

Decidable, Tag-Based Semantic Subtyping for Nominal Types, Tuples, and Unions

Julia Belyakova
Northeastern University
belyakova.y@northeastern.edu

Abstract

Text of abstract

Keywords keyword1, keyword2, keyword3

1 Introduction

Many static type systems rely on subtyping. Informally, subtyping relation $T <: S$ states that a value of type T is safe to use in the context that expects S . For example, if class `Rectangle` is a subtype of `Shape`, an instance of `Rectangle` can be passed to a function with an argument of type `Shape`.

In languages with nominal typing, subtyping can also be used for run-time dispatch, in particular, *multiple dynamic dispatch* (MDD). It allows a function to have several implementations for different types of arguments, and the most suitable implementation for a particular call is picked dynamically, based on the run-time types of all arguments. For example, consider two implementations of addition, $+(Number, Number)$ and $+(String, String)$, and the call $3+5$. In this case, language run-time should pick the implementation for numbers because $Int <: Number$ but $Int \not<: String$.

It is often convenient to think of subtyping $T <: S$ in terms of the set inclusion: “the elements of T are the subset of the elements of S ” [?]. This intuition is not always correct, but there has been research on the so-called *semantic subtyping* [?] where subtyping is defined exactly as the subset relation. Namely, types are given the set-theoretic interpretation $\llbracket \tau \rrbracket = \{v \mid \vdash v : \tau\}$, and subtyping $\tau_1 <: \tau_2$ is defined as $\llbracket \tau_1 \rrbracket \subseteq \llbracket \tau_2 \rrbracket$. In this way, a semantic definition of subtyping intertwines with a static typing relation.

However, subtyping does not always go together with static typing — it is also applicable in the context of *dynamically* typed languages. As mentioned before, subtyping can be used for multiple dynamic dispatch, and MDD is rather widespread among dynamic languages, for instance, Common Lisp, Julia, Clojure. Such languages do not prevent type errors with static type checking but detect them at run-time, by inspecting type tags associated with values. Type tags are also used during dynamic dispatch: language run-time looks at the tags attached to the arguments to find out their run-time types.

In this paper, we attempt to fill the gap between *semantic subtyping* and *dynamically* typed languages with *nominal*

$\tau \in \text{TYPE}$	$::=$	Types
	$\tau_1 \times \tau_2$	covariant pair
	$\tau_1 \cup \tau_2$	untagged union
	$cname$	concrete nominal type
	$aname$	abstract nominal type

$cname$	\in	$\{Int, Flt, Cmplx, Str\}$
$aname$	\in	$\{Real, Num\}$

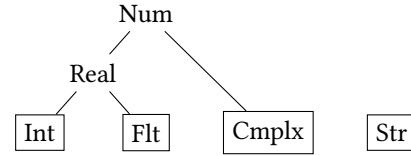


Figure 1. MINIJL: type grammar and nominal hierarchy

typing. Instead of directly interpreting types as sets of values, we interpret them as sets of *type tags* assuming each value is associated with a tag. Our contributions are as follows:

1. Tag-based semantic interpretation of types for a language with nominal types, tuples, and unions.
2. Two syntactic definitions of subtyping, declarative and reductive, along with the Coq-mechanized proofs that the definitions are equivalent and coincide with the semantic interpretation.
3. Proof of decidability of the reductive subtyping.
4. Discussion of the implications of using semantic subtyping for multiple dynamic dispatch.

2 Semantic Subtyping in MINIJL

We build the presentation around a small type language MINIJL, presented in Fig. 1. Types of MINIJL, denoted by $\tau \in \text{TYPE}$, include pairs, unions, and nominal types: $cname$ denotes *concrete* nominal types that have direct instances, and $aname$ denotes *abstract* nominal types.

