Step 4

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Abstract

In this paper I analyze the efficiency of different parsing algorithms.
First part investigates on Chomsky normal form grammars with (i) a naive top-down algorithm, (ii) a memoization algorithm, and (iii) the widely-known CYK tabulation algorithm.

Second part investigates on Linear grammars with (i) converting to Chomsky normal form then applying the CYK algorithm, (ii) directly adapting the CYK algorithm to linear grammars.

Both parts give theorical and empirical views of the algorithms' efficiency.

Introduction

Parsing is the process of analysing a string of symbols, conforming to the rules of a formal gramar. It is a fundamental computer science notion: The parser is one of the main parts of compilers and interpreters, which describe how programs are executed by the machine. Since most modern languages are context-free (and thus expressible by context-free grammars 1.1), parsers for context-free languages can be used to check that programs are syntactically correct. Having efficient parsing algorithms hence allows a faster compilation or interpretation of programs.

1 Central notions

1.1 Context-free grammar (CFG)

Parsers need set of formal rules in order to interpret the symbols of input strings, and define a syntactic relation between each other. This set of formal rules is called a grammar. A context-free grammars is a special type of grammar. Context-free grammars can express most modern programming languages.

1.1.1 Definition

A context-free grammar is defined by $G = (N, \Sigma, P, S)$, where:

1. N is a finite set, $n \in N$ is called a nonterminal element.
2. Σ is a finite set, $N \cup \Sigma = \emptyset$, $\sigma \in \Sigma$ is called a terminal element.
3. $P : N \to (N \cup \Sigma)^*$ is a finite set of production rules. The "*" symbol expresses a multiplicity. Here it means "any number of elements from N and Σ".
4. $S \in N$ called the starting symbol of G.

$L(G) = \{w \in \Sigma^* | w$ can be generated[1] with S the starting symbol$\}$ is called the "language" generated by G, and corresponds to the set of terminal strings that can be generated starting from S with production rules from P.

[1]Given a production rule $A \to \alpha$, we say that A can generate α.

Example of CFG

$$G = (N, \Sigma, P, A)$$
$$N = \{A, B\}$$
$$\Sigma = \{\alpha, \beta\}$$
$$P = \{A \to \alpha B \beta | \alpha, B \to \beta A \alpha | \beta\}$$

1.2 Chomsky normal form (CNF)

A context-free grammar G is said to be in Chomsky normal form if all of its production rules are of the form

- $A \to BC$ or
- $A \to \alpha$ or
- $S \to \epsilon$

Where A, B, C are nonterminals, S is the start symbol, ϵ denotes the empty string. Also, neither B or C may be the start symbol.
Any CFG can be transformed into a CNF grammar, with a size no larger than the square of the initial grammar's size.

1.3 Some common algorithmic techniques

Here are the main idea of the algorithmic techniques used for this assignment.

1.3.1 Tabulation

Tabulation is a dynamic programming technique that focuses on building up larger and larger subsolutions to a problem until the target solution has been reached. Each iteration uses the results of the previous ones to compute the solution faster. Tabulation algorithm are bottom-up and iterative by nature.

Example of tabulation algorithm Here is how one would use tabulation for computing fibonacci:

```
function fibo(n):
    mem[0] <- 0
    mem[1] <- 1

    for i in 2...n:
        mem[i] = mem[i-1] + mem[i-2]

    return mem[n]
```

1.3.2 Memoization

Memoization is a dynamic programming technique where the result of time-expansive function calls are stored in a data structure, in order to avoid recomputing them if the same inputs occur again. Memoization algorithm are top-own and recursive by nature.

Example of memoization algorithm Here is how one would use memoization for computing fibonacci:

```
function fibo(n):
    if n < 2:
        return n

    if mem[n] is null:
        mem[n] = fibo(n-1) + fibo(n-2)

    return mem[n]
```

1.4 Cocke-Younger-Kasami (CYK) algorithm

The CYK algorithm is a parsing algorithm: The input is G, a context free grammar in CNF and α a string, and the output is true if and only iff $\alpha \in L(G)$, $L(G)$ being the language generated by G.

It is easy to tweak the algorithm so that it returns the rules of G used for generating α.

