# Fourier series

## 1. Periodic functions and polynomials

Fourier's novel approach to study the heat equation by representing an arbitrary initial condition as series of sine and cosine functions has become one of the standard approaches to study and analyse complex-valued, periodic functions.

**Definition 1.1 (Periodicity).** A function $f\colon \mathbb{R} \to \mathbb{C}$ is called *$T$-periodic*, if there exists a $T > 0$ such that

$$f(x) = f(x+T) \qquad \forall x \in \mathbb{R}.$$

*Example 1.2 (Harmonic oscillation).* One of the simplest periodic functions is a harmonic oscillation, which can be defined by

$$f(x) := a\cos(\omega x) + b\sin(\omega x)$$

with $(a, b) \neq \mathbf{0}$ and $\omega > 0$. Using polar coordinates

$$\begin{pmatrix} a \\ b \end{pmatrix} = r \begin{pmatrix} \sin\phi \\ \cos\phi \end{pmatrix} \qquad (r > 0, 0 \le \phi \le 2\pi),$$

we here obtain the sinusoidal oscillation

$$\begin{aligned} f(x) &= r\sin(\phi)\cos(\omega x) + r\cos(\phi)\sin(\omega x) \\ &= r\sin(\omega x + \phi), \end{aligned}$$

where $r := \sqrt{a^2 + b^2}$ denotes the *amplitude*, $\phi$ the *global phase*, $\omega$ the *frequency*, and $T := 2\pi/\omega$ the *period* of $f$. ○

Although the Fourier analysis can be used to analyse arbitrary periodic function, we restrict ourselves to the case of 1-periodic functions. In so doing, we identify any 1-periodic function $f\colon \mathbb{R} \to \mathbb{C}$ on the real line with the function

$f : \mathbb{T} \to \mathbb{C}$ defined on the *torus* $\mathbb{T}$ of length one. The torus $\mathbb{T}$ can be considered as the quotient space $\mathbb{R}/\mathbb{Z}$, which turns out to be the interval $[0, 1]$ with identified endpoints, i.e. $0 \sim 1$. Figuratively, the endpoints of the interval have been stuck together.

Our main tools to analyse a real-valued, 1-periodic function are the *trigonometric series* of the form

$$x \in \mathbb{T} \mapsto \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(2\pi k x) + b_k \sin(2\pi k x),$$

where $a_k \in \mathbb{R}$ and $b_k \in \mathbb{R}$. Every trigonometric series can be interpreted as a superposition of harmonic oscillations with integer frequencies. Using $\mathrm{e}^{2\pi \mathrm{i} k x} = \cos(2\pi k x) + \mathrm{i} \sin(2\pi k x)$, we can rewrite the real trigonometric series in the complex form

$$x \in \mathbb{T} \mapsto \sum_{k=-\infty}^{\infty} c_k \, \mathrm{e}^{2\pi \mathrm{i} k x}$$

with $c_0 := a_0/2$, $c_k := 1/2\,(a_k - \mathrm{i} b_k)$, and $c_{-k} := 1/2\,(a_k + \mathrm{i} b_k)$. More generally, the coefficients $c_k$ can be arbitrary complex numbers.

If a trigonometric series has only finitely many non-zero coefficients, then the series becomes a trigonometric polynomial. More precisely, a *trigonometric polynomial* of degree $n$ at the most is always of the form

$$x \in \mathbb{T} \mapsto \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(2\pi k x) + b_k \sin(2\pi k x) \qquad \text{or} \qquad x \in \mathbb{T} \mapsto \sum_{k=-n}^{n} c_k \, \mathrm{e}^{2\pi \mathrm{i} k x}.$$

The set of trigonometric polynomials of degree less than or equal to $n$ is denoted by $\mathcal{T}_n$, the set of trigonometric series by $\mathcal{T}_\infty$.

With the help of trigonometric series and polynomials, we will mainly analyse functions from the following spaces:

- the Banach space $C(\mathbb{T})$ of all continuous functions $f : \mathbb{T} \to \mathbb{C}$ with norm

$$\|f\|_{C(\mathbb{T})} := \max\{|f(x)| : x \in \mathbb{T}\},$$

- the Banach space $C^r(\mathbb{T})$ of all $r$-times continuously differentiable functions $f : \mathbb{T} \to \mathbb{C}$ with norm

$$\|f\|C^r(\mathbb{T}) := \|f\|_{C(\mathbb{T})} + \|f^{(r)}\|_{C(\mathbb{T})},$$

- the Banach space $L^p(\mathbb{T})$ with $1 \leq p \leq \infty$ of (equivalence classes of) measurable functions $f\colon \mathbb{T} \to \mathbb{C}$ with finite norm
$$\|f\|_{L^p(\mathbb{T})} := \left(\int_0^1 |f(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \qquad (1 \leq p < \infty)$$
and
$$\|f\|_{L^\infty(\mathbb{T})} := \operatorname{ess\,sup}\{|f(x)| : x \in \mathbb{T}\},$$
- the Hilbert space $L^2(\mathbb{T})$ of all absolutely square-integrable functions $f\colon \mathbb{T} \to \mathbb{C}$ and $g\colon \mathbb{T} \to \mathbb{C}$ with inner product and norm
$$\langle f, g\rangle_{L^2(\mathbb{T})} := \int_0^1 f(x)\,\overline{g(x)} \,\mathrm{d}x, \qquad \|f\|_{L^2(\mathbb{T})} := \left(\int_0^1 |f(x)|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}.$$

For the continuously differentiable function, we clearly have $C^r(\mathbb{T}) \subset C^s(\mathbb{T})$ whenever $r \geq s$. Using Hölder's inequality, one can moreover show that the function spaces $L^p(\mathbb{T})$ are continuously embedded as
$$L^1(\mathbb{T}) \supset L^2(\mathbb{T}) \supset \cdots \supset L^\infty(\mathbb{T}).$$
In the Hilbert space setting , it holds the Cauchy–Schwarz inequality
$$|\langle f, g\rangle_{L^2(\mathbb{T})}| \leq \|f\|_{L^2(\mathbb{T})}\,\|g\|_{L^2(\mathbb{T})}$$
for all $f, g \in L^2(\mathbb{T})$. If it is clear which inner product or norm is addressed, we usually omit the index indicating the specific function space.

## 2. Complex harmonics

Writing the real-valued trigonometric series and polynomials in the complex form, we notice that ever trigonometric series or polynomial is a superposition of complex harmonics.

**DEFINITION 2.1 (COMPLEX HARMONICS).** The set of *complex harmonics* embrace all exponentials with integer frequency and is given by

$$\{e_k : k \in \mathbb{Z}\} \qquad \text{with} \qquad e_k(x) := \mathrm{e}^{2\pi \mathrm{i} k x} \qquad (x \in \mathbb{R}).$$

The essence of FOURIER analysis is now that the complex harmonics form a complete orthonormal basis of the $L^2(\mathbb{T})$, which we prove in two separate steps.

**LEMMA 2.2.** *The complex harmonics are an orthonormal system in $L^2(\mathbb{T})$.*

*Proof.* For every $k, j \in \mathbb{Z}$, we have

$$\langle e_j, e_k \rangle = \int_0^1 \mathrm{e}^{2\pi\mathrm{i}(j-k)x}\,\mathrm{d}x = \begin{cases} \frac{1}{2\pi\mathrm{i}(j-k)}\,\mathrm{e}^{2\pi\mathrm{i}(j-k)x}\Big|_0^1, & j \neq k \\ 1, & j = k \end{cases} = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}. \qquad \square$$

Since the complex harmonics form a (maybe incomplete) orthonormal basis of the HILBERT space $L^2(\mathbb{T})$, we can try to represent a square-integrable function as superposition of complex harmonics.

**DEFINITION 2.3 (FOURIER SERIES).** The *$n$th FOURIER partial sum $\mathcal{S}_n$* of $f\colon \mathbb{T} \to \mathbb{C}$ is given by the trigonometric polynomial

$$\boxed{\mathcal{S}_n[f] := \sum_{k=-n}^{n} c_k[f]\, e_k} \qquad \text{with} \qquad \boxed{c_k[f] := \langle f, e_k \rangle = \int_0^1 f(x)\,\mathrm{e}^{-2\pi\mathrm{i}kx}\,\mathrm{d}x,}$$

where $c_k$ is called the *$k$th FOURIER coefficient*. The limit $\mathcal{S}_\infty[f] := \lim_{n\to\infty} \mathcal{S}_n[f]$ is the *FOURIER series* associated with $f$.

*Remark 2.4 (Trigonometric series).* What happens if we develop an trigonometric series

$$f := \sum_{k=-\infty}^{\infty} c_k\, \mathrm{e}_k \in L^1(\mathbb{T})$$

into a FOURIER series? If the trigonometric series $f$ converges uniformly on $\mathbb{T}$, then Lemma 2.2 implies

$$c_k[f] = \left\langle \sum_{\ell=-\infty}^{\infty} c_\ell\, e_\ell, e_k \right\rangle = \sum_{\ell=-\infty}^{\infty} c_k \langle e_\ell, e_k \rangle = c_k.$$

Thus, the Fourier coefficients coincide with the coefficients of the trigonometric series, and $\mathcal{S}_\infty[f]$ reproduces $f$. Obviously, this observation holds also for trigonometric polynomials. ○

*Remark 2.5 (Arbitrary periodicity).* If $f\colon \mathbb{R} \to \mathbb{C}$ is $T$-periodic, then the function $f(\cdot/T)$ becomes obviously 1-periodic. Using this small substitution, we obtain the $T$-periodic Fourier series

$$\mathcal{S}_\infty[f] := \sum_{k=-\infty}^{\infty} c_k[f]\,\mathrm{e}^{2\pi \mathrm{i} k\cdot/T} \qquad \text{with} \qquad c_k[f] := \frac{1}{T}\int_0^T f(x)\,\mathrm{e}^{-2\pi\mathrm{i}kx/T}\,\mathrm{d}x.$$

All statements for 1-periodic functions may be easily transfered to arbitrary $T$-periodic function. ○

The Fourier series is already well defined for each function $f \in L^1(\mathbb{T})$. Moreover, the Fourier coefficients $c_k[f]$ are unique for any $f \in L^1(\mathbb{T})$. Consequently, the operator $\mathcal{S}_\infty\colon L^1(\mathbb{T}) \to \mathcal{T}_\infty$ mapping a function to a trigonometric series is injective.

**Theorem 2.6 (Uniqueness).** *Let $f \in L^1(\mathbb{T})$ be a given function with $c_k(f) = 0$ for all $k \in \mathbb{Z}$. Then $f$ is constantly zero almost everywhere, i.e. $f \equiv 0$.*

*Proof.* 1. First we consider continuous functions $f \in C(\mathbb{T})$ with zero Fourier coefficients. Due to the Weierstraß approximation theorem, we always find a sequence $(p_n)_{n=1}^\infty$ of trigonometric polynomials $p_n \in \mathcal{T}_n$ that converges uniformly to $f$, i.e.

$$\lim_{n\to\infty} \|f - p_n\|_{C(\mathbb{T})} = 0.$$

Next, we consider the antilinear, continuous functional $\phi_f\colon C(\mathbb{T}) \to \mathbb{C}$ given by

$$\phi_f(g) := \int_0^1 f(x)\overline{g(x)}\,\mathrm{d}x.$$

Since $\phi_f(e_k) = c_k[f] = 0$, and since $p_n$ is a superposition of complex harmonics, we have $\phi_f(p_n) = 0$ for all $n \in \mathbb{N}$. Further, we obtain the estimation

$$|\phi_f(f) - \phi_f(p_n)| = \left|\int_0^1 f(x)\,\overline{\left(f(x) - p_n(x)\right)}\,\mathrm{d}x\right|$$

$$\leq \|f\|_{C(\mathbb{T})} \int_0^1 |f(x) - p_n(x)| \, \mathrm{d}x$$
$$\leq \|f\|_{C(\mathbb{T})} \|f - p_n\|_{C(\mathbb{T})}.$$

Thus, $\phi_f(p_n)$ converges to $\phi_f(f)$, which further implies

$$\int_0^1 |f(x)|^2 \, \mathrm{d}x = \phi_f(f) = \lim_{n\to\infty} \phi_f(p_n) = 0.$$

Due to the continuity of $f$, we conclude $f \equiv 0$.

