Lesson 245. Functions of a Complex Variable. Examples and application

In the whole document, Ω will denote an open set of $\mathbb{C}$.

1. Analytic and holomorphic functions

1.1. Power series

1. DEFINITION. Power series
A power series is a function series of the form $\sum a_n z^n$ where z is a complex variable.

2. DEFINITION. Convergence radius
Let $(a_n)_{n\in\mathbb{N}}$ be a complex sequence, and $r > 0$. The following statements are equivalent

- $\forall s \in \mathring{\mathcal{D}}(0,r)$, the sequence $(a_n s^n)_{n\in\mathbb{N}}$ is bounded.
- $\forall s \in \mathring{\mathcal{D}}(0,r)$, the sequence $(a_n s^n)_{n\in\mathbb{N}}$ converges.
- $\forall s \in \mathring{\mathcal{D}}(0,r)$, the sequence $(|a_n s^n|)_{n\in\mathbb{N}}$ converges.
- $\forall s \in \mathring{\mathcal{D}}(0,r)$, the series $\sum a_n s^n$ converges.
- $\forall s \in \mathring{\mathcal{D}}(0,r)$, the series $\sum a_n s^n$ converges absolutely.
- $\forall s \in \mathring{\mathcal{D}}(0,r)$, the series $\sum a_n t^n$ converges uniformly over $\overline{\mathcal{D}(0,s)}$.

The convergence radius of the power series $\sum a_n z^n$ is the supremum of the $r \in \mathbb{R}_+^*$ verifying those statements. It is denoted by $R\left(\sum a_n z^n\right)$ or simply by R if there is no ambiguity.

3. COROLLARY. A power series and its formal derivative have the same convergence radius R. Hence over $\mathring{\mathcal{D}}(0,r)$, the formal derivative is the derivative of the power series.

4. EXAMPLE. $\quad e^z = \sum\limits_{n\in\mathbb{N}} \frac{z^n}{n!}$, $R = +\infty$, $\quad \log(1-z) = -\sum\limits_{n\in\mathbb{N}^*} \frac{z^n}{n}$, $\frac{1}{1-z} \sum\limits_{n\in\mathbb{N}} z^n$, $R = 1$

5. THEOREM. Abel's radial convergence
Let $(a_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$, $z \in \mathbb{C}$ so that $\sum a_n z_n$ converges, then $\sum a_n s^n$ converges uniformly over $[0,z]$

6. COROLLARY. Let $\sum a_n$, $\sum b_n$ be two convergent complex series of respective limits A and B, the Cauchy's product $\sum c_n$ of the two is convergent and its limit id AB.

7. EXAMPLE. $\sum\limits_{n>0} \frac{(-1)^n}{n} = -\log(2)$, $\quad \sum\limits_{n\geqslant 0} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$

1.2. Holomorphic functions

8. DEFINITION. Let $a \in \Omega$, $f : \Omega \longrightarrow \mathbb{C}$. f is said to be $\mathbb{C}$-differentiable in a if $\lim\limits_{Z\mapsto a} \frac{f(z)-f(a)}{z-a}$ exists. We denote this limit by $f'(a)$.

9. EXAMPLE.

- $\mathrm{Id}_{\mathbb{C}}$ is differentiable in every point of $\mathbb{C}$, and its derivative is $z \mapsto 1$.
- $z \mapsto |z|$ is not differentiable in any given point of $\mathbb{C}$.

10. PROPOSITION. Cauchy-Riemann
Let $f : \Omega \longrightarrow \mathbb{C}$ be a continuous function. We define $u,v : \Omega \longrightarrow \mathbb{C}$ by $f(x+iy) = u(x+iy) + iv(x+iy)$.
f is differentiable in $a+ib$ if and only if (u,v) is differentiable in (a,b) and

$$\begin{cases} \frac{\partial u}{\partial x}(a,b) & = & \frac{\partial v}{\partial y}(a,b) \\ \frac{\partial u}{\partial y}(a,b) & = & -\frac{\partial y}{\partial x}(a,b) \end{cases}$$

11. REMARK. This relation is the consequence of the Jacobian being a similarity matrix.

12. DEFINITION. Holomorphic functions
A continuous function $f : \Omega \longrightarrow \mathbb{C}$ is said to be holomorphic over Ω if it is $\mathbb{C}$-differentiable in any given point of Ω. We denote by $\mathcal{H}(\Omega)$ the set of holomorphic functions over Ω.

