Problem 2 and 3

Contents

Problem 2: A little more Least squares.

For avoiding unnecessary copy and paste I have implemented the function more least squares that performs all the required computations in exercises 2 and 3. The code of the function is shown at the end of the exercise.

```
clear;close all;clc;
m = 50; n = 5;

[X, labels, error, kappa_5] = moreLeastSquares(m, n);
```

Display the results for Problem 2. The variable "labels" indicate the method followed for computing the results shown below. The variable "X" contains the coefficient of the least square approximation. The variable "error" contains the norm2 error of the residuals, ie Ax - b.

```
format long
labels
X
error
```

```
labels =

    'Back Slash'    'Least Squares'    'QR factorization'    'SVD'


X =

   0.977651051050956   0.977651051050607   0.977651051050958   0.977651051050960
   0.787296437030590   0.787296437038082   0.787296437030580   0.787296437030554
 -13.726690028111138 -13.726690028144622 -13.726690028111090 -13.726690028111035
  14.993014653596203  14.993014653647393  14.993014653596127  14.993014653596131
  -3.665759654743957  -3.665759654769070  -3.665759654743924  -3.665759654743953


error =

   0.061356756239887   0.061356756239887   0.061356756239888   0.061356756239887
```

Minimum error, measured as the 2-norm of the residuals, obtained with the method To check which method stands out we check the magnitude of the log of errors.

```
log_error = log(error)
```

```
log_error =

  -2.791049987729135  -2.791049987729132  -2.791049987729119  -2.791049987729122
```

The minimum error is obtained with the method:

```
me = min(log_error);
idx = find(log_error == me);
labels{idx}
```

```
ans =

Back Slash
```

The maximum error is obtained with the method:

```
ME = max(log_error);
idx = find(log_error == ME);
labels{idx}
```

```
ans =

QR factorization
```

However the differences in the log errors being of the order of 10^-36, we could say that all of them perform similarly when looking at the norm 2 of the residuals.

Problem 3: Condition Numbers

Calculate the condition number of each method for n = 10.

```
m = 50; n = 10;
[~, ~, ~, kappa_10] = moreLeastSquares(m, n);
```

The condition numbers for the different methods and values of n are contained on the table below.

```
fprintf('Methods \t\t | \t\t 5 steps \t\t | \t\t 10 steps \n');
for it = 1:length(kappa_5)
    if it == length(kappa_5)
        fprintf('%s \t\t\t | \t\t %f \t\t | \t\t %f\n', labels{it}, kappa_5(it), kappa_10(it));
    elseif it == 3
        fprintf('%s \t | \t\t %f \t\t | \t\t %f\n', labels{it}, kappa_5(it), kappa_10(it));
    else
        fprintf('%s \t\t | \t\t %f \t\t | \t\t %f\n', labels{it}, kappa_5(it), kappa_10(it));
    end
end
```

```
Methods           |     5 steps           |     10 steps
Back Slash        |     642.667357        |     3558944.505139
Least Squares     |     413021.331912     |     12663833724895.404297
QR factorization  |     642.667357        |     3558944.505224
SVD               |     642.667357        |     3558944.505087
```

According to the rule in the homework let k be the number of digits lost and K(A) the condition number of the problem, they are related by the following formula K(A) = 10^k, thus k = log10(K(A)) The results are summarized in the table below. The worst results are for the case where we use the normal equations. This was expected as the use of the normal equations is an unstable method for doing least squares as K(A'*A) = K(A)^2 and not K(A)

```
f_digits_lost = @(k)(log10(k));
dig_lost_5 = f_digits_lost(kappa_5);
```

```
dig_lost_10 = f_digits_lost(kappa_10);

fprintf('\n\nMethods \t\t | \t\t 5 steps \t\t | \t\t 10 steps \n');
for it = 1:length(kappa_5)
    if it == length(kappa_5)
        fprintf('%s \t\t\t | \t\t %f \t\t | \t\t %f\n', labels{it}, dig_lost_5(it), dig_lost_10(it));
    elseif it == 3
        fprintf('%s \t | \t\t %f \t\t | \t\t %f\n', labels{it}, dig_lost_5(it), dig_lost_10(it));
    else
        fprintf('%s \t\t | \t\t %f \t\t | \t\t %f\n', labels{it}, dig_lost_5(it), dig_lost_10(it));
    end
end
```

```
Methods                  |                5 steps                  |                10 steps
Back Slash               |                2.807986                 |                6.551321
Least Squares            |                5.615972                 |                13.102565
QR factorization         |                2.807986                 |                6.551321
SVD                      |                2.807986                 |                6.551321
```