2. Next, we consider an arbitrary function $f \in L^1(\mathbb{T})$ with vanishing Fourier coefficients. Then the function

$$h(x) := \int_0^x f(t) \, \mathrm{d}t \qquad (x \in \mathbb{T})$$

is continuous and 1-periodic since $h(0) = h(1) = 0$. Further, $h$ is related to $f$ by $h'(x) = f(x)$. The Fourier coefficients of $h$ are given by

$$\begin{aligned} c_k[h] &= \int_0^1 h(x) \, \mathrm{e}^{-2\pi i k x} \, \mathrm{d}x \\ &= -\tfrac{1}{2\pi i k} h(x) \, \mathrm{e}^{-2\pi i k x} \Big|_0^1 + \tfrac{1}{2\pi i k} \int_0^1 h'(x) \, \mathrm{e}^{-2\pi i k x} \, \mathrm{d}x \\ &= \tfrac{1}{2\pi i k} c_k[f] = 0 \end{aligned}$$

for $k \in \mathbb{Z} \setminus \{0\}$. Hence all Fourier coefficients of the 1-periodic, continuous function $h - c_0[h]$ are zero. By the first part, $h$ and thus $f$ have to be zero almost everywhere. □

**Corollary 2.7.** *The complex harmonics are an orthonormal basis of $L^2(\mathbb{T})$.*

*Proof.* Since $L^2(\mathbb{T}) \subset L^1(\mathbb{T})$, the assertion follows from Lemma 2.2 and Theorem 2.6. □

Using EULER's formula, we may rewrite any complex FOURIER series

$$\mathcal{S}_\infty[f](x) = \sum_{k\in\mathbb{Z}} c_k[f]\, e_k = \sum_{k\in\mathbb{Z}} c_k[f]\, \mathrm{e}^{2\pi i k x}$$

into a superposition of sine and cosine functions

$$\mathcal{S}_\infty[f](x) = \frac{a_0[f]}{2} + \sum_{k\in\mathbb{N}} a_k[f]\cos(2\pi k x) + \sum_{k\in\mathbb{N}} b_k[f]\sin(2\pi k x)$$

where the coefficients are given by

$$a_k[f] := c_k[f] + c_{-k}[f] = 2\langle f, \cos(2\pi k x)\rangle$$

and

$$b_k[f] := \mathrm{i}\,(c_k[f] - c_{-k}[f]) = 2\langle f, \sin(2\pi k x)\rangle.$$

Consequently, the *real harmonics*

$$\left\{1, \sqrt{2}\cos(2\pi k x) : k \in \mathbb{N}\right\} \cup \left\{\sqrt{2}\sin(2\pi k x) : k \in \mathbb{N}\right\}$$

form also an orthonormal basis of $L^2(\mathbb{T})$.

## 3. Quadratic mean convergence

Although the FOURIER series is already well defined for absolutely integrable function, the set of square-integrable functions $L^2(\mathbb{T})$ is a much more naturel domain for the operators $\mathcal{S}_n$ and their limit $\mathcal{S}_\infty$. Using the HILBERT space structure of $L^2(\mathbb{T})$, we next show that $\mathcal{S}_n$ yields the best possible approximation of a given function by an trigonometric polynomial of order $n$ at the most.

**Theorem 3.1 (Projection property).** *The nth FOURIER partial sum $\mathcal{S}_n : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is the orthogonal projector form $L^2(\mathbb{T})$ onto $\mathcal{T}_n$, i.e.*

$$\|f - \mathcal{S}_n[f]\| = \min\{\|f - p\| : p \in \mathcal{T}_n\}$$

*for every $f \in L^2(\mathbb{T})$.*

*Proof.* 1. For $f \in L^2(\mathbb{T})$, and for an arbitrary trigonometric polynomial $p \coloneqq \sum_{k=-n}^{n} c_k\, e_k \in \mathcal{T}_n$, we have

$$\langle f, p\rangle = \left\langle f, \sum_{k=-n}^{n} c_k\, e_k \right\rangle = \sum_{k=-n}^{n} \bar{c}_k \,\langle f, e_k\rangle = \sum_{k=-n}^{n} \bar{c}_k\, c_k[f].$$

Now, the squared norm distance is bounded from below by

$$\begin{aligned}
\|f - p\|^2 = \langle f - p, f - p\rangle &= \langle f, f\rangle - \langle p, f\rangle - \langle f, p\rangle + \langle p, p\rangle \\
&= \|f\|^2 - \sum_{k=-n}^{n} c_k\, \bar{c}_k[f] + \sum_{k=-n}^{n} \bar{c}_k\, c_k[f] + \sum_{k=-n}^{n} |c_k|^2 \\
&= \|f\|^2 - \sum_{k=-n}^{n} |c_k[f]|^2 + \sum_{k=-n}^{n} |c_k[f] - c_k|^2 \qquad (3.1)\\
&\geq \|f\|^2 - \sum_{k=-n}^{n} |c_k[f]|^2,
\end{aligned}$$

where equality hold true for $c_k = c_k[f]$. The polynomial $\mathcal{S}_n[f]$ is thus the best possible approximation of $f$.

2. As discussed in Remark 2.4, the Fourier coefficients $c_k[p]$ of a trigonometric polynomial $p \coloneqq \sum_{k=-n}^{n} c_k\, e_k \in \mathcal{T}_n$ coincide with the its coefficients $c_k$. The $n$th Fourier partial sum is thus idempotent, i.e. $\mathcal{S}_n[p] = p$ and $\mathcal{S}_n[\mathcal{S}_n[f]] = \mathcal{S}_n[f]$. Further $\mathcal{S}_n$ is a self-adjoint operator, which can be seen by

$$\begin{aligned}
\langle \mathcal{S}_n[f], g\rangle = \left\langle \sum_{k=-n}^{n} c_k[f]\, e_k, g \right\rangle &= \sum_{k=-n}^{n} c_k[f]\, \bar{c}_k[g] \\
&= \left\langle f, \sum_{k=-n}^{n} c_k[g]\, e_k \right\rangle = \langle f, \mathcal{S}_n[g]\rangle
\end{aligned}$$

for all $f, g \in L^2(\mathbb{T})$. Consequently, $\mathcal{S}_n$ is an orthogonal projector. □

*Remark 3.2 (Orthogonal decomposition).* Since $\mathcal{S}_n$ is an orthogonal projector, we have $\mathcal{S}_n[f] \perp (f - \mathcal{S}_n[f])$ for any $f \in L^2(\mathbb{T})$. ○

**Corollary 3.3 (Bessel's inequality).** *For every $f \in L^2(\mathbb{T})$, we have*

$$\|\mathcal{S}_n[f]\|^2 = \sum_{k=-n}^{n} |c_k[f]|^2 \leq \|f\|^2.$$

*Proof.* Setting $p = \mathcal{S}_n[f]$ in (3.1), we have

$$0 \leq \|f - \mathcal{S}_n[f]\|^2 = \|f\|^2 - \sum_{k=-n}^{n} |c_k[f]|^2. \qquad \square$$

For a square-integrable function $f \in L^2(\mathbb{T})$, the Fourier series automatically converges in the quadratic mean. Thus, besides the $n$th Fourier partial sum, the Fourier series $\mathcal{S}_\infty[f]$ is always well defined.

**Theorem 3.4 (Quadratic mean convergence).** *For any $f \in L^2(\mathbb{T})$, the Fourier series $\mathcal{S}_n[f]$ converges to $f$ in the quadratic mean, i.e.*

$$\lim_{n \to \infty} \|f - \mathcal{S}_n[f]\| = 0.$$

*Proof.* 1. Considering the limit $n \longrightarrow \infty$ in Corollary 3.3, we see that Bessel's inequality implies

$$\sum_{k=-\infty}^{\infty} |c_k[f]|^2 \leq \|f\|^2 < \infty.$$

Since the partial sums $\sum_{k=-n}^{n} |c_k[f]|^2$ are bounded and monotonically increasing, there exists an $N_\epsilon \in \mathbb{N}$ with

$$\sum_{|k|>N_\epsilon} |c_k[f]|^2 < \epsilon$$

for every $\epsilon > 0$.

2. The sequence $(\mathcal{S}_n[f])_{n=0}^{\infty}$ is a Cauchy sequence in $L^2(\mathbb{T})$. More precisely, for $m > n \geq N_\epsilon$, we have

$$\begin{aligned} \|\mathcal{S}_m[f] - \mathcal{S}_n[f]\|^2 &= \sum_{k=-m}^{-n-1} |c_k[f]|^2 + \sum_{k=n+1}^{m} |c_k[f]|^2 \\ &\leq \sum_{|k|>N_\epsilon} |c_k[f]|^2 < \epsilon. \end{aligned}$$

Hence, $(\mathcal{S}_n)_{n=0}^{\infty}$ converges to some function $g \in L^2(\mathbb{T})$.

3. Now, we show that $g$ coincides with the given $f$. For this, we exploit that the continuity of the inner product implies

$$c_k[g] = \langle g, e_k \rangle = \lim_{n\to\infty} \langle \mathcal{S}_n[f], e_k \rangle = \langle f, e_k \rangle = c_k[f]$$

for any $k \in \mathbb{Z}$. Using Theorem 2.6, we conclude that $g$ equals $f$ almost everywhere. □

Considering the first part of the proof of Theorem 3.4, we see that the Fourier coefficients of a square-integrable function in $L^2(\mathbb{T})$ is always square-summable. Denoting the sequence of Fourier coefficients of $f$ by $c[f]$, we conclude that the Fourier coefficients $c[f]$ are included in the Hilbert space $\ell^2(\mathbb{Z})$ of all complex-valued, absolutely square-summable sequences $c := (c_k)_{k=-\infty}^{\infty}$ and $d := (d_k)_{k=-\infty}^{\infty}$ with inner product and norm

$$\langle c, d \rangle_{\ell^2(\mathbb{Z})} = \sum_{k=-\infty}^{\infty} c_k\, \bar{d}_k, \qquad \| c \|_{\ell^2(\mathbb{Z})} := \Big( \sum_{k=-\infty}^{\infty} |c_k|^2 \Big)^{\frac{1}{2}}.$$

The other way round, every sequence $c \in \ell^2(\mathbb{T})$ defines a square-integrable function via a trigonometric series.

**Theorem 3.5 (Riesz–Fischer).** *Every sequence $c \in \ell^2(\mathbb{Z})$ defines a unique function $f \in L^2(\mathbb{T})$ with $c_k[f] = c_k$ for all $k \in \mathbb{Z}$ and*

$$\lim_{n\to\infty} \| f - \mathcal{S}_n[f] \| = 0.$$

*Proof.* We consider the trigonometric polynomials $s_n := \sum_{k=-n}^{n} c_k\, e_k$, whose coefficients coincide with the given sequence $c$. Similarly to the proof of Theorem 3.4, the sequence $(s_n)_{n=0}^{\infty}$ converges to some function $f$ in $L^2(\mathbb{T})$. Using the Cauchy–Schwarz inequality, we have

$$\begin{aligned} |c_k[f] - c_k| &= |\langle f, e_k \rangle - \langle s_n, e_k \rangle| \\ &= |\langle f - s_n, e_k \rangle| \le \underbrace{\| f - s_n \|}_{\to 0}\ \underbrace{\| e_k \|}_{=1} \to 0 \end{aligned}$$

for all $k \in \mathbb{Z}$, which shows $c_k = c_k[f]$ and $s_n = \mathcal{S}_n[f]$. By Theorem 2.6, the function $f$ is uniquely defined by $c[f] = c$. □

## 4. The finite Fourier transform

There is a close relation between the Hilbert spaces $L^2(\mathbb{T})$ and $\ell^2(\mathbb{Z})$. One the one side, every function $f \in L^2(\mathbb{T})$ defines a sequence in $\ell^2(\mathbb{Z})$ by its Fourier coefficients $c[f]$. On the other hand, every sequence $c \in \ell^2(\mathbb{Z})$ defines a function in $L^2(\mathbb{T})$ by the trigonometric series $\sum_{k\in\mathbb{Z}} c_k\, e_k$. The relation between a function and its Fourier coefficients is sometimes called the finite Fourier transform.

**Definition 4.1 (Finite Fourier transform).** The *finite Fourier transform* $\mathcal{F}\colon L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ is defined by

$$\mathcal{F}[f] := c[f] \qquad (f \in L^2(\mathbb{T})).$$

Usually, the domain $L^2(\mathbb{T})$ of the finite Fourier transform is called the *time domain*. The range $\ell^2(\mathbb{Z})$ is usually referred as *frequency domain*. Further, the Fourier coefficients $c[f]$ or the image $\mathcal{F}[f]$ of a function $f$ are here denoted as the *finite spectrum* of $f$. The inverse finite Fourier transform takes the Fourier coefficients and builds a trigonometric series.

**Definition 4.2 (Inverse finite Fourier transform).** The *inverse finite Fourier transform* $\mathcal{F}^{-1}\colon \ell^2(\mathbb{Z}) \to L^2(\mathbb{T})$ is given by

$$\mathcal{F}^{-1}[c] := \sum_{k\in\mathbb{Z}} c_k\, e_k \qquad (c \in \ell^2(\mathbb{Z})).$$

For the finite Fourier transform, Bessel's inequality holds with equality.

