1 Exercise 1

1.1 Task 1

The eigenvalue problem is defined with the matrix equation

$$\underline{\underline{A}}\,\underline{x} = \lambda \underline{x}$$

or

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{x} = 0, \tag{1.1}$$

where λ denotes eigenvalue of the eigenvector $\underline{x}$. In the non-trivial case ($\underline{x} \neq 0$)

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0.$$

Evidently, any eigenvector can be multiplied with an arbitrary constant other than zero without affecting its eigenvalue, for example, normalized such that its magnitude equals one, $|\underline{x}| = 1$.

Solving the equation,

$$\begin{vmatrix} 3-\lambda & 0 \\ -9 & 6-\lambda \end{vmatrix} = (3-\lambda)(6-\lambda) + 0 \cdot 9 = 0,$$

gives the eigenvalues

(a)

$$\lambda_1 = 3, \lambda_2 = 6$$

and the eigenvectors

(b)

$$\text{for } \lambda_1 = 3: \begin{bmatrix} 3 & 0 \\ -9 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = 3 \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \Rightarrow \begin{matrix} 3x_1 = 3x_1 \\ 9x_1 = 3y_1 \end{matrix} \Rightarrow \underline{x}_1 = \begin{bmatrix} x_1 \\ 3x_1 \end{bmatrix} \quad \text{e. g. } \underline{x}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\text{for } \lambda_2 = 6: \begin{bmatrix} 3 & 0 \\ -9 & 6 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = 6 \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \Rightarrow \begin{matrix} 3x_2 = 6x_2 \\ -9x_2 + 6y_2 = 6y_2 \end{matrix} \Rightarrow \underline{x}_2 = \begin{bmatrix} 0 \\ y_2 \end{bmatrix} \quad \text{e. g. } \underline{x}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

1.2 Task 2

Conservation of momentum provides a relation between sound pressure and particle velocity:

$$\rho_0 \frac{\partial v}{\partial t} + \frac{\partial p}{\partial x} = 0.$$

Observing simple oscillations in time, $p = \hat{p}(x)e^{j\omega t}$ and $v = \hat{v}(x)e^{j\omega t}$ (notice that the phase shift between p and v is still unspecified, because $\hat{p}$ and $\hat{v}$ in this general form take complex values),

$$j\omega\rho_0 v + \frac{\partial p}{\partial x} = 0,$$

$$j\omega\rho_0\hat{v} + \frac{\partial \hat{p}}{\partial x} = 0,$$

and therefore

$$\hat{v} = -\frac{1}{j\omega\rho_0}\frac{\partial \hat{p}}{\partial x} = 0. \tag{1.2}$$

(a) The analytical solution follows from

$$\frac{\partial \hat{p}}{\partial x} = (-j0.9\text{m}^{-1})0.5e^{-j0.9\text{m}^{-1}x}\,\text{Pa}$$

and reads

$$\hat{v} = \frac{-j0.9\text{m}^{-1}0.5}{-j\omega\rho_0}e^{-j0.9\text{m}^{-1}x}\,\text{Pa} = \frac{1}{\rho_0 c_0}0.5e^{-j0.9\text{m}^{-1}x}\,\text{Pa} = \frac{\hat{p}}{\rho_0 c_0},$$

which is a simple relation between sound pressure and particle velocity of a plane wave. The two quantities are in phase. Finally, we can replace $x = 1\,\text{m}$ in the result.

(b) Numerical solution with the finite difference method can be obtained using the approximation

$$\frac{\partial \hat{p}}{\partial x} \approx \begin{cases} \frac{\hat{p}(x+\Delta x)-\hat{p}(x)}{\Delta x} & \text{for forward difference; 1}^{\text{st}}\text{-order accuracy} \\ \frac{\hat{p}(x)-\hat{p}(x-\Delta x)}{\Delta x} & \text{for backward difference; 1}^{\text{st}}\text{-order accuracy} \\ \frac{\hat{p}(x+\Delta x)-\hat{p}(x-\Delta x)}{2\Delta x} & \text{for central difference; 2}^{\text{nd}}\text{-order accuracy} \end{cases} \tag{1.3}$$

in eq. (1.2), with the given $x = 1\,\text{m}$ and $\Delta x = 0.1\,\text{m}$ or $\Delta x = 0.5\,\text{m}$.

What happens for larger values of Δx, say, $\Delta x = 3.5\,\text{m}$ or $\Delta x = 7\,\text{m}$? Note that the sound wavelength is $\lambda = 2\pi/0.9\,\text{m}^{-1} \approx 6.98\,\text{m}$.

1.3 Task 3

(b) Linear interpolation for a point i is given by

$$\frac{y_i - y_{i-1}}{x_i - x_{i-1}} = \frac{y_{i+1} - y_{i-1}}{x_{i+1} - x_{i-1}},$$

from which the value at the point equals

$$y_i = y_{i-1} + (y_{i+1} - y_{i-1})\frac{x_i - x_{i-1}}{x_{i+1} - x_{i-1}}.$$

(c) A simple polynomial interpolation tends to result in large fluctuations of values between the interpolation nodes for higher-order polynomials. Spline interpolation exhibits a more acceptable behaviour. It uses low-order (at least 3$^{\text{rd}}$-order) piecewise polynomials (n polynomials for $N = n + 1$ nodes), minimizing the bending with the condition that $\partial y/\partial x$ and $\partial^2 y/\partial x^2$ are continuous everywhere (also at the nodes). Typical boundary conditions (for $i = 0$ or $i = N$) are:

- clamped $\left(\frac{\partial y}{\partial x}\right)_i = 0$
- natural $\left(\frac{\partial^2 y}{\partial x^2}\right)_i = 0$
- periodic $\left(\frac{\partial^2 y}{\partial x^2}\right)_{i=0} = \left(\frac{\partial^2 y}{\partial x^2}\right)_{i=N}$, $\left(\frac{\partial y}{\partial x}\right)_{i=0} = \left(\frac{\partial y}{\partial x}\right)_{i=N}$, and $y_0 = y_N$.