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Custom function moreLeastSquares

```
function [X, labels, error, kappa] = moreLeastSquares(m, n)

t = linspace(0, 1, m)';
A = zeros(m,n);
P = eye(n);
for it = 1:n
    A(:, it) = polyval(P(n-it+1, :), t);
end

b = cos(4*t);

x_BS = A \ b;
x_LS = A'*A \ A'*b;

[Q, R] = qr(A, 0); % Q'*Q = Identity
R = R(1:size(R, 2), 1:end);
% normal equations A*x = b => Q*R*x = b => Q'*Q*R*x = Q'*b
% R*x = Q'*b
x_QR = backSub(R, Q'*b);

[U,S,V] = svd(A,0);
% A*x = b => (U*S*V')*x = b => U'*U*S*V'*x = U'*b =>
% V'*x = inv(S)*U'*b => x = V*inv(S)*U'*b => x = invS * (U'*b*V)
invS = diag(1./diag(S));
x_SVD = V*invS*U'*b;

labels = {'Back Slash', 'Least Squares', 'QR factorization', 'SVD'};
X = [x_BS, x_LS, x_QR, x_SVD];

col_X = size(X, 2);
error = zeros(1, col_X);
for it = 1:size(X, 2)
    error(it) = norm(A*X(:, it) - b, 2);
end


kappa = zeros(col_X, 1);
% Back-Slash
kappa(1) = cond(A);
% Normal Equations
kappa(2) = cond(A'*A);
% QR Factorization.
% By the theorem 16.1 ratio norm(x^- x)/norm(x) ~ O(K(A)eps)
```

```
kappa(3) = cond(Q*R);
% SVD
kappa(4) = cond(U*S*V');

end
```

Custom function backSub

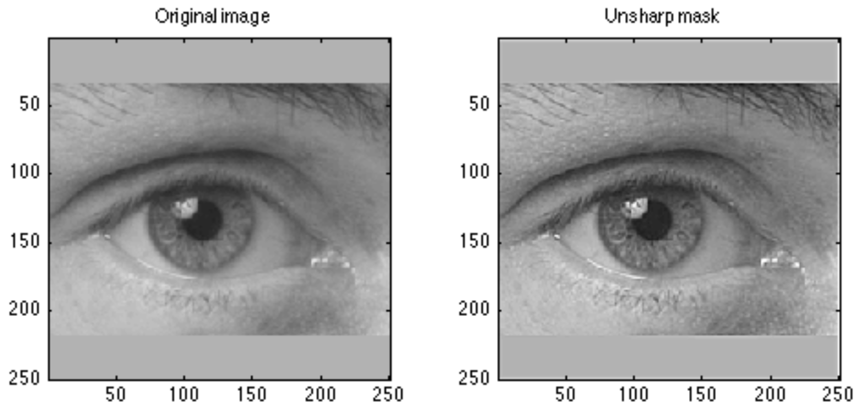
```
function x = backSub(U, b)
%BACKSUB
n = length(b);
x = zeros(n, 1);
for i=n:-1:1
   x(i) = ( b(i) - U(i, :)*x ) / U(i, i);
end

end
```

Problem 4

Contents

Problem 4: Image processing and the "unsharp mask"

