A MULTILINEAR PROOF OF JACOBI'S FORMULA

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Jacobi's formula gives the determinant of the matrix exponential in terms of the trace:

$$\det(e^{tA}) = e^{t\,\mathrm{tr}(A)}$$

The usual proof of this uses Laplace expansion of the determinant. Here is a proof that relies on multilinearity of the determinant:

Consider the function $f(t) = \det(e^{tA})$. To show that $f(t) = e^{t\,\mathrm{tr}(A)}$ it suffices to show that f satisfies the differential equation that defines $e^{t\,\mathrm{tr}(A)}$:

$$\begin{aligned} f(0) &= 1 \\ f'(t) &= \mathrm{tr}(A) f(t) \end{aligned}$$

The condition $f(0) = 1$ follows immediately, so it remains to show $f'(t)/f(t) = \mathrm{tr}(A)$. In other words, we have to show that $\det(e^{tA})' \det(e^{-tA}) = \mathrm{tr}\, A$.

To calculate the derivative of a determinant, first consider the derivative of a product:

$$(a_1 \cdot a_2 \cdots a_n)' = (a_1' \cdot a_2 \cdots a_n) + (a_1 \cdot a_2' \cdots a_n) + \cdots + (a_1 \cdot a_2 \cdots a_n')$$

The analogous derivative formula works for any multilinear function, such as the determinant, where $\det(a_1|a_2|\cdots|a_n)$ is the determinant of the matrix with columns $a_2, a_2, \cdots, a_n$:

$$\det(a_1|a_2|\cdots|a_n)' = \det(a_1'|a_2|\cdots|a_n) + \det(a_1|a_2'|\cdots|a_n) + \cdots + \det(a_1|a_2|\cdots|a_n')$$

Using this derivative formula and $(e^{tA})' = Ae^{tA}$ and $e^{tA}e^{-tA} = I$, we get:

$$\det(e^{tA})' \det(e^{-tA}) = \det(A_1|I_2|\cdots|I_n) + \det(I_1|A_2|\cdots|I_n) + \cdots + \det(I_1|I_2|\cdots|A_n) = \mathrm{tr}(A)$$

This completes the proof.

A similar argument shows that $\det(A)' \det(A^{-1}) = \mathrm{tr}(A^{-1}A')$.