Advanced Topics in Computer Graphics I - Sheet R07

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Assignment 2

In this exercise, you should analytically calculate variances for the Monte-Carlo integration of a product function when using multiple importance sampling.
We want to compute the integral

$$\int_0^1 f_1(x) f_2(x)\, dx$$

where

$$f_1(x) = 20x(x-0.5)(x-1)+1$$

$$f_2(x) = -20x(x-0.5)(x-1)+1$$

via Monte-Carlo integration.
1) Use either f1 or f2 for the importance sampling.

$$\begin{aligned}
p(x) &= \frac{f_1(x}{\int_0^1 f_1(x)\, dx} = f_1(x) \\
\Rightarrow \langle F \rangle &= \frac{1}{N} \sum_{i=1}^{N} \frac{f_1(x_i) f_2(x_i)}{f_1(x_i)} \\
&= \frac{1}{N} \sum_{i=1}^{N} f_2(x_i) \\
\Rightarrow \mathbb{V}_{f1}[F] &= \frac{1}{N} (\mathbb{E}[F^2] - \mathbb{E}[F]^2) \\
&= \frac{1}{N} \left(\int_0^1 f_2^2(x) \cdot f_1(x)\, dx - \left(\int_0^1 f_2(x) \cdot f_1(x)\, dx \right)^2 \right) \\
&= \frac{1}{N} \left(0.52381 - 0.52381^2 \right) \\
&= \frac{0.2494}{N}
\end{aligned}$$

2) Use the balance heuristic described in the lecture slides (Monte-Carlo Integration, Sampling Strategies) with an equal number of samples for both functions.

$$
\begin{aligned}
p(x) &= \frac{1}{2} \cdot \frac{f_1(x)}{\int_0^1 f_1(x)\,dx} + \frac{1}{2} \cdot \frac{f_2(x)}{\int_0^1 f_2(x)\,dx} \\
&= \frac{1}{2}\left(f_1(x) + f_2(x)\right) \\
&= \frac{1}{2} \cdot 2 = 1 \\
\Rightarrow \langle F \rangle &= \frac{1}{N} \sum_{i=1}^{N} f_1(x_i) f_2(x_i) \\
\mathbb{V}_{p(x)}[F] &= \frac{1}{N}(\mathbb{E}[F^2] - \mathbb{E}[F]^2) \\
&= \frac{1}{N}\left(\int_0^1 f_1^2(x) f_2^2(x)\,dx - \left(\int_0^1 f_1(x) f_2(x)\,dx\right)^2\right) \\
&= \frac{1}{N}\left(0.380619 - 0.52381^2\right) \\
&= \frac{0.106233}{N}
\end{aligned}
$$

3) Use the function f1 $\cdot$ f2 for the importance sampling.

$$
\begin{aligned}
p(x) &= \frac{f_1(x) f_2(x)}{\int_0^1 f_1(x) f_2(x)\,dx} \\
\Rightarrow \langle F \rangle &= \frac{1}{N} \sum_{i=1}^{N} \frac{f_1(x_i) f_2(x_i)}{f_1(x_i) f_2(x_i)} \cdot \int_0^1 f_1(x) f_2(x)\,dx \\
&= \frac{1}{N} \sum_{i=1}^{N} \int_0^1 f_1(x) f_2(x)\,dx \\
&= \int_0^1 f_1(x) f_2(x)\,dx \\
\Rightarrow \mathbb{V}_{f1f2}[F] &= 0
\end{aligned}
$$