger}]) \\
=_{1,3} & (V_1[x := V])^{\dagger}(W_1[x := V])^{\dagger}(K_1[x := V][k' := K])^{\ddagger} \\
= & ((K_1[V_1 W_1])[x := V][k' := K])^{\circ}
\end{array}
$$

$\square$

Lemma 6 (Substitution lemma)**.** *The following implications hold:*

1. ${V_1}^\natural[x := V^\natural] = (V_1[x := V])^\natural$
2. ${K_1}^\flat[x := V^\natural] = (K_1[x := V])^\flat$
3. $M^\sharp[x := V^\natural] = (M[x := V])^\sharp$
4. ${K_1}^\flat[x := V^\natural][k' := K^\flat] = (K_1[x := V][k' := K])^\flat$
5. $M^\sharp[x := V^\natural][k' := K^\flat] = (M[x := V][k' := K])^\sharp$

Proof.

1\. ${V_1}^\natural[x := V^\natural] = (V_1[x := V])^\natural$

Case $V_1 = x$:

$$\begin{aligned} x^\natural[x := V^\natural] &= x[x := V^\natural] \\ &= V^\natural \\ &= (x[x := V])^\natural \end{aligned}$$

Case $V_1 = y$:

$$\begin{aligned} y^\natural[x := V^\natural] &= y[x := V^\natural] \\ &= y \\ &= (y[x := V])^\natural \end{aligned}$$

Case $V_1 = S$:

$$\begin{aligned} S^\natural[x := V^\natural] &= \mathcal{S}[x := V^\natural] \\ &= \mathcal{S} \\ &= (S[x := V])^\natural \end{aligned}$$

Case $V_1 = \lambda xk.\, M_k$:

$$\begin{aligned} (\lambda xk.\, M_k)^\natural[x := V^\natural] &= (\lambda x.\, \mathcal{S}(\lambda k.\, \langle M_k^\sharp\rangle))[x := V^\natural] \\ &= \lambda x.\, \mathcal{S}(\lambda k.\, \langle M_k^\sharp[x := V^\natural]\rangle) \\ &= \lambda x.\, \mathcal{S}(\lambda k.\, \langle (M[x := V])_k^\sharp\rangle) \\ &= ((\lambda xk.\, M_k)[x := V])^\natural \end{aligned}$$

2\. ${K_1}^\flat[x := V^\natural] = (K_1[x := V])^\flat$

Case $K_1 = k$:

$$\begin{aligned} k^\flat[x := V^\flat] &= (k\,\texttt{[ ]})[k := K^\flat] \\ &= k\,\texttt{[ ]} \\ &= (k[x := V])^\flat \end{aligned}$$

Case $K_1 = \lambda x.\, x$:

$$\begin{aligned} (\lambda x.\, x)^\flat[x := V^\natural] &= \texttt{[ ]}[x := V^\natural] \\ &= \texttt{[ ]} \\ &= ((\lambda x.\, x)[x := V])^\flat \end{aligned}$$

Case $K_1 = \lambda x.\, N$:

$$\begin{aligned} & (\lambda x.\, N)^\flat[x := V^\natural] \\ = \;& (\mathsf{let}\ x = \texttt{[ ]}\ \mathsf{in}\ N^\sharp)[x := V^\natural] \\ = \;& \mathsf{let}\ x = \texttt{[ ]}\ \mathsf{in}\ (N^\sharp[x := V^\natural]) \\ =_3 \;& \mathsf{let}\ x = \texttt{[ ]}\ \mathsf{in}\ (N[x := V])^\sharp \\ = \;& ((\lambda x.\, N)[x := V])^\flat \end{aligned}$$