**Theorem 4.3 (Parseval's identity).** *For every $f \in L^2(\mathbb{T})$ and $g \in L^2(\mathbb{T})$, we have*

$$\langle f, g\rangle = \langle \mathcal{F}[f], \mathcal{F}[g]\rangle \qquad \text{and} \qquad \|f\| = \|\mathcal{F}[f]\|.$$

*Proof.* The first identity follows from

$$\langle f, \mathcal{S}_n[g]\rangle = \left\langle f, \sum_{k=-n}^{n} c_k[g]\, e_k \right\rangle = \sum_{k=-n}^{n} \bar{c}_k[g]\, \langle f, e_k\rangle = \sum_{k=-n}^{n} \bar{c}_k[g]\, c_k[f]$$

by considering the limit $n \to \infty$. The second identity follows immediately from the first with $g = f$. □

**Theorem 4.4.** *The finite Fourier transform $\mathcal{F}$ is a linear, bounded, and bijective operator with norm $\|\mathcal{F}\| = 1$.*

*Proof.* The finite Fourier transform is obviously linear. Further, $\mathcal{F}$ is injective by Theorem 2.6 and surjective by Theorem 3.5. The operator norm is given by

$$\|\mathcal{F}\| = \sup\left\{\frac{\|\mathcal{F}[f]\|}{\|f\|} : f \in L^2(\mathbb{T}), f \neq 0\right\} = 1,$$

where we have exploited Parseval's identity. □

*Remark 4.5 (Continuity).* The bounded norm of the finite Fourier transform is equivalent to the continuity of the linear operator. ○

The finite Fourier transform possesses an extensive computational calculus. In the following, we consider some of these calculation rules in more detail. First, we consider the effect of reflecting or conjugating the given function.

**Definition 4.6 (Reflection operator).** For every function $f : \mathbb{T} \to \mathbb{C}$, and for every sequence $c : \mathbb{Z} \to \mathbb{C}$, the *reflection operator* $\mathcal{R}$ is defined by

$$\mathcal{R}[f](x) := f(-x) \qquad \text{and} \qquad \mathcal{R}[c] := (c_{-k})_{k=-\infty}^{\infty}.$$

In the following, the conjugation is applied to a function or to a sequence pointwise.

**Definition 4.7 (Conjugation).** For every function $f : \mathbb{T} \to \mathbb{C}$, and for every sequence $c : \mathbb{Z} \to \mathbb{C}$, the *conjugation* is defined by

$$\bar{f}(x) := \overline{f(x)} \qquad \text{and} \qquad \bar{c} := (\bar{c}_k)_{k=-\infty}^{\infty}.$$

A reflection in the time domain causes a reflection in the frequency domain. The pointwise conjugation results in a conjugated reflection.

**Proposition 4.8 (Reflection/Conjugation).** *For each $f \in L^2(\mathbb{T})$, the finite Fourier transform of the reversed and conjugated function is given by*

$$\mathcal{F}[\mathcal{R}[f]] = \mathcal{R}[\mathcal{F}[f]] \qquad \text{and} \qquad \mathcal{F}[\bar{f}] = \overline{\mathcal{R}[\mathcal{F}[f]]}.$$

*Proof.* The statement follows immediately from

$$c_k[\mathcal{R}[f]] = \int_0^1 f(-x)\, \mathrm{e}^{-2\pi i k x}\, \mathrm{d}x = \int_{-1}^0 f(x)\, \mathrm{e}^{2\pi i k x}\, \mathrm{d}x = c_{-k}[f].$$

and

$$c_k[\bar{f}] = \int_0^1 \overline{f(x)}\, \mathrm{e}^{-2\pi i k x}\, \mathrm{d}x = \overline{\int_0^1 f(x)\, \mathrm{e}^{2\pi i k x}\, \mathrm{d}x} = \bar{c}_{-k}[f]. \qquad \square$$

If the function $f$ is symmetric and thus invariant with respect to the reflection, then the finite FOURIER transform has to be symmetric too.

**COROLLARY 4.9 (SYMMETRY).** *For an even function $f \in L^2(\mathbb{T})$ with $f(\cdot) = f(-\cdot)$, and for an odd $g \in L^2(\mathbb{T})$ with $g(\cdot) = -g(-\cdot)$, the FOURIER coefficients fulfil*

$$c_k[f] = c_{-k}[f] \qquad \text{and} \qquad c_k[g] = -c_{-k}[g].$$

Similarly to even and odd symmetric functions, real-valued functions are invariant with respect to the conjugation, which implies the following property.

**COROLLARY 4.10 (REAL/IMAGINARY FUNCTIONS).** *For a real function $f \in L^2(\mathbb{T})$ with $\mathfrak{I}[f] = 0$, and for an imaginary function $g \in L^2(\mathbb{T})$ with $\mathfrak{R}[f] = 0$, the FOURIER coefficients fulfil*

$$c_k[f] = \bar{c}_{-k}[f] \qquad \text{and} \qquad c_k[g] = -\bar{c}_{-k}[g].$$

If the function is conjugated symmetric, then we can combine the computation rules for reflection and conjugation to see that the disctete FOURIER transform becomes purely real- or imaginary-valued.

**COROLLARY 4.11 (CONJUGATED SYMMETRY).** *For a conjugated even function $f \in L^2(\mathbb{T})$ with $f(\cdot) = \bar{f}(-\cdot)$, and for a conjugated odd function $g \in L^2(\mathbb{T})$ with $g(\cdot) = -\bar{g}(-\cdot)$, the finite FOURIER transform fulfils*

$$\mathfrak{I}[\mathcal{F}[f]] = 0 \qquad \text{and} \qquad \mathfrak{R}[\mathcal{F}[g]] = 0.$$

Next, we consider the effect translations (shifts) and modulations of the given function under the finite FOURIER transform.

**Definition 4.12 (Translation operator).** For every function $f\colon \mathbb{T} \to \mathbb{C}$, and for every sequence $c\colon \mathbb{Z} \to \mathbb{C}$, the *translation operators* $\mathcal{T}_{x_0}$ and $\mathcal{T}_{k_0}$ with $x_0 \in \mathbb{T}$ and $k_0 \in \mathbb{Z}$ are defined by

$$\mathcal{T}_{x_0}[f](x) := f(x - x_0) \qquad \text{and} \qquad \mathcal{T}_{k_0}[c] := (c_{k-k_0})_{k=-\infty}^{\infty}.$$

The modulation of a 1-periodic function or a sequence corresponds to the multiplication with a complex harmonic point by point or component by component respectively.

**Definition 4.13 (Modulation operator).** For every function $f\colon \mathbb{T} \to \mathbb{C}$, and for every sequence $c\colon \mathbb{Z} \to \mathbb{C}$, the *modulation operator* $\mathcal{M}_{k_0}$ and $\mathcal{M}_{x_0}$ with $k_0 \in \mathbb{Z}$ and $x_0 \in \mathbb{T}$ are defined by

$$\mathcal{M}_{k_0}[f](x) := \mathrm{e}^{-2\pi \mathrm{i} k_0 x} f(x) \qquad \text{and} \qquad \mathcal{M}_{x_0}[c] := (\mathrm{e}^{-2\pi \mathrm{i} k x_0}\, c_k)_{k=-\infty}^{\infty}.$$

Notice that the finite Fourier transform transfer translations into modulations and the other way round.

**Proposition 4.14 (Translation/modulation).** *For $f \in L^2(\mathbb{T})$, $x_0 \in \mathbb{T}$, and $k_0 \in \mathbb{Z}$, the finite Fourier transform satisfies*

$$\mathcal{F}[\mathcal{T}_{x_0}[f]] = \mathcal{M}_{x_0}[\mathcal{F}[f]] \qquad \textit{and} \qquad \mathcal{F}[\mathcal{M}_{k_0}[f]] = \mathcal{T}_{-k_0}[\mathcal{F}[f]].$$

*Proof.* The Fourier coefficients of the translation may be computed by

$$c_k[\mathcal{T}_{x_0}[f]] = \int_0^1 f(x - x_0)\, \mathrm{e}^{-2\pi \mathrm{i} k x}\, \mathrm{d}x = \int_0^1 f(x)\, \mathrm{e}^{-2\pi \mathrm{i} k\,(x + x_0)}\, \mathrm{d}x = \mathrm{e}^{-2\pi \mathrm{i} k x_0} c_k[f]$$

and similarly of the modulation by

$$c_k[\mathcal{M}_{k_0}[f]] = \int_0^1 f(x)\, \mathrm{e}^{-2\pi \mathrm{i}(k + k_0)\, x}\, \mathrm{d}x = c_{k+k_0}[f]. \qquad \square$$

Finally, the differentiation in the time domain corresponds to the multiplication with a monomial in the frequency domain.

**Proposition 4.15 (Differentiation).** *Let $f \in L^2(\mathbb{T})$ be differentiable almost everywhere with $f' \in L^2(\mathbb{T})$. The finite Fourier transform is then*

$$\mathcal{F}[f'] = \mathcal{F}[Df] = \big(2\pi i k\, c_k[f]\big)_{k\in\mathbb{Z}}.$$

*Proof.* We apply integration by parts and obtain

$$c_k[f'] = \int_0^1 f'(x)\, e^{-2\pi i k x}\, dx$$

$$= f(x)\, e^{-2\pi i k x}\Big|_0^1 + 2\pi i k \int_0^1 f(x)\, e^{-2\pi i k x}\, dx = 2\pi i k\, c_k[f]. \qquad \square$$

*Remark 4.16 (Multiplication).* If the Fourier coefficients are falling fast enough such that $(k\, c_k[f])_{k\in\mathbb{Z}}$ is again contained in $\ell^2(\mathbb{Z})$, then Proposition 4.15 holds the other way around. ○

Considering the computational calculus for the finite Fourier transform, we have the following special cases:

- If $f \in L^2(\mathbb{T})$ is real-valued, then all coefficients $a_k$ and $b_k$ are real too.
- If $f \in L^2(\mathbb{T})$ is even with $f(\cdot) = f(-\cdot)$, then $f$ can be represented as a pure cosine series

$$\mathcal{S}_\infty[f](x) = c_0[f] + 2\sum_{k\in\mathbb{N}} c_k[f]\cos(2\pi k x) = \frac{a_0[f]}{2} + \sum_{k\in\mathbb{N}} a_k[f]\cos(2\pi k x).$$

- If $f \in L^2(\mathbb{T})$ is odd with $f(\cdot) = -f(-\cdot)$, then $f$ can be represented as a pure sine series

$$\mathcal{S}_\infty[f](x) = 2i\sum_{k\in\mathbb{N}} c_k[f]\sin(2\pi k x) = \sum_{k\in\mathbb{N}} b_k[f]\sin(2\pi k x).$$

## 5. Cyclic convolution

Since the finite Fourier transform is linear, the Fourier transform of a sum is the sum of the Fourier-transformed terms. Moreover, the Fourier transform converts the convolution of to functions into a product and the product of two functions into a convolution. In time domain, we here consider following (cyclic) convolution.

**Definition 5.1 (Cyclic convolution).** The *cyclic convolution* $f * g$ of two functions $f\colon \mathbb{T} \to \mathbb{C}$ and $g\colon \mathbb{T} \to \mathbb{C}$ is formally given by

$$(f * g)(x) := \int_0^1 f(y)\, g(x-y)\, \mathrm{d}y.$$

Using the substitution $y \to x - y$, we obtain

$$(f * g)(x) = \int_0^1 f(y)\, g(x-y)\, \mathrm{d}y = \int_0^1 f(x-y)\, g(y)\, \mathrm{d}y = (g * f)(x);$$

so the convolution is commutative. Furthermore, the convolution is translation invariant, which means

$$\big(f(\cdot - x_0) * g\big)(x) = \big(f * g(\cdot - x_0)\big)(x) = (f * g)(x - x_0)$$

for any $x_0 \in \mathbb{R}$. The following theorem shows that the cyclic convolution is well defined for the combination of certain function spaces.