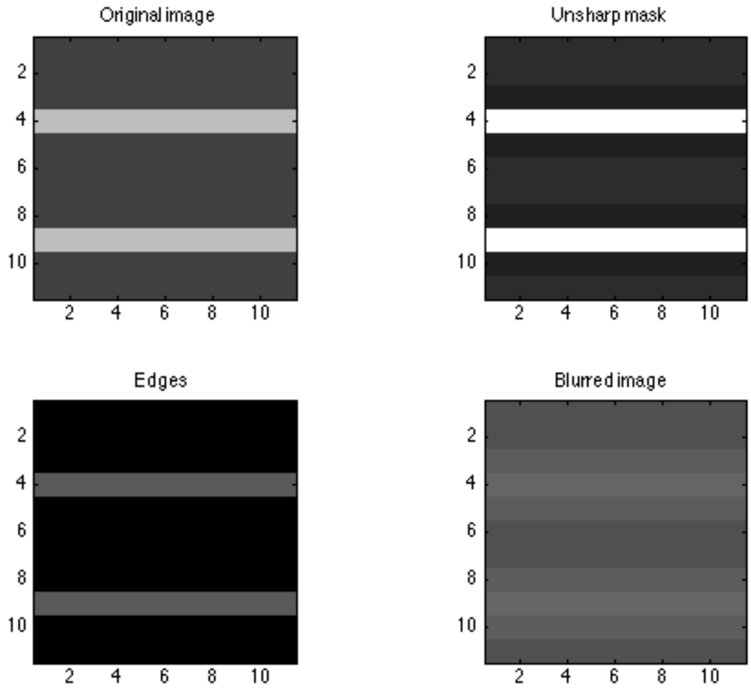
```
close all;clear;
%img_name = 'testpat_noblur.png';
img_name = 'eye.png';
[u, ublur] = blurImg(img_name, 10, 0.1);

uedges = u - ublur;
usharp = u + uedges;

figure;
subplot(1,2,1);imagesc(u);title('Original image');
caxis([0 1]);colormap(gray);axis equal, axis tight;
subplot(1,2,2);imagesc(usharp);title('Unsharp mask');
caxis([0 1]);colormap(gray);axis equal, axis tight;
```



Zoom in to see the effect on the edges

As explained in the tutorial provided, bright zones are made brighter whereas dark zones are made darker.

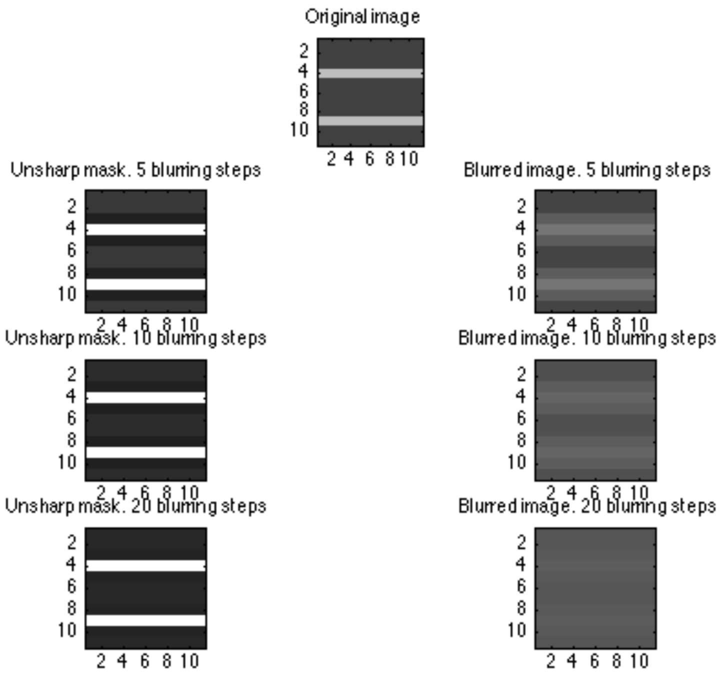
```
close all;clear;
img_name = 'testpat_noblur.png';
idx = 20:30;

[u, ublur] = blurImg(img_name, 10, 0.1);
uedges = u - ublur;
usharp = u + uedges;
figure;
title('Zoom-in testpat_noblur.png to appreaciate distorsion in the edges');
subplot(2,2, 1);imagesc(u(idx, idx));title('Original image');
caxis([0 1]);colormap(gray);axis equal, axis tight;
subplot(2,2,2);imagesc(usharp(idx, idx));title('Unsharp mask');
caxis([0 1]);colormap(gray);axis equal, axis tight;
subplot(2,2, 3);imagesc(uedges(idx, idx));title('Edges');
caxis([0 1]);colormap(gray);axis equal, axis tight;
subplot(2,2, 4);imagesc(ublur(idx, idx));title('Blurred image');
caxis([0 1]);colormap(gray);axis equal, axis tight;
```



