CPS変換メモ

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1 DS項の定義

ここでは、CPS 変換前の項、すなわち CPS 変換を行う対象の言語を示す。

1.1 構文

$$
\begin{array}{rcll}
\tau & ::= & \mathsf{Nat} \mid \mathsf{bool} \mid \tau_1 \to \tau_2 @\mathrm{cps}[\tau_3, \tau_4] & \text{型} \\
v & ::= & n \mid x \mid \lambda x.\, e \mid \mathcal{S} & \text{値} \\
e & ::= & v \mid e_1 @ e_2 \mid \langle e \rangle \mid \mathsf{let}\ x = e_1\ \mathsf{in}\ e_2 & \text{項}
\end{array}
$$

1.2 DS項の型付け規則

$$
\frac{\Gamma(x) = \tau_1}{\Gamma \vdash x : \tau_1 @\mathrm{cps}[\tau, \tau]}\ (\textrm{TVar}) \qquad \frac{}{\Gamma \vdash n : \mathsf{Nat} @\mathrm{cps}[\tau, \tau]}\ (\textrm{TNum})
$$

$$
\frac{\Gamma, x : \tau_2 \vdash e : \tau_1 @\mathrm{cps}[\tau_3, \tau_4]}{\Gamma \vdash \lambda x.\, e : (\tau_2 \to \tau_1 @\mathrm{cps}[\tau_3, \tau_4]) @\mathrm{cps}[\tau, \tau]}\ (\textrm{TFun})
$$

$$
\frac{}{\Gamma \vdash \mathcal{S} : (((\tau_3 \to \tau_4 @\mathrm{cps}[\tau, \tau]) \to \tau_1 @\mathrm{cps}[\tau_1, \tau_2]) \to \tau_3 @\mathrm{cps}[\tau_4, \tau_2]) @\mathrm{cps}[\tau, \tau]}\ (\textrm{TShift})
$$

$$
\frac{\Gamma \vdash e_1 : (\tau_2 \to \tau_1 @\mathrm{cps}[\tau_3, \tau_4]) @\mathrm{cps}[\tau_5, \tau_6] \quad \Gamma \vdash e_2 : \tau_2 @\mathrm{cps}[\tau_4, \tau_5]}{\Gamma \vdash e_1 @ e_2 : \tau_1 @\mathrm{cps}[\tau_3, \tau_6]}\ (\textrm{TApp})
$$

$$
\frac{\Gamma \vdash e : \tau_1 @\mathrm{cps}[\tau_1, \tau_2]}{\Gamma \vdash \langle e \rangle : \tau_2 @\mathrm{cps}[\tau_3, \tau_3]}\ (\textrm{TReset})
$$

$$
\frac{\Gamma \vdash e_1 : \tau_1 @\mathrm{cps}[\beta, \gamma] \quad \Gamma, x : \tau_1 \vdash e_2 : \tau_2 @\mathrm{cps}[\alpha, \beta]}{\Gamma \vdash (\mathsf{let}\ x = e_1\ \mathsf{in}\ e_2) : \tau_2 @\mathrm{cps}[\alpha, \gamma]}\ (\textrm{TLet})
$$

1.3　DS 項の代入規則

代入規則は、$e[v/x] = e'$ と表現することができ、「項 e の中に現れる変数 x を値 v に置き換える」と読む。

今回用いる代入規則は以下である。

$$\frac{}{x[v/x] = x}\ (\textsc{sVar=}) \qquad \frac{}{y[v/x] = y}\ (\textsc{sVar} \neq) \qquad \frac{}{n[v/x] = n}\ (\textsc{sNum})$$

$$\frac{\forall x.(e[v/y] = e')}{(\lambda x.\, e)[v/y] = \lambda x.\, e'}\ (\textsc{sFun}) \qquad \frac{e_1[v/x] = e'_1 \quad e_2[v/x] = e'_2}{(e_1 \mathbin{@} e_2)[v/x] = (e'_1 \mathbin{@} e'_2)}\ (\textsc{sApp})$$

$$\frac{}{\mathcal{S}[v/x] = \mathcal{S}}\ (\textsc{sShift}) \qquad \frac{e_1[v/x] = e'_1}{\langle e_1 \rangle[v/x] = \langle e'_1 \rangle}\ (\textsc{sReset})$$

$$\frac{e_1[v/y] = e'_1 \quad \forall x.(e_2[v/y] = e'_2)}{(\mathsf{let}\ x = e_1\ \mathsf{in}\ e_2)[v/y] = \mathsf{let}\ x = e'_1\ \mathsf{in}\ e'_2}\ (\textsc{sLet})$$

1.4　DS 項の簡約規則

次に、簡約規則について示す。ただし、簡約規則を示す前に、フレームを定義する必要がある。フレームを定義することによって、簡約の順序を決めることができる。

$$\begin{array}{rcl} \text{フレーム}\ F & = & [\ ] \mathbin{@} e_2 \mid v_1 \mathbin{@} [\ ] \mid \langle\, [\ ]\, \rangle \mid \mathsf{let}\ x = [\ ]\ \mathsf{in}\ e_2 \\ \text{評価文脈 (コンテキスト)}\ E & = & [\ ] \mid F \circ E \end{array}$$

$$\frac{\Gamma \vdash e_2 : \tau_2 \mathbin{@}\mathrm{cps}[\tau_4, \tau_6]}{\Gamma \vdash ([\ ] \mathbin{@} e_2) : [\ (\tau_2 \to \tau_1 @\mathrm{cps}[\tau_3, \tau_4]) @\mathrm{cps}[\tau_5, \tau_6]\ ]_{\mathsf{f}}\ \tau_1 @\mathrm{cps}[\tau_3, \tau_6]}\ (\text{F-}\textsc{App}_1)$$

$$\frac{\Gamma \vdash v_1 : (\tau_2 \to \tau_1 @\mathrm{cps}[\tau_3, \tau_4]) \mathbin{@}\mathrm{cps}[\tau_5, \tau_5]}{\Gamma \vdash (v_1 \mathbin{@} [\ ]) : [\ \tau_2 @\mathrm{cps}[\tau_4, \tau_5]\ ]_{\mathsf{f}}\ \tau_1 @\mathrm{cps}[\tau_3, \tau_5]}\ (\text{F-}\textsc{App}_2)$$

$$\frac{}{\Gamma \vdash \langle [\ ] \rangle : [\ \tau_1 @\mathrm{cps}[\tau_1, \tau_2]\ ]_{\mathsf{f}}\ \tau_2 @\mathrm{cps}[\tau_3, \tau_3]}\ (\text{F-}\textsc{Reset})$$

$$\frac{\Gamma, x : \tau_1 \vdash e_2 : \tau_2 \mathbin{@}\mathrm{cps}[\alpha, \beta]}{\Gamma \vdash \mathsf{let}\ x = [\ ]\ \mathsf{in}\ e_2 : [\ \tau_1 @\mathrm{cps}[\beta, \gamma]\ ]_{\mathsf{f}}\ \tau_2 @\mathrm{cps}[\alpha, \gamma]}\ (\text{F-}\textsc{Let})$$

[　] は、「次に計算を行う部分」である。「それ以外の部分」は、「文脈」と呼ぶ。評価文脈はフレームを何枚も重ね合わせたものであり、フレームの列として表すことができる。そのように表せるとすると、評価文脈 E に式 e を入れる関数 $plug_F$ を以下のように定義できる。

$$\begin{array}{rcl} plug_F([\ ] \mathbin{@} e_2, e_1) & = & e_1 \mathbin{@} e_2 \\ plug_F(v_1 \mathbin{@} [\ ], e_2) & = & v_1 \mathbin{@} e_2 \\ plug_F(\langle [\ ] \rangle, e_1) & = & \langle e_1 \rangle \\ plug_F(\mathsf{let}\ x = [\ ]\ \mathsf{in}\ e_2, e_1) & = & \mathsf{let}\ x = e_1\ \mathsf{in}\ e_2 \end{array}$$

次に、reset を含まないフレームである、ピュアフレームを定義する。

$$\begin{array}{rcl} \text{ピュアフレーム}\quad F_p &=& [\ ]\,@\,e_2 \mid v_1\,@\,[\ ] \mid \mathsf{let}\ x = [\ ]\ \mathsf{in}\ e_2 \\ \text{ピュアコンテキスト}\quad E_p &=& [\ ] \mid F_p \circ E_p \end{array}$$

$\boxed{\text{ピュアフレーム}\quad F_p}$

$$\frac{\Gamma \vdash e_2 : \tau_2\ @\mathrm{cps}[\tau_4, \tau_5]}{\Gamma \vdash ([\ ]\,@\,e_2) : [\ (\tau_2 \to \tau_1 @\mathrm{cps}[\tau_3, \tau_4]) @\mathrm{cps}[\tau_5, \tau_6]\ ]_{\mathsf{pf}}\ \tau_1 @\mathrm{cps}[\tau_3, \tau_6]}\ (\mathrm{F}_p\text{-}\mathrm{APP}_1)$$

$$\frac{\Gamma \vdash v_1 : (\tau_2 \to \tau_1 @\mathrm{cps}[\tau_3, \tau_4])\ @\mathrm{cps}[\tau_5, \tau_5]}{\Gamma \vdash (v_1\,@\,[\ ]) : [\ \tau_2 @\mathrm{cps}[\tau_4, \tau_5]\ ]_{\mathsf{pf}}\ \tau_1 @\mathrm{cps}[\tau_3, \tau_5]}\ (\mathrm{F}_p\text{-}\mathrm{APP}_2)$$

$$\frac{\Gamma, x : \tau_1 \vdash e_2 : \tau_2\ @\mathrm{cps}[\alpha, \beta]}{\Gamma \vdash \mathsf{let}\ x = [\ ]\ \mathsf{in}\ e_2 : [\ \tau_1 @\mathrm{cps}[\beta, \gamma]\ ]_{\mathsf{pf}}\ \tau_2 @\mathrm{cps}[\alpha, \gamma]}\ [(\mathrm{F}_p\text{-}\mathrm{LET})$$

$\boxed{\text{ピュアコンテキスト}\quad E_p}$

$$\frac{}{\Gamma \vdash [\ ] : [\ \tau_1 @\mathrm{cps}[\tau_2, \tau_3]\ ]_{\mathsf{pc}}\ \tau_1 @\mathrm{cps}[\tau_2, \tau_3]}\ (\mathrm{E}_p\text{-}\mathrm{HOLE})$$

$$\frac{\Gamma \vdash F_p : [\ \tau_4 @\mathrm{cps}[\tau_5, \tau_6]\ ]_{\mathsf{pf}}\ \tau_7 @\mathrm{cps}[\tau_8, \tau_9] \quad \Gamma \vdash E_p : [\ \tau_1 @\mathrm{cps}[\tau_2, \tau_3]\ ]_{\mathsf{pf}}\ \tau_4 @\mathrm{cps}[\tau_5, \tau_6]}{\Gamma \vdash F_p \circ E_p : [\ \tau_1 @\mathrm{cps}[\tau_2, \tau_3]\ ]_{\mathsf{pc}}\ \tau_7 @\mathrm{cps}[\tau_8, \tau_9]}\ (\mathrm{E}_p\text{-}\mathrm{FRAME})$$

