

Exercice Sheet 2 - (1)

2.1

$f \in L_p(\mathbb{T})$, $g \in L_q(\mathbb{T})$ where $1 \leq p, q, r < \infty$

$$\text{and } \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1 \Leftrightarrow \frac{r}{p} = \frac{r}{q} + 1 + r$$

$$|(f+g)(x)| = \int_0^1 |f(x-y)| |g(y)| dy$$

$$= \int_0^1 |f(x-y)|^{1+\frac{p}{r}-\frac{p}{r}} |g(y)|^{1+\frac{q}{r}-\frac{q}{r}} dy$$

$$= \int_0^1 (|f(x-y)|^p |g(y)|^q)^{\frac{1}{r}} |f(x-y)|^{1-\frac{p}{r}} |g(y)|^{1-\frac{q}{r}} dy$$

$$= \int_0^1 (|f(x-y)|^p |g(y)|^q)^{\frac{1}{r}} |f(x-y)|^{\frac{r-p}{r}} |g(y)|^{\frac{r-q}{r}} dy$$

$$\leq \int_0^1 (|f(x-y)|^p |g(y)|^q)^{\frac{1}{r}} dy \|f(x-y)\|^{\frac{r-p}{r}}_{\frac{pr}{r-p}} \|g(y)\|^{\frac{r-q}{r}}_{\frac{qr}{r-q}}$$

$$\leq \|(|f(x-y)|^p |g(y)|^q)^{\frac{1}{r}}\|_r \|f(x-y)\|^{\frac{r-p}{r}}_{\frac{pr}{r-p}} \|g(y)\|^{\frac{r-q}{r}}_{\frac{qr}{r-q}}$$

$$\|(|f(x-y)|^p |g(y)|^q)^{\frac{1}{r}}\|_r = \left(\int_0^1 |f(x-y)|^p |g(y)|^q dy \right)^{\frac{1}{r}}$$

$$\| |f(x-y)|^{\frac{r-p}{r}} \|_{\frac{pr}{r-p}} = \left(\int_0^1 |f(x-y)|^p dy \right)^{\frac{r-p}{pr}} = \|f(x-y)\|^{\frac{r-p}{r}}_p$$

$$\| |g(y)|^{\frac{r-q}{r}} \|_{\frac{qr}{r-q}} = \|g(y)\|^{\frac{r-q}{r}}_q$$

$$\|f+g\|_r^r = \int_0^1 |(f+g)(x)|^r dx \leq \int_0^1 \int_0^1 |f(x-y)|^p |g(y)|^q dy dx \|f\|_p^{\frac{r-p}{r}} \|g\|_q^{\frac{r-q}{r}}$$

$$= \|f\|_p^{\frac{r-p}{r}} \|g\|_q^{\frac{r-q}{r}} \|f\|_p^{\frac{r-p}{r}} \|g\|_q^{\frac{r-q}{r}}$$

$$= \|f\|_p^r \|g\|_q^r$$

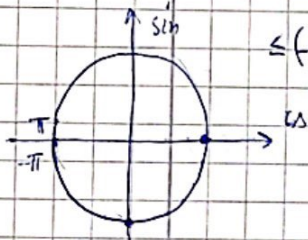
$$\Rightarrow \|f+g\|_r \leq \|f\|_p \|g\|_q$$

2.2 a) $D_N^2 f = \left(\frac{\sin((2N+1)\pi x)}{\sin(\pi x)} \right)^2$

$$= \frac{\sin^2((2N+1)\pi x)}{\sin^2(\pi x)} = \frac{1 - \cos(2\pi x(2N+1))}{1 - \cos(2\pi x)}$$

$$\|D_N^2\|_{L^2} = \left(\int_{-\pi}^{\pi} \left| \frac{1 - \cos(2\pi x(2N+1))}{1 - \cos(2\pi x)} \right|^2 dx \right)^{1/2} \leq \left(4 \int_{-\pi}^{\pi} \left| \frac{1}{1 - \cos(2\pi x)} \right|^2 dx \right)^{1/2}$$

$|f(b) - f(a)| \leq (b-a) \sup |f'(x)|$
 $\Rightarrow |\cos(0) - \cos(2\pi x)| \leq (2\pi x) \sup_{[0, 2\pi x]} |f'(x)|$
 $\leq (2\pi x) \sup_{[0, \pi x]} |\sin(x)| \leq (2\pi x)$



$$\leq \left(4 \int_{-\pi}^{\pi} \left| \frac{1}{1 - \cos(2\pi x)} \right|^2 dx \right)^{1/2}$$

$$\leq 2 \left(\int_{-\pi}^{\pi} \frac{1}{4\pi^2 x^2} dx \right)^{1/2}$$

$$\leq \frac{1}{\pi} \left(2 \int_0^{\pi} \frac{dx}{x^2} \right)^{1/2} < +\infty$$

$\Rightarrow D_N^2 \in L_2(\mathbb{T})$

b) $c_k(D_N^2) = \langle D_N^2(f), e_k \rangle$
 $= \left\langle \sum_{l=-N}^N \sum_{j=-N}^N e^{2i\pi(l+j)t}, e^{2i\pi kt} \right\rangle$
 $= \sum_{l=-N}^N \sum_{j=-N}^N \underbrace{\langle e^{2i\pi(j+l)t}, e^{2i\pi kt} \rangle}_{= \begin{cases} 1 & \text{if } j+l=k \\ 0 & \text{otherwise} \end{cases}}$

$$= \sum_{l=-N}^N \sum_{j=k-l}^N 1$$

$$= (2N+1)!$$

Exercice Sheet 2 (2)

2.3 Discrete Convolution

$$i) \quad u(k) = \sum_{j \in \mathbb{Z}} f(j) w(k-j)$$

$$f = (0, \dots, 0, f(0), f(1), f(2), 0, \dots, 0) \text{ and } w = (0, \dots, 0, w(0), w(1), w(2), 0, \dots)$$

$$u(-1) = 0$$

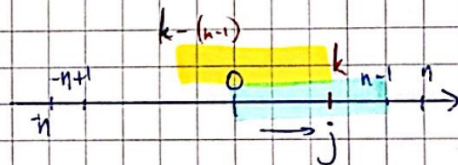
$$u(0) = f(0) w(0)$$

$$u(1) = f(0) w(1) + f(1) w(0)$$

$$u(2) = f(0) w(2) + f(1) w(1) + f(2) w(0)$$

$$\rightarrow u = (0, \dots, 0, f(0)w(0), f(0)w(1) + f(1)w(0), f(0)w(2) + f(1)w(1) + f(2)w(0), 0, \dots, 0)$$

$$ii) \quad u(n) = \sum_{j \in [0, n-1]} f(j) w(n-j)$$



$$W = \begin{pmatrix} w(n-1) & w(n-2) & \dots & w(0) \\ w(n-2) & w(n-3) & \dots & w(-1) \\ \vdots & \vdots & \ddots & \vdots \\ w(0) & w(-1) & \dots & w(-(n-2)) \\ w(-1) & w(-2) & \dots & w(-(n-1)) \\ \vdots & \vdots & \ddots & \vdots \\ w(-(n-1)) & 0 & \dots & 0 \end{pmatrix} \times f = \begin{pmatrix} f(0) \\ f(1) \\ \vdots \\ f(n-1) \end{pmatrix} = \begin{pmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \\ u(-1) \\ \vdots \\ u(-(n-1)) \end{pmatrix}$$