To simplify the development, we decided to work with a particular hierarchy of nominal types (presented in Fig. 1 as a tree) instead of a general class table. There are four concrete, leaf types (depicted in rectangles) and two abstract types in the hierarchy. Formally, the hierarchy can be represented with a list of declarations $n_1 \triangleright n_2$ read as “ n_1 is a declared subtype of n_2 ” where n is either $cname$ or $aname$. In the case

$v \in \text{VALTYPE}$	$::=$	<i>Value Types</i>
	$cname$	concrete nominal type
	$v_1 \times v_2$	pair of value types

Figure 2. Value types

$\llbracket \cdot \rrbracket : \text{TYPE} \rightarrow \mathcal{P}(\text{VALTYPE})$
$\llbracket cname \rrbracket = \{cname\}$
$\llbracket \text{Real} \rrbracket = \{\text{Int}, \text{Flt}\}$
$\llbracket \text{Num} \rrbracket = \{\text{Int}, \text{Flt}, \text{Cmplx}\}$
$\llbracket \tau_1 \times \tau_2 \rrbracket = \{v_1 \times v_2 \mid v_1 \in \llbracket \tau_1 \rrbracket, v_2 \in \llbracket \tau_2 \rrbracket\}$
$\llbracket \tau_1 \cup \tau_2 \rrbracket = \llbracket \tau_1 \rrbracket \cup \llbracket \tau_2 \rrbracket$

Figure 3. Tag-based semantic interpretation of types

of MINIJL, the hierarchy is defined as follows:

$\text{NomHrc} = [\text{Real} \triangleright \text{Num}, \text{Int} \triangleright \text{Real}, \text{Flt} \triangleright \text{Real}, \text{Cmplx} \triangleright \text{Num}]$.

Nominal hierarchy should not have cycles, and each type can have only one parent. Following the spirit of the Julia language, we do not allow concrete types to have declared subtypes, i.e. concrete types are always leafs.

Value Types Not all types of MINIJL can have direct instances, and, therefore, serve as type tags. Types that *can* be used as tags we call **value types**. Their formal definition is given in Fig. 2: value type $v \in \text{VALTYPE}$ is either a concrete nominal type or a pair of value types. For example, Flt , $\text{Int} \times \text{Int}$, and $\text{Str} \times (\text{Int} \times \text{Int})$ are all value types. Union types, just as abstract nominal types, are not value types. Even the type $\text{Int} \cup \text{Int}$ is not a value type, though it describes the same set of values as the value type Int . Note that $\text{VALTYPE} \subset \text{TYPE}$, i.e. each value type is a type.

2.1 Semantic Interpretation of Types

As mentioned in Sec. 1, we interpret types in terms of type tags, or value types, instead of values, so we call this semantic interpretation *tag-based*. Formally, the interpretation is given by the function $\llbracket \cdot \rrbracket$ that maps a type $\tau \in \text{TYPE}$ into a set of value types $s \in \mathcal{P}(\text{VALTYPE})$, as presented in Fig. 3.

The interpretation of a type states what values constitute the type: if $v \in \llbracket \tau \rrbracket$, then all instances of v , i.e. values v tagged with v , belong to τ . Thus, a *concrete nominal* type $cname$ is comprised only of its direct instances. *Abstract nominal* types do not have direct instances, but we want their interpretation to reflect the nominal hierarchy. For example, a Num value is either a concrete complex number or a real number, which, in turn, is either a concrete integer or a floating point value. Therefore, the set of values of type Num is described by the set of value types $\{\text{Cmplx}, \text{Int}, \text{Flt}\}$. More generally, the interpretation of abstract nominal types $aname$ can be given

$\frac{}{\vdash cname < cname}$ MT-CNAME		
$\frac{}{\vdash \text{Int} < \text{Real}}$ MT-INTREAL	$\frac{}{\vdash \text{Flt} < \text{Real}}$ MT-FLTREAL	
$\frac{}{\vdash \text{Int} < \text{Num}}$ MT-INTNUM	$\frac{}{\vdash \text{Flt} < \text{Num}}$ MT-FLTNUM	$\frac{}{\vdash \text{Cmplx} < \text{Num}}$ MT-CMPLXNUM
$\frac{\vdash v_1 < \tau_1 \quad \vdash v_2 < \tau_2}{\vdash v_1 \times v_2 < \tau_1 \times \tau_2}$ MT-PAIR		
$\frac{\vdash v < \tau_1}{\vdash v < \tau_1 \cup \tau_2}$ MT-UNION1	$\frac{\vdash v < \tau_2}{\vdash v < \tau_1 \cup \tau_2}$ MT-UNION2	

