HA1

Exercice 1: 1.1

(i) $1 \leq p < r \leq \infty$

L^r embedded in L^p ?

$f \in L^p$

So $\|f\|_p = \left(\int_\Omega |f|^p\right)^{1/p} \leq \left(\int_{|f|>1} |f|^p\right)^{1/p} + \left(\int_{|f|<1} |f|^p\right)^{1/p}$

$\|f\|_p^p = \int_\Omega |f|^p = \int_{|f|>1} |f|^p + \int_{|f|<1} |f|^p$

$\leq \int_{|f|>1} |f|^r + \int_{|f|<1} 1\,dx$

$\leq \|f\|_r^r + \mu(\Omega)$

(ii) Show that $l^p \subset l^r$

$(x_i)_{i\in\mathbb{N}} \in l^p \rightarrow \sum_{i\in\mathbb{N}} |x_i|^p < \infty$

So there is a rank N from which $|u_k| < 1$ $(k \geq N)$ and so $|u_k|^r \leq |u_k|^p$

Thus, $\sum_{i=N+1}^{+\infty} |x_i|^r < \sum_{i=N+1}^{+\infty} |x_i|^p \leq \|x\|_p < +\infty$

Then $(x_i) \in l^r$.

(iii)

(1.2) Fourier coefficient

(i) $f(x) = 4\cos(2\pi 9x)$

$$c_k(f) = \langle f(x), e^{2i\pi kx} \rangle = \int_0^1 4\cos(2\pi 9x) e^{-2i\pi kx} dx$$

$$= \frac{4}{2}\left(\int_0^1 e^{-2i\pi kx + 2\pi 9xi} dx + \int_0^1 e^{-2i\pi kx - 2\pi 9xi} dx\right)$$

$$= 2\left(\frac{1}{2\pi 9i - 2i\pi k}\left(e^{2\pi i(9-k)} - 1\right) + \frac{1}{-2i\pi k - 2\pi 9i}\left(e^{-2i\pi(k+9)} - 1\right)\right)$$

$$= \frac{2}{2\pi i}\frac{1}{81-k^2}\left((k+9)\left(e^{2\pi i(9-k)} - 1\right) - (9-k)\left(e^{-2i\pi k(k+9)} - 1\right)\right)$$

$\cos(2\pi 9) = 1$ et $\frac{e^{ia}+e^{-ia}}{2} = \cos(a)$

$\sin(2\pi 9) = 0$ et $\frac{e^{+ia}-e^{-ia}}{2i} = \sin(a)$

$$= \frac{1}{\pi i}\frac{1}{81-k^2}\left(-2k + k\left(e^{-2i\pi k}\left(e^{2\pi i 9} + e^{-2i\pi 9}\right)\right) + 9\left(e^{-2i\pi k}\left(e^{2\pi i 9} - e^{-2i\pi 9}\right)\right)\right)$$

$$= \frac{1}{\pi i}\frac{1}{81-k^2}\left(-2k + 2ke^{-2i\pi k}\right)$$

$$= \frac{2k}{\pi i \times (81-k^2)}\left(e^{-2i\pi k} - 1\right)$$

1.2.

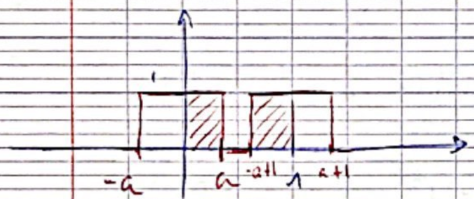
Exercice 1 (p2)

(ii) $f(x) = 6\sin(2\pi 9x)$

De même on trouve,

$$c_k(f) = \frac{3\times 9}{\pi}\,\frac{(e^{-2\pi i k}-1)}{81-k^2}$$

(iii) $f(x) = \begin{cases} 1 \text{ if } x\in(-a,a) \\ 0 \text{ otherwise} \end{cases}$, where f is continued 1-periodically and $0<a<1/2$.



$$c_k = \int_0^1 f(x)e^{-2i\pi kx}dx = \int_0^a e^{-2i\pi kx}dx + \int_{-a+1}^{1} e^{-2i\pi kx}dx \qquad x' = x+a-1$$

$$= -\frac{1}{2i\pi k}\left(e^{-2i\pi ka}-1\right) + \int_0^a e^{-2i\pi k(x'+1-a)}dx'$$

$$= -\frac{1}{2i\pi k}\left(e^{-2i\pi ka}-1\right) + e^{2i\pi k(1-a)}\cdot -\frac{1}{2i\pi k}\left(e^{-2i\pi ka}-1\right)$$

$$= \left(e^{-2i\pi k(1-a)}+1\right)\frac{1}{2i\pi k}\left(\cancel{1} - e^{-2i\pi ka}\right)$$