**Theorem 5.2 (Cyclic convolution).**

(i) *For $f \in L^p(\mathbb{T})$ with $1 \le p \le \infty$ and $g \in L^1(\mathbb{T})$, the convolution $f * g$ exists almost everywhere and $f * g$ is contained in $L^p(\mathbb{T})$. Further, it holds* Young's inequality

$$\|f * g\|_{L^p(\mathbb{T})} \le \|f\|_{L^p(\mathbb{T})}\, \|g\|_{L^1(\mathbb{T})}.$$

(ii) *For $f \in L^p(\mathbb{T})$ and $g \in L^q(\mathbb{T})$ with $1 \le p, q \le \infty$ and $1/p + 1/q = 1$, the convolution $f * g$ exists for every $x \in \mathbb{T}$ and $f * g$ is contained in $C(\mathbb{T})$. It holds*

$$\|f * g\|_{C(\mathbb{T})} \le \|f\|_{L^p(\mathbb{T})} \|g\|_{L^q(\mathbb{T})}.$$

(iii) *For $f \in L^p(\mathbb{T})$ and $g \in L^q(\mathbb{T})$ with $1/p + 1/q = 1/r + 1$ and $1 \le p, q, r \le \infty$, the convolution $f * g$ exists almost everywhere and $f * g$ is contained in $L^r(\mathbb{T})$.*

*Further, it holds the* generalized Young inequality

$$\|f * g\|_{L^r(\mathbb{T})} \leq \|f\|_{L^p(\mathbb{T})} \|g\|_{L^q(\mathbb{T})}.$$

*Proof.* (i) The cases $p = 1$ and $p = \infty$ are straightforward and left as exercise. For $p \in (1, \infty)$ with $1/p + 1/q = 1$, Hölder's inequality yields

$$\begin{aligned}
|(f * g)(x)| &\leq \int_0^1 |f(y)| \underbrace{|g(x-y)|}_{=|g|^{1/p}|g|^{1/q}} \,\mathrm{d}y \\
&\leq \left(\int_0^1 |f(y)|^p \, |g(x-y)| \,\mathrm{d}y\right)^{\frac{1}{p}} \left(\int_0^1 |g(x-y)| \,\mathrm{d}x\right)^{\frac{1}{q}} \\
&= \|g\|_{L^1(\mathbb{T})}^{1/q} \left(\int_0^1 |f(y)|^p \, |g(x-y)| \,\mathrm{d}y\right)^{\frac{1}{p}}.
\end{aligned}$$

Note that both sides of the inequality may be infinite. Using this estimate and Fubini's theorem, we get

$$\begin{aligned}
\|f * g\|_{L^p(\mathbb{T})}^p &\leq \|g\|_{L^1(\mathbb{T})}^{p/q} \int_0^1 \int_0^1 |f(y)|^p \, |g(x-y)| \,\mathrm{d}y \,\mathrm{d}x \\
&= \|g\|_{L^1(\mathbb{T})}^{p/q} \int_0^1 |f(y)|^p \int_0^1 |g(x-y)| \,\mathrm{d}x \,\mathrm{d}y \\
&= \|g\|_{L^1(\mathbb{T})}^{1+p/q} \|f\|_{L^p(\mathbb{T})}^p = \|g\|_{L^1(\mathbb{T})}^p \|f\|_{L^p(\mathbb{T})}^p.
\end{aligned}$$

(ii) For $f \in L^p(\mathbb{T})$ and $g \in L^q(\mathbb{T})$ with $1/p + 1/q = 1$ and $p > 1$, Hölder's inequality immediately gives us

$$\begin{aligned}
|(f * g)(x)| &\leq \left(\int_0^1 |f(x-y)|^p \,\mathrm{d}y\right)^{\frac{1}{p}} \left(\int_0^1 |g(y)|^q \,\mathrm{d}y\right)^{\frac{1}{q}} \\
&\leq \|f\|_{L^p(\mathbb{T})} \|g\|_{L^q(\mathbb{T})}
\end{aligned}$$

The neglected case $p = 1$ follows analogously. To conclude that the convolution is a continuous function, we consider

$$|(f * g)(x+t) - (f * g)(x)| \leq \|f(\cdot + t) - f\|_{L^p(\mathbb{T})} \|g\|_{L^q(\mathbb{T})}.$$

Since the translation is continuous in the $L^p(\mathbb{T})$ norm, the right-hand side goes to zero for $t \longrightarrow 0$. (The uniformly continuous function are dense in $L^p(\mathbb{T})$.)

(iii) The last statement is left as exercise. □

*Counterexample 5.3.* In general, the convolution of an $f \in L^p(\mathbb{T})$ function and an $g \in L^1(\mathbb{T})$ function has not to be defined pointwise and thus has not to be continuous. For exemple, we consider the function

$$g(y) := \begin{cases} |y|^{-3/4} & \text{for } y \in [-1/2, 1/2] \setminus \{0\}, \\ 0 & \text{for } y = 0. \end{cases}$$

Now, the convolution $g * g$ is well defined for all $x \in [-1/2, 1/2] \setminus \{0\}$. For $x = 0$, the convolution however becomes infinite since

$$\int_{-1/2}^{1/2} g(y)\, g(-y)\, \mathrm{d}y = \int_{-1/2}^{1/2} |y|^{-3/2}\, \mathrm{d}y = \infty. \qquad \circ$$

Because the convolution of two $L^1(\mathbb{T})$-functions is again a $L^1(\mathbb{T})$-function, we may compute its Fourier coefficients.

**Lemma 5.4 (Fourier convolution).** *For $f, g \in L^1(\mathbb{T})$, the Fourier coefficients of the convolution are given by*

$$c_k(f * g) = c_k(f)\, c_k(g).$$

*Proof.* Using the 1-periodicity of the second function $g$, we obtain

$$\begin{aligned} c_k(f * g) &= \int_0^1 \int_0^1 f(y)\, g(x-y)\, \mathrm{d}y\, \mathrm{e}^{-2\pi \mathrm{i} k x}\, \mathrm{d}x \\ &= \int_0^1 \int_0^1 f(y) \mathrm{e}^{-2\pi \mathrm{i} k y} g(x-y) \mathrm{e}^{-2\pi \mathrm{i} k (x-y)}\, \mathrm{d}y\, \mathrm{d}x \\ &= \int_0^1 f(y) \mathrm{e}^{-2\pi \mathrm{i} k y}\, \mathrm{d}y \int_0^1 g(x) \mathrm{e}^{-2\pi \mathrm{i} k x}\, \mathrm{d}x = c_k(f)\, c_k(g). \end{aligned}$$ □

**Theorem 5.5 (Fourier convolution).** *For $f, g \in L^2(\mathbb{T})$, the finite* Fourier *transform fulfils*

$$\mathcal{F}[f * g] = \mathcal{F}[f] \cdot \mathcal{F}[g].$$

*Proof.* The statement immediately follows from Lemma 5.4. □

Interestingly, the Fourier convolution theorem in Theorem 5.5 holds the other way around too. For this, we define the convolution of two sequences as follows.

**Definition 5.6 (Convolution of sequences).** The *convolution* $c * d$ of two sequences $c \in \ell^2(\mathbb{Z})$ and $d \in \ell^2(\mathbb{Z})$ is formally given by

$$\boxed{(c * d)_k := \sum_{\ell \in \mathbb{Z}} c_\ell \, d_{k-\ell}.}$$

**Theorem 5.7 (Fourier convolution).** *For $f, g \in L^2(\mathbb{T})$ with $f \cdot g \in L^2(\mathbb{T})$, the finite* Fourier *transform fulfils*

$$\mathcal{F}[fg] = \mathcal{F}[f] * \mathcal{F}[g].$$

*Proof.* We consider the Fourier series corresponding to the convolution of the finite Fourier transform

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \sum_{\ell\in\mathbb{Z}} c_\ell[f] \, c_{k-\ell}[g] \, \mathrm{e}^{2\pi i k x} &= \sum_{k\in\mathbb{Z}} \sum_{\ell\in\mathbb{Z}} c_\ell[f] \, e_\ell(x) \, c_{k-\ell}[g] \, e_{k-\ell}(x) \\ &= \Big(\sum_{k\in\mathbb{Z}} c_k[f] \, e_k(x)\Big) \Big(\sum_{\ell\in\mathbb{Z}} c_\ell[g] \, e_\ell(x)\Big) = f \cdot g.\end{aligned}$$

Thus, the convolution of the Fourier coefficients corresponds to the multiplication of the functions by Theorem 2.6. □

Besides the Fourier transform, the convolution plays an important role in the context of kernel functions.

*Example 5.8 (Dirichlet kernel).* For $n \in \mathbb{N}_0$, the *$n$th Dirichlet kernel* is defined by

$$D_n(x) := \sum_{k=-n}^{n} \mathrm{e}^{2\pi i k x} \qquad (x \in \mathbb{R}). \tag{5.1}$$

By Euler's formula, it follows

$$D_n(x) = 1 + 2\sum_{k=1}^{n} \cos(2\pi k x)\,.$$

Obviously, $D_n \in \mathcal{T}_n$ is real-valued and even. For $x \in (0, 1/2]$ and $n \in \mathbb{N}$, we can express $\sin(\pi x)\, D_n(x)$ as telescope sum

$$\begin{aligned}
\sin(\pi x)\, D_n(x) &= \sin(\pi x) + \sum_{k=1}^{n} 2\, \cos(2\pi k x)\, \sin(\pi x) \\
&= \sin(\pi x) + \sum_{k=1}^{n} \big( \sin((2k+1)\pi x) - \sin((2k-1)\pi x)\big) \\
&= \sin((2n+1)\pi x)\,.
\end{aligned}$$

Thus, the $n$th Dirichlet kernel can be represented as a fraction

$$D_n(x) = \frac{\sin\big((2n+1)\pi x\big)}{\sin(\pi x)} \qquad (x \in [-1/2, 1/2)\setminus\{0\}) \tag{5.2}$$

with $D_n(0) = 2n + 1$. The Fourier coefficients of $D_n$ are

$$c_k(D_n) = \begin{cases} 1 & k = -n, \dots, n, \\ 0 & |k| > n\,. \end{cases}$$

For $f \in L^1(\mathbb{T})$ with Fourier coefficients $c_k(f)$, we obtain by Lemma 5.4 that

$$f * D_n = \sum_{k=-n}^{n} c_k(f)\, \mathrm{e}^{2\pi i k \cdot} = \mathcal{S}_n f,$$

which is just the $n$th Fourier partial sum of $f$ and hence its orthogonal projection onto the space of trigonometric polynomials $\mathcal{T}_n$. By the following calculations, the Dirichlet kernel fulfils

$$\|D_n\|_{L^1(\mathbb{T})} = \int_0^1 |D_n(x)|\, \mathrm{d}x \geq \frac{4}{\pi^2}\, \ln n\,. \tag{5.3}$$

Since $\sin(\pi x) \leq \pi x$ for $x \in [0, 1/4)$, equation (5.2) implies

$$\|D_n\|_{L^1(\mathbb{T})} = 2\int_0^1 \frac{|\sin((2n+1)\pi x)|}{\sin(\pi x)}\, \mathrm{d}x \geq 2\int_0^1 \frac{|\sin((2n+1)\pi x)|}{\pi x}\, \mathrm{d}x.$$

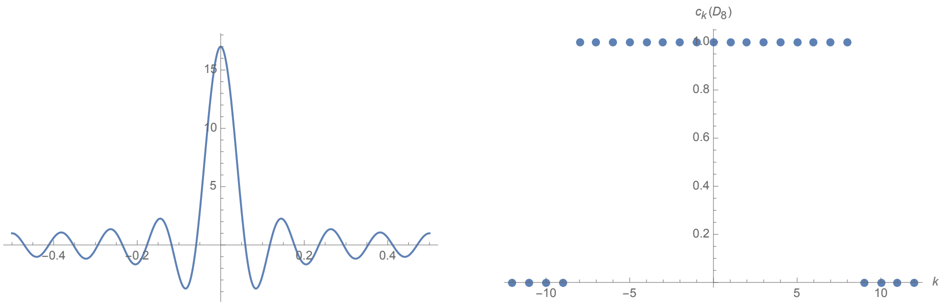



**Figure 5.1.:** The DIRICHLET kernel $D_8(x)$ (left) and its FOURIER coefficients $c_k(D_8)$ (right).

The substitution $y = (2n+1)\pi x$ now results in

$$\begin{aligned}
\|D_n\|_{L^1(\mathbb{T})} &\geq \frac{2}{\pi} \int\limits_0^{(n+1/2)\pi} \frac{|\sin y|}{y}\,\mathrm{d}y \\
&\geq \frac{2}{\pi} \sum_{k=1}^{n} \int\limits_{(k-1)\pi}^{k\pi} \frac{|\sin y|}{y}\,\mathrm{d}y \geq \frac{2}{\pi} \sum_{k=1}^{n} \int\limits_{(k-1)\pi}^{k\pi} \frac{|\sin y|}{k\pi}\,\mathrm{d}y \\
&= \frac{4}{\pi^2} \sum_{k=1}^{n} \frac{1}{k} \geq \frac{4}{\pi^2} \int\limits_1^{n+1} \frac{1}{x}\,\mathrm{d}x \geq \frac{4}{\pi^2} \ln n.
\end{aligned}$$

The Dirichlet kernel $D_8$ and its FOURIER coefficients $c_k(D_8)$ are shown in Figure 5.1. ○

*Example 5.9 (FEJÉR kernel).* The *nth FEJÉR kernel* is defined by

$$\begin{aligned}
F_n(x) &:= \frac{1}{n+1} \sum_{j=0}^{n} D_j(x) = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) \mathrm{e}^{2\pi \mathrm{i} k x} \\
&= \frac{1}{n+1} \left(\frac{\sin(\pi(n+1)x)}{\sin(\pi x)}\right)^2.
\end{aligned}$$

TheFOURIER coefficients $c_k(F_n)$ are

$$c_k(F_n) = \begin{cases} 1 - \frac{|k|}{n+1} & \text{for } k = -n, \ldots, n, \\ 0 & \text{otherwise.} \end{cases}$$