1.4.1 Naive top-down implementation

Let G be the grammar, Σ the set of nonterminals symbols of G, P the rules of G, α the string to parse

and $S \in \Sigma$ the starting symbol of G.
Here is how *parse(S, α)* behaves:

- If $|\alpha| = 1$: Return true if $(S \to \alpha) \in P$ else return false
- If $|\alpha| > 1$: For each $(S \to AB) \in P$, and for each possible partition of α into two parts β and δ, return true if both *parse(A, β)* and *parse(B, δ)* return true. If no such combination of (A, B, β, δ) is found, return false

This algorithm consists of a recursive enumeration of each rule applying possibility.

```
function naive(G: the grammar, S: the
     starting symbol, alpha[i, j]: the
     string):
    if j = 1:
        return G.contains(S -> alpha)

    for each P production rule of G, of
         the form S -> AB:
        for k = 0...j:
            if naive(G, A, alpha[i, k]) and
                 naive(G, B, alpha[k, j]):
                return true

    return false
```

Complexity: The exact time cost is hard to compute, but we can get a worst case lower time bound which will be sufficient for comparing this algorithm with the other ones:
Given an instance of size n, if we consider the two recursive calls of depth $n-1$ (rooted in the instances $[1, n-1]$, and $[2, n]$), we get a recursion tree of height n and of branching factor 2. We can hence express our lower bound as follows:

$$T(n) > n + 2(n-1)|G| + 4(n-2)|G| + \cdots + 2^{n-1}|G|$$

$$\Leftrightarrow T(n) > |G| \sum_{i=0}^{n-1} 2^i (n-i)$$

$$\leftrightarrow T(n) = \Omega(2^n)$$

1.4.2 Top-down implementation with memoization

The Top-down implementation of the CYK algorithm works just like the naive implementation but maintains a global three-dimensional boolean table Z.
Each time a recursive call such as "*parse(A, $\alpha[i, j]$)*" is made (A being the starting symbol, and α the string to parse from index i to j), the result is stored in $Z[A, i, j]$, so that if the result is needed again, it can be accessed in constant time.
That programming technique is called memoization and has the following characteristics:

- It stops on the first positive result, instead of going through all of the possible subproblems, which can save time
- It has a recursive nature, which makes it easy to write but can cause excessive memory use with unadapted compilers
- It only requires a small adaptation of the naive top-down method, which is maintain and use the global table Z

Pseudocode Here is the pseudocode for the top-down implementation:

```
global Z
function TD(G: the grammar, S: the starting
     symbol, alpha[i, j]: the string):
    if j = 1:
        return G.contains(S -> alpha)

    for each P production rule of G, of the
         form S -> AB:
        for k = 0...j:
            if Z[A, i, k] is null:
                Z[A, i, k] <- TD(G, A, alpha[i,
                     k]

            if Z[B, k, j] is null:
                Z[B, k, j] <- TD(G, B, alpha[k,
                     j]

            if Z[A, i, k] and T[B, k, j]:
                return true
    return false
```

Complexity: At most, there will be as many recursive calls as there are cells in T, i.e. $|G|n^2$. Since the body of the function, except the recursive calls has a time complexity of $O(n)$, the total time complexity is $T(n) = O(n) * O(|G|n^2) = O(|G|n^3) = O(n^3)$

1.4.3 Bottom-up implementation with tabulation

Let G be the grammar, S the starting symbol, α the string to parse.

The following bottom-up implementation of the CYK algorithm is the original version that can be found in textbooks. The idea is to maintain a table of substrings that can be parsed by rules of the grammar, storing the possible starting symbol(s). The iterations start at substring size 1, and end at substring size $|\alpha|$. If S is stored in the table for the size $|\alpha|$, $\alpha \in L(G)$. Note that previous iterations are used for building the next ones.
Let P be the table of substrings. $P[a, b, c]$ is true iff the substring of α starting at index c and of length a can be parsed with the production rule of index b.
That programming technique is called tabulation and has the following characteristics:

- It has an iterative nature, which does not cause excessive memory use with any compiler, but it is usually harder to program
- It computes all of the possible problems, which can cost a bit of compututational time (without increasing the asymptotic complexity), but allows to give instantly the solution to any subproblem afterwards ($\bigcup_b P[a, b, c] \equiv parse(\alpha[c, c + a])$.

Pseudocode Here is the pseudocode for the bottom-up implementation:

```
function TD(G: the grammar, alpha[1, n]: the
    string):
    R[1, r] <- G.nonterminal_symbols // with
        R[1] the starting symbol of G
    let P[n, n, r] an array of boolean with
        all values initialized to false
    for s in 1...n:
        for each production rule R[v] -> a[s]:
            P[1, v, s] <- true

    for l in 2...n:
        for s in 1...n-l+1:
            for p in 1...l-1:
                for each production rule R[a]
                    -> R[b]R[c]:
                    if P[p, s, b] and P[l-p,
                        s+p, c]:
                        P[l, s, a] <- true

    return P[n, 1, 1]
```

Complexity: Most of the work is done in four imbricated loops, three of them are bounded by n, and the last one bounded by $|G|$. We hence have $T(n) = O(n^3|G|) = O(n^3)$

2 Empirical measurements

Given the respective theorical complexities, we should expect a running time growing extremely fast as n grows for the naive algorithm, and similar running times for the top-down and the bottom-up dynamic programming variants.

