Advanced Topics in Computer Graphics I - Sheet R10

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Assignment 2

a)

$$p_{HG}(\mu|g) = \frac{1-g^2}{2} \cdot (1+g^2-2g\mu)^{-\frac{3}{2}}$$

$$\begin{aligned}
\mathbb{E}[\mu] &= \int_{-1}^{1} \mu \cdot p_{HG}(\mu|g) d\mu \\
&= \int_{-1}^{1} \mu \cdot \frac{1-g^2}{2} \cdot (1+g^2-2g\mu)^{-\frac{3}{2}} d\mu \\
&= \frac{1-g^2}{2} \cdot \int_{-1}^{1} \mu \cdot (1+g^2-2g\mu)^{-\frac{3}{2}} d\mu \\
&\text{substitute } t = 1+g^2-2g\mu \Leftrightarrow \mu = \frac{1+g^2-t}{2g} \text{ with } dt = -2g\, d\mu \Leftrightarrow d\mu = \frac{1}{-2g} dt \\
&= \frac{1-g^2}{2} \cdot \int_{(1+g)^2}^{(1-g)^2} \frac{1+g^2-t}{2g} \cdot t^{-\frac{3}{2}} \cdot \frac{1}{-2g} dt \\
&= \frac{1-g^2}{2} \cdot -\frac{1}{4g^2} \cdot \int_{(1+g)^2}^{(1-g)^2} (1+g^2-t) \cdot t^{-\frac{3}{2}} dt \\
&= -\frac{1-g^2}{8g^2} \cdot \int_{(1+g)^2}^{(1-g)^2} (1+g^2) t^{-\frac{3}{2}} - t^{-\frac{1}{2}} dt \\
&= -\frac{1-g^2}{8g^2} \cdot \int_{(1+g)^2}^{(1-g)^2} (1+g^2) t^{-\frac{3}{2}} dt - \int_{(1+g)^2}^{(1-g)^2} t^{-\frac{1}{2}} dt \\
&= -\frac{1-g^2}{8g^2} \cdot \left[\left((1+g^2) \cdot \int_{(1+g)^2}^{(1-g)^2} t^{-\frac{3}{2}} dt \right) - \int_{(1+g)^2}^{(1-g)^2} t^{-\frac{1}{2}} dt \right] \\
&\text{with } \int t^{-\frac{3}{2}} dt = -2t^{-\frac{1}{2}} \text{ and } \int t^{-\frac{1}{2}} dt = 2t^{\frac{1}{2}} \\
&= -\frac{1-g^2}{8g^2} \cdot \left[\left((1+g^2) \cdot \left[-2t^{-\frac{1}{2}} \right]_{(1+g)^2}^{(1-g)^2} \right) - \left[2t^{\frac{1}{2}} \right]_{(1+g)^2}^{(1-g)^2} \right] \\
&= -\frac{1-g^2}{8g^2} \cdot \left[\left((1+g^2) \cdot \left[-2((1-g)^2)^{-\frac{1}{2}} + 2((1+g)^2)^{-\frac{1}{2}} \right] \right) - \left[2((1-g)^2)^{\frac{1}{2}} - 2((1+g)^2)^{\frac{1}{2}} \right] \right] \\
&= -\frac{1-g^2}{8g^2} \cdot \left[\left((1+g^2) \cdot \left[-2((1-g)^2)^{-\frac{1}{2}} + 2((1+g)^2)^{-\frac{1}{2}} \right] \right) - 2((1-g)^2)^{\frac{1}{2}} + 2((1+g)^2)^{\frac{1}{2}} \right] \\
&= -\frac{1-g^2}{8g^2} \cdot \left[(1+g^2) \cdot \left[-2 \cdot \frac{1}{|1-g|} + 2 \cdot \frac{1}{|1+g|} \right] - 2 \cdot |1-g| + 2 \cdot |1+g| \right] \\
&\text{for } g \in [-1,1] : 1-g \geq 0 \Rightarrow |1-g| = 1-g \text{ and } 1+g \geq 0 \Rightarrow |1+g| = 1+g \\
&= -\frac{1-g^2}{8g^2} \cdot \left[(1+g^2) \cdot \left[-2 \cdot \frac{1}{1-g} + 2 \cdot \frac{1}{1+g} \right] - 2 \cdot (1-g) + 2 \cdot (1+g) \right] \\
&= -\frac{1-g^2}{8g^2} \cdot \left[(1+g^2) \cdot \left[\frac{-4g}{1-g^2} \right] + 4g \right] \\
&= -\frac{1-g^2}{8g^2} \cdot -\frac{8g^3}{1-g^2} \\
&= g
\end{aligned}$$

b)

- $g = -1$: a dirac delta in the opposite direction of ω_i, purely back-scattering
- $g = 0$: isotropic scattering, uniformly distributed on the sphere
- $g = 1$: a dirac delta in the same direction of ω_i, purely forward-scattering