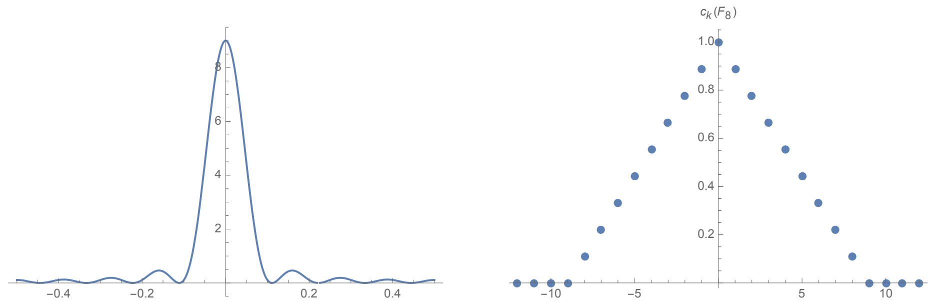



**Figure 5.2.:** The FEJÉR kernel $F_8(x)$ (left) and its FOURIER coefficients $c_k(F_8)$ (right).

The convolution of $f$ with a FEJÉR kernel $F_n$ is given by

$$(f * F_n)(x) = \sum_{k=-n}^{n} \left(1 - \left|\tfrac{k}{n+1}\right|\right) c_k(f) \mathrm{e}^{2\pi\mathrm{i}kx}.$$

Hence, convolving $f$ with a FEJÉR kernel $F_n$ is a weighted $n$th Fourier partial sum of $f$. For the FEJÉR kernel, we further have $F_n(x) = |F_n(x)|$ and hence

$$\boxed{\|F_n\|_{L^1(\mathbb{T})} = \int_0^1 |F_n(x)|\,\mathrm{d}x = \int_0^1 F_n(x)\,\mathrm{d}x = c_0(F_n) = 1.}$$

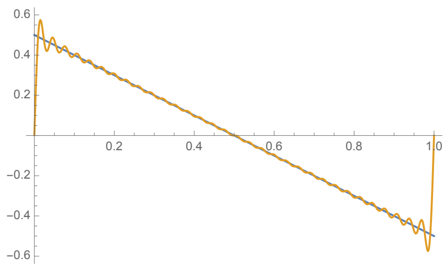
The FEJÉR kernel $F_8(x)$ and its Fourier coefficients $c_k(F_8)$ are shown in Figure 5.2. For the 1-periodic function $f(x) = 1/2 - x$ with $x \in (-1/2, 1/2]$ the Figure 5.3 on the facing page shows both convolutions $f * D_{32}$ and $f * F_{32}$. ○

*Example 5.10 (DE LA VALLÉE POUSSIN kernel).* The *$n$th DE LA VALLÉE POUSSIN kernel* $V_{2n}$ for $n \in \mathbb{N}_0$ is defined by

$$V_{2n} = \frac{1}{n}\sum_{j=n}^{2n-1} D_j = 2\,F_{2n-1} - F_{n-1} = \sum_{k=-2n+1}^{2n-1} c_k(V_{2n})\,\mathrm{e}^{2\pi\mathrm{i}k\cdot}$$

with the FOURIER coefficients

$$c_k(V_{2n}) = \begin{cases} 2 - \frac{|k|}{n} & |k| = n+1, \dots, 2n-1, \\ 1 & |k| = 0, \dots, n, \\ 0 & |k| > 2n. \end{cases}$$ ○



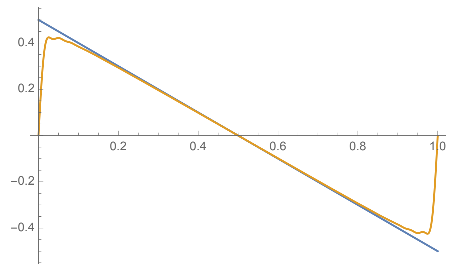




**Figure 5.3.:** Convolving the saw-tooth function with the $n$th Dirichlet kernel ($n = 32$, left). The approximation is quite good except at the jump. Convolution of the saw-tooth function with the $n$th Fejér kernel ($n = 32$, right). The approximation is not as good as using the Dirichlet kernel, however it behaves in a certain sense 'nicer' at the jump.

By Theorem 5.2, the convolution of two $L^1(\mathbb{T})$-functions is again a function in $L^1(\mathbb{T})$. The space $L^1(\mathbb{T})$ forms together with the addition and the convolution a so-called Banach algebra. Unfortunately, there does not exist an identity element with respect to the convolution $*$, i.e. there is no function $g \in L^1(\mathbb{T})$ such that $f * g = f$ for all $f \in L^1(\mathbb{T})$. As a remedy, we can define approximate identities.

**Definition 5.11 (Approximate identity).** A sequence $(K_n)_{n\in\mathbb{N}}$ of functions $K_n \in L^1(\mathbb{T})$ is called an *approximate identity* or a *summation kernel*, if it satisfies

(i) $\int_0^1 K_n(x)\,\mathrm{d}x = 1$ for all $n \in \mathbb{N}$,

(ii) $\|K_n\|_{L^1(\mathbb{T})} = \int_0^1 |K_n(x)|\,\mathrm{d}x \le C < \infty$ for all $n \in \mathbb{N}$,

(iii) $\lim_{n\to\infty} \left(\int_{-1/2}^{-\delta} + \int_\delta^{1/2}\right) |K_n(x)|\,\mathrm{d}x = 0$ for each $0 < \delta < 1/2$.

**Theorem 5.12.** *For all $f \in C(\mathbb{T})$, an approximate identity $(K_n)_{n\in\mathbb{N}}$ satisfies*

$$\lim_{n\to\infty} \|K_n * f - f\|_{C(\mathbb{T})} = 0.$$

*Proof.* Since a continuous function is uniformly continuous on a compact interval, for all $\varepsilon > 0$ there exists $\delta > 0$ so that for all $|u| < \delta$

$$\|f(\cdot - u) - f\|_{C(\mathbb{T})} < \varepsilon\,. \tag{5.4}$$

Using the first property of the approximate identity, we obtain

$$\begin{aligned}
\|K_n * f - f\|_{C(\mathbb{T})} &= \sup_{x\in\mathbb{T}} \left| \int_0^1 f(x-u)\, K_n(u)\, \mathrm{d}u - f(x) \right| \\
&= \sup_{x\in\mathbb{T}} \left| \int_0^1 \big(f(x-u) - f(x)\big)\, K_n(u)\, \mathrm{d}u \right| \\
&\le \sup_{x\in\mathbb{T}} \int_0^1 |f(x-u) - f(x)|\, |K_n(u)|\, \mathrm{d}u \\
&= \sup_{x\in\mathbb{T}} \left( \int_{-1/2}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{1/2} \right) |f(x-u) - f(x)|\, |K_n(u)|\, \mathrm{d}u\,.
\end{aligned}$$

Due to the bound (5.4), the right-hand side can be estimated as

$$\varepsilon \int_{-\delta}^{\delta} |K_n(u)|\, \mathrm{d}u + \sup_{x\in\mathbb{T}} \left( \int_{-1/2}^{-\delta} + \int_{\delta}^{1/2} \right) |f(x-u) - f(x)|\, |K_n(u)|\, \mathrm{d}u\,.$$

By the Properties (ii) and (iii) of the summation kernel $K_n$, we obtain

$$\|K_n * f - f\|_{C(\mathbb{T})} \le \varepsilon\, C + \varepsilon^2.$$

for sufficiently large $n \in \mathbb{N}$. Since $\varepsilon > 0$ can be chosen arbitrarily small, this yields the assertion. □

*Example 5.13 (Dirichlet kernel).* The sequence $(D_n)_{n\in\mathbb{N}}$ of Dirichlet kernels is no approximate identity, since $\|D_n\|_{L^1(\mathbb{T})}$ is not uniformly bounded for all $n \in \mathbb{N}$. Indeed, we will see in the next section that $S_n f = D_n * f$ does in general not converge uniformly to $f \in C(\mathbb{T})$ for $n \to \infty$. ○

*Example 5.14 (Fejér kernel).* The sequence $(F_n)_{n\in\mathbb{N}}$ of Fejér kernels possesses by definition the first two properties of an approximate identity. The third property is also fulfilled since

$$\left( \int_{-1/2}^{-\delta} + \int_{\delta}^{1/2} \right) F_n(x)\, \mathrm{d}x = 2 \int_{\delta}^{1/2} F_n(x)\, \mathrm{d}x$$

$$= \frac{2}{n+1} \int_{\delta}^{1/2} \left( \frac{\sin(\pi(n+1)x)}{\sin(\pi x)} \right)^2 \mathrm{d}x$$
$$\leq \frac{2}{n+1} \int_{\delta}^{1/2} \frac{1}{4x^2} \mathrm{d}x = \frac{1}{2(n+1)} \left( \frac{1}{\delta} - 2 \right).$$

The right-hand side tends to zero as $n \longrightarrow \infty$ so that $(F_n)_{n \in \mathbb{N}}$ is an approximate identity. ○

## 6. Pointwise and uniform convergence

In one of the previous sections, we have proben that the Fourier series of an arbitrary function $f \in L^2(\mathbb{T})$ converges in $L^2(\mathbb{T})$, which means

$$\lim_{n\to\infty} \| S_n[f] - f \|_{L^2(\mathbb{T})} = 0.$$

In the case of an $L^2(\mathbb{T})$ function, we cannot expect stronger convergence but what happens for continuous functions? Does the Fourier series here converges pointwise everywhere or even uniformly?

In fact, many mathematicians like Riemann, Weierstraß, and Dedekind conjectured over long time that the Fourier series of a continuous function converges pointwise everywhere. Unfortunately, we have in general neither pointwise nor uniform convergence. A concrete counterexample was found by Du Bois–Reymond in 1876 and was quite a remarkable surprise. They constructed a (real-valued) function $f \in C(\mathbb{T})$ such that

$$\lim_{n\to\infty} \sup |S_n f(0)| = \infty.$$

Using the principle of uniform boundedness, we will at least show that such a function has to exist.

**Theorem 6.1 (Banach–Steinhaus).** *Let $X$ be a Banach space with a dense subset $D \subset X$, and let $Y$ a normed space. The bounded linear operators $T_n : X \longrightarrow Y$ converges pointwise to the bounded linear operator $T : X \longrightarrow Y$, i.e.*

$$T(f) = \lim_{n\to\infty} T_n(f) \qquad \text{for all } f \in X,$$

*if and only if*

(i) $\|T_n\|_{X\to Y} \le C < \infty$ *for all* $n \in \mathbb{N}$,

(ii) $\lim_{n\to\infty} T_n(p) = T(p)$ *for all* $p \in D$.

**THEOREM 6.2 (EXISTENCE OF NON-CONVERGING SEQUENCE).** *There exists a function $f \in C(\mathbb{T})$ whose FOURIER series does not converge pointwise.*

*Proof.* We assume the contrary, which means that $S_n[f](x) \to f(x)$ converges pointwise everywhere for all continuous functions $f \in C(\mathbb{T})$. In order to apply the theorem of BANACH–STEINHAUS, we choose $X := C(\mathbb{T})$, $Y := \mathbb{C}$, and $D := \cup_{n=0}^{\infty} \mathcal{T}_n$. Now, we consider the pointe valuation of the $n$th FOURIER sum and of the function itself at zero. Hence, we consider

$$T_n(f) := S_n[f](0) \qquad \text{and} \qquad T(f) := f(0),$$

where we could also choose any other point $x \in \mathbb{T}$.

Due to the assumed pointwise convergence of the FOURIER partial sums, we have

$$\sup_n |T_n(f)| = \sup_n |S_n[f](0)| < \infty \qquad \forall f \in C(\mathbb{T})$$

such that the operator sequence $(T_n)_{n\in\mathbb{N}}$ converges pointwise too. Consequently, the operator sequence has to be uniformly bounded, i.e.

$$\sup_n \| T_n \|_{C(\mathbb{T})\to\mathbb{C}} < \infty,$$

by the theorem of BANACH–STEINHAUS.

Let us compute the norm of $T_n$ explicitly. First, we notice

$$|T_n(f)| = |S_n[f](0)| = |(D_n * f)(0)| \le \|f\|_{C(\mathbb{T})} \|D_n\|_{L^1(\mathbb{T})}$$

for any $f \in C(\mathbb{T})$ by Theorem 5.2 so that $\|T_n\|_{C(\mathbb{T})\to\mathbb{C}} \le \|D_n\|_{L^1(\mathbb{T})}$. To verify the opposite inequality, we define the functions

$$f_{n,\epsilon} := \frac{\overline{D_n}}{|D_n| + \epsilon} \in C(\mathbb{T}),$$

with $\epsilon > 0$. The functions $f_{n,\epsilon}$ are bounded by one such that $\|f_{n,\epsilon}\|_{C(\mathbb{T})} \le 1$. Applying the operators $T_n$ to $f_{n,\epsilon}$, we observe

$$|T_n f_{n,\epsilon}| = |(D_n * f_{n,\epsilon})(0)| = \int_0^1 \frac{|D_n(x)|^2}{|D_n(x)| + \epsilon} \, \mathrm{d}x \ge \int_0^1 \frac{|D_n(x)|^2 - \epsilon^2}{|D_n(x)| + \epsilon} \, \mathrm{d}x$$

$$= \int_0^1 |D_n(x)| - \epsilon \, \mathrm{d}x = (\|D_n\|_{L^1(\mathbb{T})} - \epsilon) \geq (\|D_n\|_{L^1(\mathbb{T})} - \epsilon) \, \|f_{n,\epsilon}\|_{C(\mathbb{T})}.$$

Since $\epsilon$ can be arbitrary small, this implies $\|T_n\|_{C(\mathbb{T})\to\mathbb{C}} \geq \|D_n\|_{L^1(\mathbb{T})}$, and the operator norm of $T_n$ is simply $\|D_n\|_{L^1(\mathbb{T})}$.