2.1 Comparison with theoritical running time estimations

2.1.1 Well-balanced parentheses

The grammar used for the next tests only accepts well balanced parentheses:

$$S \to SS|LA|LR$$
$$A \to SR$$
$$L \to ($$
$$R \to)$$

$\frac{n}{2}$ opening followed by $\frac{n}{2}$ closing, i.e. (((...)))
The naive complexity is too important for it to be on the graph.
You may find the figure at appendix 0.4.

$\frac{n}{2}$ pairs, i.e. '()()...'
You may find the figure at appendix 0.5.

Here, both the naive and the top-down variants show a linear complexity, whilst the bottom-up variant has a quadratic complexity. This probably comes from the fact that the bottom-up variant performs the parsing for every possible substring while the others algorithms stop at the first positive recursive call.

For the naive and the top-down DP algorithms, if the first recursive call (i.e. $k = 0$) is positive each time, the cost is linear, since the recursion tree is equivalent to a basic for-each loop.

It is interesting to note that, in terms of effective computation time, the naive version is slightly faster, since there is no additional table management:
You may find the figure at appendix 0.6.

closing parenthesis then $\frac{n}{2} - 1$ pairs, i.e. ')()()...'
The naive complexity is too important for it to be on the graph.
You may find the figure at appendix 0.7.

$\frac{n}{2} - 1$ pairs, then opening parenthesis, i.e. '()()...('
The naive complexity is too important for it to be on the graph.
You may find the figure at appendix 0.8.

2.1.2 Stupid grammar

The grammar used for the next tests does not generate any word ($L(G) = \emptyset$) but the parsing algorithms can still be applied on it. Here is how it looks:

$$S \to ST|TS$$
$$T \to a$$

several a's, i.e. 'aa...'
The naive complexity is too important for it to be on the graph.
You may find the figure at appendix 0.9.

3 Linear grammars

3.1 Turn it into Chomsky normal form

This is the easy solution: Transform each rule that does not match CNF into CNF rule:

$$A \to B\alpha \equiv A \to BC, C \to \alpha$$
$$A \to \alpha B \equiv A \to CB, C \to \alpha$$

Since each initial rule generates at most 2 CNF ones, the size of the generated CNF grammar is $O(2|G|) = O(|G|)$, $|G|$ being grammar's size. The transformation's time complexity is $O(|G|)$.

Since the CYK algorithm has a time complexity of $O(n^3|G|)$, and since $|G|$ can be considered unchanged after the transformation, the total parsing time complexity is $O(|G|) + O(n^3|G|) = O(n^3|G|)$.

3.2 Adapting the CYK algorithm

It is possible to adapt the CYK algorithm to parse linear grammars, changing the way the global three-dimensional table is updated. The idea is that instead of testing all the possible string partitions, we check partitions of size $(1, n-1)$ for rules of form $A \to \alpha B$ and partitions of size $(n-1, 1)$ for rules of form $A \to B\alpha$.

Formally, the table is filled as follows:
Let $G = (N, \Sigma, P, S)$, N being the nonterminal symbols, Σ the terminal symbols, P the production rules, and S the starting symbol. let t a global three-dimensional table, s the parsed string and $n = |s|$.
$t[i,j] = \bigcup_{A \in N} \{A | A \to s[i]t[i-1, j+1] \in P$
$\vee A \to t[i-1, j]s[i+j-1] \in P\}$
for $i \in [1, n], j \in [1, n-i+1]$
If $S \in t[1, n] \quad s \in L(G)$, else $s \notin L(G)$.

Since the algorithm iterates over a n^2 sized array, and since the whole grammar is iterated for each cell, $T(n) = O(n^2|G|) < O(n^3|G|)$. It is hence worth it to adapt the CYK algorithm instead of turning G into CNF.

3.3 Empirical measurements

It is interesting to compare the two precedent methods to verify that both follow the expected behavior.

3.3.1 abc

The grammar used for the next tests only accepts strings of form $a^k b^l c^k \ \forall k \in \mathbb{N} \ \forall l \in \mathbb{N}\backslash 0$:

$$G = (N, \Sigma, P, A)$$
$$P = \left\{ \begin{array}{l} S \rightarrow Ac \\ S \rightarrow b \\ A \rightarrow aS \\ A \rightarrow aB \\ B \rightarrow bS \end{array} \right\}$$

Inputs from $L(G)$
Here are the measured time complexities for strings from $L(G)$.
You may find the figure at appendix 0.10.