1.3. Analytic functions

13. DEFINITION. Analytic function
A function $f : \Omega \longrightarrow \mathbb{C}$ is said to be analytic over Ω if it can be expressed as a power series on the neighbourhood of any given point of Ω. We denote by $\mathcal{A}(\Omega)$ the set of analytic functions over Ω. Consequently, $\mathcal{A}(\Omega) \subset \mathcal{H}(\Omega)$.

14. PROPOSITION. If $\sum a_n z^n$ has R for convergence radius, $z \mapsto \sum\limits_{n\in\mathbb{N}} a_n z^n \in \mathcal{A}(\mathring{\mathcal{D}}(0,R))$

15. THEOREM. Principle of isolated zeroes
Let $f \in \mathcal{A}(\Omega)$, if the set $\{z \in \Omega | f(z) = 0\}$ contains a limit point and Ω is connected, $f = O$.

16. EXAMPLE.

- The Fourier transform of $f : x \mapsto e^{-\pi x^2}$ is f.
- $\forall z$ s. t. $\mathrm{Re}(z) > 0$, $\quad \Gamma(z+1) = z\Gamma(z)$

17. THEOREM. Maximum principle
Let $f \in \mathcal{A}(\Omega)$ and continuous on $\overline{\Omega}$, if Ω is bounded, forall $z \in \Omega$,

$$|f(z)| \leqslant \sup_{z\in\partial\Omega} |f(z)|$$

18. EXAMPLE. If $f \in \mathcal{A}(\Omega)$ and $\exists a \in \Omega$ s. t. $\forall z \in \Omega, |f(z)| \leqslant |f(a)|$, then f is constant.

2. Cauchy's theory

2.1. Line integrals

19. DEFINITION. A curve of $\mathbb{C}$ is a function $\gamma : [0,1] \longrightarrow \mathbb{C}$ which is continuous and piecewise-smooth. A curve γ is said to be a loop if $\gamma(0) = \gamma(1)$. Hence any loop can be seen as a piecewise-smooth function of $\mathbb{S}$ to $\mathbb{C}$. We denote by γ^* the range of the curve.

20. EXAMPLE. Let $r > 0$, $\gamma_r : t \in [0,1] \mapsto r\, e^{2i\pi t}$ is a loop.

21. DEFINITION. Line integral
Let $f : \Omega \longrightarrow \mathbb{C}$ be a mesurable function and $\gamma : [0,1] \longrightarrow \Omega$ be a curve, the line integral of f along γ is defined by

$$\int_{\gamma}^{f(z)} z \, \mathrm{d}\lambda = \int_0^1 f \circ \gamma(t) \cdot \gamma'(t) \mathrm{d}t$$

22. EXAMPLE. The Fourier transform of $f : x \mapsto e^{-\pi x^2}$ is f.

23. DEFINITION. Winding number

Let γ be a loop, and $a \in \mathbb{C} \setminus \gamma^*$, the winding number of γ around a is the number

$$\mathrm{Ind}(\gamma,a) := \frac{1}{2i\pi} \int_\gamma \frac{\mathrm{d}z}{z-a}$$

24. Remark. Intuitively the winding number of a loop around a point is the number of times the loop travels counterclockwise around the point, i.e., the curve's number of turns, e.g. the winding number of $z \mapsto e^{2ik\pi z}$ around zero is k.

2.2. Cauchy's formula around a convex set

In this subsection, we consider Ω as convex, and A be a finite subset of Ω.

