Fast Coalgebraic Bisimilarity Minimization

(POPL'23)

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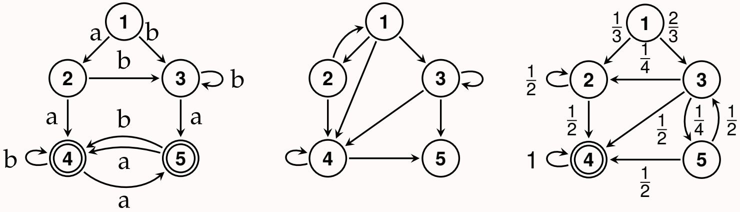
The Automaton Zoo

Deterministic finite automata, tree automata, (labeled) transition systems, weighted and probabilistic automata, Markov decision processes, ...



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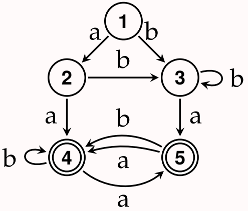

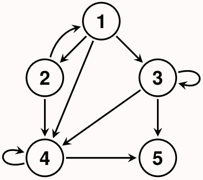

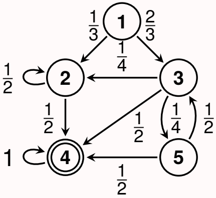

Automaton Minimization

Find and merge behaviorally equivalent states

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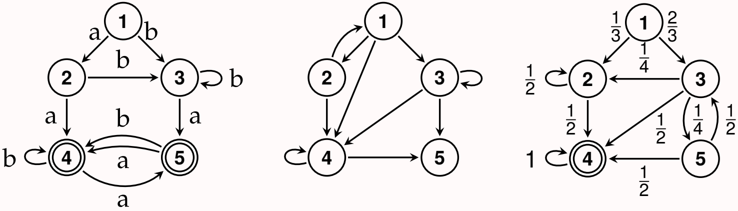

Automaton Minimization

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Coalgebraic Bisimilarity Minimization

Algorithms that work for a general class of F-automata

Our contribution

a **fast** and **general** algorithm for minimizing automata

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- *General:* works for any computable coalgebra
- *Decent asymptotic complexity:* $\mathcal{O}(\phi_F \cdot m \log n)$
- *Fast in practice:* no penalty for generality
- *Low memory usage:* important for large automata

Examples of Coalgebraic Automata

Automaton type	**Equivalence**	**Functor** $F(X)$
DFA	Language Equivalence	$2 \times A^X$
Transition Systems	Strong Bisimilarity	$\mathcal{P}(X)$
LTS	Strong Bisimilarity	$\mathcal{P}(A \times X)$
Weighted Systems	Weighted Bisimilarity	$M^{(X)}$
Markov Chain	Probabilistic Bisimilarity	$A \times \mathcal{D}(X)$
MDP	Probabilistic Bisimilarity	$\mathcal{P}(\mathcal{D}(X))$
Weighted Tree Automata	Backwards Bisimilarity	$M^{(\Sigma X)}$
Monotone Neigh. Frames	Monotone Bisimilarity	$\mathcal{N}(X)$
$\vdots$	$\vdots$	$\vdots$

Automaton types compose: $F \circ G, F + G, F \times G, \ldots$

<table>
<tr><th>DFA</th><th>Transition system</th><th>Markov chain</th></tr>
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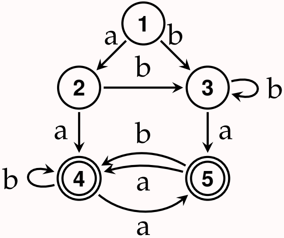
<td>
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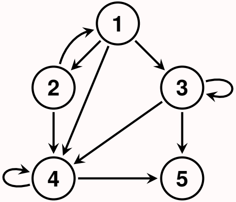
<td>
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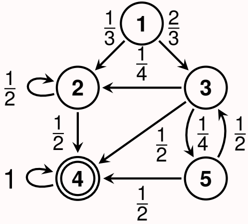
<td>
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$\mathbf{4} \mapsto (\mathsf{T}, \{\mathbf{4}\colon 1\})$
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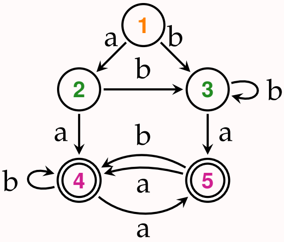	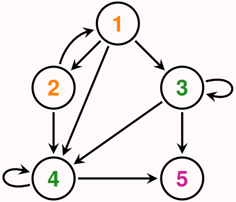	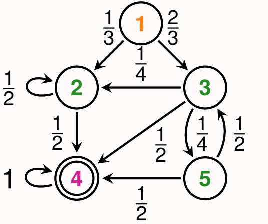
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$2 \equiv 3, 4 \equiv 5$	$1 \equiv 2, 3 \equiv 4$	$2 \equiv 3 \equiv 5$

What is coalgebraic bisimilarity minimization?