$\boxed{\text{ピュアフレーム同士の関係}\quad F_p \cong_{\mathsf{c}} F_p}$

$$\frac{}{([\ ]\,@\,e_2) \cong_{\mathsf{f}} ([\ ]\,@\,e_2)}\ (\cong_{\mathsf{pf}}\text{-}\mathrm{APP}_1)$$

同様に、関数 $plug_{F_p}$ を定義する。

$$\begin{array}{rcl} plug_{F_p}([\ ]\,@\,e_2, e_1) &=& e_1\,@\,e_2 \\ plug_{F_p}(v_1\,@\,[\ ], e_2) &=& v_1\,@\,e_2 \\ plug_{F_p}(\mathsf{let}\ x = [\ ]\ \mathsf{in}\ e_2, e_1) &=& \mathsf{let}\ x = e_1\ \mathsf{in}\ e_2 \end{array}$$

関数 $plug_{E_p}$ は以下のように定義できる。

$$\begin{array}{rcl} plug_{E_p}([\ ], e_1) &=& e_1 \\ plug_{E_p}(F_p \circ E_p, e_2) &=& plug_{F_p}(F_p, plug_{E_p}(E_p, e_1)) \end{array}$$

以上より、簡約規則は以下のように表せる。

$$\frac{e_1[v_2/x] = e_1'}{(\lambda x.\, e_1)\,@\,v_2 \rightsquigarrow e_1'}\ (\mathrm{RBETA}) \qquad \frac{e_1 \rightsquigarrow e_2}{plug_F(F, e_1) \rightsquigarrow plug_F(F, e_2)}\ (\mathrm{RFRAME})$$

$$\frac{E_{p_1} \cong_{\mathsf{c}} E_{p_2}}{\langle E_{p_1}[\ \mathcal{S}\,@\,v_2\ ]\rangle \rightsquigarrow \langle v_2\,@\,(\lambda y.\, \langle E_{p_2}[\ y\ ]\rangle)\rangle}\ (\mathrm{RSHIFT}) \qquad \frac{}{\langle v_1 \rangle \rightsquigarrow v_1}\ (\mathrm{RRESET})$$

$$\frac{e_2[v_1/x] = e_2'}{\mathsf{let}\ x = v_1\ \mathsf{in}\ e_2 \rightsquigarrow e_2'}\ (\mathrm{RLET})$$

2 CPS 項の定義

CPS 変換後の項を示す。

ここで、$\overline{\lambda}.$ や $\overline{@}$ のように、上付きの線が書かれているものは、static な項。また、$\underline{\lambda}.$ や $\underline{@}$ のように、下付きの線が描かれているものは、dynamic な項と呼ぶ。

2.1 CPS 項の構文

$$
\begin{array}{rcll}
\tau & ::= & \mathsf{Nat} \mid \mathsf{bool} \mid \tau_1 \to \tau_2 & \text{型} \\
v & ::= & n \mid x \mid \underline{\lambda} x.\, \underline{\lambda} k.\, e \mid \mathcal{S} & \text{値} \\
\mathcal{S} & ::= & \underline{\lambda} w.\, \underline{\lambda} k.\, (w \mathbin{\underline{@}} (\underline{\lambda} a.\, \underline{\lambda} k'.\, k' \mathbin{\underline{@}} (k \mathbin{\underline{@}} a))) \mathbin{\underline{@}} (\underline{\lambda} m.\, m) & \texttt{shift} \\
e & ::= & v \mid e_1 \mathbin{\underline{@}} e_2 \mid \underline{\mathsf{let}}\ x = e_1\ \underline{\mathsf{in}}\ e_2 & \text{項}
\end{array}
$$

2.2 CPS 項の型付け規則

$$
\frac{\Gamma(x) = \tau_1}{\Gamma \vdash x : \tau_1}\ (\textsc{TVar}_\mathsf{C}) \qquad \frac{}{\Gamma \vdash n : \mathsf{Nat}}\ (\textsc{TNum}_\mathsf{C}) \qquad \frac{\Gamma, x : \tau_2 \vdash e : \tau_1}{\Gamma \vdash \underline{\lambda} x.\, \underline{\lambda} k.\, e : \tau_2 \to \tau_1}\ (\textsc{TFun}_\mathsf{C})
$$

$$
\frac{}{\Gamma \vdash \mathcal{S} : ((\tau_1 \to (\tau_2 \to \tau_3) \to \tau_3) \to (\tau_4 \to \tau_4) \to \tau_5) \to (\tau_1 \to \tau_2) \to \tau_5}\ (\textsc{TShift}_\mathsf{C})
$$

$$
\frac{\Gamma \vdash e_1 : \tau_2 \to \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1 \mathbin{\underline{@}} e_2 : \tau_1}\ (\textsc{TApp}_\mathsf{C}) \qquad \frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma, x : \tau_1 \vdash e_2 : \tau_2}{\Gamma \vdash (\mathsf{let}\ x = e_1\ \mathsf{in}\ e_2) : \tau_2}\ (\textsc{TLet}_\mathsf{C})
$$

2.3 CPS 項の代入規則

$$
\frac{}{x[v/x] = x}\ (\textsc{sVar=}_\mathsf{C}) \qquad \frac{}{y[v/x] = y}\ (\textsc{sVar} \neq_\mathsf{C}) \qquad \frac{}{n[v/x] = n}\ (\textsc{sNum}_\mathsf{C})
$$

$$
\frac{\forall x.(e[v/y] = e')}{(\lambda x.\, e)[v/y] = \lambda x.\, e'}\ (\textsc{sFun}_\mathsf{C}) \qquad \frac{e_1[v/x] = e_1' \quad e_2[v/x] = e_2'}{(e_1 \mathbin{@} e_2)[v/x] = (e_1' \mathbin{@} e_2')}\ (\textsc{sApp}_\mathsf{C})
$$

$$
\frac{}{\mathcal{S}[v/x] = \mathcal{S}}\ (\textsc{sShift}) \qquad \frac{e_1[v/y] = e_1' \quad \forall x.(e_2[v/y] = e_2')}{(\mathsf{let}\ x = e_1\ \mathsf{in}\ e_2)[v/y] = \mathsf{let}\ x = e_1'\ \mathsf{in}\ e_2'}\ (\textsc{sLet})
$$

2.4 CPS 項の簡約規則

$$\frac{e_1[v_2/x] = e'_1}{(\lambda x.\, e_1) \mathbin{@} v_2 \rightsquigarrow e'_1}\ (\textsc{eqBeta}_{\mathsf{C}}) \qquad \frac{e_2[v_1/x] = e'_2}{\mathsf{let}\ x = v_1\ \mathsf{in}\ e_2 \rightsquigarrow e'_2}\ (\textsc{eqLet}_{\mathsf{C}})$$

$$\frac{}{((\underline{\lambda} wk.\, (w \mathbin{\underline{@}} (\underline{\lambda} ak'.\, k' \mathbin{\underline{@}} (k \mathbin{\underline{@}} a))) \mathbin{\underline{@}} (\underline{\lambda} m.\, m)) \mathbin{\underline{@}} v_2) \mathbin{\underline{@}} k \rightsquigarrow (v_2 \mathbin{\underline{@}} (\underline{\lambda} a.\, \underline{\lambda} k'.\, k' \mathbin{\underline{@}} (k \mathbin{\underline{@}} a))) \mathbin{\underline{@}} (\underline{\lambda} m.\, m)}\ (\textsc{eqShift}_{\mathsf{C}})$$

$$\frac{e_1 \rightsquigarrow e'_1}{e_1 \mathbin{\underline{@}} e_2 \rightsquigarrow e'_1 \mathbin{\underline{@}} e_2}\ (\textsc{eqApp1}_{\mathsf{C}}) \qquad \frac{e_2 \rightsquigarrow e'_2}{v_1 \mathbin{\underline{@}} e_2 \rightsquigarrow v_1 \mathbin{\underline{@}} e'_2}\ (\textsc{eqApp2}_{\mathsf{C}})$$

$$\frac{e_1 \rightsquigarrow e'_1}{\underline{\mathsf{let}}\ \ x = e_1\ \ \underline{\mathsf{in}}\ \ e_2 \rightsquigarrow \underline{\mathsf{let}}\ \ x = e'_1\ \ \underline{\mathsf{in}}\ \ e_2}\ (\textsc{eqLet1}_{\mathsf{C}}) \qquad \frac{e_2 \rightsquigarrow e'_2}{\underline{\mathsf{let}}\ \ x = e_1\ \ \underline{\mathsf{in}}\ \ e_2 \rightsquigarrow \underline{\mathsf{let}}\ \ x = e_1\ \ \underline{\mathsf{in}}\ \ e'_2}\ (\textsc{eqLet2}_{\mathsf{C}})$$

