Exercise Sheet 3

due date: May 02, 2021, 23:59

Exercise 3.1.

Using Fourier series prove the convergence of the following series and calculate the limit.

(i) $\sum\limits_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$

(ii) $\sum\limits_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$

(iii) $\sum\limits_{k=1}^{\infty} \frac{1}{k^2}$

Hint: Consider Exercise 1.3.

Exercise 3.2.

(i) Let $f \in L^1(\mathbb{T})$, and let $S_n f$ be the n-th Fourier sum of f. The arithmetic mean of the first n Fourier sums is denoted by

$$\sigma_n\,(x) := \frac{1}{n+1} \sum_{k=0}^{n} \left[S_k f\right](x)\,.$$

Show that σ_n is given by the convolution

$$\sigma_n\,(x) = \int_0^1 f\,(x-u)\,F_n\,(u)\,du,$$

where F_n is the n-th Fejer kernel.

(ii) Show that there exist a sequence $\{f_n\}_{n=1}^{\infty} \subset C(\mathbb{T}) \setminus \{0\}$ and a constant $c > 0$ which is independent of n such that

$$c\,\|f_n\|_{C(\mathbb{T})} \|D_n\|_{L^1(\mathbb{T})} \leq \|f_n * D_n\|_{C(\mathbb{T})}$$

Exercise 3.3.

Let $f \in L^1(\mathbb{T})$ and suppose that f is differentiable at $x_0 \in [0,1]$. Show that in this case

$$\sum_{k=-n}^{n} c_k(f)\,\mathrm{e}^{2\pi i k x_0} \to f(x_0) \quad \text{as} \quad n \to \infty.$$

Exercise 3.4.

Consider the 1-periodic function which is defined by

$$f\,(x) := \ln\left(2\cos(\pi x)\right)$$

for $x \in (-\frac{1}{2}, \frac{1}{2})$. Note that f is not defined at $x = \pm\frac{2k+1}{2}$ where $k \in \mathbb{N}_0$, and that

$$\lim_{x \to \frac{1}{2}+} f\,(x) = \lim_{x \to \frac{1}{2}-} f\,(x) = -\infty.$$

Show that $f \in L^1(\mathbb{T})$, and determine the Fourier coefficients a_k of f where $k \in \mathbb{N}_0$. Plot the functions $f(x)$ and $[S_7 f](x)$.

Hint: In order to compute the Fourier coefficients, the expansion

$$\ln(1+z) = z - \tfrac{z^2}{2} + \tfrac{z^3}{3} - \tfrac{z^4}{4} + \cdots$$

can be used, which converges for $|z| \leq 1$ and $z \neq -1$.