3\. $M^\sharp[x := V^\natural] = (M[x := V])^\sharp$

Case $M = K_1V_1$:

$$\begin{aligned} & (K_1V_1)^\sharp[x := V^\natural] \\ = \;& ({K_1}^\flat[{V_1}^\natural])[x := V^\natural] \\ = \;& ({K_1}^\flat[x := V^\natural])[{V_1}^\natural[x := V^\natural]] \\ =_{1,2} \;& (K_1[x := V])^\flat[(V_1[x := V])^\natural] \\ = \;& ((K_1V_1)[x := V])^\sharp \end{aligned}$$

Case $M = V_1W_1K_1$:

$$\begin{aligned} & (V_1W_1K_1)^\sharp[x := V^\natural] \\ = \;& {K_1}^\flat[{V_1}^\natural {W_1}^\natural][x := V^\natural] \\ = \;& ({K_1}^\flat[x := V^\natural])[({V_1}^\natural[x := V^\natural])({W_1}^\natural[x := V^\natural])] \\ =_{1,2} \;& (K_1[x := V])^\flat[(V_1[x := V])^\natural (W_1[x := V])^\natural] \\ = \;& ((V_1W_1K_1)[x := V])^\sharp \end{aligned}$$

Case $M = K_1M_1$:

$$\begin{aligned} & (K_1M_1)^\sharp[x := V^\natural] \\ = \;& ({K_1}^\flat[\langle {M_1}^\sharp\rangle])[x := V^\natural] \\ = \;& ({K_1}^\flat[x := V^\natural])[\langle {M_1}^\sharp\rangle[x := V^\natural]] \\ =_{1,3} \;& (K_1[x := V])^\flat[\langle (M_1[x := V])^\sharp\rangle] \\ = \;& ((K_1M_1)[x := V])^\sharp \end{aligned}$$

4\. ${K_1}^\flat[x := V^\natural][k' := K^\flat] = (K_1[x := V][k' := K])^\flat$

Case $K_1 = k$:

$$\begin{aligned} & k^\flat[x := V^\natural][k' := K^\flat] \\ = \;& (k\,\texttt{[ ]})[x := V^\natural][k' := K^\flat] \\ = \;& K^\flat\,\texttt{[ ]} \\ = \;& (k[x := V][k' := K])^\flat \end{aligned}$$

Case $K_1 = \lambda x.\, N_\Delta$:

$$\begin{aligned} & (\lambda x.\, N)^\flat[x := V^\natural][k' := K^\flat] \\ = \;& (\mathsf{let}\ x = \texttt{[ ]}\ \mathsf{in}\ N^\sharp)[x := V^\natural][k' := K^\flat] \\ = \;& \mathsf{let}\ x = \texttt{[ ]}\ \mathsf{in}\ (N_\Delta^\sharp[x := V^\natural][k' := K^\flat]) \\ =_5 \;& \mathsf{let}\ x = \texttt{[ ]}\ \mathsf{in}\ (N[x := V][k' := K])_\Delta^\sharp \\ = \;& ((\lambda x.\, N_\Delta)[x := V][k' := K])^\flat \end{aligned}$$

Case $K_1 = \lambda x.\, x$:

$$\begin{array}{ll} & (\lambda x.\, x)^{\flat}[x := V^{\natural}][k' := K^{\flat}] \\ = & \texttt{[ ]}[x := V^{\natural}][k' := K^{\flat}] \\ = & \texttt{[ ]} \\ = & ((\lambda x.\, x)[x := V][k' := K])^{\flat} \end{array}$$

5. $M^{\sharp}[x := V^{\natural}][k' := K^{\flat}] = (M[x := V][k' := K])^{\sharp}$

Case $M = K_1 V_1$:

$$\begin{array}{ll} & (K_1 V_1)^{\sharp}[x := V^{\natural}][k' := K^{\flat}] \\ = & ({K_1}^{\flat}[{V_1}^{\natural}])[x := V^{\natural}][k' := K^{\flat}] \\ = & ({K_1}^{\flat}[x := V^{\natural}][k' := K^{\flat}])[{V_1}^{\natural}[x := V^{\natural}]] \\ =_{1,4} & (K_1[x := V][k' := K])^{\flat}[(V_1[x := V])^{\natural}] \\ = & ((K_1 V_1)[x := V][k' := K])^{\sharp} \end{array}$$