2.5 CPS 変換の定式化

η redex を作らない、one-pass の CPS 変換の定義を示す。

$$\begin{aligned}
[\![n]\!]_v &= n \\
[\![x]\!]_v &= x \\
[\![\lambda x.\, e]\!]_v &= \underline{\lambda} x.\, \underline{\lambda} k.\, [\![e]\!]' \mathbin{\overline{@}} k \\
[\![\mathcal{S}]\!]_v &= \underline{\lambda} wk.\, (w \mathbin{\underline{@}} (\underline{\lambda} ak'.\, k' \mathbin{\underline{@}} (k \mathbin{\underline{@}} a))) \mathbin{\underline{@}} (\underline{\lambda} m.\, m) \\
\\
[\![v]\!] &= \overline{\lambda} \kappa.\, \kappa \mathbin{\overline{@}} [\![v]\!]_v \\
[\![e_1 \mathbin{@} e_2]\!] &= \overline{\lambda} \kappa.\, [\![e_1]\!] \mathbin{\overline{@}} (\overline{\lambda} m.\, [\![e_2]\!] \mathbin{\overline{@}} (\overline{\lambda} n.\, (m \mathbin{\underline{@}} n) \mathbin{\underline{@}} (\underline{\lambda} a.\, \kappa \mathbin{\overline{@}} a))) \\
[\![\langle e \rangle]\!] &= \overline{\lambda} \kappa.\, \underline{\mathsf{let}}\ \ x = [\![e]\!] \mathbin{\overline{@}} (\overline{\lambda} m.\, m)\ \ \underline{\mathsf{in}}\ \ \kappa \mathbin{\overline{@}} x \\
&= \overline{\lambda} \kappa.\, ((\underline{\lambda} x.\, \kappa \mathbin{\overline{@}} x) \mathbin{\underline{@}} ([\![e]\!] \mathbin{\overline{@}} (\overline{\lambda} m.\, m))) \\
[\![\mathsf{let}\ x = e_1\ \mathsf{in}\ e_2]\!] &= \overline{\lambda} \kappa.\, [\![e_1]\!] \mathbin{\overline{@}} (\overline{\lambda} m.\, \underline{\mathsf{let}}\ \ x = m\ \ \underline{\mathsf{in}}\ \ [\![e_2]\!] \mathbin{\overline{@}} \kappa) \\
&= \overline{\lambda} \kappa.\, [\![e_1]\!] \mathbin{\overline{@}} (\overline{\lambda} m.\, ((\underline{\lambda} x.\, [\![e_2]\!] \mathbin{\overline{@}} \kappa) \mathbin{\underline{@}} m)) \\
\\
[\![v]\!]' &= \overline{\lambda} k.\, k \mathbin{\underline{@}} [\![v]\!]_v \\
[\![e_1 \mathbin{@} e_2]\!]' &= \overline{\lambda} \kappa.\, [\![e_1]\!] \mathbin{\overline{@}} (\overline{\lambda} m.\, [\![e_2]\!] \mathbin{\overline{@}} (\overline{\lambda} n.\, (m \mathbin{\underline{@}} n) \mathbin{\underline{@}} k)) \\
[\![\langle e \rangle]\!]' &= \overline{\lambda} k.\, \underline{\mathsf{let}}\ \ x = [\![e]\!] \mathbin{\overline{@}} (\overline{\lambda} m.\, m)\ \ \underline{\mathsf{in}}\ \ k \mathbin{\underline{@}} x \\
&= \overline{\lambda} k.\, k \mathbin{\underline{@}} ([\![e]\!] \mathbin{\overline{@}} (\overline{\lambda} m.\, m)) \\
[\![\mathsf{let}\ x = e_1\ \mathsf{in}\ e_2]\!]' &= \overline{\lambda} k.\, [\![e_1]\!] \mathbin{\overline{@}} (\overline{\lambda} m.\, \underline{\mathsf{let}}\ \ x = m\ \ \underline{\mathsf{in}}\ \ [\![e_2]\!]' \mathbin{\overline{@}} k) \\
&= \overline{\lambda} k.\, [\![e_1]\!] \mathbin{\overline{@}} (\overline{\lambda} m.\, (\underline{\lambda} x.\, [\![e_2]\!]' \mathbin{\overline{@}} k) \mathbin{\underline{@}} m)
\end{aligned}$$

3 schematic な継続

5 節で CPS 変換の正当性の証明をするが、このとき CPS 項が受け取る static な継続は schematic な継続とする。

定義 3.1: (schematic な継続)

$v_1[v/x] = v_1'$ について、$(\kappa \overline{@} v_1)[v/y] = \kappa \overline{@} v_1'$ を満たすとき、κ は schematic であるという。

4 補題の証明

CPS 変換の証明を行う前に、必要な補題をいくつか証明する。

4.1 CPS 項に関する代入補題の証明

補題 4.1.1: (eSubstV)

$v_1[v/x] = v_1'$ のとき、$[\![v_1]\!]_v[[\![v]\!]_v/x] = [\![v_1']\!]_v$

証明.

$v = x$ **のとき**

$$\begin{aligned} [\![x]\!]_v[[\![v]\!]_v/x] &= x[[\![v]\!]_v/x] \\ &= [\![v]\!]_v & \texttt{(sVar=)} \end{aligned}$$

$v = y$ **のとき**

$$\begin{aligned} [\![y]\!]_v[[\![v]\!]_v/x] &= y[[\![v]\!]_v/x] \\ &= [\![y]\!]_v & \texttt{(sVar} \neq \texttt{)} \end{aligned}$$

$v = \lambda x.\, e$ **のとき**

$$\begin{aligned} [\![\lambda x.\, e]\!]_v[[\![v]\!]_v/x] &\equiv (\underline{\lambda} x.\, \underline{\lambda} k.\, [\![e]\!]' \mathbin{\overline{@}} k)[[\![v]\!]_v/x] \\ &= \underline{\lambda} x.\, \underline{\lambda} k.\, [\![e[v/x]]\!]' \mathbin{\overline{@}} k \\ &= \underline{\lambda} x.\, \underline{\lambda} k.\, [\![e']\!]' \mathbin{\overline{@}} k & (\textbf{補題 4.1.3 } ekSubst') \\ &\equiv [\![\lambda x.\, e']\!]_v \end{aligned}$$

$v = \mathcal{S}$ **のとき**

$$\begin{aligned} [\![\mathcal{S}]\!]_v[[\![v]\!]_v/x] &= (\underline{\lambda} wk.\, (w \mathbin{\underline{@}} (\underline{\lambda} ak'.\, k' \mathbin{\underline{@}} (k \mathbin{\underline{@}} a))) \mathbin{\underline{@}} (\underline{\lambda} m.\, m))[[\![v]\!]_v/x] \\ &= [\![\mathcal{S}]\!]_v \end{aligned}$$

□

補題 4.1.2: (e κ Subst)

$e_1[v/x] = e_2$ かつ $\kappa_1[\llbracket v \rrbracket_v/x] = \kappa_2$ のとき、$(\llbracket e_1 \rrbracket \,\overline{@}\, \kappa_1)[\llbracket v \rrbracket_v/x] = \llbracket e_2 \rrbracket \,\overline{@}\, \kappa_2$

証明.

$e_1 = v_1$ **(値) のとき**

$v_1[v/x] = v_2$ とすると、

$$
\begin{array}{rcll}
(\text{与式}) & \equiv & (\llbracket v_1 \rrbracket \,\overline{@}\, \kappa_1)[\llbracket v \rrbracket_v/x] & \\
& \equiv & (\kappa_1 \,\overline{@}\, \llbracket v_1 \rrbracket_v)[\llbracket v \rrbracket_v/x] & \\
& = & (\kappa_1[\llbracket v \rrbracket_v/x]) \,\overline{@}\, (\llbracket v_1 \rrbracket_v[\llbracket v \rrbracket_v/x]) & \\
& = & \kappa_2 \,\overline{@}\, \llbracket v_2 \rrbracket_v & (\textbf{補題 4.1.1 } eSubstV) \\
& \equiv & \llbracket v_2 \rrbracket \,\overline{@}\, \kappa_2 &
\end{array}
$$

e_1 **が App のとき**

$(e_1 \,@\, e_2)[v/x] = e'_1 \,@\, e'_2$ とすると、

$$
\begin{array}{rcll}
(\text{与式}) & \equiv & (\llbracket e_1 \,@\, e_2 \rrbracket \,\overline{@}\, \kappa_1)[\llbracket v \rrbracket_v/x] & \\
& \equiv & (\llbracket e_1 \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, \llbracket e_2 \rrbracket \,\overline{@}\, (\overline{\lambda} n.\, (m \,\underline{@}\, n) \,\underline{@}\, (\underline{\lambda} a.\, \kappa_1 \,\overline{@}\, a))))[\llbracket v \rrbracket_v/x] & \\
& = & (\llbracket e_1 \rrbracket[\llbracket v \rrbracket_v/x]) \,\overline{@}\, ((\overline{\lambda} m.\, \llbracket e_2 \rrbracket \,\overline{@}\, (\overline{\lambda} n.\, (m \,\underline{@}\, n) \,\underline{@}\, (\underline{\lambda} a.\, \kappa_1 \,\overline{@}\, a)))[\llbracket v \rrbracket_v/x]) & \\
& = & \llbracket e'_1 \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, (\llbracket e_2 \rrbracket[\llbracket v \rrbracket_v/x]) \,\overline{@}\, ((\overline{\lambda} n.\, (m \,\underline{@}\, n) \,\underline{@}\, (\underline{\lambda} a.\, \kappa_1 \,\overline{@}\, a))[\llbracket v \rrbracket_v/x])) & (I.H.) \\
& = & \llbracket e'_1 \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, \llbracket e'_2 \rrbracket \,\overline{@}\, (\overline{\lambda} n.\, (m \,\underline{@}\, n) \,\underline{@}\, (\underline{\lambda} a.\, \kappa_2 \,\overline{@}\, a))) & (I.H.) \\
& \equiv & \llbracket e'_1 \,@\, e'_2 \rrbracket \,\overline{@}\, \kappa_2 &
\end{array}
$$

e_1 **が Reset のとき**

$\langle e \rangle[v/x] = \langle e' \rangle$ とすると、

$$
\begin{array}{rcll}
(\text{与式}) & \equiv & (\llbracket \langle e \rangle \rrbracket \,\overline{@}\, \kappa_1)[\llbracket v \rrbracket_v/x] & \\
& \equiv & ((\underline{\lambda} c.\, \kappa_1 \,\overline{@}\, c) \,\underline{@}\, (\llbracket e \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, m)))[\llbracket v \rrbracket_v/x] & \\
& = & ((\underline{\lambda} c.\, \kappa_1 \,\overline{@}\, c)[\llbracket v \rrbracket_v/x]) \,\underline{@}\, ((\llbracket e \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, m))[\llbracket v \rrbracket_v/x]) & \\
& = & (\underline{\lambda} c.\, \kappa_2 \,\overline{@}\, c) \,\underline{@}\, (\llbracket e' \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, m)) & (I.H.) \\
& \equiv & \llbracket \langle e' \rangle \rrbracket \,\overline{@}\, \kappa_2 &
\end{array}
$$

e_1 **が Let のとき**

$(\mathsf{let}\ x = e_1\ \mathsf{in}\ e_2)[v/x] = \mathsf{let}\ x = e'_1\ \mathsf{in}\ e'_2$ とすると、

$$
\begin{array}{rcll}
(\text{与式}) & \equiv & \mathsf{let}\ x = e_1\ \mathsf{in}\ e_2 \,\overline{@}\, \kappa_1[\llbracket v \rrbracket_v/x] & \\
& \equiv & (\llbracket e_1 \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, (\underline{\lambda} c.\, \llbracket e_2 \rrbracket \,\overline{@}\, \kappa_1) \,\underline{@}\, m))[\llbracket v \rrbracket_v/x] & \\
& = & (\llbracket e_1 \rrbracket[\llbracket v \rrbracket_v/x]) \,\overline{@}\, (\overline{\lambda} m.\, (\underline{\lambda} c.\, \llbracket e_2 \rrbracket[\llbracket v \rrbracket_v/x] \,\overline{@}\, \kappa_1[\llbracket v \rrbracket_v/x]) \,\underline{@}\, m) & \\
& = & \llbracket e'_1 \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, (\underline{\lambda} c.\, \llbracket e'_2 \rrbracket \,\overline{@}\, \kappa_2) \,\underline{@}\, m) & (I.H.) \\
& \equiv & \llbracket \mathsf{let}\ x = e'_1\ \mathsf{in}\ e'_2 \rrbracket \,\overline{@}\, \kappa_2 &
\end{array}
$$

$\square$

補題 4.1.3: (ekSubst')

$e[v/x] = e'$ のとき、$(\llbracket e \rrbracket' \,\overline{@}\, k)[\llbracket v \rrbracket_v / x] = \llbracket e' \rrbracket' \,\overline{@}\, k$

証明.