25. Lemma. Goursat
Let $\Delta \subset \Omega$ be a triangle and $f : \Omega \longrightarrow \mathbb{C}$ be a continuous fonction, holomorphic over $\Omega \setminus A$, then $\int_\Delta f = 0$.

26. Proposition. Let $f : \Omega \longrightarrow \mathbb{C}$ be a continuous fonction, holomorphic over $\Omega \setminus A$ and γ be a curve in Ω, then

- there exists $F \in \mathcal{H}(\Omega)$ s. t. $F' = f$.
- if γ is a loop, $\int_\gamma f = 0$

27. Theorem. Cauchy's formula
Let γ be a loop in Ω and $f \in \mathcal{H}(\Omega)$, $\forall z \in \Omega \setminus \gamma^*$,

$$\mathrm{Ind}(\gamma,z) f(z) = \frac{1}{2i\pi} \int_\gamma \frac{f(s)}{s-z} \mathrm{d}s$$

furthermore,

$$\mathrm{Ind}(\gamma,z) f^{(n)}(z) = \frac{n!}{2i\pi} \int_\gamma \frac{f(s)}{(s-z)^n} \mathrm{d}s$$

.

28. Example. The Fourier transforms of $f : x \mapsto e^{-\pi x^2}$ and $f : z \mapsto \frac{1}{1+x^2}$ are f and $\xi \mapsto \pi\, \varepsilon^{-2\pi|\xi|}$.

2.3. Consequences

29. Proposition. $\mathcal{A}(\Omega) = \mathcal{H}(\Omega)$

30. Theorem. Weierstraß
Let $(f_n)_{n\in\mathbb{N}}$ be an uniformly convergent sequence of holomorphic functions over Ω, which limit is denoted by f, then $f \in \mathcal{H}(\Omega)$ and $f'_n \xrightarrow{uc} f'$.

31. Theorem. Morera
Let $f : \Omega \longrightarrow \mathbb{C}$, if for every triangle Δ, $\int_\Delta f = 0$, then $f \in H(\Omega)$.

3. Dirichlet series

3.1. Definition and basic properties

32. Definition. Dirichlet series
Let $(\lambda_n)_{n\in\mathbb{N}} \in \mathbb{R}^n$ be an unbound growing sequence and $(a_n)_{n\in\mathbb{N}^*} \in \mathbb{C}^n$. The associated Dirichlet series is the function series $\sum a_n\, e^{-\lambda_n z}$.

33. Remark. By Dirichlet series, we commonly mean a Dirichlet series where $\lambda_n = \log(n)$, which gives a function of the form $\sum \frac{a_n}{n^z}$.

34. Example. The zeta function defined by $\zeta(s) = \sum_{n\in\mathbb{N}^*} \frac{1}{n^s}$ is the classic example for a Dirichlet series. A less classic one is the eta function $\eta(s) = \sum_{n\in\mathbb{N}^*} \frac{(-1)^{n+1}}{n^s}$ which, thanks to the radial convergence of Abel, converges uniformly on the neighbourhood of 1^+.

35. Definition. Abscissæ of convergence

- The abscissa of convergence of f is $\sigma_c := \inf\left\{\sigma \in \mathbb{R} | \forall s, \mathrm{Re}(s) > \sigma, \sum a_n\, e^{-\lambda_n s} \text{ converges } \right\}$.
  - If $\sum a_n$ is convergent, then $\sigma_c = \limsup\limits_{n\longrightarrow\infty} \frac{\log\left|\sum a_n\right|}{\lambda_n}$
  - If $\sum a_n$ is divergent, then $\sigma_c = \limsup\limits_{n\longrightarrow\infty} \frac{\log|a_1+\cdots+a_n|}{\lambda_n}$
- The abscissa of absolute convergence of f is $\sigma_a := \inf\left\{\sigma \in \mathbb{R} | \forall s, \mathrm{Re}(s) > \sigma, \sum a_n\, e^{-\lambda_n s}\right.$ abs
  - If $\sum a_n$ is convergent, then $\sigma_a = \limsup\limits_{n\longrightarrow\infty} \frac{\log\left(\sum|a_n|\right)}{\lambda_n}$
  - If $\sum a_n$ is divergent, then $\sigma_a = \limsup\limits_{n\longrightarrow\infty} \frac{\log(|a_1|+\cdots+|a_n|)}{\lambda_n}$
- The abscissa of holomorphy of f is $\sigma_h := \inf\{\sigma \in \mathbb{R} | f \in \mathcal{H}(\{s | \mathrm{Re}(s) > \sigma\})\}$.

