HA1

Exercice 1: 1.1

(i) $1 \leq p < r \leq \infty$

L^r embedded in L^p ? what about $r=\infty$

$f \in L^p$

So $\|f\|_p = \left(\int_\Omega |f|^p\right)^{1/p} \leq \left(\int_{|f|>1} |f|^p\right)^{1/p} + \left(\int_{|f|<1} |f|^p\right)^{1/p}$

$\|f\|_p^p = \int_\Omega |f|^p = \int_{|f|>1} |f|^p + \int_{|f|<1} |f|^p$ ✓

$\leq \int_{|f|>1} |f|^r + \int_{|f|<1} 1\,dx$

$\leq \|f\|_r^r + \mu(\Omega)$ ✓

(ii) Show that $\ell^p \subset \ell^r$

$(x_i)_{i\in\mathbb{N}} \in \ell^p \rightarrow \sum_{i\in\mathbb{N}} |x_i|^p < \infty$

So there is a rank N from which $|u_k| \leq 1$ $(k \geq N)$ and so $|u_k|^r \leq |u_k|^p$

what is u_k

Thus, $\sum_{i=N+1}^{+\infty} |x_i|^r < \sum_{i=N+1}^{+\infty} |x_i|^p \leq \|x\|_p < +\infty$

Then $(x_i) \in \ell^r$. ✓

(iii)

1.2 Fourier coefficient

(i) $f(x) = 4\cos(2\pi 9x)$

$c_k(f) = \langle f(x), e^{2i\pi kx} \rangle = \int_0^1 4\cos(2\pi 9x) e^{-2i\pi kx} dx$

$\rightarrow f = 4\left(\frac{e^{2\pi i 9x} + e^{-2\pi i 9x}}{2}\right)$

$= \frac{4}{2}\left(\int_0^1 e^{-2i\pi kx + 2\pi 9xi} dx + \int_0^1 e^{-2i\pi kx - 2\pi 9xi} dx\right)$

$\rightarrow c_9 = c_{-9} = 2$

$= 2\left(\frac{1}{2\pi 9i - 2i\pi k}\left(e^{2\pi i(9-k)} - 1\right) + \frac{1}{-2i\pi k - 2\pi 9i}\left(e^{-2i\pi(k+9)} - 1\right)\right)$

$c_k = 0 \quad |k| \neq 9$

$= \frac{2}{2\pi i} \frac{1}{81 - k^2}\left((k+9)\left(e^{2\pi i(9-k)} - 1\right) - (9-k)\left(e^{-2i\pi k(k+9)} - 1\right)\right)$

$\cos(2\pi 9) = 1$ et $\frac{e^{ia} + e^{-ia}}{2} = \cos(a)$

$\sin(2\pi 9) = 0$ et $\frac{e^{+ia} - e^{-ia}}{2i} = \sin(a)$

$= \frac{1}{\pi i} \frac{1}{81 - k^2}\left(-2k + k\left(e^{-2i\pi k}\left(e^{2\pi i 9} + e^{-2i\pi 9}\right)\right) + 9\left(e^{-2i\pi k}\left(e^{2\pi i 9} - e^{-2i\pi 9}\right)\right)\right)$

$= \frac{1}{\pi i} \frac{1}{81 - k^2}\left(-2k + 2ke^{-2i\pi k}\right)$

$= \frac{2k}{\pi i \times (81 - k^2)}\left(e^{-2i\pi k} - 1\right)$

1.2

Exercice 1 (p2)

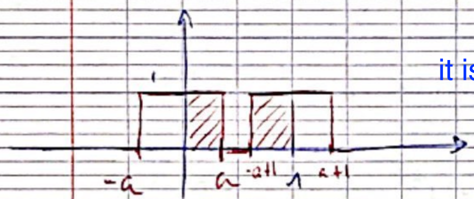
(ii) $f(x) = 6\sin(2\pi 9x)$

De même on trouve,

$$c_k(f) = \frac{3\times 9}{\pi} \frac{\left(e^{-2\pi i k}-1\right)}{81-k^2}$$

(iii) $f(x) = \begin{cases} 1 \text{ if } x \in (-a,a) \\ 0 \text{ otherwise} \end{cases}$, where f is continued 1-periodically and $0 < a < 1/2$.

it is one periodic



$$c_k = \int_0^1 f(x) e^{-2i\pi k x} dx = \int_0^a e^{-2i\pi k x} dx + \int_{-a+1}^{1} e^{-2i\pi k x} dx = \int_{-a}^{a} e^{-2\pi i k x} dx$$

