I have a collection of N similar items. I want to find out how many sets of 6 items are fully Compatible. "Compatible" is a comparison that I can do easily pairwise and is commutative but not reflexive (so it is not transitive)

- Compatible$(A, B) \rightarrow$ Compatible(B, A)
- But A is not Compatible with A
- Compatible(A, B) and Compatible$(B, C) \neq$ Compatible(A, C)

That's it really. There's a physical puzzle that I'm working with (hint: it has to do with a cube) but I've been trying to generalize because what I'm doing is still in bad time complexity. Below I've outlined my process.

<u>Step 1:</u>

The worst case is obviously checking $\binom{N}{6}$ sets where maybe not all $\binom{N}{6}$ sets are Compatible. To verify a set is fully Compatible takes checking every sub pair (order doesn't matter) which is 15 comparisons. $\binom{N}{6}$ sets at $\binom{6}{2}$ checks per set is $\binom{N}{6}\binom{6}{2}$ checks in order to learn how many sets of 6 from N are Compatible.

A lot of wasted comparisons. The main source of wasted time that I identified was that if a set $\{A, B, C, D, E, F\}$ is not Compatible because $\{A, B\}$ is not Compatible (therefore ruining the Compatibility of the set) then every set containing both those items will also fail. That's $\binom{N-2}{4}$ guaranteed failures *<u>per</u>* incompatible pair of which there could be up to $\binom{N}{2}$ (if all of them). Which is a lot of things I will want to avoid.

<u>Step 2:</u>

That above might be enough for you to know of an algorithm that can avoid repeated inCompatible pairs. I went a different direction with the idea of hash buckets. I made 6 non-Compatible buckets. All items in bucket **A** are inCompatible with each other, **B** are inCompatible with each other, and so on through bucket **F**. These 6 buckets are roughly equally sized at $\frac{N}{6}$. Because I'm trying to make a set of 6 items, and I have 6 buckets that are inCompatible with themselves, I know that exactly one element in every fully Compatible set comes from each bucket.

The worst case is choosing one from each bucket: $\left(\frac{N}{6}\right)^6$ and then doing all $\binom{6}{2}$ checks at once which is only a little better than the original. This, however, doesn't take into account the problem in Step 1 where $\{A, B\}$ inCompatibility affects choosing the rest of the set.

I don't know how to show that mathematically, but I trust that skipping over subsets that I know don't work (such as pairs $\{a \in \mathbf{A}, \mathbf{b} \in \mathbf{B}\}$ if not Compatible) definitely reduces the total work.

<u>Trying to predict successful sets</u>

When I tried to calculate the asymptotic simplification due to skipping subsets I know don't work, I thought that representing probabilities would help. I found that

$$\text{Probability(Compatible(} a \in N, b \in N \text{))} \sim 33\%$$

Also, conveniently:

$$\forall S_1, S_2 \in \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}, \mathbf{F} \mid S_1 \neq S_2 : \text{Probability(Compatible(} a \in S_1, b \in S_2 \text{))} \sim 33\%$$

Which is to say that if I choose two items randomly and don't already know they were in the same bucket, there is a 1 in 3 chance that the items are compatible. This leaves $\left(\frac{N}{6} \times \frac{1}{3}\right)^6$ successes as an upper limit. At first I thought that this 33% chance meant that I could say choosing from a third bucket is 11% chance and so on

Buckets:	A	B	C	D	E	F
# Compatible choices in bucket given choosing left to right	$\frac{N}{6}$	$\left(\frac{N}{6} \times \frac{1}{3}\right)$	$\left(\frac{N}{6}\right)\left(\frac{1}{3}\right)^2$	$\left(\frac{N}{6}\right)\left(\frac{1}{3}\right)^3$	$\left(\frac{N}{6}\right)\left(\frac{1}{3}\right)^4$	$\left(\frac{N}{6}\right)\left(\frac{1}{3}\right)^5$

$= \left(\frac{N}{6}\right)^6 \left(\frac{1}{3}\right)^{\Sigma 5}$ valid solutions

I can also say though, that, given a valid $\{A, B, C, D, E\}$, there is actually 1 and only 1 Compatible *F*. This is a hidden constraint from the physical puzzle I'm trying to solve, but is responsible for the largest simplification in complexity I've discussed so far. $O(N^6) \rightarrow O(N^5)$