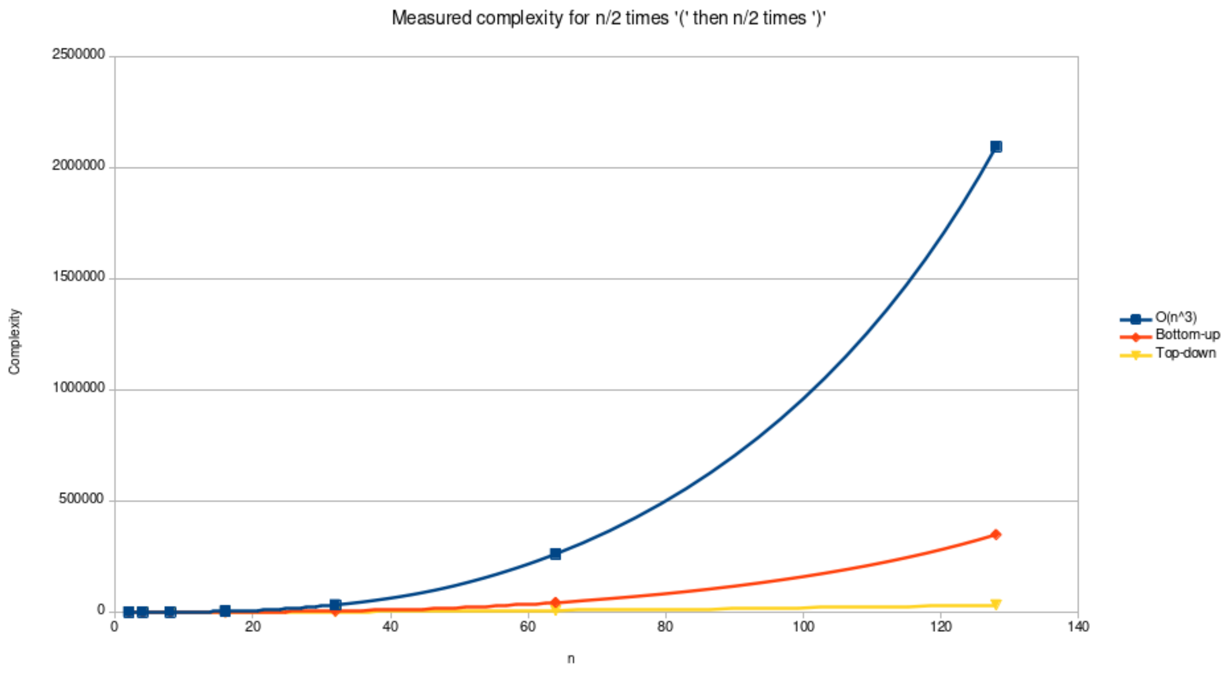
As expected, the adapted CYK algorithm's is $O(n^2|G|)$ for the given inputs, while turning G into a CNF grammar is $O(n^3|G|)$.

Inputs not from $L(G)$
Here are the measured time complexity for random strings not from $L(G)$, i.e. $\{\alpha = (a+b+c)^n | \alpha \notin L(G)\}$.
You may find the figure at appendix 0.11.

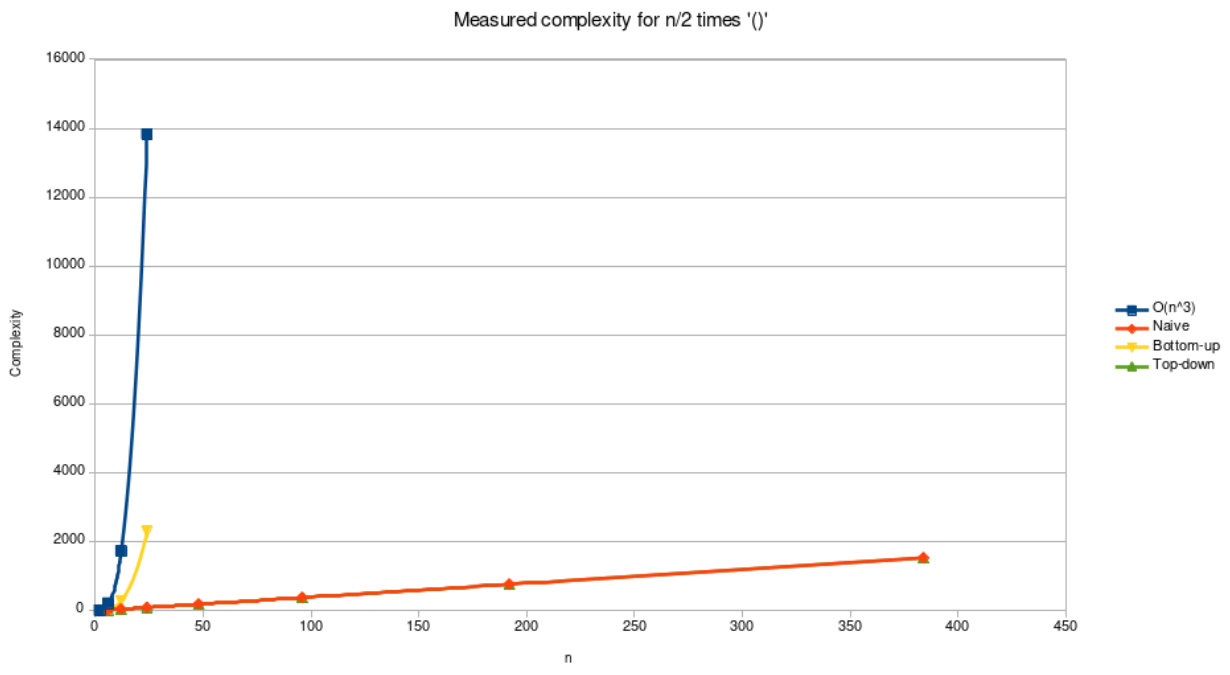
With wrong inputs, The measured complexities preserves the same behavior. It is hence indeed more efficient to adapt the CYK algorithm.