$e = v$ **(値) のとき**

$v[v/x] = v'$ とすると、

$$\begin{array}{rcll}
(\text{与式}) & \equiv & (\llbracket v \rrbracket' \,\overline{@}\, k)[\llbracket v_2 \rrbracket_v / x] & \\
& \equiv & (k \,\underline{@}\, \llbracket v \rrbracket_v)[\llbracket v_2 \rrbracket_v / x] & \\
& = & (k[\llbracket v_2 \rrbracket_v / x]) \,\underline{@}\, (\llbracket v \rrbracket_v [\llbracket v_2 \rrbracket_v / x]) & \\
& = & k \,\underline{@}\, \llbracket v' \rrbracket_v & (\text{補題 } \mathbf{4.1.1}\ eSubstV) \\
& \equiv & \llbracket v' \rrbracket' \,\overline{@}\, k &
\end{array}$$

e **が App のとき**

$(e_1 \,@\, e_2)[v/x] = e_1' \,@\, e_2'$ とすると、

$$\begin{array}{rcl}
(\text{与式}) & \equiv & (\llbracket e_1 \,@\, e_2 \rrbracket' \,\overline{@}\, k)[\llbracket v \rrbracket_v / x] \\
& \equiv & (\llbracket e_1 \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, \llbracket e_2 \rrbracket \,\overline{@}\, (\overline{\lambda} n.\, (m \,\underline{@}\, n) \,\underline{@}\, k)))[\llbracket v \rrbracket_v / x] \\
& = & (\llbracket e_1 \rrbracket [\llbracket v \rrbracket_v / x]) \,\overline{@}\, ((\overline{\lambda} m.\, \llbracket e_2 \rrbracket \,\overline{@}\, (\overline{\lambda} n.\, (m \,\underline{@}\, n) \,\underline{@}\, k))[\llbracket v \rrbracket_v / x]) \\
& = & \llbracket e_1[v/x] \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, \llbracket e_2[v/x] \rrbracket \,\overline{@}\, ((\overline{\lambda} n.\, (m \,\underline{@}\, n) \,\underline{@}\, k)[\llbracket v \rrbracket_v / x])) \\
& = & \llbracket e_1' \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, \llbracket e_2' \rrbracket \,\overline{@}\, (\overline{\lambda} n.\, (m \,\underline{@}\, n) \,\underline{@}\, k)) \\
& \equiv & \llbracket e_1' \,@\, e_2' \rrbracket' \,\overline{@}\, k
\end{array}$$

e **が Reset のとき**

$\langle e \rangle [v/x] = \langle e' \rangle$ とすると、

$$\begin{array}{rcll}
(\text{与式}) & = & (\llbracket \langle e \rangle \rrbracket' \,\overline{@}\, k)[\llbracket v \rrbracket_v / x] & \\
& = & (k \,\underline{@}\, (\llbracket e \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, m)))[\llbracket v \rrbracket_v / x] & \\
& = & k \,\underline{@}\, (\llbracket e[v/x] \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, m)) & \\
& = & k \,\underline{@}\, (\llbracket e' \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, m)) & (\text{補題 } \mathbf{4.1.2}\ e\kappa Subst) \\
& = & \llbracket \langle e' \rangle \rrbracket' \,\overline{@}\, k &
\end{array}$$

e **が Let のとき**

$(\mathsf{let}\ x = e_1\ \mathsf{in}\ e_2)[v/x] = \mathsf{let}\ x = e_1'\ \mathsf{in}\ e_2'$ とすると、

$$\begin{array}{rcll}
(\text{与式}) & = & (\llbracket \mathsf{let}\ x = e_1\ \mathsf{in}\ e_2 \rrbracket' \,\overline{@}\, k)[\llbracket v \rrbracket_v / x] & \\
& = & (\llbracket e_1 \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, (\underline{\lambda} c.\, \llbracket e_2 \rrbracket' \,\overline{@}\, k) \,\underline{@}\, m))[\llbracket v \rrbracket_v / x] & \\
& = & (\llbracket e_1 \rrbracket [\llbracket v \rrbracket_v / x]) \,\overline{@}\, (\overline{\lambda} m.\, (\underline{\lambda} c.\, (\llbracket e_2 \rrbracket' [\llbracket v \rrbracket_v / x]) \,\overline{@}\, k) \,\underline{@}\, m) & \\
& = & \llbracket e_1 \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, (\underline{\lambda} c.\, (\llbracket e_2 \rrbracket' [\llbracket v \rrbracket_v / x]) \,\overline{@}\, k) \,\underline{@}\, m) & (\text{補題 } \mathbf{4.1.2}\ e\kappa Subst) \\
& = & \llbracket e_1 \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, (\underline{\lambda} c.\, \llbracket e_2 \rrbracket' \,\overline{@}\, k) \,\underline{@}\, m) & (I.H.) \\
& = & \llbracket \mathsf{let}\ x = e_1'\ \mathsf{in}\ e_2' \rrbracket' \,\overline{@}\, k &
\end{array}$$

□

補題 4.1.4: (κ Subst)

schematic な κ ($\kappa[v/k] = \kappa'$) について、$(\llbracket e \rrbracket \,\overline{@}\, \kappa)[v/k] = e \,\overline{@}\, \kappa'$ が成り立つ

証明.

$e = v_1$ **(値) のとき**

$$
\begin{array}{rcll}
(与式) & = & (\llbracket v_1 \rrbracket \,\overline{@}\, \kappa)[v/x] & \\
& \equiv & (\kappa \,\overline{@}\, \llbracket v_1 \rrbracket_v)[v/k] & \\
& = & (\kappa[v/k]) \,\overline{@}\, (\llbracket v_1 \rrbracket_v [v/k]) & \\
& = & \kappa' \,\overline{@}\, \llbracket v_1 \rrbracket_v & \\
& \equiv & \llbracket v_1 \rrbracket \,\overline{@}\, \kappa' &
\end{array}
$$

$e = e_1 \,@\, e_2$ **(App) のとき**

$$
\begin{array}{rcll}
(与式) & = & (\llbracket e_1 \,@\, e_2 \rrbracket \,\overline{@}\, \kappa)[v/x] & \\
& \equiv & (\llbracket e_1 \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, \llbracket e_2 \rrbracket \,\overline{@}\, (\overline{\lambda} n.\, (m \,\underline{@}\, n) \,\underline{@}\, (\underline{\lambda} a.\, \kappa \,\overline{@}\, a))))[v/x] & \\
& = & (\llbracket e_1 \rrbracket [v/x]) \,\overline{@}\, ((\overline{\lambda} m.\, \llbracket e_2 \rrbracket \,\overline{@}\, (\overline{\lambda} n.\, (m \,\underline{@}\, n) \,\underline{@}\, (\underline{\lambda} a.\, \kappa \,\overline{@}\, a)))[v/x]) & \\
& = & \llbracket e_1 \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, (\llbracket e_2 \rrbracket [v/x]) \,\overline{@}\, ((\overline{\lambda} n.\, (m \,\underline{@}\, n) \,\underline{@}\, (\underline{\lambda} a.\, \kappa \,\overline{@}\, a))[v/x])) & (I.H.) \\
& = & \llbracket e_1 \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, \llbracket e_2 \rrbracket \,\overline{@}\, (\overline{\lambda} n.\, (m \,\underline{@}\, n) \,\underline{@}\, (\underline{\lambda} a.\, \kappa' \,\overline{@}\, a))) & (I.H.) \\
& \equiv & \llbracket e_1 \,@\, e_2 \rrbracket \,\overline{@}\, \kappa' &
\end{array}
$$

$e = \langle e \rangle$ **(Reset) のとき**

$$
\begin{array}{rcll}
(与式) & = & (\llbracket \langle e \rangle \rrbracket \,\overline{@}\, \kappa)[v/x] & \\
& \equiv & (\underline{\lambda} c.\, \kappa \,\overline{@}\, c) \,\underline{@}\, (\llbracket e \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, m))[v/x] & \\
& = & ((\underline{\lambda} c.\, \kappa \,\overline{@}\, c)[v/x]) \,\underline{@}\, (\llbracket e \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, m)) & \\
& = & (\underline{\lambda} c.\, \kappa' \,\overline{@}\, c) \,\underline{@}\, (\llbracket e \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, m)) & \\
& \equiv & \llbracket \langle e \rangle \rrbracket \,\overline{@}\, \kappa' &
\end{array}
$$

e **が Let のとき**

$$
\begin{array}{rcll}
(与式) & = & (\llbracket \mathsf{let}\ x = e_1\ \mathsf{in}\ e_2 \rrbracket \,\overline{@}\, \kappa)[v/x] & \\
& \equiv & (\llbracket e_1 \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, (\underline{\lambda} c.\, \llbracket e_2 \rrbracket \,\overline{@}\, \kappa) \,\underline{@}\, m))[v/x] & \\
& = & \llbracket e_1 \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, ((\underline{\lambda} c.\, \llbracket e_2 \rrbracket \,\overline{@}\, \kappa)[v/x]) \,\underline{@}\, m) & (I.H.) \\
& = & \llbracket e_1 \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, (\underline{\lambda} c.\, \llbracket e_2 \rrbracket \,\overline{@}\, \kappa') \,\underline{@}\, m) & (I.H.) \\
& \equiv & \llbracket \mathsf{let}\ x = e_1\ \mathsf{in}\ e_2 \rrbracket \,\overline{@}\, \kappa' &
\end{array}
$$

□

補題 4.1.5: (kSubst')

$k[v/x] = k'$ のとき、$(\llbracket e \rrbracket' \mathbin{\overline{@}} k)[v/k] = \llbracket e \rrbracket' \mathbin{\overline{@}} k'$ が成り立つ

証明.

