Exercise Sheet 1

due date: April 18, 2021, 23:59

Exercise 1.1.

Let μ be a finite measure on a measurable space $(\Omega, \mathfrak{A})$ and let $1 \le p < r \le \infty$.

(i) Show that $L^r(\Omega, \mathfrak{A}, \mu)$ is continuously embedded in $L^p(\Omega, \mathfrak{A}, \mu)$, i.e. that the mapping

$$\mathrm{id}\colon L^r(\Omega, \mathfrak{A}, \mu) \to L^p(\Omega, \mathfrak{A}, \mu), \quad f \mapsto f$$

is well-defined and continuous.

(ii) Denote by ℓ^p the Banach space of all sequences with finite p-Norm, i.e.

$$\ell^p = \{(x_i)_{i\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \sum_{i\in\mathbb{N}} |x_i|^p < \infty\}.$$

Show that $\ell^p \subset \ell^r$.

(iii) Find counterexamples that the following statements are not true:

$$L^1(\mathbb{T}) \subseteq L^2(\mathbb{T}), \quad \ell^2 \subseteq \ell^1, \quad L^1(\mathbb{R}) \subseteq L^2(\mathbb{R}) \quad \text{and} \quad L^2(\mathbb{R}) \subseteq L^1(\mathbb{R}).$$

Exercise 1.2.

Compute the Fourier coefficients $c_k(f) = \langle f,\ e^{2\pi i k\cdot}\rangle$, $k \in \mathbb{Z}$, of the following functions:

(i) $f(x) = 4\cos(2\pi\ 9x)$,

(ii) $f(x) = 6\sin(2\pi\ 9x)$,

(iii) $f(x) = \begin{cases} 1 & \text{if } x \in [-a, a), \\ 0 & \text{otherwise,} \end{cases}$ where f is continued 1-periodically and $0 < a < \frac{1}{2}$.

Exercise 1.3.

Find the Fourier series of the 1-periodic functions

(i) $g(x) = \frac{1}{2} - x, \quad x \in [0, 1)$,

(ii) $g(x) = -x, \quad x \in [-\frac{1}{2}, \frac{1}{2})$,

(iii) $g(x) = x - x^2, \quad x \in [0, 1)$,

(iv) $g(x) = \frac{1}{2} - x^2, \quad x \in [0, 1)$.

Write complex and real Fourier series!

Exercise 1.4.

The function

$$D_n(x) := \sum_{k=-n}^{n} e^{2\pi i k x}$$

is called **n–th Dirichlet kernel**.

(i) Write a Python script that plots D_n in the interval $[-\frac{1}{2}, \frac{1}{2}]$ with step size 0.001 and print out the respective plots for $n = 8$ and $n = 16$.

(ii) Show that for all $x \in \mathbb{R}$ it holds

$$D_n(x) = \frac{\sin(\pi(2n+1)x)}{\sin(\pi x)}.$$

(iii) Prove that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} D_n(x)\,dx = 1, \quad \text{and} \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} |D_n(x)|\,dx \geq \frac{4}{\pi^2}\ln(n).$$