Exercice Sheet 2 · (1).

2.1 $f \in L_p(\mathbb{T})$, $g \in L_q(\mathbb{T})$ where $1 \leq p, q, r < +\infty$

and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1 \Leftrightarrow \frac{r}{p} + \frac{r}{q} = 1 + r$ Case $r = \infty$?

$$|(f*g)(x)| = \int_0^1 |f(x-y)||g(y)|dy$$

$$= \int_0^1 |f(x-y)|^{1+\frac{p}{r}-\frac{p}{r}} |g(y)|^{1+\frac{q}{r}-\frac{q}{r}} dy$$

Are these numbers positive?why?

$$= \int_0^1 \left(|f(x-y)|^p |g(y)|^q\right)^{1/r} |f(x-y)|^{1-\frac{p}{r}} |g(x-y)|^{1-\frac{q}{r}} dy$$

$$= \int_0^1 \left(|f(x-y)|^p |g(y)|^q\right)^{1/r} |f(x-y)|^{\frac{r-p}{r}} |g(y)|^{\frac{r-q}{r}} dy$$

$$\leq \int_0^1 \left(|f(x-y)|^p |g(y)|^q\right)^{1/r} dy \; \left\| |f(x-y)|^{\frac{r-p}{r}} \right\|_{\frac{pr}{r-p}} \; \left\| |g(y)|^{\frac{r-q}{r}} \right\|_{\frac{qr}{r-q}}$$

$$\leq \left\| \left(|f(x-y)|^p |g(y)|^q\right)^{1/r} \right\|_r \; \left\| f(x-y)^{\frac{r-p}{r}} \right\|_{\frac{pr}{r-p}} \; \left\| g(y)^{\frac{r-q}{r}} \right\|_{\frac{qr}{r-q}}$$

$$\left\| \left(|f(x-y)|^p |g(y)|^q\right)^{1/r} \right\|_r = \left(\int_0^1 |f(x-y)|^p |g(y)|^q dy\right)^{1/r}$$

$$\left\| |f(x-y)|^{\frac{r-p}{r}} \right\|_{\frac{pr}{r-p}} = \left(\int_0^1 |f(x-y)|^p dy\right)^{\frac{r-p}{pr}} = \| f(x-y) \|_p^{\frac{r-p}{r}}$$

$$\left\| |g(y)|^{\frac{r-q}{r}} \right\|_{\frac{qr}{r-q}} = \| g(y) \|_q^{\frac{r-q}{r}}$$ ✓

$$\| f*g \|_r^r = \int_0^1 |(f*g)(x)|^r dx \leq \int_0^1 \int_0^1 |f(x-y)|^p |g(y)|^q dy\,dx \times \|f\|_p^{r-p} \|g\|_q^{r-q}$$

$$= \|f\|_p^p \|g\|_q^q \|f\|_p^{r-p} \|g\|_q^{r-q}$$

$$= \|f\|_p^r \|g\|_q^r .$$

$$\Rightarrow \| f*g \|_r \leq \|f\|_p \|g\|_q$$

2.2

a) $$D_N^2(x) = \left(\frac{\sin((2N+1)\pi x)}{\sin(\pi x)} \right)^2$$ This function is 1-periodic

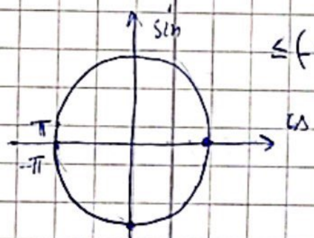
$$= \frac{\sin^2((2N+1)\pi x)}{\sin^2(\pi x)} = \frac{1-\cos(2\pi x(2N+1))}{1-\cos(2\pi x)}$$

$$\|D_N^2\|_{L_2} = \left(\int_{-\pi}^{\pi} \left| \frac{1-\cos(2\pi x(2N+1))}{1-\cos(2\pi x)} \right|^2 dx \right)^{1/2} \leq \left(4 \int_{-\pi}^{\pi} \left| \frac{1}{1-\cos(2\pi x)} \right| dx \right)^{1/2}$$