Zoom in to check the effect of the time steps

The longer the time the more the effect of the method fades away. For the unsharp we observe that increasing the time steps results in that a greater surface is darkened, not only the edges. The resulting blurred image is of the same grey color.

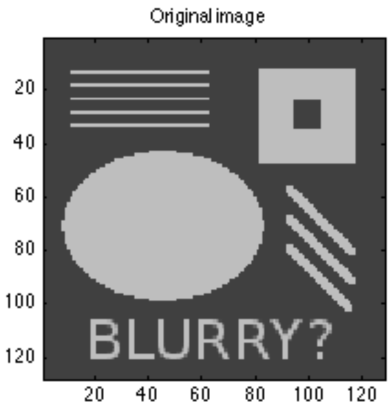
```
steps = [5, 10 , 20];

figure;
title('Effect of different time steps');
subplot(4,1, 1);imagesc(u(idx, idx));title('Original image');
caxis([0 1]);colormap(gray);axis equal, axis tight;
for it = 1:length(steps)
    [u, ublur] = blurImg(img_name, steps(it), 0.1);
    uedges = u - ublur;
    usharp = u + uedges;

    subplot(4,2, 3 + 2*(it-1));imagesc(usharp(idx, idx));
    title(sprintf('Unsharp mask. %d blurring steps', steps(it)));
    caxis([0 1]);colormap(gray);axis equal, axis tight;
    subplot(4,2, 4 + 2*(it-1));imagesc(ublur(idx, idx));
    title(sprintf('Blurred image. %d blurring steps', steps(it)));
    caxis([0 1]);colormap(gray);axis equal, axis tight;
end
```

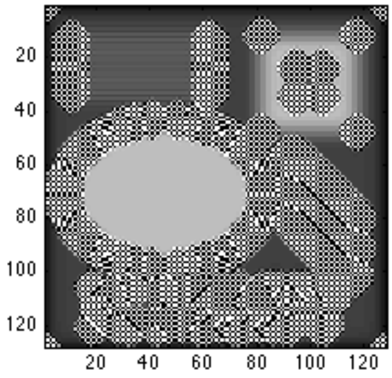


Increasing the time step

Changing the dt to 0.5 we observe that the values of ublur do not lie within the [0, 1] limit any more. This is due to the conditional stability of the explicit scheme

```
img_name = 'testpat_noblur.png';
[u, ublur] = blurImg(img_name, 10, 0.5);
figure;
subplot(1,2,1);imagesc(u);title('Original image');
caxis([0 1]);colormap(gray);axis equal, axis tight;
subplot(1,2,2);imagesc(ublur);title('');
caxis([0 1]);colormap(gray);axis equal, axis tight;
```





Maximum and minimum values of the original image

```
max(max(u)), min(min(u))
```

```
ans =

   0.749019607843137


ans =

   0.250980392156863
```

Maximum and minimum values of the original image. Values out of bounds.

```
max(max(ublur)), min(min(ublur))
```

```
ans =

     2.879130675551471e+03


ans =

    -2.849341674325981e+03
```

Problem 5

Fitting elipses via least squares

Contents

- Using manually input points

Using the points in section b

```
y = [3;-2;3;2;-2;-4;0;0];
x = [3;1;0;-1;-2;0;-2;2];
```