Appendices

0.4

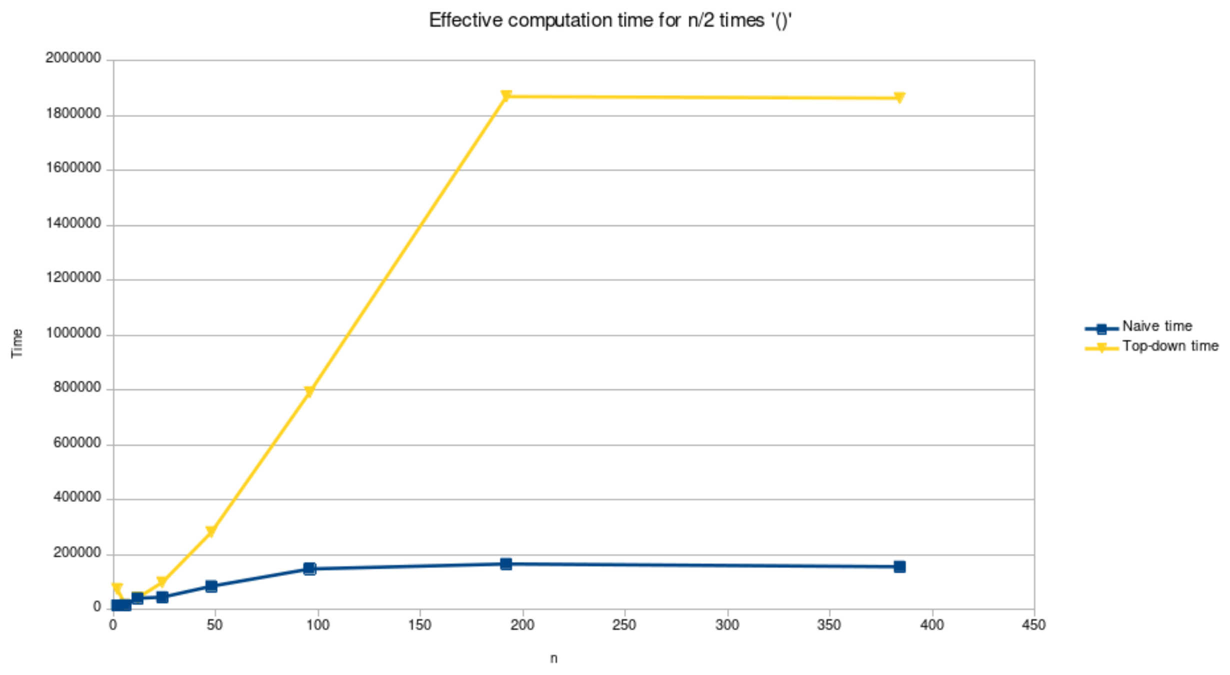


0.5

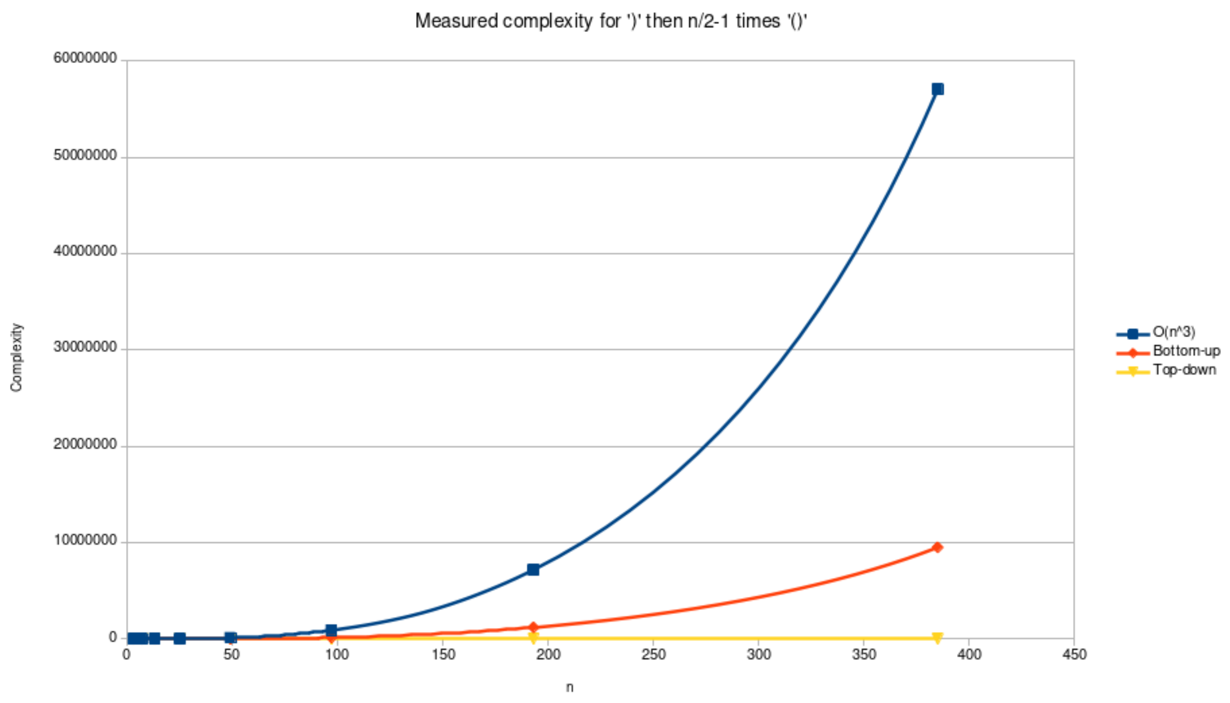


Measured complexity for n/2 times '()'
Complexity
n
16000
14000
12000
10000
8000
6000
4000
2000
0
0
50
100
150
200
250
300
350
400
450
O(n^3)
Naive