In Example 5.8, we have already estimated the norm of the Dirichlet kernel. Plugging the computed lower bound, we thus obtain

$$\|T_n\|_{C(\mathbb{T})\to\mathbb{C}} = \|D_n\|_{L^1(\mathbb{T})} \geq \frac{4}{\pi^2} \ln n,$$

which contradicts the uniform boundness of the operators. Consequently, the $n$th Fourier partial sum cannot converge pointwise for every continuous function. □

Although we cannot get pointwise converges for every continuous function, we will show that at least for an suitable subset of functions pointwise convergence is still possible. The corresponding results mainly rely on the so-called Riemann–Lebesgue lemma.

**Lemma 6.3 (Riemann–Lebesgue).** *For every absolutely integrable function $f \in L^1([a,b])$ on a finite or infinite interval $[a,b]$ with $-\infty \leq a < b \leq \infty$, we have*

$$\lim_{|v|\to\infty} \int_a^b f(x) \, \mathrm{e}^{-2\pi \mathrm{i} x v} \, \mathrm{d}x = 0.$$

*Proof.* 1. For the characteristic function $\mathbb{1}_{[\alpha,\beta]}$ of a finite interval $[\alpha,\beta] \subseteq [a,b]$, we have

$$\left| \int_a^b \mathbb{1}_{[\alpha,\beta]}(x) \mathrm{e}^{-2\pi \mathrm{i} x v} \, \mathrm{d}x \right| = \left| -\frac{1}{2\pi \mathrm{i} v} (\mathrm{e}^{-2\pi \mathrm{i} v \beta} - \mathrm{e}^{-2\pi \mathrm{i} v \alpha}) \right| \leq \frac{2}{2\pi |v|} \qquad (v \neq 0).$$

This becomes arbitrarily small as $|v| \to \infty$ so that characteristic functions and also finite linear combinations of characteristic functions (i.e. step functions) fulfill the assertion.

2. The set of step functions is dense in $L^1([a,b])$, i.e. for any $\epsilon > 0$ and $f \in L^1([a,b])$, there exists a step function $\varphi$ such that

$$\|f - \varphi\|_{L^1([a,b])} = \int_a^b |f(x) - \varphi(x)| \, \mathrm{d}x < \epsilon.$$

Estimating the Fourier integral by

$$\left|\int_a^b f(x)\mathrm{e}^{-2\pi\mathrm{i}xv}\,\mathrm{d}x\right| \le \left|\int_a^b (f(x)-\varphi(x))\mathrm{e}^{-2\pi\mathrm{i}xv}\,\mathrm{d}x\right| + \left|\int_a^b \varphi(x)\mathrm{e}^{-2\pi\mathrm{i}xv}\,\mathrm{d}x\right|$$
$$\le \epsilon + \left|\int_a^b \varphi(x)e^{-2\pi\mathrm{i}xv}\,\mathrm{d}x\right|$$

we obtain the assertion. □

Replacing the exponential in the above proof by a sine or cosine function, we notice that the lemma of Riemann–Lebesgue remains valid for trigonmetric functions too.

**Lemma 6.4 (Riemann–Lebesgue).** *For every absolutely integrable function $f \in L^1([a,b])$ on a finite or infinite interval $[a,b]$ with $-\infty \le a < b \le \infty$, we have*

$$\lim_{|v|\to\infty} \int_a^b f(x)\sin(2\pi xv)\,\mathrm{d}x = 0, \quad \textit{and} \quad \lim_{|v|\to\infty} \int_a^b f(x)\cos(2\pi xv)\,\mathrm{d}x = 0.$$

Choosing the interval $[0,1]$, we further see that the Fourier coefficients of an one-periodic, absolutely integrable function tends to zero.

**Corollary 6.5.** *For every $f \in L^1(\mathbb{T})$, the Fourier coefficients satisfy*

$$\lim_{|k|\to\infty} c_k[f] = 0.$$

The convergence of the Fourier series of $f$ in a certain point $x_0 \in \mathbb{T}$ only depends on the function values in a neighbourhood of $x_0$.

**Theorem 6.6 (Riemann's localization principle).** *For $f \in L^1(\mathbb{T})$, the Fourier series converges in $x_0 \in \mathbb{T}$ to some number $c \in \mathbb{C}$, i.e.*

$$\lim_{n\to\infty} S_n f(x_0) = c,$$

*if and only if there exists a $\delta \in (0, 1/2)$ with*

$$\boxed{\lim_{n\to\infty} \int_0^{\delta} \big(f(x_0 - t) + f(x_0 + t) - 2c\big)\, D_n(t)\, \mathrm{d}t = 0.}$$

*Proof.* Since the Dirichlet kernel $D_n \in C(\mathbb{T})$ is even, we get

$$S_n[f](x_0) = \left(\int_{-1/2}^{0} + \int_0^{1/2}\right) f(x_0 - t)\, D_n(t)\, \mathrm{d}t = \int_0^{1/2} \big(f(x_0 - t) + f(x_0 + t)\big)\, D_n(t)\, \mathrm{d}t.$$

Exploiting $\int_0^{1/2} D_n(t)\, \mathrm{d}t = 1/2$, we conclude further

$$S_n[f](x_0) - c = \int_0^{1/2} \big(f(x_0 - t) + f(x_0 + t) - 2c\big)\, D_n(t)\, \mathrm{d}t.$$

Incorporating the explicite form of the Dirichlet kernel $D_n(t) = \frac{\sin \pi(2n+1)t}{\sin \pi t}$ from Example 5.8, and applying Lemma 6.4, we obtain

$$\lim_{n\to\infty} \int_{\delta}^{1} \frac{f(x_0 - t) + f(x_0 + t) - 2c}{\sin \pi t} \sin(\pi(2n+1)t)\, \mathrm{d}t = 0.$$

Thus, the pointwise convergens of the Fourier series in $x_0$ is equivalent to

$$\lim_{n\to\infty} S_n[f](x_0) - c = \lim_{n\to\infty} \int_0^{\delta} \big(f(x_0 - t) + f(x_0 + t) - 2c\big)\, D_n(t)\, \mathrm{d}t,$$

which finishes the proof. □

In the following, we will see that for frequently appearing classes of functions stronger convergence results can be guaranteed. First, we consider piecewise continuous functions.

**Definition 6.7 (Piecewise continuity).** A function $f$ is *piecewise continuous* if it is continuous on $\mathbb{T}$ except for finitely many points and if the right-hand and left-hand limit

$$f(x_0^+) := \lim_{\epsilon\to 0} f(x_0 + |\epsilon|) \quad \text{and} \quad f(x_0^-) := \lim_{\epsilon\to 0} f(x_0 - |\epsilon|)$$

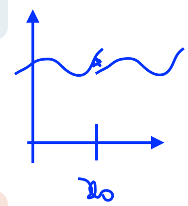
exists everywhere.

In the case $f(x_0^-) \neq f(x_0^+)$, the piecewise continuously function $f : \mathbb{T} \to \mathbb{C}$ has a *jump discontinuity* at $x_0$ with jump height $|f(x_0^+) - f(x_0^-)|$. This definition may be extended to differentiability too.

**Definition 6.8 (Piecewise differentiability).** A piecewise continuous function $f$ is *piecewise continuously differentiable* if it is continuously differentiable on $\mathbb{T}$ except for finitely many points and if the right-hand and left-hand derivative

$$f'(x_0^+) := \lim_{h\to 0} \frac{f(x_0 + |h|) - f(x_0^+)}{|h|} \quad \text{and} \quad f'(x_0^-) := \lim_{h\to 0} \frac{f(x_0 - |h|) - f(x_0^+)}{|h|}$$

exists everywhere.



On the basis of Riemann's localization principle, we obtain the following pointwise convergence of the Fourier series of a piecewise continuously differentiable function.

**Theorem 6.9 (Dirichlet–Jordan).** *Let $f : \mathbb{T} \to \mathbb{C}$ be piecewise continuously differentiable. For every $x_0 \in \mathbb{T}$, the Fourier series then converges as*

$$\lim_{n\to\infty} S_n[f](x_0) = \tfrac{1}{2}\big(f(x_0^+) + f(x_0^-)\big).$$

*Proof.* Due to the piecewise continuous differentiability of $f$, there exists a small $\delta \in (0, 1/2)$ such that $f$ is continuously differentiable on $[x_0 - \delta, x_0 + \delta]\backslash\{x_0\}$. Further, the right-hand and left-hand derivative exist everywhere and are finite such

$$M := \max_{x\in\mathbb{T}}\{|f'(x^+)|, |f'(x^-)|\}.$$

is well defined. Employing the mean value theorem, we further conclude

$$|f(x_0 + x) - f(x_0^+)| \leq xM \quad \text{and} \quad |f(x_0 - x) - f(x_0^-)| \leq xM$$

for all $x \in (0, \delta]$. This implies

$$\int_0^\delta \frac{|f(x_0 - x) + f(x_0 + x) - f(x_0^+) - f(x_0^-)|}{x}\, \mathrm{d}x \leq 2M\delta < \infty.$$

Since $|\frac{x}{\sin \pi x}| \leq \frac{1}{2}$ for all $x \in (0, 1/2)$, the function

$$h(x) := \frac{f(x_0 - x) + f(x_0 + x) - f(x_0^+) - f(x_0^-)}{x} \frac{x}{\sin(\pi x)}$$

is absolutely integrable on $[0, \delta]$. Using the lemma of Riemann–Lebesgue, we find

$$\lim_{N\to\infty} \int_0^\delta h(x) \sin(\pi(2N+1)x)\, \mathrm{d}x = 0.$$

With the Riemann's localization principle from Theorem 6.6, the assertion follows with $c = 1/2\,(f(x_0^+) + f(x_0^-))$. □

**Corollary 6.10 (Dirichlet–Jordan).** *Let $f\colon \mathbb{T} \to \mathbb{C}$ be piecewise continuously differentiable. If $f$ is continuous at $x_0$ then*

$$\lim_{n\to\infty} S_n f(x_0) = f(x_0).$$

Next, we consider the uniform convergence of a Fourier series. A useful criterion for this kind of convergence is the absolute summability of the Fourier coefficients.

**Theorem 6.11 (Uniform convergence).** *Let $f \in C(\mathbb{T})$ be given a continuous function with*

$$\sum_{k\in\mathbb{Z}} |c_k(f)| < \infty.$$

*Then the Fourier series of $f$ converges uniformly.*

*Proof.* Since $|c_k(f)\mathrm{e}^{2\pi i k x}| = |c_k(f)|$, and since $\sum_{k\in\mathbb{Z}} |c_k(f)|$ converges, the Weierstraß M-test yields the absolute and uniform convergence of the Fourier series to some function $g$. Because of the continuity of the $n$th Fourier partial sum, the uniform limit theorem stats that the function $g$ is additionally continuous too; so $f$ and $g$ have to coincide. □

**Theorem 6.12 (Uniform convergence).** *Let $f \in C^1(\mathbb{T})$ be a continuously differentiable function. Then the Fourier series of $f$ converges uniformly.*

*Proof.* Due to Proposition 4.15, we have $c_k[f'] = 2\pi i k\, c_k[f]$, which allows us to rewrite the sum of the absolute Fourier coefficients into

$$\sum_{k\in\mathbb{Z}} |c_k[f]| = |c_0[f]| + \frac{1}{2\pi} \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \frac{1}{|k|} |c_k[f']|.$$

Expanding and rearranging the square

$$\left(|c_k[f']| - \tfrac{1}{|k|}\right)^2 = |c_k[f']|^2 - 2\,\tfrac{|c_k[f']|}{|k|} + \tfrac{1}{|k|^2} \geq 0,$$

we can estimate the summands on the right-hand side by

$$\tfrac{1}{|k|}\,|c_k[f']| \leq \tfrac{1}{2}\left(|c_k[f']|^2 + \tfrac{1}{|k|^2}\right).$$

Using Parseval's identity for $f' \in C(\mathbb{T}) \subset L^2(\mathbb{T})$, we find $\sum_{k\in\mathbb{Z}} |c_k[f']|^2 = \|f'\|^2_{L^2(\mathbb{T})}$. Together with $\sum_{k=1}^{\infty} 1/k^2 = \pi^2/6$, this leads us to

$$\sum_{k\in\mathbb{Z}} |c_k[f]| \leq |c_0[f]| + \tfrac{1}{4\pi} \|f'\|^2_{L^2(\mathbb{T})} + \tfrac{\pi}{12} \leq \infty.$$

Now, the assertion follows form Theorem 6.11. □

In practice, the following convergence result of Fourier series for a sufficiently smooth, periodic function is very useful.