Case $M = K_1 M_1$:

$$\begin{array}{ll} & (K_1 M_1)^{\sharp}[x := V^{\natural}][k' := K^{\flat}] \\ = & ({K_1}^{\flat}[\langle {M_1}^{\sharp} \rangle])[x := V^{\natural}][k' := K^{\flat}] \\ = & ({K_1}^{\flat}[x := V^{\natural}][k' := K^{\flat}])[\langle {M_1}^{\sharp} \rangle[x := V^{\natural}]] \\ =_{3,4} & (K_1[x := V][k' := K])^{\flat}[\langle (M_1[x := V])^{\sharp} \rangle] \\ = & ((K_1 M_1)[x := V][k' := K])^{\sharp} \end{array}$$

Case $M = V_1 W_1 K_1$:

$$\begin{array}{ll} & (V_1 W_1 K_1)^{\sharp}[x := V^{\natural}][k' := K^{\flat}] \\ = & {K_1}^{\flat}[{V_1}^{\natural} {W_1}^{\natural}][x := V^{\natural}][k' := K^{\flat}] \\ = & ({K_1}^{\flat}[x := V^{\natural}][k' := K^{\flat}])[({V_1}^{\natural}[x := V^{\natural}])({W_1}^{\natural}[x := V^{\natural}])] \\ =_{1,4} & (K_1[x := V][k' := K])^{\flat}[(V_1[x := V])^{\natural}(W_1[x := V])^{\natural}] \\ = & ((V_1 W_1 K_1)[x := V][k' := K])^{\sharp} \end{array}$$

$\square$

Theorem 7 (Single-step reduction preservation by $\star$, $\dagger$, and $\ddagger$)**.** *The following implications hold:*

1. *if $M \to M'$ then $M^\circ \to M'^\circ$*
2. *if $\langle M \rangle \to \langle M' \rangle$ then $M^\circ \to M'^\circ$*
3. *if $\langle M \rangle \to V$ then $M^\circ \to V^\dagger$*
4. *if $V \to V'$ then $V^\dagger \to V'^\dagger$*
5. *if $K \to K'$ then $K^\ddagger \to K'^\ddagger$*

Proof. Mutual structural induction on the first term and invert the reduction relation.

1. if $M \to M'$ then $M^\circ \to M'^\circ$

Case $(\beta.v)$ **:**

$$\begin{array}{ll} & (K[(\lambda x.\, \mathcal{S}(\lambda k.\, \langle M \rangle))V])^\circ \\ = & (\lambda x.\, \mathcal{S}(\lambda k.\, \langle M_k \rangle))^\dagger V^\dagger K^\ddagger \\ = & (\lambda xk.\, M^\circ) V^\dagger K^\ddagger \\ \to_{(\beta.v)} & M^\circ[x := V^\dagger][k' := K^\ddagger] \\ =_{(Lemma 5.5)} & (M[x := V][k' := K])^\circ \end{array}$$

Case $(\beta.let)$ **:**

$$\begin{array}{ll} & (\mathsf{let}\ x = V\ \mathsf{in}\ M)^\circ \\ = & (\mathsf{let}\ x = [\ ]\ \mathsf{in}\ M)^\ddagger V^\dagger \\ = & (\lambda x.\, M^\circ) V^\dagger \\ \to_{(\beta.let)} & M^\circ[x := V^\dagger] \\ =_{(Lemma 5.3)} & (M[x := V])^\circ \end{array}$$

