2 Exercise 2

2.1 Task 1

Eigenmodes are solutions of the homogeneous Helmholtz equation. The one-dimensional homogeneous Helmholtz equation is

$$\frac{\partial^2 \hat{p}}{\partial x^2} + k^2 \hat{p} = 0 \tag{2.1}$$

and the plane-wave ansatz for its solution is

$$\hat{p}(x) = \hat{p}_+ e^{-jkx} + \hat{p}_- e^{jkx},$$

with three unknowns – complex amplitudes $\hat{p}_+$ and $\hat{p}_-$ and real and positive wave number k. The constant amplitudes $\hat{p}_+$ and $\hat{p}_-$ are associated with the forward (in the direction of increasing x) and backward propagating plane waves, for the usual time dependence $e^{j\omega t}$.

Boundary conditions for a closed pipe are

$$\left(\frac{\partial \hat{p}}{\partial x}\right)_{x=0} = \left(\frac{\partial \hat{p}}{\partial x}\right)_{x=l} = 0.$$

Using

$$\frac{\partial \hat{p}}{\partial x} = -jk\hat{p}_+ e^{-jkx} + jk\hat{p}_- e^{jkx},$$

the boundary conditions give

$$\text{for } x = 0 : -jk\hat{p}_+ + jk\hat{p}_- = 0 \Rightarrow \hat{p}_+ = \hat{p}_-$$

$$\text{for } x = l : -jk\hat{p}_+(e^{-jkl} - e^{jkl}) = -2k\hat{p}_+ \sin(kl) = 0 \Rightarrow k = k_m = \frac{m\pi}{l} \text{ with } m = 1, 2, 3...$$

The wave number thus takes discrete values due to the boundary conditions. (In an unbounded free space it is a continuous parameter.)

(a) The lowest five eigenfrequencies are

$$f_m = \frac{c_0}{\lambda_m} = \frac{c_0 k_m}{2\pi} = \frac{mc_0}{2l}.$$

For $c_0 = 343\,\mathrm{m/s}$ and $l = 1\,\mathrm{m}$: $f_1 = 171.5\,\mathrm{Hz}$, $f_2 = 343\,\mathrm{Hz}$, $f_3 = 514.5\,\mathrm{Hz}$, $f_4 = 686\,\mathrm{Hz}$, $f_5 = 857.5\,\mathrm{Hz}$.

(b) The eigenfunctions (modes) are specified down to an arbitrary constant (compare with the task in section 1.1 in Exercise 1) by the solution which follows from above,

$$\hat{p}_m(x) = \hat{p}_{+m}(e^{-jk_m x} + e^{jk_m x}) = 2\hat{p}_{+m}\cos(k_m x) = 2\hat{p}_{+m}\cos\left(\frac{m\pi x}{l}\right).$$

The amplitude $\hat{p}_{+m}$ remains unknown and depends on the strength of the source of waves in a particular case (not on the mode). The general solution of the Helmholtz equation is given as the (infinite) sum

$$\hat{p}(x) = \sum_{m=1}^{\infty} \hat{p}_m(x).$$

2.2 Task 2

We discretize the space along the right-oriented x-axis with N equidistant ($\Delta x = h$) points x_i, where $i = 1, 2, ...N$. The points x_1 and x_N lie at the left and right end of the pipe, respectively, and the pipe is parallel to the x-axis.

(a) For convenience, we also add two virtual points outside the computational domain of the pipe: x_0 at the distance h before (left from) x_1 and x_{N+1} at the distance h after (right from) x_N. The 2^{nd}-order accurate central difference scheme gives

$$\left(\frac{\partial^2 \hat{p}}{\partial x^2}\right)_i \approx \frac{\hat{p}_{i-1} - 2\hat{p}_i + \hat{p}_{i+1}}{h^2},$$

while the boundary conditions at the pipe ends read

$$\left(\frac{\partial \hat{p}}{\partial x}\right)_1 = 0 \Rightarrow \hat{p}_0 = \hat{p}_2 \quad \text{and} \quad \left(\frac{\partial \hat{p}}{\partial x}\right)_N = 0 \Rightarrow \hat{p}_{N+1} = \hat{p}_{N-1}.$$

These follow from the central difference scheme (recall eq. (1.3) from Exercise 1), giving also 2^{nd}-order accuracy.

