Exercise Sheet 5

due date: May 16, 2021, 23:59

Exercise 5.1.

(i) Show that the set of all circulant matrices $\mathrm{Circ}(N) := \{C(g) : g \in \mathbb{C}^N\}$ is a commutative ring with the operations of matrix addition and matrix multiplication. Which elements are units and which ones are zero divisors?

(ii) The discrete derivative Δf of a vector $f \in \mathbb{C}^N$ is defined by $(\Delta f)_n = f_{n+1} - f_n$, where f is continued periodically. The discrete derivative of order $s \in \mathbb{N}$ is defined recursively by $\Delta^s f = \Delta(\Delta^{s-1} f)$. Show that for every two vectors $f, g \in \mathbb{C}^N$, it holds that

$$\Delta^s(f * g) = (\Delta^s f) * g = f * (\Delta^s g).$$

(iii) From the first part we know that for $f, g \in \mathbb{C}^N$, there exists $h \in \mathbb{C}^N$ such that $C(h) = C(f)C(g)$. Find an explicit expression for h.

Exercise 5.2.

The aim of this exercise is to get a better understanding of the Fourier transform of a signal. To do so, implement the following tasks in Python:

(i) Create the following five signals $f_i \in \mathbb{R}^{1024}$, $i = 1, \dots, 5$:

   a) a one-periodic sine wave uniformly sampled on $[-\frac{1}{2}, \frac{1}{2}]$, yielding f_1,

   b) a one-periodic sine wave uniformly sampled on $[0, 1]$, yielding f_2,

   c) add a sine wave with 100 times the frequency and $\frac{1}{10}$ of the amplitude of f_1 to f_1, yielding f_3,

   d) add a sine wave with 100 times the frequency and $\frac{1}{10}$ of the amplitude of f_2 to f_2, yielding f_4,

   e) add a linear increasing signal starting in $(-\frac{1}{2}, \frac{1}{2})$ and going to $(\frac{1}{2}, 1)$ to f_3, yielding f_5.

(ii) For each signal, compute the discrete Fourier transform $\hat{f}_i \in \mathbb{C}^{1024}$, $i = 1, \dots, 5$. Plot the signal, its Fourier spectrum (absolute value of the Fourier transform or alternatively, the logarithm of one plus the absolute value) and its Fourier phase. Analyse your results and explain them.

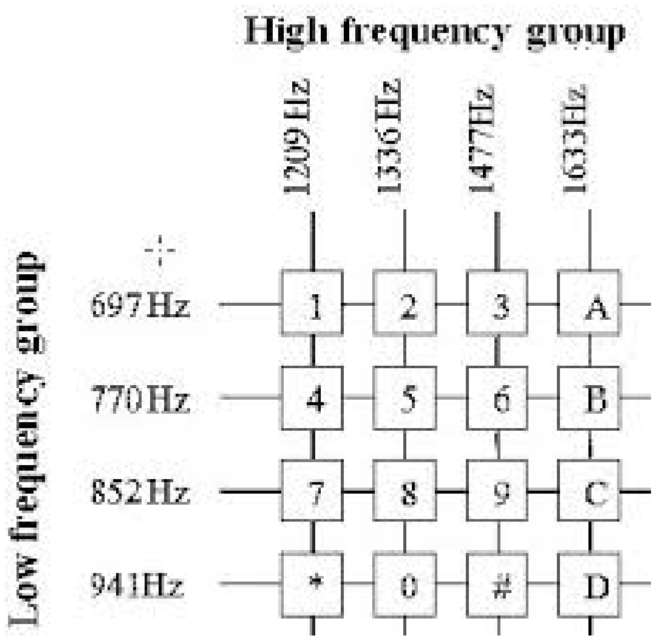
(iii) Filter out the high frequency sine from the Fourier transform of the signals f_3 and f_5 and display the inverse Fourier transform. Make sure you don't get any complex values after the inverse Fourier transform! Interpret your results. Do you get the "clean" sine wave back? If not, why?

Hint: The commands `fft`, `ifft` and `fftshift` in the package `numpy.fft` might be helpful.

Exercise 5.3.

The DTMF telephone keypad is laid out in a matrix of push buttons in which each row represents the low frequency component and each column represents the high frequency component of the DTMF signal. Pressing a key sends a combination of the row and column frequencies. For example, the key 1 produces a superimposition of tones of 697 and 1209 hertz. The tones are then decoded using Fourier analysis to determine the keys pressed by the user. Three numbers are dialed and the corresponding

signals are uniformly sampled (thought as sum of $e^{2\pi ikx}$) at 4096 points x in the interval $[0, 1]$, yielding the signals `signal_k`, $k = 1, 2, 3$. The result is provided in the file `telefonsignal.mat` on the homepage. Find out which keys are pressed! Why couldn't we choose a smaller power of two to generate the signals?
Hint: You can load the content of .mat files in Python using the command `scipy.io.loadmat`.



Exercise 5.4.

Load the signal `gong.mat` and listen to it (in Python you can write the signal to an audio file using the function `write` from the package `scipy.io.wavfile`). Set the amplitude of the highest and lowest 5000 frequencies to zero and listen to the resulting new signal.