$$|f(b) - f(a)| \leq (b-a) \sup |f'(y)|$$

$$\Leftrightarrow |\cos(0) - \cos(2\pi x)| \leq (-2\pi x) \sup_{[0;2\pi x]} |f'(x)|$$

$$\leq (-2\pi x) \sup_{[0;\pi x]} |\sin(t)| \leq (-2\pi x)$$



$$\leq \left(4 \int_{-\pi}^{\pi} \left| \frac{1}{1-\cos(2\pi x)} \right|^2 dx \right)^{1/2}$$

$$\leq 2 \left(\int_{-\pi}^{\pi} \frac{1}{4\pi^2 x^2} dx \right)^{1/2}$$

$$\leq \frac{1}{\pi} \left(2 \int_0^{\pi} \frac{dx}{x^2} \right)^{1/2} < +\infty.$$

$$\longrightarrow D_N^2 \in L_2(\mathbb{T}).$$

b) $$c_k(D_N^2) = \langle D_N^2(t), e_k \rangle$$

$$= \left\langle \sum_{l=-N}^{N} \sum_{j=-N}^{N} e^{2i\pi(l+j)t}, e^{2i\pi k t} \right\rangle$$

$$= \sum_{l=-N}^{N} \sum_{j=-N}^{N} \underbrace{\langle e^{2i\pi(j+l)}, e^{2i\pi k t} \rangle}_{= \begin{cases} 1 & \text{if } j+l=k \\ 0 & \text{otw.} \end{cases}}$$

you can count how many times e^{2pi i k} shows up in D_n^2

$$= \sum_{l=-N}^{N} \sum_{j=k-l} 1 \; ? = \begin{cases} & |k| \leq 2N \\ & |k| > 2N \end{cases}$$

$$= (2N+1) \; ?$$

and the connection between c_k(D_n^2) and c_k(F_n)?

Exercise Sheet 2 (2)

2.3 Discrete Convolution

i) $u(k) = \sum'_{j \in \mathbb{Z}} f(j)\, w(k-j)$

$f = (0, \dots, 0, f(0), f(1), f(2), 0, \dots, 0)$ and $w = (0, -, 0, w(0), w(1), w(2), 0 -)$

$u(-1) = 0$

$u(0) = f(0)\,w(0)$

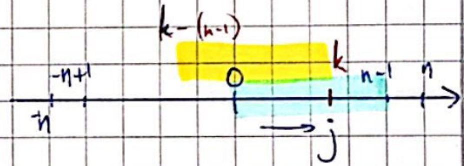
$u(1) = f(0)\,w(1) + f(1)\,w(0)$

$u(2) = f(0)\,w(2) + f(1)\,w(1) + f(2)\,w(0)$

u_3 and u_4 are also non-zero

$\rightarrow u = (0, -, 0, f(0)w(0), f(0)w(1) + f(1)w(0), f(0)w(2) + f(1)w(1) + f(2)w(0), 0 - 0)$

ii) $u(0) = \sum'_{j \in [0, n-1]} f(j)\, w(-j)$



$$W = \begin{pmatrix} w(n-1) & w(n-2) & \cdots & w(0) \\ w(n-2) & w(n-3) & \cdots & w(-1) \\ \vdots & & & \vdots \\ w(0) & w(-1) & \cdots & w(-(n-1)) \\ w(-1) & w(-2) & \cdots\ w(-(n-1)) & 0 \\ \vdots & & & \vdots \\ w(-(n-1)) & 0 & \cdots & 0 \end{pmatrix} \times f = \begin{pmatrix} f(0) \\ f(1) \\ \vdots \\ \vdots \\ f(n-1) \end{pmatrix} \quad \begin{pmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \\ u(-1) \\ \vdots \\ u(-(n-1)) \end{pmatrix}$$

the order of the rows in W and u must be changed
so we have u=(u_(-n+1)…u0…..u_(n-1))