Bottom-up
Top-down

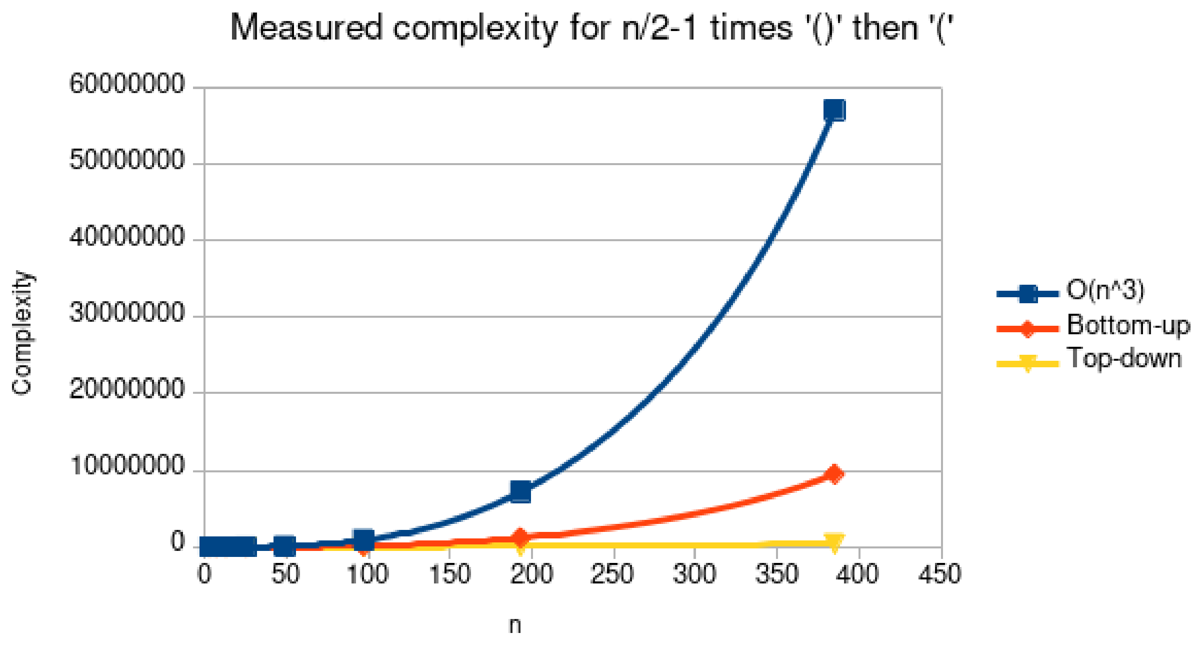
0.6


Effective computation time for n/2 times '()'
Time
n
Naive time
Top-down time
2000000
1800000
1600000
1400000
1200000
1000000
800000
600000
400000
200000
0
0
50
100
150
200
250
300
350
400
450