The input:

- a functor $F(X)$ – describes automaton type
- a coalgebra $t : C \to F(C)$ – the automaton

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- a functor $F(X)$ – describes automaton type
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The output:

- a partition $p : C \to C'$
  – the equivalence classes of bisimilar states
- s.t. $p(x) = p(y) \implies Fp(t(x)) = Fp(t(y))$
- $|C'|$ as small as possible

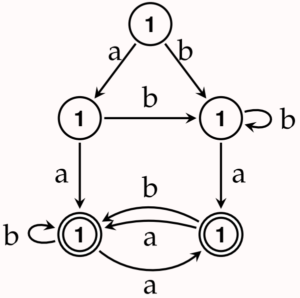
Sketch of our algorithm

1. Assume all states are equivalent
2. Split equivalence classes by *signature* (*normalised* outgoing transitions)
3. Iterate until convergence

Key points

- Only recompute signatures of *changed* states
- Never loop over *unchanged* states

Our algorithm: Minimizing a DFA



Algorithm

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Iterate:

1. Pick equivalence class & compute missing signatures.
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Until all states have signatures.

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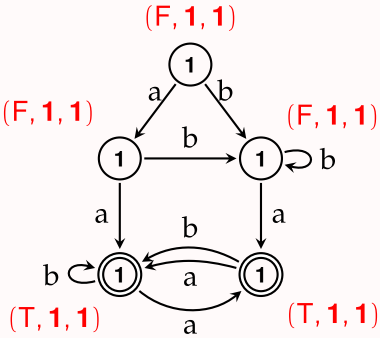

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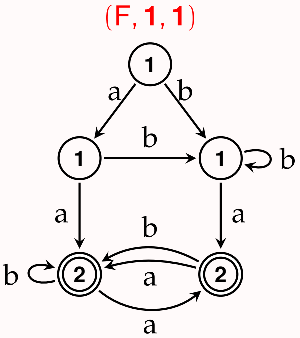

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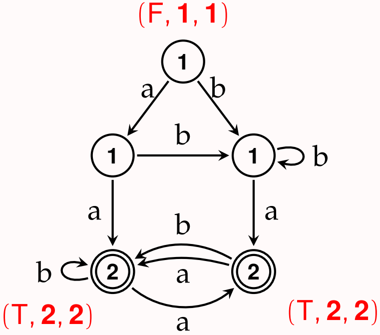

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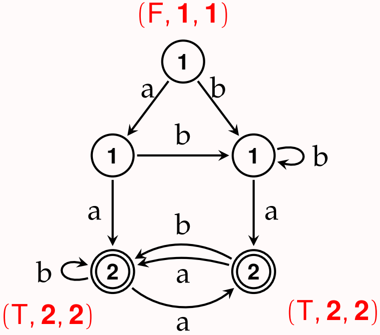

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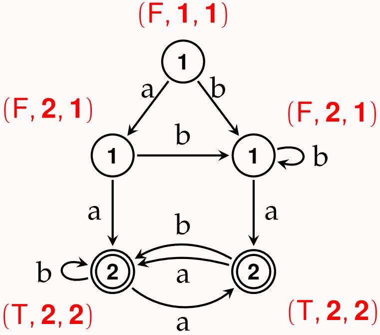

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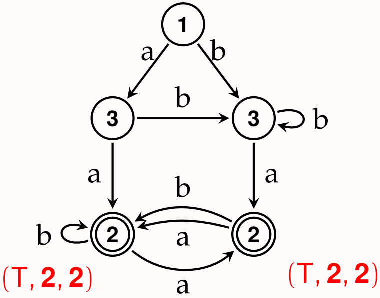

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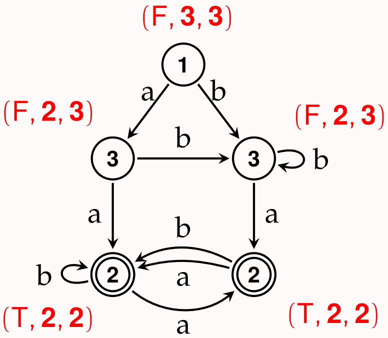

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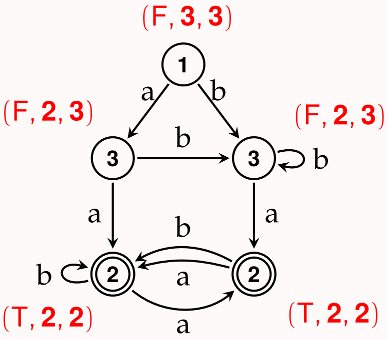