1.3

(i) $g(x) = \frac{1}{2} - x \quad , x\in[0,1)$

$$S_\infty[g] = \sum_{k=-\infty}^{+\infty} c_k(g)\, e_k$$

$$c_k(g) = \int_0^1\left(\frac{1}{2}-x\right)e^{-2i\pi kx} = \frac{1}{2}\,\frac{1}{-2i\pi k}\left(e^{-2i\pi k}-1\right) - \int_0^1 x e^{-2i\pi kx}$$

$x \longrightarrow 1$

$e^{-2i\pi kx} \longrightarrow \frac{1}{-2i\pi k}e^{-2i\pi kx}$

$$= \quad '' \quad - \left(\frac{1}{-2i\pi k}e^{-2i\pi k} + \frac{1}{(2i\pi k)^2}\left(e^{-2i\pi k}-1\right)\right)$$

$= \left(\text{to simplify ?}\right)$

$=$

(ii) $g(x) = -x \, , \, x \in [-1/2, 1/2)$

$$c_k[g] = \int_{-1/2}^{1/2} -x e^{-2i\pi k x} dx = -\left(\frac{1}{2}e^{-i\pi k} + \frac{1}{2}e^{i\pi k} - \frac{1}{(2i\pi k)^2}\left[e^{-2i\pi k x}\right]_{-1/2}^{1/2}\right)$$

$$= -\left(\cos(\pi k) + \frac{1}{2i\pi k^2}\frac{1}{2i}\left(e^{\pi k i} - e^{-\pi k i}\right)\right)$$

$$= -\left((-1)^k + \frac{1}{2i\pi k^2}\cancel{\sin(\pi k)}\right)$$

$$= (-1)^{k+1}$$

$$S_\infty[g] = \sum_{k=-\infty}^{+\infty}{}' (-1)^{k+1} e_k$$

(iii) $g(x) = x - x^2 \, , \, x \in [0,1)$.

$$c_k[g] = \int_0^1 x e^{-2i\pi k x} - \int_0^1 x^2 e^{-2i\pi k x}$$

$x^2 \to 2x$

$e \to \frac{1}{\cdot} e$

$$= \int_0^1 x e^{-2i\pi k x} - \left(-\frac{1}{2\pi k i}e^{-2i\pi k} + 2\int_0^1 x e^{-2i\pi k x} dx\right)$$

$$= -\int_0^1 x e^{-2i\pi k x} + \frac{1}{2\pi k i}e^{-2i\pi k}$$

$$= \frac{1}{2\pi k i}e^{-2i\pi k} + \frac{1}{(2i\pi k)^2}\left(e^{-2i\pi k} - 1\right) + \frac{1}{2\pi k i}e^{-2i\pi k}$$

$$= \frac{1}{\pi k i}e^{-2i\pi k} + \frac{1}{(2i\pi k)^2}\left(e^{-2i\pi k} - 1\right) = \cdots$$

Exercice 1 (p3)

1.3

(iii) $S_\infty[g] = \ldots$

(iv) $g(x) = \frac{1}{2} - x^2, \; x \in (0,1)$.

$$c_k(g) = \frac{-1}{2 \times 2i\pi k}\left(e^{-2i\pi k} - 1\right) + \frac{1}{2\pi k i} e^{-2i\pi k} + \left\{ \frac{2}{2\pi k i} e^{-2i\pi k} + \frac{1}{(2i\pi k)^2}\left(e^{-2i\pi k} - 1\right)\right)$$

$= \ldots$

$= \ldots$

$S_\infty[g] = \ldots$

1.4

(i) See github link in comment.

(ii) $$D_n(x) = \sum_{k=-n}^{n} e^{2i\pi k x} = \sum_{k=-n}^{n} \left(e^{2\pi i x}\right)^k$$

$$= e^{-2\pi i x n} \frac{e^{2\pi i x(2n+1)} - 1}{e^{2\pi i x} - 1}$$

$$= \frac{e^{\pi i x(2n+2)} - e^{-2\pi i x n}}{e^{2\pi i x} - 1}$$

$$= \frac{e^{\pi i x}}{e^{\pi i x}} \frac{e^{\pi i x(2n+1)} - e^{-\pi i x(2n+1)}}{e^{\pi i x} - e^{-\pi i x}}$$

$$= \frac{\sin\left((2n+1)\pi x\right)}{\sin(\pi x)}$$