e **が値** $(e = v_1)$ **のとき**

$$\begin{aligned}
(\text{与式}) &= \llbracket v_1 \rrbracket' \mathbin{\overline{@}} k[v/x] \\
&\equiv (k \mathbin{\underline{@}} \llbracket v_1 \rrbracket_v)[v/k] \\
&= k' \mathbin{\underline{@}} \llbracket v_1 \rrbracket_v \\
&\equiv \llbracket v_1 \rrbracket' \mathbin{\overline{@}} k'
\end{aligned}$$

e **が App** $(e = e_1 @ e_2)$ **のとき**

$$\begin{aligned}
(\text{与式}) &= (\llbracket e_1 @ e_2 \rrbracket' \mathbin{\overline{@}} k)[v/x] \\
&\equiv (\llbracket e_1 \rrbracket \mathbin{\overline{@}} (\overline{\lambda} m.\, \llbracket e_2 \rrbracket \mathbin{\overline{@}} (\overline{\lambda} n.\, (m \mathbin{\underline{@}} n) \mathbin{\underline{@}} k)))[v/x] \\
&= (\llbracket e_1 \rrbracket[v/x]) \mathbin{\overline{@}} ((\overline{\lambda} m.\, \llbracket e_2 \rrbracket \mathbin{\overline{@}} (\overline{\lambda} n.\, (m \mathbin{\underline{@}} n) \mathbin{\underline{@}} k))[v/x]) \\
&= \llbracket e_1 \rrbracket \mathbin{\overline{@}} (\overline{\lambda} m.\, (\llbracket e_2 \rrbracket[v/x]) \mathbin{\overline{@}} ((\overline{\lambda} n.\, (m \mathbin{\underline{@}} n) \mathbin{\underline{@}} k)[v/x])) && (\textbf{補題 4.1.4 } \kappa Subst) \\
&= \llbracket e_1 \rrbracket \mathbin{\overline{@}} (\overline{\lambda} m.\, \llbracket e_2 \rrbracket \mathbin{\overline{@}} (\overline{\lambda} n.\, (m \mathbin{\underline{@}} n) \mathbin{\underline{@}} v)) && (\textbf{補題 4.1.4 } \kappa Subst) \\
&\equiv (\llbracket e_1 @ e_2 \rrbracket') \mathbin{\overline{@}} v
\end{aligned}$$

e **が Reset** $(e = \langle e \rangle)$ **のとき**

$$\begin{aligned}
(\text{与式}) &= (\llbracket \langle e \rangle \rrbracket' \mathbin{\overline{@}} k)[v/x] \\
&\equiv (k \mathbin{\underline{@}} (\llbracket e \rrbracket \mathbin{\overline{@}} (\overline{\lambda} m.\, m)))[v/x] \\
&= k' \mathbin{\underline{@}} (\llbracket e \rrbracket \mathbin{\overline{@}} (\overline{\lambda} m.\, m)) \\
&\equiv \llbracket \langle e \rangle \rrbracket' \mathbin{\overline{@}} k'
\end{aligned}$$

e **が Let** $(e = \textbf{let } c = e_1 \textbf{ in } e_2)$ **のとき**

$$\begin{aligned}
(\text{与式}) &= (\llbracket \text{let } c = e_1 \text{ in } e_2 \rrbracket' \mathbin{\overline{@}} k)[v/x] \\
&\equiv (\llbracket e_1 \rrbracket \mathbin{\overline{@}} (\overline{\lambda} m.\, (\underline{\lambda} c.\, \llbracket e_2 \rrbracket' \mathbin{\overline{@}} k) \mathbin{\underline{@}} m))[v/x] \\
&= \llbracket e_1 \rrbracket \mathbin{\overline{@}} (\overline{\lambda} m.\, ((\underline{\lambda} c.\, \llbracket e_2 \rrbracket' \mathbin{\overline{@}} k)[v/x]) \mathbin{\underline{@}} m) && (\textbf{補題 4.1.4 } \kappa Subst) \\
&= \llbracket e_1 \rrbracket \mathbin{\overline{@}} (\overline{\lambda} m.\, (\underline{\lambda} c.\, \llbracket e_2 \rrbracket' \mathbin{\overline{@}} k') \mathbin{\underline{@}} m) && (I.H.) \\
&\equiv \llbracket \text{let } x = e_1 \text{ in } e_2 \rrbracket' \mathbin{\overline{@}} k'
\end{aligned}$$

□

補題 4.1.6: (eSubst)

schematic な κ について、$e_1[v/x] = e_2$ のとき、$(\llbracket e_1 \rrbracket \,\overline{@}\, \kappa)[\llbracket v \rrbracket_v / x] = \llbracket e_2 \rrbracket \,\overline{@}\, \kappa$

証明.

$e_1 = v_1$ **(値) のとき**

$v_1[v/x] = v_2$ とすると、

$$
\begin{aligned}
(\text{与式}) &\equiv (\llbracket v_1 \rrbracket \,\overline{@}\, \kappa_1)[\llbracket v \rrbracket_v / x] \\
&\equiv (\kappa \,\overline{@}\, \llbracket v_1 \rrbracket_v)[\llbracket v \rrbracket_v / x] \\
&= (\kappa[\llbracket v \rrbracket_v / x]) \,\overline{@}\, (\llbracket v_1 \rrbracket_v [\llbracket v \rrbracket_v / x]) \\
&= \kappa \,\overline{@}\, \llbracket v_2 \rrbracket_v && (\textbf{補題 4.1.1 } \mathit{eSubstV}) \\
&\equiv \llbracket v_2 \rrbracket \,\overline{@}\, \kappa
\end{aligned}
$$

e_1 **が App のとき**

$(e_1 \,@\, e_2)[v/x] = e_1' \,@\, e_2'$ とすると、

$$
\begin{aligned}
(\text{与式}) &\equiv (\llbracket e_1 \,@\, e_2 \rrbracket \,\overline{@}\, \kappa)[\llbracket v \rrbracket_v / x] \\
&\equiv (\llbracket e_1 \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, \llbracket e_2 \rrbracket \,\overline{@}\, (\overline{\lambda} n.\, (m \,\underline{@}\, n) \,\underline{@}\, (\underline{\lambda} a.\, \kappa \,\overline{@}\, a))))[\llbracket v \rrbracket_v / x] \\
&= (\llbracket e_1 \rrbracket [\llbracket v \rrbracket_v / x]) \,\overline{@}\, ((\overline{\lambda} m.\, \llbracket e_2 \rrbracket \,\overline{@}\, (\overline{\lambda} n.\, (m \,\underline{@}\, n) \,\underline{@}\, (\underline{\lambda} a.\, \kappa \,\overline{@}\, a)))[\llbracket v \rrbracket_v / x]) \\
&= \llbracket e_1' \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, (\llbracket e_2 \rrbracket [\llbracket v \rrbracket_v / x]) \,\overline{@}\, ((\overline{\lambda} n.\, (m \,\underline{@}\, n) \,\underline{@}\, (\underline{\lambda} a.\, \kappa_1 \,\overline{@}\, a))[\llbracket v \rrbracket_v / x])) && (\textbf{補題 4.1.2 } \mathit{e\kappa Subst}) \\
&= \llbracket e_1' \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, \llbracket e_2' \rrbracket \,\overline{@}\, (\overline{\lambda} n.\, (m \,\underline{@}\, n) \,\underline{@}\, (\underline{\lambda} a.\, \kappa \,\overline{@}\, a))) && (\textbf{補題 4.1.2 } \mathit{e\kappa Subst}) \\
&\equiv \llbracket e_1' \,@\, e_2' \rrbracket \,\overline{@}\, \kappa
\end{aligned}
$$

e_1 **が Reset のとき**

$\langle e \rangle [v/x] = \langle e' \rangle$ とすると、

$$
\begin{aligned}
(\text{与式}) &\equiv (\llbracket \langle e \rangle \rrbracket \,\overline{@}\, \kappa)[\llbracket v \rrbracket_v / x] \\
&\equiv ((\underline{\lambda} c.\, \kappa \,\overline{@}\, c) \,\underline{@}\, (\llbracket e \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, m)))[\llbracket v \rrbracket_v / x] \\
&= ((\underline{\lambda} c.\, \kappa \,\overline{@}\, c)[\llbracket v \rrbracket_v / x]) \,\underline{@}\, ((\llbracket e \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, m))[\llbracket v \rrbracket_v / x]) \\
&= (\underline{\lambda} c.\, \kappa \,\overline{@}\, c) \,\underline{@}\, (\llbracket e' \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, m)) && (I.H.) \\
&\equiv \llbracket \langle e' \rangle \rrbracket \,\overline{@}\, \kappa
\end{aligned}
$$

e_1 **が Let のとき**

$(\mathsf{let}\ x = e_1\ \mathsf{in}\ e_2)[v/x] = \mathsf{let}\ x = e_1'\ \mathsf{in}\ e_2'$ とすると、

$$
\begin{aligned}
(\text{与式}) &\equiv \mathsf{let}\ x = e_1\ \mathsf{in}\ e_2 \,\overline{@}\, \kappa_1 [\llbracket v \rrbracket_v / x] \\
&\equiv (\llbracket e_1 \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, (\underline{\lambda} c.\, \llbracket e_2 \rrbracket \,\overline{@}\, \kappa) \,\underline{@}\, m))[\llbracket v \rrbracket_v / x] \\
&= (\llbracket e_1 \rrbracket [\llbracket v \rrbracket_v / x]) \,\overline{@}\, ((\overline{\lambda} m.\, (\underline{\lambda} c.\, \llbracket e_2 \rrbracket \,\overline{@}\, \kappa) \,\underline{@}\, m)[\llbracket v \rrbracket_v / x]) \\
&= \llbracket e_1' \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, ((\underline{\lambda} c.\, \llbracket e_2 \rrbracket \,\overline{@}\, \kappa)[\llbracket v \rrbracket_v / x]) \,\underline{@}\, m) && (\textbf{補題 4.1.2 } \mathit{e\kappa Subst}) \\
&= \llbracket e_1' \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, (\underline{\lambda} c.\, \llbracket e_2' \rrbracket \,\overline{@}\, \kappa) \,\underline{@}\, m) && (I.H.) \\
&\equiv \llbracket \mathsf{let}\ x = e_1'\ \mathsf{in}\ e_2' \rrbracket \,\overline{@}\, \kappa
\end{aligned}
$$

□

4.2 $[\![\cdot]\!]'$ と $[\![\cdot]\!]$ の関係性についての補題の証明

補題 4.2.1: (correctCont)

任意の項 e と schematic な 継続 κ_1, κ_2 について、$(\kappa_1 \,\overline{@}\, v) \sim (\kappa_2 \,\overline{@}\, v)$ が成り立つならば、$[\![e]\!] \,\overline{@}\, \kappa_1 \sim [\![e]\!] \,\overline{@}\, \kappa_2$ が成り立つ

証明.