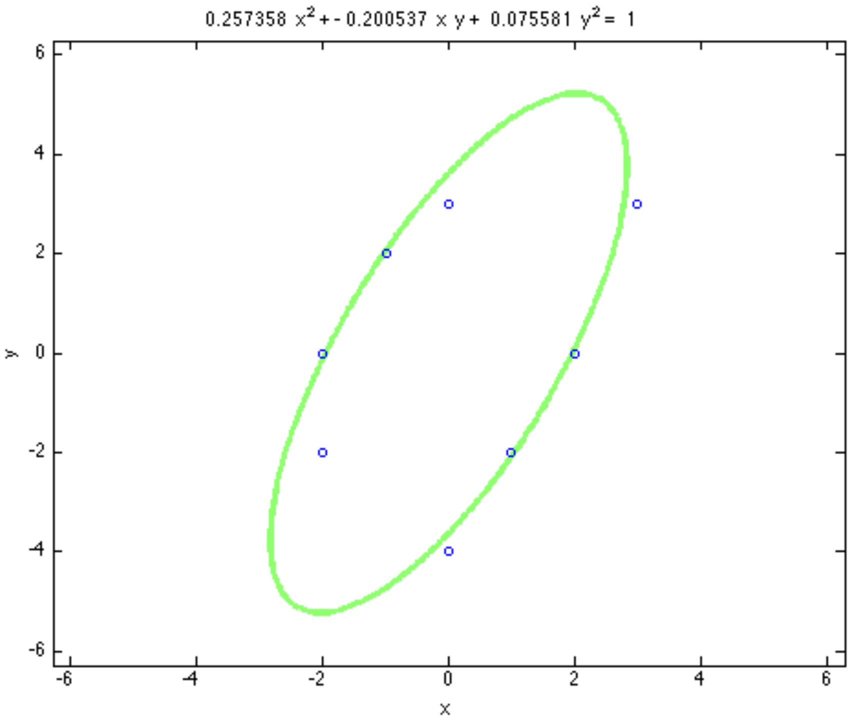
Displaying the equation of the fitted elipse

```
[b, c, d] = ellipse(x, y);
fprintf('\nEllipse: %f*x^2 + %f*x*y + %f*y^2 = 1\n', b, c, d);
```

```
Ellipse: 0.257358*x^2 + -0.200537*x*y + 0.075581*y^2 = 1
```

I have done two plots the first one using ezplot

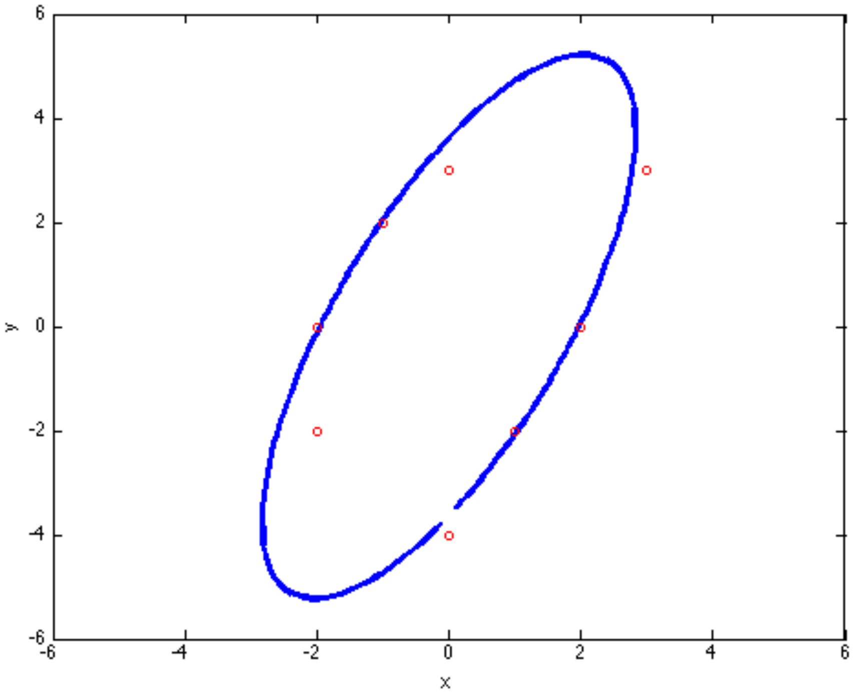
```
figure;
el = ezplot(sprintf('%f*x^2 + %f*x*y + %f*y^2 = 1', b, c, d));
set(el, 'LineWidth', 3)
hold on;
plot(x, y, 'o', 'MarkerSize', 5);
xlabel('x'); ylabel('y');
```