Figure 4. Matching relation in MINIJL

in then following way:

$$\llbracket aname \rrbracket = \{cname \mid cname \triangleright^* aname\},$$

where the relation $n_1 \triangleright^* n_2$ means that n_1 is transitively a declared subtype of n_2 :

$$\frac{n_1 \triangleright n_2 \in \text{NomHrc}}{n_1 \triangleright^* n_2} \quad \frac{n_1 \triangleright^* n_2 \quad n_2 \triangleright^* n_3}{n_1 \triangleright^* n_3}.$$

Finally, *pairs* and *unions* are interpreted set-theoretically.

Once we have the interpretation of types, we define **tag-based semantic subtyping** as the subset relation:

$$\tau_1 \stackrel{\text{sem}}{<} \tau_2 \stackrel{\text{def}}{=} \llbracket \tau_1 \rrbracket \subseteq \llbracket \tau_2 \rrbracket.$$

2.2 Syntactic Model of Semantic Subtyping

Let us unroll the definition of semantic subtyping¹:

$$\tau_1 \stackrel{\text{sem}}{<} \tau_2 \stackrel{\text{def}}{=} \forall v. (v \in \llbracket \tau_1 \rrbracket \implies v \in \llbracket \tau_2 \rrbracket). \quad (1)$$

Its main ingredient is the relation $v \in \llbracket \tau \rrbracket$, which has a syntactic counterpart – an equivalent² inductive, syntax-directed relation $\vdash v < \tau$ defined in Fig. 4. We call the latter **matching relation**, read “tag v matches type τ ”.

Using the matching relation, we define a syntactic model of semantic subtyping $\models \tau_1 \subseteq \tau_2$ as follows:

$$\models \tau_1 \subseteq \tau_2 \stackrel{\text{def}}{=} \forall v. (\vdash v < \tau_1 \implies \vdash v < \tau_2). \quad (2)$$

Because $v \in \llbracket \tau \rrbracket$ and $\vdash v < \tau$ are equivalent, definitions (1) and (2) are also equivalent:

$$\models \tau_1 \subseteq \tau_2 \iff \tau_1 \stackrel{\text{sem}}{<} \tau_2. \quad (3)$$

¹In set theory, the subset relation $X \subseteq Y \stackrel{\text{def}}{=} \forall x. (x \in X \implies x \in Y)$.

²It is easy to show by induction on τ that the matching relation is sound and complete with respect to the member-of-interpretation relation, i.e. $\forall v. (\vdash v < \tau \iff v \in \llbracket \tau \rrbracket)$.

We will later use the syntactic model 2 to prove that alternative, syntactic definitions of subtyping coincides with the semantic interpretation.

3 Syntactic Definitions of Subtyping

While the semantic approach does provides us with a useful intuition, we need to be able to build the subtyping algorithm in order to use subtyping. However, neither of the definitions (1), (2) can be directly computed because of the quantification $\forall v$. Therefore, we provide another, *syntactic* definition of subtyping, which is equivalent to the semantic one and also straightforward to implement.

We do this in two steps. First, we give an inductive definition that is handy to reason about, called *declarative* subtyping. We prove it equivalent to the semantic subtyping using the syntactic model discussed above. Second, we provide a *reductive*, syntax-directed definition of subtyping and prove it equivalent to the declarative definition (and, hence, the semantic one as well). We prove that the reductive subtyping is decidable, that is, for any two types τ_1 and τ_2 , it is possible to prove that τ_1 either is a subtype of τ_2 or is not. Since the Coq-proof is constructive, it essentially gives an algorithm to decide subtyping. However, it is also possible to devise the algorithm as a straightforward recursive function.