The discretization replaces the homogeneous Helmholtz equation (2.1), which is a partial differential equation, with the system of N equations

$$\begin{bmatrix} -\frac{2}{h^2}+k^2 & \frac{2}{h^2} & 0 & 0 & \dots & & \\ \frac{1}{h^2} & -\frac{2}{h^2}+k^2 & \frac{1}{h^2} & 0 & \dots & & \\ 0 & \frac{1}{h^2} & -\frac{2}{h^2}+k^2 & \frac{1}{h^2} & 0 & \dots & \\ \dots & & & \dots & & & \dots \\ & \dots & 0 & \frac{1}{h^2} & -\frac{2}{h^2}+k^2 & \frac{1}{h^2} & 0 \\ & & \dots & 0 & \frac{1}{h^2} & -\frac{2}{h^2}+k^2 & \frac{1}{h^2} \\ & & \dots & 0 & 0 & \frac{2}{h^2} & -\frac{2}{h^2}+k^2 \end{bmatrix} \begin{bmatrix} \hat{p}_1 \\ \hat{p}_2 \\ \hat{p}_3 \\ \dots \\ \hat{p}_{N-2} \\ \hat{p}_{N-1} \\ \hat{p}_N \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dots \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\frac{1}{h^2}\left(\begin{bmatrix} -2 & 2 & 0 & 0 & \dots & & \\ 1 & -2 & 1 & 0 & \dots & & \\ 0 & 1 & -2 & 1 & 0 & \dots & \\ \dots & & & \dots & & & \dots \\ & \dots & 0 & 1 & -2 & 1 & 0 \\ & & \dots & 0 & 1 & -2 & 1 \\ & & \dots & 0 & 0 & 2 & -2 \end{bmatrix} + k^2 \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & & \\ 0 & 1 & 0 & 0 & \dots & & \\ 0 & 0 & 1 & 0 & 0 & \dots & \\ \dots & & & \dots & & & \dots \\ & \dots & 0 & 0 & 1 & 0 & 0 \\ & & \dots & 0 & 0 & 1 & 0 \\ & & \dots & 0 & 0 & 0 & 1 \end{bmatrix}\right) \begin{bmatrix} \hat{p}_1 \\ \hat{p}_2 \\ \hat{p}_3 \\ \dots \\ \hat{p}_{N-2} \\ \hat{p}_{N-1} \\ \hat{p}_N \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dots \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

It has the same form, $(\underline{\underline{A}} + k^2\underline{\underline{I}})\underline{x} = 0$, as eq. (1.1) in Exercise 1. Without a source term on the right-hand side of the (homogeneous) Helmholtz equation, we are solving an eigenvector/eigenvalue problem with eigenvalues $-k_m^2$ and eigenvectors $[\hat{p}_{m1}\hat{p}_{m2}...\hat{p}_{mN}]^T$ (which are specified down to an arbitrary non-zero constant). In Task 1 in section 2.1 we solved the problem analytically and here we are solving its discretized form, with the linear differential operator $\partial^2/\partial x^2$ replaced by the matrix $\underline{\underline{A}}$.