Using the tan(theta) trick

```
theta = linspace(-0.49*pi, 0.49*pi)';
gen_x = @(theta)(1./sqrt(b + c * tan(theta) + d*tan(theta).^2));
gen_y = @(theta)(tan(theta)./sqrt(b + c * tan(theta) + d*tan(theta).^2));
x_gen = gen_x(theta);
y_gen = gen_y(theta);

figure;
plot([x_gen; -x_gen], [y_gen; -y_gen],'LineWidth', 3);
hold on;
```

```
plot(x, y, 'ro', 'MarkerSize', 5);
xlabel('x'); ylabel('y');axis([-6 6 -6 6]);
```



Using manually input points

```
load('points.mat');
```

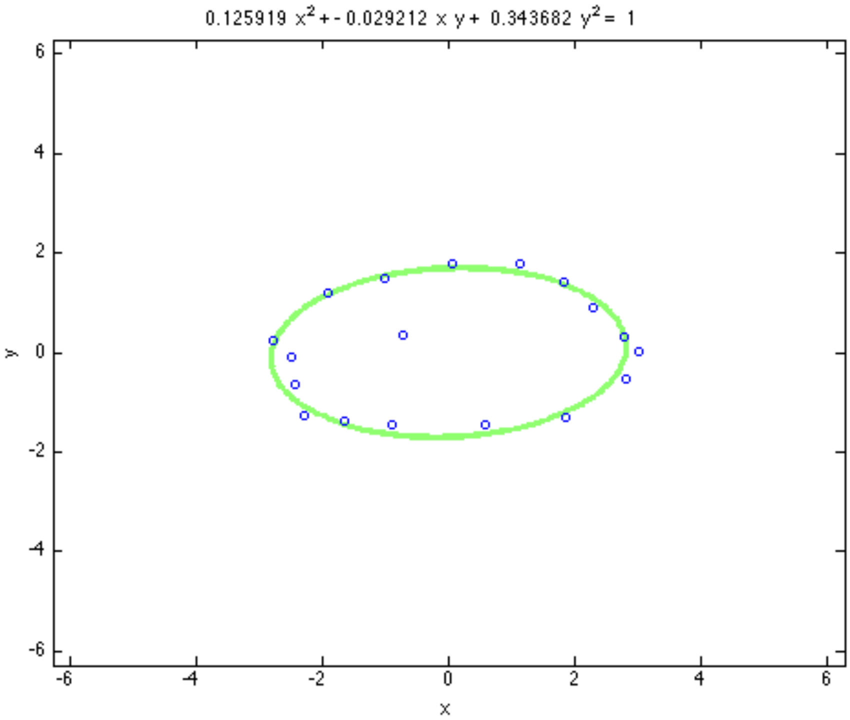
The fitted ellipse is

```
[b, c, d] = ellipse(x, y);
fprintf('\nEllipse : %f*x^2 + %f*x*y + %f*y^2 = 1\n', b, c, d);
```

```
Ellipse : 0.125919*x^2 + -0.029212*x*y + 0.343682*y^2 = 1
```

Plot of the fitted ellipse

```
figure;
el = ezplot(sprintf('%f*x^2 + %f*x*y + %f*y^2 = 1', b, c, d));
set(el, 'LineWidth', 3)
hold on;
plot(x, y, 'o', 'MarkerSize', 5);
xlabel('x'); ylabel('y');
```



Custom function ellipse

```
function [b, c, d] = ellipse(x, y)

if size(x, 2) ~= 1 || size(y, 2) ~= 1
    error('x and y should be column vectors')
end

A = [x.^2, x.*y, y.^2];
b = ones(size(x));

coeffs = A\b;
b = coeffs(1);
c = coeffs(2);
d = coeffs(3);
```