3.1 Declarative Subtyping

Declarative definition of subtyping is provided in Fig. 5. The definition is mostly comprised of the standard rules of syntactic subtyping for unions and pairs. Namely, reflexivity and transitivity (SD-REFL and SD-TRANS), subtyping of pairs (SD-PAIRS), and subtyping of unions (SD-UNIONL, SD-UNIONR1, SD-UNIONR2). Though SD-UNIONR* rules are seemingly very strict, transitivity allows us to derive judgments such as $\vdash \text{Int} \leq \text{Str} \cup \text{Real}$ via $\vdash \text{Int} \leq \text{Real}$ and $\vdash \text{Real} \leq \text{Str} \cup \text{Real}$. Note that we do need the syntactic definition of subtyping to be *reflexive* and *transitive* because so is semantic subtyping, which is just the subset relation.

Semantic subtyping also forces us to add rules for distributing pairs over unions, SD-DISTR1 and SD-DISTR2. For example, consider two types, $(\text{Str} \times \text{Int}) \cup (\text{Str} \times \text{Flt})$ and $\text{Str} \times (\text{Int} \cup \text{Flt})$. They have the same semantic interpretation — $\{\text{Str} \times \text{Int}, \text{Str} \times \text{Flt}\}$. Therefore, we should be able to derive the equivalence of the types using the declarative definition, i.e., the declarative subtyping should hold in both directions. One direction is trivial:

$$\frac{\vdash \text{Str} \leq \text{Str} \quad \vdash \text{Int} \leq \text{Int} \cup \text{Flt} \quad \dots}{\vdash \text{Str} \times \text{Int} \leq \text{Str} \times (\text{Int} \cup \text{Flt}) \quad \vdash \text{Str} \times \text{Flt} \leq \dots} \quad \vdash (\text{Str} \times \text{Int}) \cup (\text{Str} \times \text{Flt}) \leq \text{Str} \times (\text{Int} \cup \text{Flt})$$

But the other direction,

$$\vdash \text{Str} \times (\text{Int} \cup \text{Flt}) \leq (\text{Str} \times \text{Int}) \cup (\text{Str} \times \text{Flt}),$$

$$\begin{array}{c} \frac{}{\vdash \tau \leq \tau} \text{SD-REFL} \quad \frac{\vdash \tau_1 \leq \tau_2 \quad \vdash \tau_2 \leq \tau_3}{\vdash \tau_1 \leq \tau_3} \text{SD-TRANS} \\[10pt] \frac{}{\vdash \text{Int} \leq \text{Real}} \text{SD-INTREAL} \quad \frac{}{\vdash \text{Flt} \leq \text{Real}} \text{SD-FLTREAL} \\[10pt] \frac{}{\vdash \text{Real} \leq \text{Num}} \text{SD-REALNUM} \quad \frac{}{\vdash \text{Cmplx} \leq \text{Num}} \text{SD-CMPLXNUM} \\[10pt] \boxed{\frac{}{\vdash \text{Real} \leq \text{Int} \cup \text{Flt}} \text{SD-REALUNION}} \quad \boxed{\frac{}{\vdash \text{Num} \leq \text{Real} \cup \text{Cmplx}} \text{SD-NUMUNION}} \\[10pt] \frac{\vdash \tau_1 \leq \tau'_1 \quad \vdash \tau_2 \leq \tau'_2}{\vdash \tau_1 \times \tau_2 \leq \tau'_1 \times \tau'_2} \text{SD-PAIR} \\[10pt] \frac{\vdash \tau_1 \leq \tau' \quad \vdash \tau_2 \leq \tau'}{\vdash \tau_1 \cup \tau_2 \leq \tau'} \text{SD-UNIONL} \\[10pt] \frac{}{\vdash \tau_1 \leq \tau_1 \cup \tau_2} \text{SD-UNIONR1} \quad \frac{}{\vdash \tau_2 \leq \tau_1 \cup \tau_2} \text{SD-UNIONR2} \\[10pt] \frac{}{\vdash (\tau_{11} \cup \tau_{12}) \times \tau_2 \leq (\tau_{11} \times \tau_2) \cup (\tau_{12} \times \tau_2)} \text{SD-DISTR1} \\[10pt] \frac{}{\vdash \tau_1 \times (\tau_{21} \cup \tau_{22}) \leq (\tau_1 \times \tau_{21}) \cup (\tau_1 \times \tau_{22})} \text{SD-DISTR2} \end{array}$$