(b) The 4th-order accurate central difference scheme gives

$$\left(\frac{\partial^2\hat{p}}{\partial x^2}\right)_i \approx \frac{-\frac{1}{12}\hat{p}_{i-2} + \frac{4}{3}\hat{p}_{i-1} - \frac{5}{2}\hat{p}_i + \frac{4}{3}\hat{p}_{i+1} - \frac{1}{12}\hat{p}_{i+2}}{h^2},$$

after adding the points x_{-1} and x_{N+2} before x_0 and after x_{N+1}, respectively, also at the distances h. The boundary conditions at the pipe ends read as before,

$$\left(\frac{\partial\hat{p}}{\partial x}\right)_1 = 0 \Rightarrow \hat{p}_0 = \hat{p}_2 \quad \text{and} \quad \left(\frac{\partial\hat{p}}{\partial x}\right)_N = 0 \Rightarrow \hat{p}_{N+1} = \hat{p}_{N-1} \tag{2.2}$$

leading to

$$\left(\frac{\partial^2\hat{p}}{\partial x^2}\right)_2 \approx \frac{-\frac{1}{12}\hat{p}_2 + \frac{4}{3}\hat{p}_1 - \frac{5}{2}\hat{p}_2 + \frac{4}{3}\hat{p}_3 - \frac{1}{12}\hat{p}_4}{h^2} = \frac{\frac{4}{3}\hat{p}_1 - \frac{31}{12}\hat{p}_2 + \frac{4}{3}\hat{p}_3 - \frac{1}{12}\hat{p}_4}{h^2}$$

$$\left(\frac{\partial^2\hat{p}}{\partial x^2}\right)_{N-1} \approx \frac{-\frac{1}{12}\hat{p}_{N-3} + \frac{4}{3}\hat{p}_{N-2} - \frac{5}{2}\hat{p}_{N-1} + \frac{4}{3}\hat{p}_N - \frac{1}{12}\hat{p}_{N-1}}{h^2} = \frac{-\frac{1}{12}\hat{p}_{N-3} + \frac{4}{3}\hat{p}_{N-2} - \frac{31}{12}\hat{p}_{N-1} + \frac{4}{3}\hat{p}_N}{h^2}$$

$$\left(\frac{\partial^2\hat{p}}{\partial x^2}\right)_1 \approx \frac{\hat{p}_0 - 2\hat{p}_1 + \hat{p}_2}{h^2} = \frac{-2\hat{p}_1 + 2\hat{p}_2}{h^2}$$

$$\left(\frac{\partial^2\hat{p}}{\partial x^2}\right)_N \approx \frac{\hat{p}_{N-1} - 2\hat{p}_N + \hat{p}_{N+1}}{h^2} = \frac{2\hat{p}_{N-1} - 2\hat{p}_N}{h^2}.$$

Note that eq. (2.2) gives only 2nd-order accuracy at the boundaries. The matrix $\underline{\underline{A}}$ now equals

$$\underline{\underline{A}} = \frac{1}{h^2}\begin{bmatrix}
-2 & 2 & 0 & 0 & ... & & & & & \\
\frac{4}{3} & -\frac{31}{12} & \frac{4}{3} & -\frac{1}{12} & 0 & 0 & ... & & & \\
-\frac{1}{12} & \frac{4}{3} & -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} & 0 & 0 & ... & & \\
0 & -\frac{1}{12} & \frac{4}{3} & -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} & 0 & ... & & \\
... & & & & ... & & & & ... & \\
& ... & 0 & -\frac{1}{12} & \frac{4}{3} & -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} & 0 & \\
& ... & 0 & 0 & -\frac{1}{12} & \frac{4}{3} & -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} & \\
& & ... & 0 & 0 & -\frac{1}{12} & \frac{4}{3} & -\frac{31}{12} & \frac{4}{3} & \\
& & & & ... & 0 & 0 & 2 & -2 &
\end{bmatrix}.$$

(c) Table 2.1 shows the numerically estimated eigenfrequency $f_5 = 857.5\,\text{Hz}$ for different number of points N and the two considered orders of accuracy. The value converges to the exact solution for both orders of accuracy, but the 4th-order scheme converges faster with the increasing number of points.

Table 2.1: Numerical estimation of the eigenfrequency $f_5 = 857.5\,\mathrm{Hz}$.

	2^{nd} order	4^{th} order
N = 10	752.73 Hz	804.91 Hz
N = 100	856.60 Hz	857.48 Hz
N = 500	857.46 Hz	857.50 Hz