We have by definition $\sigma_h \leqslant \sigma_c \leqslant \sigma_a$, the width of the strip L verifies

$$0 \leqslant \sigma_a - \sigma_c \leqslant L = \limsup \frac{\log n}{\lambda_n}$$

36. Theorem. Landau
If $\forall n \in \mathbb{N}^*$, $a_n \geqslant 0$, then σ_c is a singular point of f, hence $\sigma_h = \sigma_c$

3.2. Analytic extension of Dirichlet series

37. Theorem. Unicity of Dirichlet's development
Let f and g be two Dirichlet series that coincide on an open set of $\mathbb{C}$, then the sequences defining them are the same.

38. Example. The vector space induced by the sequences $(e^k_n)_n = (n^{-1-\frac{i}{k}})_n$ for $k \in \mathbb{N}^*$ is dense in the Hilbert space $l_2(\mathbb{N}^*)$.

39. Proposition. If $\sigma_c(f) < \infty$, $F(z) = \sum a_n e^{-\mu_n z}$ where $\mu_n = e^{\lambda_n}$ verifies $\sigma_c(F) \leqslant 0$, and $f(s) = \frac{1}{\Gamma(s)} \int_0^\infty f(x) x^{s-1} \mathrm{d}x$.

40. Remark. F is in fact the reverse Mellin transform of f.

41. Theorem. Hardy-Fekete
If $\sigma_c < \infty$ and F can be prolonged into a meromorphic function in zero, if we set q to be the order of the pole in 0, then f can be prolonged into a meromorphic functions with only simple poles included in $\{1, \ldots, q\}$.

42. Example. ζ can be prolonged by a meromorphic function with a simple pole in 1.

3.3. Application on prime numbers theory

43. Definition. Let us define some useful functions for the following properties and theorems.

- The prime-counting function $\pi : x \in \mathbb{R}_+ \mapsto |\mathscr{P} \cap [0,x]|$.
- The lambda function $\Lambda : n \in \mathbb{N}^* \mapsto \log p$ if $n = p^k$ else 0.

44. PROPOSITION. $\forall x \geqslant 0, 1 < \sigma \leqslant 2$,

$$\sum_{n>0} \Lambda(n)e^{-nx} = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \Gamma(\sigma+it)\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)}x^{-(\sigma+it)}\mathrm{d}t$$

45. PROPOSITION. If $\sigma_c \leqslant 0$, $(a_n)_{n\in\mathbb{N}^*} \in \mathbb{R}_+^{\mathbb{N}^*}$, and $\forall p \in \mathbb{N}^*$, $l_p = \lim\limits_{n\to\infty} \frac{f(p\sigma)}{f(\sigma)}$ so that $l_2 \neq 0$, then $\exists! \alpha \geqslant 0$ so that $\forall p \in \mathbb{N}^*$, $l_p = p^{-\alpha}$ and

$$\lim_{n\to\infty} \frac{1}{f(\lambda_n^{-1})} \sum_{0<k\leqslant n} a_k = \frac{1}{\Gamma(\alpha+1)}$$

46. LEMMA. $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{0<k\leqslant n} \Lambda(k) = 1$

47. THEOREM. Prime numbers

$$\pi(x) \underset{x\to\infty}{\sim} \frac{x}{\log x}$$