Figure 5. Declarative subtyping for MINIJL

cannot be derived without SD-DISTR2 rule: $\text{Str} \times (\text{Int} \cup \text{Flt})$ is not a subtype of either $\text{Str} \times \text{Int}$ or $\text{Str} \times \text{Flt}$, so we cannot apply SD-UNIONR* rules³.

The most interesting part of the definition hides in subtyping of nominal types. There are four obvious rules coming directly from the nominal hierarchy. For instance, SD-REALNUM mirrors the fact that $\text{Real} \triangleright \text{Num} \in \text{NomHrc}$. But there are also new rules, SD-REALUNION and SD-NUMUNION, (highlighted in Fig. 5), which are dictated by the semantic subtyping. For example, we need SD-REALUNION to prove the equivalence of types $\text{Int} \cup \text{Flt}$ and Real , which have the same interpretation $\{\text{Int}, \text{Flt}\}$.

3.2 Correctness of Declarative Subtyping

In order to show that the declarative subtyping is equivalent to the semantic one, we need to prove that the former is sound and complete with respect to the latter one, that is:

$$\forall \tau_1, \tau_2. (\vdash \tau_1 \leq \tau_2 \iff \tau_1 \stackrel{\text{sem}}{<} \tau_2).$$

³Transitivity does not help in this case.

As discussed in Sec. 2.2, the syntactic model $\models \tau_1 \subseteq \tau_2$ is equivalent to the semantic subtyping. So for proofs, we will be using the model.

Theorem 1 (Correctness of Declarative Subtyping).

$$\forall \tau_1, \tau_2. (\vdash \tau_1 \leq \tau_2 \iff \models \tau_1 \subseteq \tau_2)$$

In order to prove the theorem, we need several auxiliary observations. Let us recall the definition of $\models \tau_1 \subseteq \tau_2$:

$$\models \tau_1 \subseteq \tau_2 \stackrel{\text{def}}{=} \forall v. (\vdash v < \tau_1 \implies \vdash v < \tau_2).$$

The first thing to note is that subtyping a value type coincides with matching:

$$\forall v, \tau. (\vdash v \leq \tau \iff \vdash v < \tau). \quad (4)$$

Having that, it is easy to prove the *soundness* direction of T. 1.

Lemma 1 (Soundness of Declarative Subtyping).

$$\forall \tau_1, \tau_2. [\vdash \tau_1 \leq \tau_2 \implies \forall v. (\vdash v < \tau_1 \implies \vdash v < \tau_2)]$$

Proof. We know $\vdash v < \tau_1$ and $\vdash \tau_1 \leq \tau_2$. We need to show that $\vdash v < \tau_2$. First, we apply (4) to $\vdash v < \tau_1$ and $\vdash v < \tau_2$. Now it suffices to show that $\vdash v \leq \tau_2$ follows from $\vdash v \leq \tau_1$ and $\vdash \tau_1 \leq \tau_2$, which is trivially true by SD-TRANS. \square

Lemma 2 (Completeness of Declarative Subtyping).

$$\forall \tau_1, \tau_2. (\models \tau_1 \subseteq \tau_2 \implies \vdash \tau_1 \leq \tau_2)$$

This direction of T. 1 is more challenging. The key observation here is that L. 2 can be shown for τ_1 of the form $v_1 \cup v_2 \cup \dots \cup v_n$ (we omit parenthesis because union is associative). In this case, in the definition of $\models \tau_1 \subseteq \tau_2$ the only v s that match τ_1 and τ_2 are v_i . By (4) we know that matching implies subtyping, so we also have $\vdash v_i \leq \tau_2$. From the latter, it is easy to show that $\vdash \tau_1 \leq \tau_2$ because τ_1 is just a union of value types, and subtyping of the left-hand side union amounts to subtyping its components, according to the SD-UNIONL rule.