**Theorem 6.13 (Bernstein).** *Let $f \in C^r(\mathbb{T})$ be an $r$-times differentiable function with $r \in \mathbb{N}$. Then, for $n > 1$, the approximation error of the nth Fourier partial sum is bounded by*

$$\|f - S_n f\|_{C(\mathbb{T})} \leq c\, \|f^{(r)}\|_{C(\mathbb{T})} \frac{\ln n}{n^r},$$

*where $c > 0$ is a constant independent of $f$ and $n$.*

*Proof.* 1. Remembering that the $n$th Fourier partial sum can be written as convolution with the $n$th Dirichlet kernel in Example 5.8, and recalling the explicit representation (5.2), we can express the $n$th Fourier partial sum of the $r$th derivative $f^{(r)}$ by

$$\mathcal{S}_n[f^{(r)}](x) = \int_{-1/2}^{1/2} f^{(r)}(x-y)\, \frac{\sin((2n+1)\,\pi y)}{\sin(\pi y)}\, \mathrm{d}y$$

$$= \int_0^{1/2} \left(f^{(r)}(x+y) + f^{(r)}(x-y)\right) \frac{\sin((2n+1)\,\pi y)}{\sin(\pi y)}\, \mathrm{d}y,$$

where we split the integral into two parts and use the substitution $y \mapsto -y$ for the negative interval.

Next, we estimate the absolute value of this Fourier sum by

$$\begin{aligned}
|\mathcal{S}_n[f^{(r)}](x)| &\leq 2\|f^{(r)}\|_{C(\mathbb{T})} \int_0^{1/2} \frac{|\sin((2n+1)\,\pi y)|}{\sin(\pi y)}\, \mathrm{d}y \\
&\leq \|f^{(r)}\|_{C(\mathbb{T})} \int_0^{1/2} \frac{|\sin((2n+1)\,\pi y)|}{y}\, \mathrm{d}y \\
&< \|f^{(r)}\|_{C(\mathbb{T})} \int_0^{(n+1/2)\pi} \frac{|\sin(y)|}{y}\, \mathrm{d}y \\
&< \|f^{(r)}\|_{C(\mathbb{T})} \left(1 + \int_1^{(n+1/2)\pi} \frac{|\sin(y)|}{y}\, \mathrm{d}y\right) \\
&\leq \|f^{(r)}\|_{C(\mathbb{T})} \left(1 + \ln((n+1/2)\pi)\right).
\end{aligned}$$

Consequently, for a convenient constant $c_1 > 0$, we have the estimate

$$\|\mathcal{S}_n[f^{(r)}]\|_{C(\mathbb{T})} \leq c_1 \|f^{(r)}\|_{C(\mathbb{T})} \ln(n) \tag{6.1}$$

for all $n > 1$.

Turning to the Fourier partial sums of $f$, we first notice that these converge uniformly by Theorem 6.12. Using the computation rule in Proposition 4.15, we obtain

$$\begin{aligned}
f - \mathcal{S}_n[f] &= \sum_{k=n+1}^{\infty} c_k[f]\, e_k + c_{-k}[f]\, e_{-k} \\
&= \sum_{k=n+1}^{\infty} \frac{1}{(2\pi \mathrm{i} k)^r} \left(c_k[f^{(r)}]\, e_k + (-1)^r\, c_{-k}[f^{(r)}]\, e_{-k}\right).
\end{aligned} \tag{6.2}$$

2. For even smoothness $r = 2s$ with $s \in \mathbb{N}$, we rearrange (6.2) to

$$f - \mathcal{S}_n[f] = (-1)^s \sum_{k=n+1}^{\infty} \frac{1}{(2\pi k)^r} \left(c_k[f^{(r)}]\, e_k + c_{-k}[f^{(r)}]\, e_{-k}\right)$$

$$= (-1)^s \sum_{k=n+1}^{\infty} \frac{1}{(2\pi k)^r} \left(\mathcal{S}_k[f^{(r)}] - \mathcal{S}_{k-1}[f^{(r)}]\right).$$

In order to compute an upper bound for the difference, we apply the identity

$$\sum_{k=n+1}^{N} a_k \left(b_k - b_{k-1}\right) = a_N b_N - a_{n+1} b_n + \sum_{k=n+1}^{N-1} \left(a_k - a_{k+1}\right) b_k, \tag{6.3}$$

which obviously holds true for all complex numbers $a_k$ and $b_k$ and $N > n$. Choosing $a_k = (2\pi k)^{-r}$ and $b_k = \mathcal{S}_k[f^{(r)}]$, and considering the limit $N \to \infty$, we receive

$$f - \mathcal{S}_n[f] = \frac{(-1)^{s+1}}{(2\pi(n+1))^r} \mathcal{S}_n[f^{(r)}] + (-1)^s \sum_{k=n+1}^{\infty} \left(\frac{1}{(2\pi k)^r} - \frac{1}{(2\pi(k+1))^r}\right) \mathcal{S}_k[f^{(r)}]$$

since (6.1) implies

$$\frac{\| \mathcal{S}_N[f^{(r)}] \|_{C(\mathbb{T})}}{(2\pi N)^r} \le c_1 \| f^{(r)} \|_{C(\mathbb{T})} \frac{\ln(N)}{(2\pi N)^r} \to 0$$

for $N \to \infty$. Using (6.1) once more, the approximation error is bounded by

$$\| f - \mathcal{S}_n[f] \|_{C(\mathbb{T})} \le \frac{c_1}{(2\pi)^r} \| f^{(r)} \|_{C(\mathbb{T})} \left(\frac{\ln(n)}{(n+1)^r} + \sum_{k=n+1}^{\infty} \left(\frac{\ln(k)}{k^r} - \frac{\ln(k)}{(k+1)^r}\right)\right).$$

In order to estimate the sum in the right-hand side, we use the identity in (6.3) with $a_k = \ln(k)$ and $b_k = -(k+1)^{-r}$ and $N \to \infty$, which yields

$$\sum_{k=n+1}^{\infty} \left(\frac{1}{k^r} - \frac{1}{(k+1)^r}\right) \ln(k) = \frac{\ln(n+1)}{(n+1)^r} + \sum_{k=n+1}^{\infty} \frac{\ln(1 + 1/k)}{(k+1)^r}.$$

Due to $\ln(1 + 1/k) < 1/k$, we further have

$$\sum_{k=n+1}^{\infty} \frac{\ln(1 + 1/k)}{(k+1)^r} < \sum_{k=n+1}^{\infty} \frac{1}{k(k+1)^r} < \sum_{k=n+1}^{\infty} \frac{1}{k^{r+1}} < \int_n^{\infty} \frac{1}{x^{r+1}} \, \mathrm{d}x = \frac{1}{r\, n^r}$$

and thus

$$\| f - \mathcal{S}_n[f] \|_{C(\mathbb{T})} \le \frac{c_1}{(2\pi)^r} \| f^{(r)} \|_{C(\mathbb{T})} \frac{\ln(n+1) + 1/r}{n^r},$$

which gives us the assertion for even $r$.

3. For odd smoothness $r = 2s + 1$ with $s \in \mathbb{N}_0$, we rearrange (6.2) to

$$
\begin{aligned}
f - \mathcal{S}_n[f] &= (-1)^s \,\mathrm{i} \sum_{k=n+1}^{\infty} \frac{1}{(2\pi k)^r} \left(c_{-k}[f^{(r)}]\, e_{-k} - c_k[f^{(r)}]\, e_k\right) \\
&= (-1)^s \sum_{k=n+1}^{\infty} \frac{1}{(2\pi k)^r} \left(\tilde{\mathcal{S}}_k[f^{(r)}] - \tilde{\mathcal{S}}_{k-1}[f^{(r)}]\right)
\end{aligned}
$$

with the $n$th partial sum of the *conjugate* FOURIER *series*

$$
\tilde{\mathcal{S}}_n[f^{(r)}] \coloneqq \mathrm{i} \sum_{k=1}^{n} c_{-k}[f^{(r)}]\, e_{-k} - c_k[f^{(r)}]\, e_k.
$$

Next, we derive an upper bound of the conjugate FOURIER partial sum similar to (6.1) by rewriting $\tilde{\mathcal{S}}_n[f^{(r)}]$ as convolution with an appropriate function.

For this, we first rewrite the summands as

$$
\begin{aligned}
&\mathrm{i}\left(c_{-k}[f^{(r)}]\, e_{-k}(x) - c_k[f^{(r)}]\, e_k(x)\right) \\
&\quad = \mathrm{i}\left(\mathrm{e}^{-2\pi \mathrm{i} kx} \int_{-1/2}^{1/2} f^{(r)}(y)\, \mathrm{e}^{2\pi \mathrm{i} ky}\, \mathrm{d}y - \mathrm{e}^{2\pi \mathrm{i} kx} f^{(r)}(y) \int_{-1/2}^{1/2} \mathrm{e}^{-2\pi \mathrm{i} ky}\, \mathrm{d}y\right) \\
&\quad = \mathrm{i}\left(\int_{-1/2}^{1/2} f^{(r)}(y) \left(\mathrm{e}^{2\pi \mathrm{i} k(y-x)} - \mathrm{e}^{-2\pi \mathrm{i} k(y-x)}\right) \mathrm{d}y\right) \\
&\quad = -2 \int_{-1/2}^{1/2} f^{(r)}(y)\, \sin(2\pi k(y-x))\, \mathrm{d}y \\
&\quad = -2 \int_{-1/2}^{1/2} f^{(r)}(x+y)\, \sin(2\pi ky)\, \mathrm{d}y \\
&\quad = -2 \int_{0}^{1/2} \left(f^{(r)}(x+y) - f^{(r)}(x-y)\right) \sin(2\pi ky)\, \mathrm{d}y,
\end{aligned}
$$

where we again split up the integral and use the substitution $y \mapsto -y$ for the negative interval. Using the formula for the partial sums of the geometric series to compute the sum of sines

$$\begin{aligned}\sum_{k=1}^{n} \sin(2\pi k y) &= \frac{1}{2\mathrm{i}} \sum_{k=1}^{n} \mathrm{e}^{2\pi \mathrm{i} k y} - \mathrm{e}^{-2\pi \mathrm{i} k y} \\ &= \frac{1}{2\mathrm{i}} \Big( \frac{\mathrm{e}^{2\pi \mathrm{i}(n+1)y} - 1}{\mathrm{e}^{2\pi \mathrm{i} y} - 1} - \frac{\mathrm{e}^{-2\pi \mathrm{i}(n+1)y} - 1}{\mathrm{e}^{-2\pi \mathrm{i} y} - 1} \Big) \\ &= \frac{1}{2\mathrm{i}} \Big( \frac{\mathrm{e}^{2\pi \mathrm{i}(n+1/2)y} - \mathrm{e}^{-\pi \mathrm{i} y}}{\mathrm{e}^{\pi \mathrm{i} y} - \mathrm{e}^{-\pi \mathrm{i} y}} - \frac{\mathrm{e}^{-2\pi \mathrm{i}(n+1/2)y} - \mathrm{e}^{\pi \mathrm{i} y}}{\mathrm{e}^{-\pi \mathrm{i} y} - \mathrm{e}^{\pi \mathrm{i} y}} \Big) \\ &= \frac{\cos(\pi y) - \cos(2\pi(n+1/2)y)}{2 \sin(\pi y)}\end{aligned}$$

for all $y \in \mathbb{R} \setminus \mathbb{Z}$, we obtain

$$\tilde{\mathcal{S}}_n[f^{(r)}](x) = -\int_0^{1/2} \big(f^{(r)}(x+y) - f^{(r)}(x-y)\big) \, \frac{\cos(\pi y) - \cos(2\pi(n+1/2)y)}{\sin(\pi y)} \, \mathrm{d}y.$$

On the basis of the sum-to-product identity for cosine, the absolute value of the conjugate FOURIER partial sum is bounded by

$$\begin{aligned}|\tilde{\mathcal{S}}_n[f^{(r)}](x)| &\le 2 \, \|f^{(r)}\|_{C(\mathbb{T})} \int_0^{1/2} \frac{|\cos(\pi y) - \cos(2\pi(n+1/2)y)|}{\sin(\pi y)} \, \mathrm{d}y \\ &= 4 \, \|f^{(r)}\|_{C(\mathbb{T})} \int_0^{1/2} \frac{|\sin(n\pi y)\sin((n+1)\pi y)|}{\sin(\pi y)} \, \mathrm{d}y \\ &< 4 \, \|f^{(r)}\|_{C(\mathbb{T})} \int_0^{1/2} \frac{|\sin((n+1)\pi y)|}{\sin(\pi y)} \, \mathrm{d}y.\end{aligned}$$

Similarly to step 1, the conjugate FOURIER partial sum is bounded from above by

$$\|\tilde{\mathcal{S}}_n[f^{(r)}]\|_{C(\mathbb{T})} \le c_1 \|f^{(r)}\|_{C(\mathbb{T})} \ln(n) \tag{6.4}$$

for an appropriate constant $c_1 > 0$.