e **が値** $(e = v_1)$ **のとき**

$$\begin{aligned}
(\text{左式}) &\equiv [\![v]\!] \,\overline{@}\, \kappa_1 \\
&\equiv \kappa_1 \,\overline{@}\, [\![v]\!]_v \\
&\sim \kappa_2 \,\overline{@}\, [\![v]\!]_v \\
&\equiv [\![v]\!] \,\overline{@}\, \kappa_2
\end{aligned}$$

e **が App** $(e = e_1 \,@\, e_2)$ **のとき**

$$\begin{aligned}
(\text{左式}) &\equiv [\![e_1 \,@\, e_2]\!] \,\overline{@}\, \kappa_1 \\
&\equiv [\![e_1]\!] \,\overline{@}\, (\overline{\lambda} m.\, [\![e_2]\!] \,\overline{@}\, (\overline{\lambda} n.\, (m \,\underline{@}\, n) \,\underline{@}\, (\underline{\lambda} a.\, \kappa_1 \,\overline{@}\, a))) \\
&\sim [\![e_1]\!] \,\overline{@}\, (\overline{\lambda} m.\, [\![e_2]\!] \,\overline{@}\, (\overline{\lambda} n.\, (m \,\underline{@}\, n) \,\underline{@}\, (\underline{\lambda} a.\, \kappa_2 \,\overline{@}\, a))) \qquad (I.H.) \\
&\equiv [\![e_1 \,@\, e_2]\!] \,\overline{@}\, \kappa_2
\end{aligned}$$

e **が Reset** $(e = \langle e \rangle)$ **のとき**

$$\begin{aligned}
(\text{左式}) &\equiv [\![\langle e \rangle]\!] \,\overline{@}\, \kappa_1 \\
&\equiv (\underline{\lambda} c.\, \kappa_1 \,\overline{@}\, c) \,\underline{@}\, ([\![e]\!] \,\overline{@}\, (\overline{\lambda} m.\, m)) \\
&\sim (\underline{\lambda} c.\, \kappa_2 \,\overline{@}\, c) \,\underline{@}\, ([\![e]\!] \,\overline{@}\, (\overline{\lambda} m.\, m)) \qquad (I.H.) \\
&\equiv [\![\langle e \rangle]\!] \,\overline{@}\, \kappa_2
\end{aligned}$$

e **が Let** $(e = \textbf{let}\ x = e_1\ \textbf{in}\ e_2)$ **のとき**

$$\begin{aligned}
(\text{左式}) &\equiv [\![\mathsf{let}\ x = e_1\ \mathsf{in}\ e_2]\!] \,\overline{@}\, \kappa_1 \\
&\equiv [\![e_1]\!] \,\overline{@}\, (\overline{\lambda} m.\, (\underline{\lambda} c.\, [\![e_2]\!] \,\overline{@}\, \kappa_1) \,\underline{@}\, m) \\
&\sim [\![e_1]\!] \,\overline{@}\, (\overline{\lambda} m.\, (\underline{\lambda} c.\, [\![e_2]\!] \,\overline{@}\, \kappa_2) \,\underline{@}\, m) \qquad (I.H.) \\
&\equiv [\![\mathsf{let}\ x = e_1\ \mathsf{in}\ e_2]\!] \,\overline{@}\, \kappa_2
\end{aligned}$$

□

補題 4.2.2: (correctEtaEta')

$\llbracket e \rrbracket' \,\overline{@}\, (\underline{\lambda} a.\, \kappa \,\overline{@}\, a) \leadsto^{*} \llbracket e \rrbracket \,\overline{@}\, \kappa$ が成り立つ

証明.

e **が値** $(e = v_1)$ **のとき**

$$
\begin{array}{rcl}
(\text{左式}) & \equiv & \llbracket v \rrbracket' \,\overline{@}\, (\underline{\lambda} a.\, \kappa \,\overline{@}\, a) \\
& \equiv & (\overline{\lambda} k.\, k \,\underline{@}\, \llbracket v \rrbracket_v) \,\overline{@}\, (\underline{\lambda} a.\, \kappa \,\overline{@}\, a) \\
& \equiv & (\underline{\lambda} a.\, \kappa \,\overline{@}\, a) \,\underline{@}\, \llbracket v \rrbracket_v \\
& \leadsto & \kappa \,\overline{@}\, \llbracket v \rrbracket_v \\
& \equiv & \llbracket v \rrbracket \,\overline{@}\, \kappa
\end{array}
$$

e **が App** $(e = e_1 \,@\, e_2)$ **のとき**

$$
\begin{array}{rcl}
(\text{左式}) & \equiv & \llbracket e_1 \,@\, e_2 \rrbracket' \,\overline{@}\, (\underline{\lambda} a.\, \kappa \,\overline{@}\, a) \\
& \equiv & \llbracket e_1 \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, \llbracket e_2 \rrbracket \,\overline{@}\, (\overline{\lambda} n.\, (m \,\underline{@}\, n) \,\underline{@}\, (\underline{\lambda} a.\, \kappa \,\overline{@}\, a))) \\
& \equiv & \llbracket e_1 \,@\, e_2 \rrbracket \,\overline{@}\, \kappa
\end{array}
$$

e **が Reset** $(e = \langle e \rangle)$ **のとき**

$$
\begin{array}{rcl}
(\text{左式}) & \equiv & \llbracket \langle e \rangle \rrbracket' \,\overline{@}\, (\underline{\lambda} a.\, \kappa \,\overline{@}\, a) \\
& \equiv & (\underline{\lambda} a.\, \kappa \,\overline{@}\, a) \,\underline{@}\, (\llbracket e \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, m)) \\
& \equiv & \llbracket \langle e \rangle \rrbracket \,\overline{@}\, \kappa
\end{array}
$$

e **が Let** $(e = \textbf{let}\ x = e_1\ \textbf{in}\ e_2)$ **のとき**

$$
\begin{array}{rcll}
(\text{左式}) & \equiv & \llbracket \mathsf{let}\ x = e_1\ \mathsf{in}\ e_2 \rrbracket' \,\overline{@}\, (\underline{\lambda} a.\, \kappa \,\overline{@}\, a) & \\
& \equiv & \llbracket e_1 \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, (\underline{\lambda} c.\, \llbracket e_2 \rrbracket' \,\overline{@}\, (\underline{\lambda} a.\, \kappa \,\overline{@}\, a)) \,\underline{@}\, m) & \\
& \sim & \llbracket e_1 \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, (\underline{\lambda} c.\, \llbracket e_2 \rrbracket \,\overline{@}\, \kappa) \,\underline{@}\, m) & (I.H.) \\
& \equiv & \llbracket \mathsf{let}\ x = e_1\ \mathsf{in}\ e_2 \rrbracket \,\overline{@}\, \kappa &
\end{array}
$$

□

4.3 ピュアコンテキストに関する代入補題

補題 4.3.1: (subst-context)

任意のピュアコンテキスト con について、$E_{\mathsf{con}}[\ x\ ][v/x] = E_{\mathsf{con}}[\ v\ ]$ が成り立つ

証明.

con が Hole のとき

$$\begin{array}{rcl} (\text{左式}) & \equiv & x[v/x] \\ & = & v \end{array}$$

con が Frame (App_1 e_2) のとき

$$\begin{array}{rcl} (\text{左式}) & \equiv & (x \mathbin{@} e_2)[v/x] \\ & = & v \mathbin{@} e_2 \end{array}$$

con が Frame (App_2 v_1) のとき

$$\begin{array}{rcl} (\text{左式}) & \equiv & (v_1 \mathbin{@} x)[v/x] \\ & = & v_1 \mathbin{@} v \end{array}$$

con が Frame (Let e_2) のとき

$$\begin{array}{rcl} (\text{左式}) & \equiv & (\mathsf{let}\ c = x\ \mathsf{in}\ e_2)[v/x] \\ & = & \mathsf{let}\ c = v\ \mathsf{in}\ e_2 \end{array}$$

□

4.4 Shift に関する補題

補題 4.4.1: (contextContE)

$[\![E_{p_1}[\ \mathcal{S} @ v\ ]]\!] \overline{@} \kappa \equiv [\![\mathcal{S} @ v]\!]' \overline{@} (\underline{\lambda} a. [\![E_{p_2}[\ a\ ]]\!] \overline{@} \kappa)$ が成り立つことを証明する

証明.

p_1, p_2 **が Hole のとき**

$$\begin{aligned}
(\text{左式}) &\equiv [\![\mathcal{S} @ v]\!] \overline{@} \kappa \\
&\equiv [\![\mathcal{S}]\!] \overline{@} (\overline{\lambda} m. [\![v]\!] \overline{@} (\overline{\lambda} n. (m \underline{@} n) \underline{@} (\underline{\lambda} a. \kappa \overline{@} a))) \\
&\equiv [\![\mathcal{S} @ v]\!]' \overline{@} (\underline{\lambda} a. \kappa \overline{@} a) \\
&\equiv [\![\mathcal{S} @ v]\!]' \overline{@} (\underline{\lambda} a. [\![a]\!] \overline{@} \kappa) \\
&\equiv [\![\mathcal{S} @ v]\!]' \overline{@} (\underline{\lambda} a. [\![E_{p_2}[\ a\ ]]\!] \overline{@} \kappa)
\end{aligned}$$

p_1, p_2 **が Frame ($\mathbf{App_1}$ $\mathbf{e_2}$) のとき**

$$\begin{aligned}
(\text{左式}) &\equiv [\![E_{p_1'}[\ \mathcal{S} @ v\ ] @ e_2]\!] \overline{@} \kappa \\
&\equiv [\![E_{p_1'}[\ \mathcal{S} @ v\ ]]\!] \overline{@} (\overline{\lambda} m. [\![e_2]\!] \overline{@} (\overline{\lambda} n. (m \underline{@} n) \underline{@} (\underline{\lambda} a. \kappa \overline{@} a))) \\
&\equiv [\![\mathcal{S} @ v]\!]' \overline{@} (\underline{\lambda} a. [\![E_{p_2'}[\ a\ ]]\!] \overline{@} (\overline{\lambda} m. [\![e_2]\!] \overline{@} (\overline{\lambda} n. (m \underline{@} n) \underline{@} (\underline{\lambda} a. \kappa \overline{@} a)))) \qquad (I.H.) \\
&\equiv [\![\mathcal{S} @ v]\!]' \overline{@} (\underline{\lambda} a. [\![E_{p_2'}[\ a\ ] @ e_2]\!] \overline{@} \kappa) \\
&\equiv [\![\mathcal{S} @ v]\!]' \overline{@} (\underline{\lambda} a. [\![E_{p_2}[\ a\ ]]\!] \overline{@} \kappa)
\end{aligned}$$