Normal Form We say that a type $\tau \equiv v_1 \cup v_2 \cup \dots \cup v_n$ is in **normal form** and denote this fact by $\text{InNF}(\tau)$ (formal definition of InNF is given in Fig. 8, App. A). For each type τ , there is an equivalent normalized type that can be computed with the function NF defined in Fig. 6 (the auxiliary function un_prs can be found in Fig. 9, App. A). Note that abstract nominal types are unfolded into unions of all their value subtypes. A pairs gets rewritten into a union of value pairs, thus producing a type in the disjunctive normal form.

Using the fact that every type can be normalized, and that declarative subtyping is complete for normalized types, we can finally prove L. 2.

Lemma 3 (Properties of the Normal Form).

$$\forall \tau. (\text{InNF}(\text{NF}(\tau)) \wedge \vdash \tau \leq \text{NF}(\tau) \wedge \vdash \text{NF}(\tau) \leq \tau)$$

Lemma 4 (Completeness for Normalized Types).

$$\forall \tau_1, \tau_2 \mid \text{InNF}(\tau_1). (\models \tau_1 \subseteq \tau_2 \implies \vdash \tau_1 \leq \tau_2)$$

$\text{NF} : \text{TYPE}$	\rightarrow	TYPE
$\text{NF}(\text{cname})$	$=$	cname
$\text{NF}(\text{Real})$	$=$	$\text{Int} \cup \text{Flt}$
$\text{NF}(\text{Num})$	$=$	$\text{Int} \cup \text{Flt} \cup \text{Cmplx}$
$\text{NF}(\tau_1 \times \tau_2)$	$=$	$\text{un_prs}(\text{NF}(\tau_1), \text{NF}(\tau_2))$
$\text{NF}(\tau_1 \cup \tau_2)$	$=$	$\text{NF}(\tau_1) \cup \text{NF}(\tau_2)$

Figure 6. Computing normal form of MINIJL types

Lemma 5. $\forall \tau_1, \tau_2. (\models \tau_1 \subseteq \tau_2 \implies \models \text{NF}(\tau_1) \subseteq \tau_2)$

Proof (Lemma 2). We know $\models \tau_1 \subseteq \tau_2$, and we need to show $\vdash \tau_1 \leq \tau_2$. First, we apply L. 5 to $\models \tau_1 \subseteq \tau_2$, and then L. 4, this gives us $\vdash \text{NF}(\tau_1) \leq \tau_2$. Using L. 3 and SD-TRANS, we can show $\vdash \tau_1 \leq \tau_2$. \square

3.3 Reductive Subtyping

The declarative definition is not syntax-directed because it involves the SD-TRANS rule, which requires “coming up” with a middle type τ_2 . For instance, in order to show

$$\vdash \text{Str} \times \text{Real} \leq (\text{Str} \times \text{Int}) \cup (\text{Str} \times \text{Str}) \cup (\text{Str} \times \text{Flt}),$$

we need to apply transitivity several times, in particular, with the middle type $\text{Str} \times (\text{Int} \cup \text{Flt})$.