2. if $\langle M \rangle \to \langle M' \rangle$ then $M^\circ \to M'^\circ$

Case $(\beta.\mathcal{S})$ **:**

$$\begin{array}{ll} & (K[\mathcal{S}W])^\circ \\ = & \mathcal{S}^\dagger W^\dagger K^\ddagger \\ \to_{(\beta.\mathcal{S})} & W^\dagger(\lambda yk.\, k(K^\ddagger y))(\lambda x.\, x) \\ = & W^\dagger(\lambda y.\, \mathcal{S}(\lambda k.\, \langle k \langle K[y] \rangle \rangle))^\dagger [\ ]^\ddagger \\ = & ([\ ][W(\lambda y.\, \mathcal{S}(\lambda k.\, \langle k \langle K[y] \rangle \rangle))])^\circ \\ = & (W(\lambda y.\, \mathcal{S}(\lambda k.\, \langle k \langle K[y] \rangle \rangle)))^\circ \end{array}$$

3. if $\langle M \rangle \to V$ then $M^\circ \to V^\dagger$

Case $(\beta.\mathcal{R})$ **:**

$$\begin{array}{lll} V^\circ & = & ([\ ][V])^\circ \\ & = & [\ ]^\ddagger V^\dagger \\ & = & (\lambda x.\, x) V^\dagger \\ & \to_{(\beta.\mathcal{R})} & V^\dagger \end{array}$$

4. if $V \to V'$ then $V^\dagger \to V'^\dagger$

Case $(\eta.v)$ **:**

$$\begin{array}{ll} & (\lambda x.\, \mathcal{S}(\lambda k.\, \langle k(Vx) \rangle))^\dagger \\ = & \lambda xk.\, (k(Vx))^\circ \\ = & \lambda xk.\, V^\dagger x^\dagger k^\ddagger \\ = & \lambda xk.\, V^\dagger xk \\ \to_{(\eta.v)} & V^\dagger \end{array}$$

5. if $K \to K'$ then $K^\ddagger \to K'^\ddagger$

Case $(\eta.let)$ **:**

$$\begin{array}{ll} & (\mathsf{let}\ x = [\ ]\ \mathsf{in}\ K[x])^\ddagger \\ = & \lambda x.\, (K[x])^\circ \\ = & \lambda x.\, K^\ddagger x \\ \to_{(\eta.let)} & K^\ddagger \end{array}$$

$\square$

Theorem 8 (Single-step preservation by $\#$, $\sharp$, $\natural$, and $\flat$)**.** *The following implications hold:*

1. *if* $M_k \to M'_k$ *then* $M_k^\sharp \to {M'}_k^\sharp$
2. *if* $M_\bullet \to M'_\bullet$ *then* $\langle M_\bullet^\sharp \rangle \to \langle {M'}_\bullet^\sharp \rangle$
3. *if* $M \to V$ *then* $\langle M^\sharp \rangle \to V^\natural$
4. *if* $V \to V'$ *then* $V^\natural \to {V'}^\natural$
5. *if* $K \to K'$ *then* $K_\Delta^\flat \to {K'}_\Delta^\flat$

Proof. Mutual structural induction on the first term and then each case by inversion on single-step reduction.

1. if $M_k \to M'_k$ then $M_k^\sharp \to {M'}_k^\sharp$

Case $(\beta.v)$ **:**

$$
\begin{array}{ll}
 & ((\lambda xk.\, M_k) V K_\Delta)^\sharp \\
= & K_\Delta^\flat[(\lambda xk.\, M_k)^\natural V^\natural] \\
= & K_\Delta^\flat[(\lambda x.\, \mathcal{S}(\lambda k.\, \langle M_k^\sharp \rangle)) V^\natural] \\
\to_{(\beta.v)} & M_k^\sharp[x := V^\natural][k' := K^\flat] \\
=_{(Lemma 6.5)} & (M_k[x := V][k' := K])^\sharp
\end{array}
$$

