Harmonic Analysis : Exercice Sheet 5 (1)

(i) Lets take $f, g \in \mathbb{C}^N$.

We have to prove that $C(g) + C(f) = C(f) + C(g)$

and $C(g)C(f) = C(f)C(g)$.

$C(g) + C(f) = C(f) + C(g) \longrightarrow$ definition of matrix addition.

$C(f)C(g) = C(g)C(f)$

$$C(g) = \begin{pmatrix} f & T_N g & T_N^2 g & \cdots & T_N^{N-1} g \end{pmatrix} = \begin{pmatrix} R_N g^T \\ R_N^2 g^T \\ \vdots \\ R_N^N g^T \end{pmatrix}$$

$$C(f)C(g) = \begin{pmatrix} R_N f^T \times g & R_N f\, T_N g & \cdots & R_N f\, T_N^{N-1} g \\ R_N^2 f^T \times g & & & \\ \vdots & & & \\ R_N^N f \times g & \cdots & \cdots & R_N^N f\, T_N^{N-1} g \end{pmatrix} = \sum_{i=0}^{N-1} T_N^i g \; R_N^{i+1} g^T$$

We have to show that for $i \in [\![1, N]\!]$ and $j \in [\![0, N-1]\!]$ $\sum R_N^i f^T T_N^{j+1} g = \sum R_N^i g\, T_N^{j+1} f^T$.

$R_N^i(g) = (g_{i-1}, \ldots, g_0, g_{N-1}, \ldots, g_i)^T$ and $T_N^j(g) = (g_{N-j}, g_{N-1-j}, \ldots, g_{N-1}, g_0, \ldots, g_{N-j+1})$

$$f \times R_N g^T = \begin{pmatrix} f_0 \\ \vdots \\ f_{N-1} \end{pmatrix} (g_0, g_{N-1}, \ldots, g_1) = \begin{pmatrix} f_0 g_0 & f_0 g_{N-1} & \cdots & f_0 g_1 \\ f_1 g_0 & & & \\ \vdots & & & \\ f_{N-1} g_0 & & & f_{N-1} g_1 \end{pmatrix}$$

$g \times R_N f^T = ($

??

Units is $f = (1, 0, \ldots, 0)$ so that $C(f) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} = I_N$.

Zeros

(ii)

$\Delta^s (f*g) = (\Delta^s f)*g = f*(\Delta^s g)$.

$\Delta f(n) = f(n+1) - f(n)$.

$\forall n \in [\![0, N]\!]$: By reccurence.

let say $s=0$ $\quad \Delta^0 (f*g)(n) = (f*g)(n) = \begin{cases} (\Delta^0 f)*g\ (n) \\ f*(\Delta^0 g)(n) \end{cases}$.

let suppoze it is true for $s=k$, let show it's also true for $s=k+1$.

$$\Delta^{k+1}(f*g) = \Delta\left(\Delta^k (f*g)\right)$$

$$= \Delta((\Delta^k f)*g) \quad \text{and} = \Delta(f*(\Delta^k g))$$

$$= \Delta(\Delta^k f)*g \quad \text{and} = f*\Delta(\Delta^k g)$$

$$= \Delta^{k+1} f*g \quad \text{and} = f*\Delta^{k+1} g.$$

We then showed by reccurence that $\Delta^s (f*g) = (\Delta^s f)*g = f*\Delta^s g$ $\forall n \in \mathbb{N}$.

(iii) ?

(5.2) see github link in comment (but I couldn't do the (iii)...)

(5.3) $1-8-3$.

$f_{ech} \geq 2 \times f_{max} = 2 \times 1633 = 3266$ Hz and $2041 < 3266$.