Fig. 7 presents the syntax-directed reductive definition of subtyping, which can be easily turned into a subtyping algorithm.⁴ Some of the inductive rules are similar to the declarative definition, e.g. subtyping of pairs (SR-PAIR) and the left-hand side union (SR-UNIONL). The differing rules are **highlighted**. Instead of the general reflexivity rule SD-REFL, the reductive definition has reflexivity only for concrete nominal types (SR-BASEREFL). General transitivity is also gone; instead, it gets incorporated into subtyping of nominal types (SR-INTNUM, SR-FLTNUM) and the right-hand side union (SR-UNIONR1, SR-UNIONR2), as well as the rule for using the normal form (SR-NF). Note that the last rule also takes care of distributivity.

With the reductive subtyping, the example above can be derived as follows:

$$\frac{\frac{\vdash \text{Str} \times \text{Int} \leq (\text{Str} \times \text{Int}) \dots \quad \vdash \text{Str} \times \text{Flt} \leq \dots (\text{Str} \times \text{Flt})}{\vdash (\text{Str} \times \text{Int}) \cup (\text{Str} \times \text{Flt}) \leq (\text{Str} \times \text{Int}) \cup \dots \cup (\text{Str} \times \text{Flt})}}{\vdash \text{Str} \times \text{Real} \leq (\text{Str} \times \text{Int}) \cup (\text{Str} \times \text{Str}) \cup (\text{Str} \times \text{Flt})}.$$

In the very bottom, we use SR-NF, and then SR-UNIONL in the level above. To complete the top of derivation, we would also need to use SR-UNIONR*, SR-PAIR, and SR-BASEREFL.

Theorem 2 (Correctness of Reductive Subtyping).

$$\forall \tau_1, \tau_2. (\vdash \tau_1 \leq_R \tau_2 \iff \vdash \tau_1 \leq \tau_2)$$

⁴The definition is not deterministic, though. For example, there are two ways to derive $\vdash \text{Str} \times (\text{Int} \cup \text{Flt}) \leq_R \text{Str} \times (\text{Int} \cup \text{Flt})$: either by immediately applying SR-PAIR, or by first normalizing the left-hand side with SR-NF.

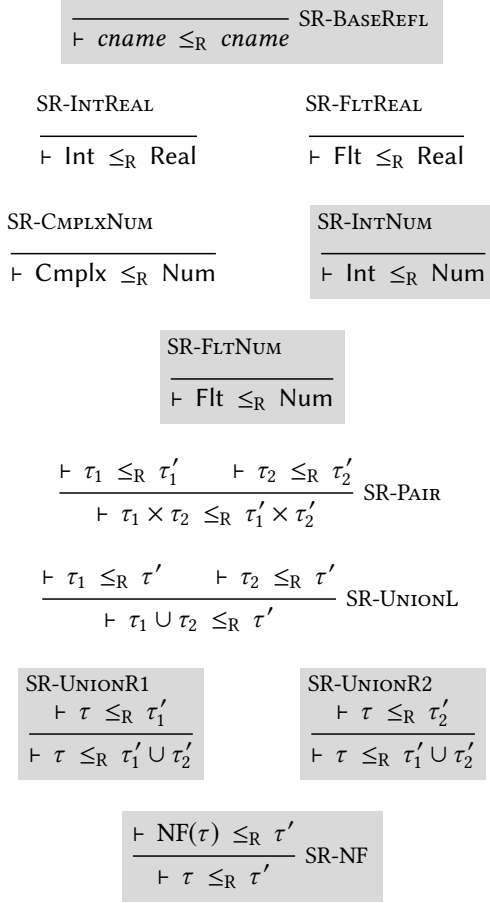


Figure 7. Reductive subtyping for MINIJL

It is relatively easy to show by induction on a derivation of $\vdash \tau_1 \leq_R \tau_2$ that the reductive subtyping is sound: for each case we build a corresponding derivation of $\vdash \tau_1 \leq \tau_2$. Most of the reductive rules have direct declarative counterparts. In the case of SR-*NUM and SR-UNIONR*, we need to additionally use transitivity. Finally, in the case of SR-NF, the induction hypothesis gives us $\vdash \text{NF}(\tau_1) \leq \tau_2$, so we can use L. 3 and SD-TRANS to derive $\vdash \tau_1 \leq \tau_2$.