p_1, p_2 **が Frame ($\mathbf{App_2}$ $\mathbf{v_1}$) のとき**

$$\begin{aligned}
(\text{左式}) &\equiv [\![v_1 @ E_{p_1'}[\ \mathcal{S} @ v\ ]]\!] \overline{@} \kappa \\
&\equiv [\![v_1]\!] \overline{@} (\overline{\lambda} m. [\![E_{p_1'}[\ \mathcal{S} @ v\ ]]\!] \overline{@} (\overline{\lambda} n. (m \underline{@} n) \underline{@} (\underline{\lambda} a. \kappa \overline{@} a))) \\
&\equiv [\![E_{p_1'}[\ \mathcal{S} @ v\ ]]\!] \overline{@} (\overline{\lambda} n. ([\![v_1]\!]_v \underline{@} n) \underline{@} (\underline{\lambda} a. \kappa \overline{@} a)) \\
&\equiv [\![\mathcal{S} @ v]\!]' \overline{@} (\underline{\lambda} a. [\![E_{p_2'}[\ a\ ]]\!] \overline{@} (\overline{\lambda} n. ([\![v_1]\!]_v \underline{@} n) \underline{@} (\underline{\lambda} a. \kappa \overline{@} a))) \qquad (I.H.) \\
&\equiv [\![\mathcal{S} @ v]\!]' \overline{@} (\underline{\lambda} a. [\![v_1 @ E_{p_2'}[\ a\ ]]\!] \overline{@} \kappa) \\
&\equiv [\![\mathcal{S} @ v]\!]' \overline{@} (\underline{\lambda} a. [\![E_{p_2}[\ a\ ]]\!] \overline{@} \kappa)
\end{aligned}$$

p_1, p_2 **が Frame (Let $\mathbf{e_2}$) のとき**

$$\begin{aligned}
(\text{左式}) &\equiv [\![\mathsf{let}\ x = E_{p_1'}[\ \mathcal{S} @ v\ ]\ \mathsf{in}\ e_2]\!] \overline{@} \kappa \\
&\equiv [\![E_{p_1'}[\ \mathcal{S} @ v\ ]]\!] \overline{@} (\overline{\lambda} m. (\underline{\lambda} c. [\![e_2]\!] \overline{@} \kappa) \underline{@} m) \\
&\equiv [\![\mathcal{S} @ v]\!]' \overline{@} (\underline{\lambda} a. [\![E_{p_2'}[\ a\ ]]\!] \overline{@} (\overline{\lambda} m. (\underline{\lambda} c. [\![e_2]\!] \overline{@} \kappa) \underline{@} m)) \qquad (I.H.) \\
&\equiv [\![\mathcal{S} @ v]\!]' \overline{@} (\underline{\lambda} a. [\![\mathsf{let}\ x = E_{p_2'}[\ a\ ]\ \mathsf{in}\ e_2]\!] \overline{@} \kappa) \\
&\equiv [\![\mathcal{S} @ v]\!]' \overline{@} (\underline{\lambda} a. [\![E_{p_2}[\ a\ ]]\!] \overline{@} \kappa)
\end{aligned}$$

□

5　CPS 変換の正当性の証明

この節では、CPS 変換の正当性の証明として、CPS 変換が項の簡約関係を保存することを示す。

5.1　変換の証明

定理 5.1: (CPS 変換の正当性の証明)

任意の項 e, e' について $e \to e'$ が成り立つならば、任意の schematic な継続 κ について $\llbracket e \rrbracket \,\overline{@}\, \kappa \rightsquigarrow^* e' \,\overline{@}\, \kappa$

これは、以下のような図を意味する。

$$
\begin{tikzcd}
e \arrow[rr, "\mathsf{Reduce}"] \arrow[d, "\mathsf{CPS}"'] & & e' \arrow[d, "\mathsf{CPS}"] \\
\llbracket e \rrbracket \arrow[dr, "\mathsf{Substitute}\ \kappa"'] & & \llbracket e' \rrbracket \arrow[dl, "\mathsf{Substitute}\ \kappa"] \\
& \llbracket e \rrbracket \,\overline{@}\, \kappa =_\beta \llbracket e' \rrbracket \,\overline{@}\, \kappa &
\end{tikzcd}
$$

この図にある Reduce の部分について、RBeta、RFrame (App_1)、RFrame (App_2)、RReset、RShift のケースについて場合分けをして帰納的に解く。

5.1.1　RBeta のケース

RBeta のケースでの証明を行う。

$$
\begin{tikzcd}
(\lambda x.\, e_1) \mathbin{@} v_2 \arrow[rr, "\mathsf{RBeta}"] \arrow[d, "\mathsf{CPS}"'] & & e_2 \arrow[d, "\mathsf{CPS}"] \\
\llbracket (\lambda x.\, e_1) \mathbin{@} v_2 \rrbracket \arrow[dr, "\mathsf{Substitute}\ \kappa"'] & & \llbracket e_2 \rrbracket \arrow[dl, "\mathsf{Substitute}\ \kappa"] \\
& \llbracket (\lambda x.\, e_1) \mathbin{@} v_2 \rrbracket \,\overline{@}\, \kappa =_\beta \llbracket e_2 \rrbracket \,\overline{@}\, \kappa &
\end{tikzcd}
$$

$$
\begin{array}{cll}
 & \llbracket (\lambda x.\, e_1) \mathbin{@} v_2 \rrbracket \,\overline{@}\, \kappa & \\
= & ((\underline{\lambda} x.\, \underline{\lambda} k.\, \llbracket e_1 \rrbracket' \,\overline{@}\, k) \,\underline{@}\, \llbracket v_2 \rrbracket_v) \,\underline{@}\, (\underline{\lambda} a.\, \kappa \,\overline{@}\, a) & \\
\rightsquigarrow & (\underline{\lambda} k.\, \llbracket e_2 \rrbracket' \,\overline{@}\, k) \,\underline{@}\, (\underline{\lambda} a.\, \kappa \,\overline{@}\, a) & (\text{補題 1.1 } \texttt{ekSubst'}) \\
\rightsquigarrow & \llbracket e_2 \rrbracket' \,\underline{@}\, (\underline{\lambda} a.\, \kappa \,\overline{@}\, a) & (\text{補題 1.2 } \texttt{kSubst'}) \\
\rightsquigarrow^* & \llbracket e_2 \rrbracket \,\overline{@}\, \kappa & (\text{補題 1.3 } \texttt{cpsEtaEta'})
\end{array}
$$

5.1.2 RFrame($\mathbf{App}_1$) のケース

$$
\begin{tikzcd}
e_1 \mathbin{@} e_2 \arrow[rr, "\mathsf{RFrame(App_1\ e_2)}"] \arrow[d, "\mathsf{CPS}"'] & & e_1' \mathbin{@} e_2 \arrow[d, "\mathsf{CPS}"] \\
\llbracket e_1 \mathbin{@} e_2 \rrbracket \arrow[dr, "\mathsf{Substitute}\ \kappa"'] & & \llbracket e_1' \mathbin{@} e_2 \rrbracket \arrow[dl, "\mathsf{Substitute}\ \kappa"] \\
& \llbracket e_1 \mathbin{@} e_2 \rrbracket \mathbin{\overline{@}} \kappa =_\beta \llbracket e_1' \mathbin{@} e_2 \rrbracket \mathbin{\overline{@}} \kappa &
\end{tikzcd}
$$

$$
\begin{array}{lll}
 & \llbracket e_1 \mathbin{@} e_2 \rrbracket \mathbin{\overline{@}} \kappa & \\
\equiv & \llbracket e_1 \rrbracket \mathbin{\overline{@}} (\overline{\lambda} m.\, \llbracket e_2 \rrbracket \mathbin{\overline{@}} (\overline{\lambda} n.\, (m \mathbin{\underline{@}} n) \mathbin{\underline{@}} (\underline{\lambda} a.\, \kappa \mathbin{\overline{@}} a))) & \\
\rightsquigarrow & \llbracket e_1' \rrbracket \mathbin{\overline{@}} (\overline{\lambda} m.\, \llbracket e_2 \rrbracket \mathbin{\overline{@}} (\overline{\lambda} n.\, (m \mathbin{\underline{@}} n) \mathbin{\underline{@}} (\underline{\lambda} a.\, \kappa \mathbin{\overline{@}} a))) & (I.H.) \\
\equiv & \llbracket e_1' \mathbin{@} e_2 \rrbracket \mathbin{\overline{@}} \kappa &
\end{array}
$$

5.1.3 RFrame($\mathbf{App}_2$) のケース

$$
\begin{tikzcd}
v_1 \mathbin{@} e_2 \arrow[rr, "\mathsf{RFrame(App_2\ v_1)}"] \arrow[d, "\mathsf{CPS}"'] & & v_1 \mathbin{@} e_2' \arrow[d, "\mathsf{CPS}"] \\
\llbracket v_1 \mathbin{@} e_2 \rrbracket \arrow[dr, "\mathsf{Substitute}\ \kappa"'] & & \llbracket v_1 \mathbin{@} e_2' \rrbracket \arrow[dl, "\mathsf{Substitute}\ \kappa"] \\
& \llbracket v_1 \mathbin{@} e_2 \rrbracket \mathbin{\overline{@}} \kappa =_\beta \llbracket v_1 \mathbin{@} e_2' \rrbracket \mathbin{\overline{@}} \kappa &
\end{tikzcd}
$$

$$
\begin{array}{lll}
 & \llbracket v_1 \mathbin{@} e_2 \rrbracket \mathbin{\overline{@}} \kappa & \\
\equiv & \llbracket v_1 \rrbracket \mathbin{\overline{@}} (\overline{\lambda} m.\, \llbracket e_2 \rrbracket \mathbin{\overline{@}} (\overline{\lambda} n.\, (m \mathbin{\underline{@}} n) \mathbin{\underline{@}} (\underline{\lambda} a.\, \kappa \mathbin{\overline{@}} a))) & \\
\equiv & \llbracket e_2 \rrbracket \mathbin{\overline{@}} (\overline{\lambda} n.\, (\llbracket v_1 \rrbracket_v \mathbin{\underline{@}} n) \mathbin{\underline{@}} (\underline{\lambda} a.\, \kappa \mathbin{\overline{@}} a)) & \\
\rightsquigarrow & \llbracket e_2' \rrbracket \mathbin{\overline{@}} (\overline{\lambda} n.\, (\llbracket v_1 \rrbracket_v \mathbin{\underline{@}} n) \mathbin{\underline{@}} (\underline{\lambda} a.\, \kappa \mathbin{\overline{@}} a)) & (I.H.) \\
\equiv & \llbracket v_1 \mathbin{@} e_2' \rrbracket \mathbin{\overline{@}} \kappa &
\end{array}
$$