$x' = x + a - 1$

$$= -\frac{1}{2i\pi k}\left(e^{-2i\pi k a} - 1\right) + \int_0^a e^{-2i\pi k (x' + 1 - a)} dx' = \frac{\sin(2\pi k a)}{k\pi}$$

$$= -\frac{1}{2i\pi k}\left(e^{-2i\pi k a} - 1\right) + e^{-2i\pi k (1-a)} \cdot -\frac{1}{2i\pi k}\left(e^{-2i\pi k a} - 1\right) \quad k \neq 0$$

$$= \left(e^{-2i\pi k (1-a)} + 1\right) \frac{1}{2i\pi k}\left(\cancel{k} - e^{-2i\pi k a}\right)$$

$$k = 0 \rightarrow c_0 = \int_{-a}^{a} 1\, dx = 2a$$

1.3

(i) $g(x) = \frac{1}{2} - x$, $x \in [0,1)$

$$S_\infty[g] = \sum_{k=-\infty}^{+\infty} c_k(g) e_k$$

$$c_k(g) = \int_0^1 \left(\frac{1}{2} - x\right) e^{-2i\pi k x} = \frac{1}{2} \frac{1}{-2i\pi k}\left(e^{-2i\pi k} - 1\right) - \int_0^1 x e^{-2i\pi k x}$$

$x \rightarrow 1$

$e^{-2i\pi k x} \rightarrow \frac{1}{-2i\pi k} e^{-2i\pi k x}$

$$= \quad '' \quad - \left(\frac{1}{-2i\pi k} e^{-2i\pi k} + \frac{1}{(2i\pi k)^2}\left(e^{-2i\pi k} - 1\right)\right)$$

Use Euler's formula to simplify it

$= \left(\text{to simplify ?}\right) \quad c_k = \begin{cases} 0 & k=0 \\ \dfrac{1}{2\pi i k} & k \neq 0 \end{cases}$

$=$

(ii) $g(x) = -x \,,\; x \in [-1/2, 1/2)$

$$c_k[g] = \int_{-1/2}^{1/2} -x e^{-2i\pi k x}\,dx = -\left(\frac{\frac{1}{2}e^{-i\pi k} + \frac{1}{2}e^{i\pi k}}{2\pi i k} - \frac{1}{(2i\pi k)^2}\left[e^{-2i\pi k x}\right]_{-1/2}^{1/2}\right)$$

$$= -\left(\underline{\underline{\cos(\pi k)}} + \frac{1}{2i\pi k^2}\,\frac{1}{2i}\left(e^{\pi k i} - e^{-\pi k i}\right)\right)$$

$$= -\left(\underline{\underline{(-1)^k}} + \frac{1}{2i\pi k^2}\,\cancel{\sin(\pi k)}\right)$$

$$= \frac{(-1)^{k+1}}{2\pi k}$$

$$S_\infty[g] = \sum_{k=-\infty}^{+\infty}{}' (-1)^{k+1} e_k \,\alpha$$

(iii) $g(x) = x - x^2 \,,\; x \in [0,1)$.

$x^2 \to 2x$

$e \to \frac{1}{\cdot} e$

$$c_k[g] = \int_0^1 x e^{-2i\pi k x} - \int_0^1 x^2 e^{-2i\pi k x}$$

$$= \int_0^1 x e^{-2i\pi k x} - \left(-\frac{1}{2\pi k i} e^{-2i\pi k} + 2\int_0^1 x e^{-2i\pi k x}\,dx\right)$$

$$= -\int_0^1 x e^{-2i\pi k x} + \frac{1}{2\pi k i} e^{-2i\pi k}$$

$$= \frac{1}{2\pi k i} e^{-2i\pi k} + \frac{1}{(2i\pi k)^2}\left(e^{-2i\pi k} - 1\right) + \frac{1}{2\pi k i} e^{-2i\pi k}$$

$$= \frac{1}{\pi k i} e^{-2i\pi k} + \frac{1}{(2i\pi k)^2}\left(e^{-2i\pi k} - 1\right) = \ldots$$

Exercice 1 (p3)

1.3

(iii) $S_\infty[g] = \ldots$

(iv) $g(x) = \frac{1}{2} - x^2 \; , x \in (0,1)$.

$$c_k(g) = \frac{-1}{2 \times 2i\pi k}\left(e^{-2i\pi k} - 1\right) + \frac{1}{2\pi k i} e^{-2i\pi k} + \left\{ \frac{2}{2\pi k i} e^{-2i\pi k} + \frac{1}{(2i\pi k)^2}\left(e^{-2i\pi k} - 1\right)\right.$$

$= \ldots$

$= \ldots$

$S_\infty[g] = \ldots$

line 25 in your file must be replaced with
plt.plot(x, d_re)
the imaginary part de_im must be close to zero, and it contain only the rounding error.

1.4

(i) See github link in comment. ✓

(ii) $D_n(x) = \sum_{k=-n}^{n} e^{2i\pi k x} = \sum_{k=-n}^{n} \left(e^{2\pi i x}\right)^k$

$$= e^{-2\pi i x n} \frac{e^{2\pi i x(2n+1)} - 1}{e^{2\pi i x} - 1}$$

$$= \frac{e^{\pi i x(2n+2)} - e^{-2\pi i x n}}{e^{2\pi i x} - 1}$$

$$= \frac{e^{\pi i x}}{e^{\pi i x}} \frac{e^{\pi i x(2n+1)} - e^{-\pi i x(2n+1)}}{e^{\pi i x} - e^{-\pi i x}}$$

Good!

$$= \frac{\sin\left((2n+1)\pi x\right)}{\sin(\pi x)}$$ ✓