The challenging part of the proof is to show completeness. For this, we need to prove that the reductive definition is *reflexive*, *transitive*, and *distributive*. To prove transitivity and distributivity, we need several auxiliary statements:

1. $\vdash \tau \leq_R \tau' \implies \vdash \text{NF}(\tau) \leq_R \tau'$,
2. $\vdash \text{NF}(\tau_1) \leq_R \text{NF}(\tau_2) \wedge \vdash \text{NF}(\tau_2) \leq_R \tau_3$
 $\implies \vdash \text{NF}(\tau_1) \leq_R \tau_3$,
3. $\vdash \text{NF}(\tau) \leq_R \text{NF}(\tau') \implies \vdash \tau \leq_R \tau'$.

Having all the facts, we can prove completeness by induction on a derivation of $\vdash \text{NF}(\tau_1) \leq \tau_2$. For details, the reader can refer to the full Coq-proof.

Theorem 3 (Decidability of Reductive Subtyping).

$$\forall \tau_1, \tau_2. (\vdash \tau_1 \leq_R \tau_2 \quad \vee \quad \neg \vdash \tau_1 \leq \tau_2)$$

By induction on a derivation of $\text{InNF}(\tau_1)$, we can show that the reductive subtyping is decidable for $\tau_1 \mid \text{InNF}(\tau_1)$. In this case, τ_1 is either a value type or a union of normalized types. If it is a value type, subtyping is equivalent to matching, and matching is decidable. If τ_1 is a union $\tau_a \cup \tau_b$, we use induction hypothesis for components and the fact that subtyping union on the left-hand side $\vdash \tau_a \cup \tau_b \leq_R \tau'$ implies $\vdash \tau_a \leq_R \tau'$ and $\vdash \tau_b \leq_R \tau'$. Since $\text{InNF}(\text{NF}(\tau_1))$, we can use the decidability of $\vdash \text{NF}(\tau_1) \leq_R \tau_2$ to prove decidability of $\vdash \tau_1 \leq_R \tau_2$.

4 Discussion: Semantic Subtyping and Multiple Dynamic Dispatch

Acknowledgments

References

$$\frac{}{\text{InNF}(v)} \text{NF-VALTYPE} \qquad \frac{\text{InNF}(\tau_1) \quad \text{InNF}(\tau_2)}{\text{InNF}(\tau_1 \cup \tau_2)} \text{NF-UNION}$$

Figure 8. Normal form of types in MINIJL

$$\begin{aligned} \text{NF} : \text{TYPE} &\rightarrow \text{TYPE} \\ \text{NF}(\text{cname}) &= \text{cname} \\ \text{NF}(\text{Real}) &= \text{Int} \cup \text{Flt} \\ \text{NF}(\text{Num}) &= \text{Int} \cup \text{Flt} \cup \text{Cmplx} \\ \text{NF}(\tau_1 \times \tau_2) &= \text{un_prs}(\text{NF}(\tau_1), \text{NF}(\tau_2)) \\ \text{NF}(\tau_1 \cup \tau_2) &= \text{NF}(\tau_1) \cup \text{NF}(\tau_2) \\ \text{un_prs} : \text{TYPE} \times \text{TYPE} &\rightarrow \text{TYPE} \\ \text{un_prs}(\tau_{11} \cup \tau_{12}, \tau_2) &= \text{un_prs}(\tau_{11}, \tau_2) \cup \text{un_prs}(\tau_{12}, \tau_2) \\ \text{un_prs}(\tau_1, \tau_{21} \cup \tau_{22}) &= \text{un_prs}(\tau_1, \tau_{21}) \cup \text{un_prs}(\tau_1, \tau_{22}) \\ \text{un_prs}(\tau_1, \tau_2) &= \tau_1 \times \tau_2 \end{aligned}$$

Figure 9. Computing normal form of MINIJL types

A Appendix: Normal Forms

B Appendix

Text of appendix ...