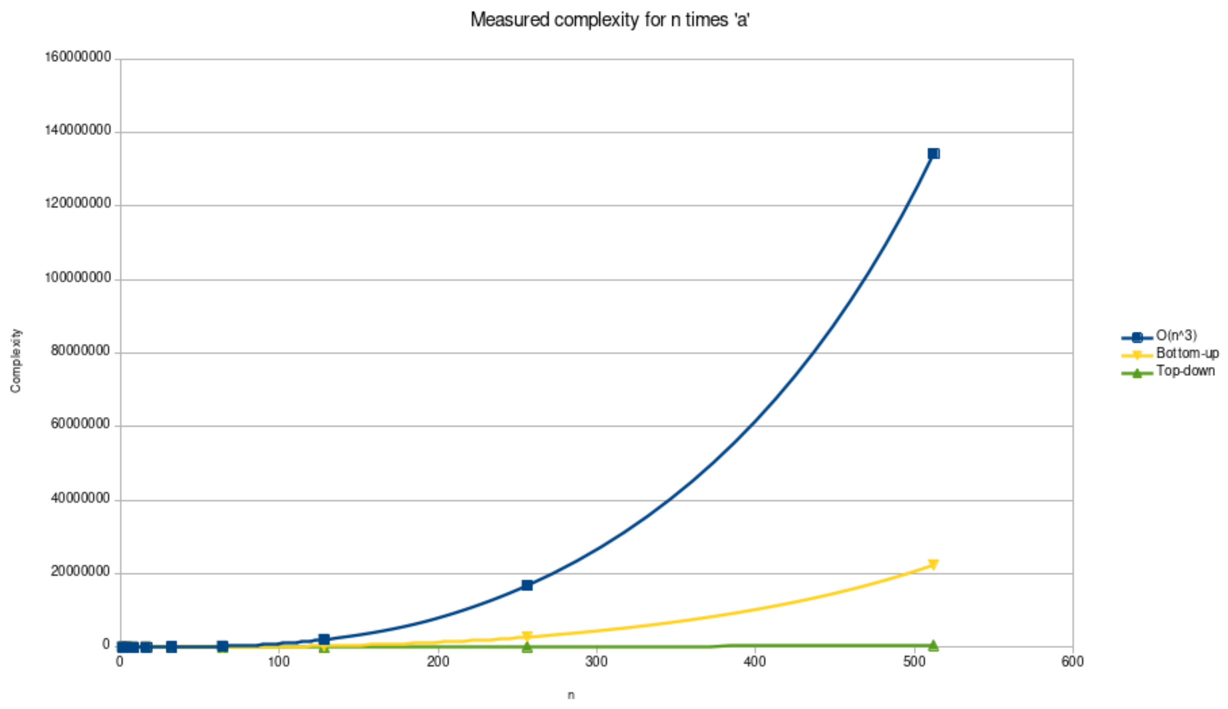
0.7


Measured complexity for ')' then n/2-1 times '()'
Complexity
n
60000000
50000000
40000000
30000000
20000000
10000000
0
0
50
100
150
200
250
300
350
400
450
O(n^3)
Bottom-up
Top-down