5.1.4 RFrame(Let) のケース

$$
\begin{tikzcd}
\textsf{let } x = e_1 \textsf{ in } e_2 \arrow[rr, "\textsf{RFrame(Let e}_2\textsf{)}"] \arrow[d, "\textsf{CPS}"'] & & \textsf{let } x = e_1' \textsf{ in } e_2 \arrow[d, "\textsf{CPS}"] \\
\llbracket \textsf{let } x = e_1 \textsf{ in } e_2 \rrbracket \arrow[dr, "\textsf{Substitute } \kappa"'] & & \llbracket \textsf{let } x = e_1' \textsf{ in } e_2 \rrbracket \arrow[dl, "\textsf{Substitute } \kappa"] \\
& \llbracket \textsf{let } x = e_1 \textsf{ in } e_2 \rrbracket \,\overline{@}\, \kappa =_\beta \llbracket \textsf{let } x = e_1' \textsf{ in } e_2 \rrbracket \,\overline{@}\, \kappa &
\end{tikzcd}
$$

$$
\begin{array}{cll}
 & \llbracket \textsf{let } x = e_1 \textsf{ in } e_2 \rrbracket \,\overline{@}\, \kappa & \\
\equiv & \llbracket e_1 \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, (\underline{\lambda} c.\, \llbracket e_2 \rrbracket \,\overline{@}\, \kappa) \,\underline{@}\, m) & \\
\rightsquigarrow & \llbracket e_1' \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, (\underline{\lambda} c.\, \llbracket e_2 \rrbracket \,\overline{@}\, \kappa) \,\underline{@}\, m) & (I.H.) \\
\equiv & \llbracket \textsf{let } x = e_1' \textsf{ in } e_2 \rrbracket \,\overline{@}\, \kappa &
\end{array}
$$

5.1.5 RFrame(Reset) のケース

$$
\begin{tikzcd}
\langle e \rangle \arrow[rr, "\textsf{RFrame(Reset e)}"] \arrow[d, "\textsf{CPS}"'] & & \langle e' \rangle \arrow[d, "\textsf{CPS}"] \\
\llbracket \langle e \rangle \rrbracket \arrow[dr, "\textsf{Substitute } \kappa"'] & & \llbracket \langle e' \rangle \rrbracket \arrow[dl, "\textsf{Substitute } \kappa"] \\
& \llbracket \langle e \rangle \rrbracket \,\overline{@}\, \kappa =_\beta \llbracket \langle e' \rangle \rrbracket \,\overline{@}\, \kappa &
\end{tikzcd}
$$

$$
\begin{array}{cll}
 & \llbracket \langle e \rangle \rrbracket \,\overline{@}\, \kappa & \\
\equiv & (\underline{\lambda} c.\, \kappa \,\overline{@}\, x) \,\underline{@}\, (\llbracket e \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, m)) & \\
\rightsquigarrow & (\underline{\lambda} c.\, \kappa \,\overline{@}\, x) \,\underline{@}\, (\llbracket e' \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, m)) & (I.H.) \\
\equiv & \llbracket \langle e' \rangle \rrbracket \,\overline{@}\, \kappa &
\end{array}
$$

5.1.6 RLet のケース

$$
\begin{tikzcd}
\textsf{let } x = v_1 \textsf{ in } e_2 \arrow[rr, "\textsf{RLet}"] \arrow[d, "\textsf{CPS}"'] & & e_2' \arrow[d, "\textsf{CPS}"] \\
\llbracket \textsf{let } x = v_1 \textsf{ in } e_2 \rrbracket \arrow[dr, "\textsf{Substitute } \kappa"'] & & \llbracket e_2' \rrbracket \arrow[dl, "\textsf{Substitute } \kappa"] \\
& \llbracket \textsf{let } x = v_1 \textsf{ in } e_2 \rrbracket \,\overline{@}\, \kappa =_\beta \llbracket e_2' \rrbracket \,\overline{@}\, \kappa &
\end{tikzcd}
$$

$$
\begin{array}{cl}
 & \llbracket \textsf{let } x = v_1 \textsf{ in } e_2 \rrbracket \,\overline{@}\, \kappa \\
\equiv & \llbracket v_1 \rrbracket \,\overline{@}\, (\overline{\lambda} m.\, (\underline{\lambda} x.\, \llbracket e_2 \rrbracket \,\overline{@}\, \kappa) \,\underline{@}\, m) \\
\equiv & (\underline{\lambda} x.\, \llbracket e_2 \rrbracket \,\overline{@}\, \kappa) \,\underline{@}\, \llbracket v_1 \rrbracket_v \\
\rightsquigarrow & \llbracket e_2' \rrbracket \,\overline{@}\, \kappa
\end{array}
$$

5.1.7 RReset のケース

$$
\begin{tikzcd}
\langle v \rangle \arrow[rr, "\mathsf{RReset}"] \arrow[d, "\mathsf{CPS}"'] & & v \arrow[d, "\mathsf{CPS}"] \\
\llbracket \langle v \rangle \rrbracket \arrow[dr, "\mathsf{Substitute}\ \kappa"'] & & \llbracket v \rrbracket \arrow[dl, "\mathsf{Substitute}\ \kappa"] \\
& \llbracket \langle v \rangle \rrbracket \mathbin{\overline{@}} \kappa =_\beta \llbracket v \rrbracket \mathbin{\overline{@}} \kappa &
\end{tikzcd}
$$

$$
\begin{array}{ll}
& \llbracket \langle v \rangle \rrbracket \mathbin{\overline{@}} \kappa \\
\equiv & (\underline{\lambda} c.\, \kappa \mathbin{\overline{@}} c) \mathbin{\underline{@}} (\llbracket v \rrbracket \mathbin{\overline{@}} (\overline{\lambda} m.\, m)) \\
\equiv & (\underline{\lambda} c.\, \kappa \mathbin{\overline{@}} c) \mathbin{\underline{@}} \llbracket v \rrbracket_v \\
\leadsto & \kappa \mathbin{\overline{@}} \llbracket v \rrbracket_v \\
\equiv & \llbracket v \rrbracket \mathbin{\overline{@}} \kappa
\end{array}
$$

5.1.8 RShift のケース

$$
\begin{tikzcd}
\langle E_{p_1} \texttt{[}\ \mathcal{S} @ v\ \texttt{]} \rangle \arrow[rr, "\mathsf{RShift}"] \arrow[d, "\mathsf{CPS}"'] & & \langle v @ (\lambda y.\, \langle E_{p_2} \texttt{[}\ y\ \texttt{]} \rangle) \rangle \arrow[d, "\mathsf{CPS}"] \\
\llbracket \langle E_{p_1} \texttt{[}\ \mathcal{S} @ v\ \texttt{]} \rangle \rrbracket \arrow[dr, "\mathsf{Substitute}\ \kappa"'] & & \llbracket \langle v @ (\lambda y.\, \langle E_{p_2} \texttt{[}\ y\ \texttt{]} \rangle) \rangle \rrbracket \arrow[dl, "\mathsf{Substitute}\ \kappa"] \\
& \llbracket \langle E_{p_1} \texttt{[}\ \mathcal{S} @ v\ \texttt{]} \rangle \rrbracket \mathbin{\overline{@}} \kappa =_\beta \llbracket \langle v @ (\lambda y.\, \langle E_{p_2} \texttt{[}\ y\ \texttt{]} \rangle) \rangle \rrbracket \mathbin{\overline{@}} \kappa &
\end{tikzcd}
$$

$$
\begin{array}{lll}
& \llbracket \langle E_{p_1} \texttt{[}\ \mathcal{S} @ v\ \texttt{]} \rangle \rrbracket \mathbin{\overline{@}} \kappa & \\
\equiv & (\underline{\lambda} c.\, \kappa \mathbin{\overline{@}} c) \mathbin{\underline{@}} (\llbracket E_{p_1} \texttt{[}\ \mathcal{S} @ v\ \texttt{]} \rrbracket \mathbin{\overline{@}} (\overline{\lambda} m.\, m)) & \\
\equiv & (\underline{\lambda} c.\, \kappa \mathbin{\overline{@}} c) \mathbin{\underline{@}} (\llbracket \mathcal{S} @ v \rrbracket' \mathbin{\overline{@}} (\underline{\lambda} a.\, \llbracket E_{p_2} \texttt{[}\ a\ \texttt{]} \rrbracket \mathbin{\overline{@}} (\overline{\lambda} m.\, m))) & (\text{補題 } \mathbf{4.4.1}\ \mathit{contextContE}) \\
\equiv & (\underline{\lambda} c.\, \kappa \mathbin{\overline{@}} c) \mathbin{\underline{@}} (((\underline{\lambda} wk.\, (w \mathbin{\underline{@}} (\underline{\lambda} ak'.\, k' \mathbin{\underline{@}} (k \mathbin{\underline{@}} a))) \mathbin{\underline{@}} (\underline{\lambda} m.\, m)) \mathbin{\underline{@}} \llbracket v \rrbracket_v) \mathbin{\overline{@}} (\underline{\lambda} a.\, \llbracket E_{p_2} \texttt{[}\ a\ \texttt{]} \rrbracket \mathbin{\overline{@}} (\overline{\lambda} m.\, m))) & \\
\leadsto & (\underline{\lambda} c.\, \kappa \mathbin{\overline{@}} c) \mathbin{\underline{@}} ((\llbracket v \rrbracket_v \mathbin{\underline{@}} (\underline{\lambda} ak'.\, k' \mathbin{\underline{@}} ((\underline{\lambda} a.\, \llbracket E_{p_2} \texttt{[}\ a\ \texttt{]} \rrbracket \mathbin{\overline{@}} (\overline{\lambda} m.\, m)) \mathbin{\underline{@}} a))) \mathbin{\underline{@}} (\underline{\lambda} m.\, m)) & (\texttt{eqShiftc}) \\
\leadsto & (\underline{\lambda} c.\, \kappa \mathbin{\overline{@}} c) \mathbin{\underline{@}} ((\llbracket v \rrbracket_v \mathbin{\underline{@}} (\underline{\lambda} ak'.\, k' \mathbin{\underline{@}} (\llbracket E_{p_2} \texttt{[}\ a\ \texttt{]} \rrbracket \mathbin{\overline{@}} (\overline{\lambda} m.\, m)))) \mathbin{\underline{@}} (\underline{\lambda} m.\, m)) & \\
\equiv & \underline{\mathsf{let}}\ \ x = (\llbracket v \rrbracket_v \mathbin{\underline{@}} (\underline{\lambda} ak'.\, \llbracket \langle E_{p_2} \texttt{[}\ a\ \texttt{]} \rangle \rrbracket' \mathbin{\overline{@}} k')) \mathbin{\underline{@}} (\underline{\lambda} m.\, m)\ \ \underline{\mathsf{in}}\ \ \kappa \mathbin{\overline{@}} x & \\
\equiv & \underline{\mathsf{let}}\ \ x = (\llbracket v \rrbracket_v \mathbin{\underline{@}} \llbracket \lambda a.\, E_{p_2} \texttt{[}\ a\ \texttt{]} \rrbracket_v) \mathbin{\underline{@}} (\underline{\lambda} m.\, m)\ \ \underline{\mathsf{in}}\ \ \kappa \mathbin{\overline{@}} x & \\
\equiv & \llbracket \langle v @ (\lambda y.\, \langle E_{p_2} \texttt{[}\ y\ \texttt{]} \rangle) \rangle \rrbracket \mathbin{\overline{@}} \kappa &
\end{array}
$$