Due to this bound, the identity in (6.3) with $a_k = (2\pi k)^{-r}$, $b_k = \tilde{\mathcal{S}}_k[f^{(r)}]$, and $N \to \infty$ yields

$$f - \mathcal{S}_n[f] = \frac{(-1)^{s+1}}{(2\pi(n+1))^r} \, \tilde{\mathcal{S}}_n[f^{(r)}] + (-1)^s \sum_{k=n+1}^{\infty} \Big( \frac{1}{(2\pi k)^r} - \frac{1}{(2\pi(k+1))^r} \Big) \, \tilde{\mathcal{S}}_k[f^{(r)}].$$

Exploiting the inequality (6.4) once again, we derive

$$\|f - \mathcal{S}_n[f]\|_{C(\mathbb{T})} \leq \frac{c_1}{(2\pi)^r} \|f^{(r)}\|_{C(\mathbb{T})} \left( \frac{\ln(n)}{(n+1)^r} + \sum_{k=n+1}^{\infty} \left( \frac{\ln(k)}{k^r} - \frac{\ln(k)}{(k+1)^r} \right) \right).$$

Proceeding exactly as in step 2, we obtain the wanted inequality for odd $r$ as well, which finishes the proof. □

Roughly speaking, the theorem says the following:

*The smoother a function $f$ is, the faster its FOURIER coefficients $c_k(f)$ tend to zero as $|k| \to \infty$ and the faster its FOURIER series converges uniformly to $f$.*

## 7. GIBBS PHENOMENON

For a piecewise continuously differentiable function $f : \mathbb{T} \to \mathbb{C}$ with a jump discontinuity at $x_0 \in \mathbb{R}$, the Theorem of DIRICHLET–JORDAN states

$$\lim_{n\to\infty} \mathcal{S}_n[f](x_0) = \frac{f(x_0^-) + f(x_0^+)}{2}.$$

Clearly, the FOURIER series of $f$ cannot converge uniformly in any small neighborhood of $x_0$, because the uniform limit of the continuous functions $\mathcal{S}_n[f]$ would be continuous. The *GIBBS phenomenon* exactly describes this bad convergence behavior of the FOURIER sums near a jump discontinuity, where $\mathcal{S}_n[f]$ overshoots and undershoots $f$.

In the following, we study GIBBS phenomenon on the 1-periodic sawtooth wave $g$ given by

$$g(x) := \begin{cases} \frac{1}{2} - x, & x \in (0,1), \\ 0, & x = 0. \end{cases} \tag{7.1}$$

Looking at the FOURIER partial sums of the sawtooth wave in Figure 7.1 on the next page, we see that the overshooting is not vanishing by increasing the number of FOURIER coefficients.

It can be shown and is left as an exercise that the $n$th FOURIER partial sum of $g$ reads

$$\mathcal{S}_n[g](x) = \frac{1}{\pi} \sum_{k=1}^{n} \frac{1}{k} \sin(2\pi k x).$$

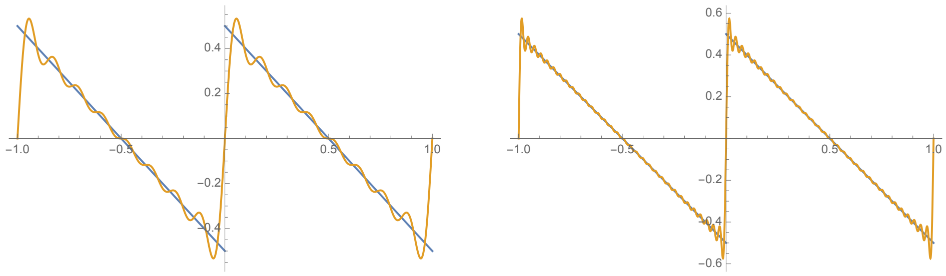



**Figure 7.1.:** Illustration of the Gibbs phenomenon: The approximation using $\mathcal{S}_8[g]$ on two periods (left) yields an overshooting, which is still visible for the case $n = 32$ (right).

We are interested in the approximation error $S_n[g]-g$ of this $n$th Fourier partial sum near the jump location at $x_0 = 0$. If we now differentiate $\mathcal{S}_n[g]$, then we obtain the the Dirichlet kernel form Example 5.8

$$D_n(t) = 1 + 2\sum_{k=1}^{n} \cos(2\pi k t) = \frac{\sin(\pi(2n+1)t)}{\sin(\pi t)}$$

up to the constant 1. The other way round, by integration, we thus have

$$\mathcal{S}_n[g](x) = \int_0^x D_n(t)\,\mathrm{d}t - x = \int_0^x \frac{\sin(\pi(2n+1)t)}{\sin(\pi t)}\,\mathrm{d}t - x \tag{7.2}$$

Next, we extand $1/\sin(\pi t)$ to $1/\sin(\pi t) - 1/\pi t + 1/\pi t$, which leads us to

$$\mathcal{S}_n[g](x) = \int_0^x \frac{\sin(\pi(2n+1)t)}{\pi t}\,\mathrm{d}t + \int_0^x h(t)\sin(\pi(2n+1)t)\,\mathrm{d}t - x, \tag{7.3}$$

where the function $h$ is defined by

$$h(t) := \begin{cases} \frac{1}{\sin \pi t} - \frac{1}{\pi t}, & t \in \left[-\frac{1}{2}, \frac{1}{2}\right] \setminus \{0\}, \\ 0, & t = 0. \end{cases}$$

Here the function $h$ is contained in $C^1([-\frac{1}{2}, \frac{1}{2}])$, which can be validated by using L'Hôpital's rule. We consider both summands in (7.3) separately. Integration by parts yields

$$\int_0^x h(t)\sin(\pi(2n+1)t)\,\mathrm{d}t = -\frac{1}{\pi(2n+1)}h(t)\cos(\pi(2n+1)t)\Big|_0^x$$

$$+ \frac{1}{\pi(2n+1)} \int_0^x h'(t)\cos(\pi(2n+1)t)\,\mathrm{d}t$$
$$= \mathcal{O}(n^{-1}) \qquad \text{as } n \longrightarrow \infty.$$

Incorporating the sine integral

$$\boxed{\mathrm{Si}(y) := \int_0^y \frac{\sin t}{t}\,\mathrm{d}t} \qquad (0 \le y < \infty),$$

we finally obtain

$$\mathcal{S}_n[g](x) = \frac{1}{\pi}\,\mathrm{Si}(\pi(2n+1)x) - x + \mathcal{O}(n^{-1}), \qquad \text{as } n \longrightarrow \infty.$$

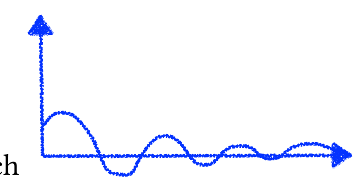
Since we are interested in the limit $n \longrightarrow \infty$, we study the limit of the sine integral.

**Lemma 7.1 (Sine integral).** *The limit of the sine integral is*

$$\boxed{\lim_{y\to\infty} \mathrm{Si}(y) = \frac{\pi}{2}}.$$

*Proof.* Dividing the sine integral into integrals over intervals of length $\pi$. we have

$$\int_0^\infty \frac{\sin t}{t}\,\mathrm{d}t = \sum_{k=0}^{\infty} \int_{k\pi}^{(k+1)\pi} \frac{\sin(t)}{t}\,\mathrm{d}t = \sum_{k=0}^{\infty} A_k \quad \text{where} \quad A_k := \int_{k\pi}^{(k+1)\pi} \frac{\sin t}{t}\,\mathrm{d}t.$$



It is easy to check that $\mathrm{sgn}(A_k) = (-1)^k$, $|A_k| > |A_{k+1}|$ and $\lim_{k\to\infty} A_k = 0$ such that the above sum converges by Leibniz's alternating series test. For $x = 1/2$ we obtain from (7.2) and (7.3) the relation

$$\int_0^{1/2} D_n(t)\,\mathrm{d}t - \frac{1}{2} = \int_0^{1/2} \frac{\sin(\pi(2n+1)t)}{\pi t}\,\mathrm{d}t + \int_0^{1/2} h(t)\sin(\pi(2n+1)t))\,\mathrm{d}t - \frac{1}{2}$$

and further

$$\frac{1}{2} = \frac{1}{\pi} \int_0^{\pi(n+1/2)} \frac{\sin t}{t}\,\mathrm{d}t + \int_0^{1/2} h(t)\sin(\pi(2n+1)t))\,\mathrm{d}t.$$

Since the second integral vanishes for $n \to \infty$ as discussed above, the limit of the right-hand side yields the assertion

$$\frac{\pi}{2} = \lim_{n\to\infty} \int_0^{\pi(n+1/2)} \frac{\sin t}{t}\,\mathrm{d}t = \int_0^{\infty} \frac{\sin t}{t}\,\mathrm{d}t. \qquad \square$$

By the properties of the $A_k$, we notice that the sine integral is bounded by

$$\max_{y\in[0,\infty)} \operatorname{Si}(y) = \operatorname{Si}\pi = 1.8519\ldots.$$

Hence, for $x \in (0, 1/2)$, it follows

$$\begin{aligned} \mathcal{S}_n[g](x) - g(x) &= \frac{1}{\pi}\operatorname{Si}(\pi(2n+1)x) - x + \mathcal{O}(n^{-1}) - \left(\frac{1}{2} - x\right) \\ &= \frac{1}{\pi}\operatorname{Si}(\pi(2n+1)x) - \frac{1}{2} + \mathcal{O}(n^{-1}). \end{aligned}$$

This error attains its maximum for $x = 1/(2n+1)$ and thus is getting closer and closer to the jump discontinuity for larger $n$. Furthermore, the maximum error is explicitly given by

$$\mathcal{S}_n[g]\left(\tfrac{1}{2n+1}\right) - g\left(\tfrac{1}{2n+1}\right) = \tfrac{1.8519\ldots}{\pi} - \tfrac{1}{2} + \mathcal{O}(n^{-1}) = 0.08947\ldots + \mathcal{O}(n^{-1}).$$

Hence, the error is preserved in size for large $n$ although the position $\frac{1}{2n+1}$ converges to $x_0 = 0$. Since $g$ and $\mathcal{S}_n[g]$ are both odd 1-periodic functions, we have an overshooting of $\mathcal{S}_n[g]$ at both sides of the jump discontinuity of approximately 9% of the jump height. This behavior and hence the error do not change with growing $n$ und is typical for the convergence of $\mathcal{S}_n$ near a jump discontinuity.

A general description of the Gibbs phenomenon is given by the following theorem.

**Theorem 7.2 (Phenomenon of Gibbs–Wilbraham).** *Let $f$ be a 1-periodic, piecewise continuously differentiable function and $x_0$ be a jump discontinuity of $f$ with height $h_0 := f(x_0^+) - f(x_0^-) > 0$. Let $(x_0 - 2\delta, x_0 + 2\delta)$ with $0 < \delta < \frac{1}{4}$ be an interval, where $f$ has no further discontinuities. Then the maximal approximation error is given by*

$$\boxed{\begin{aligned} \lim_{n\to\infty} \max_{x\in(x_0,x_0+\delta]} \mathcal{S}_n[f](x) - f(x) &= \left(\tfrac{1}{\pi}\operatorname{Si}(\pi) - \tfrac{1}{2}\right) h_0 \approx 0.089 h_0, \\ \lim_{n\to\infty} \max_{x\in[x_0-\delta,x_0)} f(x) - \mathcal{S}_n[f](x) &= \left(\tfrac{1}{\pi}\operatorname{Si}(\pi) - \tfrac{1}{2}\right) h_0 \approx 0.089 h_0. \end{aligned}}$$

*Let $x_0 + \delta_n$ be the maximizer of $\mathcal{S}_n[f] - f$ on $(x_0, x_0 + \delta]$. Then the positions satisfy*

$$\lim_{n \to \infty} (n\,\delta_n) = \tfrac{1}{2}.$$

*Proof.* Without loss of generality, assume $x_0 = 0$. Now, we consider the function $h : x \mapsto f(x) - h_0\, g(x)$, where $g$ is the sawtooth wave from (7.1). The function $h$ hast no points of discontinuity on $[x_0 - \delta, x_0 + \delta]$. Hence, the Theorem of Dirichlet–Jordan implies that $\mathcal{S}_n[h]$ converges uniformly in $[x_0 - \delta, x_0 + \delta]$ to $h$ for $n \to \infty$. Considering the limit of

$$\mathcal{S}_n[f] - f = \mathcal{S}_n[h] - h + h_0\,(\mathcal{S}_n[g] - g),$$

we obtain the assertion. □