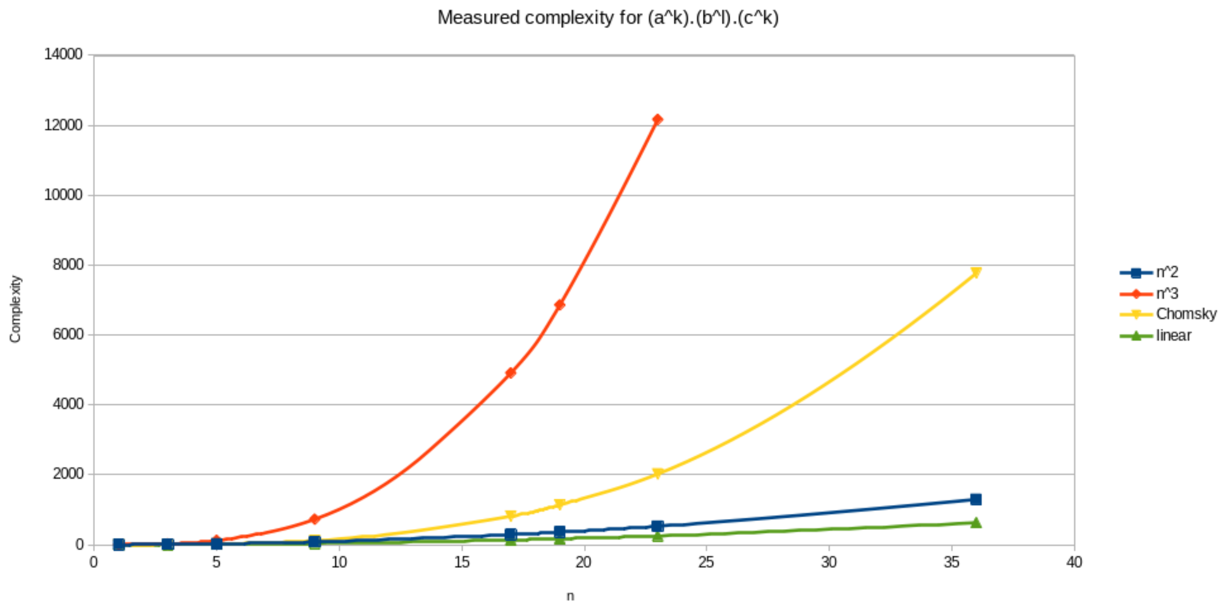
0.8


Measured complexity for n/2-1 times '()' then '('
Complexity
60000000
50000000
40000000
30000000
20000000
10000000
0
0
50
100
150
200
250
300
350
400
450
n
O(n^3)
Bottom-up
Top-down

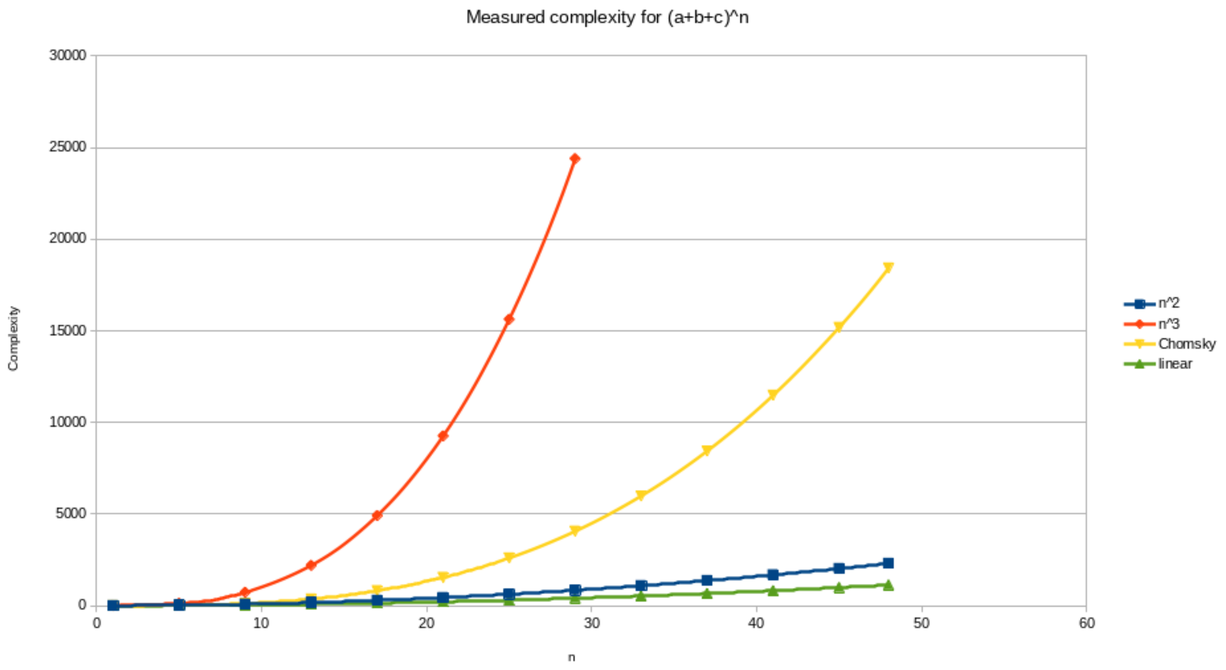
0.9


Measured complexity for n times 'a'
Complexity
n
160000000
140000000
120000000
100000000
80000000
60000000
40000000
20000000
0
0
100
200
300
400
500
600
O(n^3)
Bottom-up
Top-down

0.10


Measured complexity for (a^k).(b^l).(c^k)
Complexity
n
n^2
n^3
Chomsky
linear
0
2000
4000
6000
8000
10000
12000
14000
0
5
10
15
20
25
30
35
40

0.11


Measured complexity for (a+b+c)^n
Complexity
n
30000
25000
20000
15000
10000
5000
0
0
10
20
30
40
50
60
n^2
n^3
Chomsky
linear