Case $(\beta.let)$ **:**

$$
\begin{array}{ll}
 & ((\lambda x.\, M) V)^\sharp \\
= & (\lambda x.\, M)^\flat [V^\natural] \\
= & (\mathsf{let}\ x = \texttt{[ ]}\ \mathsf{in}\ M^\sharp)[V^\natural] \\
= & \mathsf{let}\ x = V^\natural\ \mathsf{in}\ M^\sharp \\
\to_{(\beta.let)} & M^\sharp[x := V^\natural] \\
=_{(Lemma 6.3)} & (M[x := V])^\sharp
\end{array}
$$

2. if $M_\bullet \to M'_\bullet$ then $\langle M_\bullet^\sharp \rangle \to \langle {M'}_\bullet^\sharp \rangle$

Case $(\beta.\mathcal{S})$ **:**

$$
\begin{array}{ll}
 & \langle (SW J_\bullet)^\sharp \rangle \\
= & \langle J_\bullet^\flat [S^\natural W^\natural] \rangle \\
= & \langle J_\bullet^\flat [\mathcal{S} W^\natural] \rangle \\
\to_{(\beta.\mathcal{S})} & \langle W^\natural (\lambda y.\, \mathcal{S}(\lambda k.\, \langle k \langle J_\bullet[y] \rangle \rangle)) \rangle \\
= & \langle W^\natural (\lambda y.\, \mathcal{S}(\lambda k.\, \langle k^\flat[\langle J_\bullet[y] \rangle] \rangle)) \rangle \\
= & \langle W^\natural (\lambda y.\, \mathcal{S}(\lambda k.\, \langle k^\flat[\langle (J_\bullet y)^\sharp \rangle] \rangle)) \rangle \\
= & \langle W^\natural (\lambda y.\, \mathcal{S}(\lambda k.\, \langle (k(J_\bullet y))^\sharp \rangle)) \rangle \\
= & \langle \texttt{[ ]} [W^\natural (\lambda yk.\, k(J_\bullet y))^\natural] \rangle \\
= & \langle (\lambda x.\, x)^\flat [W^\natural (\lambda yk.\, k(J_\bullet y))^\natural] \rangle \\
= & \langle (W(\lambda yk.\, k(J_\bullet y))(\lambda x.\, x))^\sharp \rangle
\end{array}
$$

3. if $M \to V$ then $\langle M^\sharp \rangle \to V^\natural$

Case $(\beta.\mathcal{R})$ **:**

$$
\begin{array}{ll}
 & \langle ((\lambda x.\, x) V)^\sharp \rangle \\
= & \langle (\lambda x.\, x) V^{\natural\flat} \rangle \\
= & \langle \texttt{[ ]} V^\natural \rangle \\
= & \langle V^\natural \rangle \\
\to_{(\beta.\mathcal{R})} & V^\natural
\end{array}
$$

4. if $V \to V'$ then $V^\natural \to {V'}^\natural$

Case $(\eta.v)$ **:**

$$
\begin{array}{ll}
 & (\lambda xk.\, V x k)^\natural \\
= & \lambda x.\, \mathcal{S}(\lambda k.\, \langle (V x k)^\sharp \rangle) \\
= & \lambda x.\, \mathcal{S}(\lambda k.\, \langle K_k^\flat [V^\natural x^\natural] \rangle) \\
= & \lambda x.\, \mathcal{S}(\lambda k.\, \langle k(V^\natural x) \rangle) \\
\to_{(\eta.v)} & V^\natural
\end{array}
$$

5. if $K \to K'$ then $K_\Delta^\flat \to {K'}_\Delta^\flat$

Case $(\eta.let)$ **:**

$$
\begin{array}{ll}
 & (\lambda x.\, K_\Delta x)^\flat \\
= & \mathsf{let}\ x = \texttt{[ ]}\ \mathsf{in}\ (K_\Delta x)^\sharp \\
= & \mathsf{let}\ x = \texttt{[ ]}\ \mathsf{in}\ K_\Delta^\flat x \\
\to_{(\eta.let)} & K_\Delta^\flat
\end{array}
$$

□