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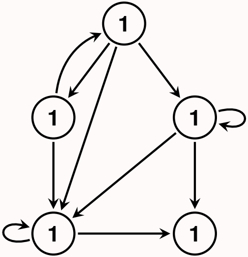

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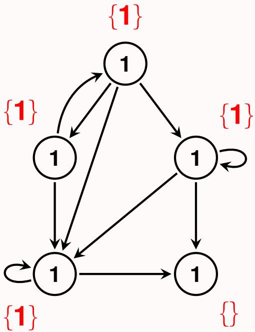

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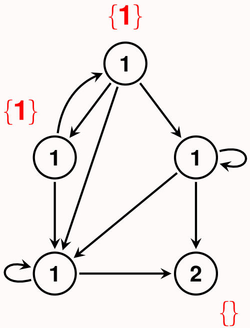

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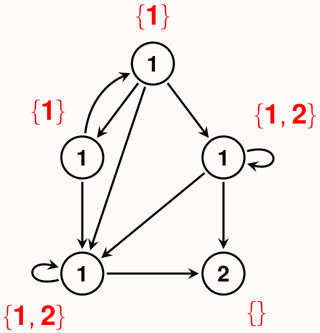

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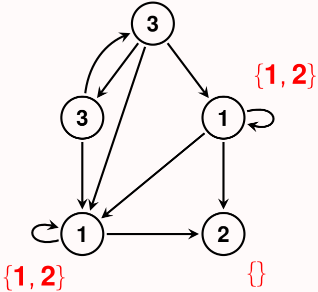

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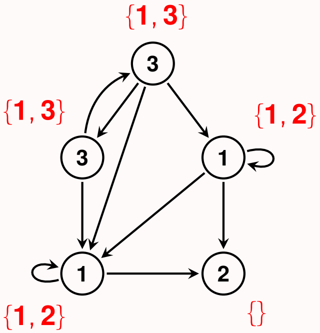

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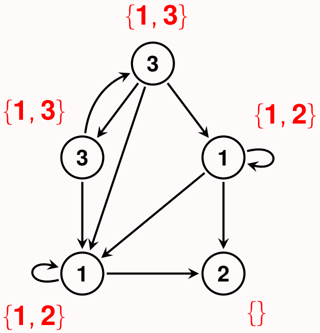

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The general picture

1. *Pick equivalence class with missing signatures:*

A	A						
1	1	1	1	1	1	1	1

2. *Compute missing signatures:*

A	A	B	B	B	B	C	A
1	1	1	1	1	1	1	1

3. *Assign new state numbers:*

A	A	B	B	B	B	C	A
2	2	1	1	1	1	3	2

4. *Remove invalid signatures from predecessors*

What we need from the automaton

- Set of states C
- Predecessors of each state $\mathsf{pred} : C \to \mathcal{P}(C)$
- **Procedure to (re)compute signatures**
  $\mathsf{sig} : (C \to \mathbb{N}) \to (C \to F(\mathbb{N}))$

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Key: *re-use old state number for*
largest new equivalence class
Invalidates fewer signatures

Complexity: $O(m \log n)$ signature computations

Hopcroft's trick

State	1	2	3	4	5	6	7	8	9
Iteration 1	1	1	1	1	1	1	1	1	1

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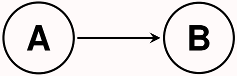
Iteration 4	2	3	3	6	1	1	4	4	5
Iteration 5	2	3	3	6	1	7	4	4	5

Each state's number changes $O(\log n)$ *times!*

Why $O(m \log n)$ signature recomputations?



An edge **A** $\rightarrow$ **B** may cause a signature recomputation of **A** when **B**'s state number changes.

Time complexity: $O(\Phi_F \cdot m \log n)$

Key ingredient: *never touch unchanged states*

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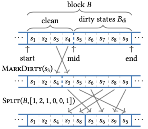
Key ingredient: *never touch unchanged states*

- n-way partition refinement data structure
- also tracks invalid signatures
- uses radix sort & bucket sort for n-way split

Algorithm 5 Refinable partition data structure with n-way split

```
procedure MarkDirty(s)
    ▹ Determine the block data ◃
    B := block_of[s]
    j := state2loc[s]
    (start, mid, end) := blocks[B]
    ▹ Do nothing if already dirty ◃
    if mid ≤ j then return
    ▹ Add to worklist if first dirty state ◃
    if mid = end then worklist.add(B)
    ▹ Swap s with the last clean state ◃
    s' := loc2state[mid − 1]
    state2loc[s'] := j
    state2loc[s] := mid
    loc2state[j] := s'
    loc2state[mid] := s
    ▹ Move marker to make s dirty ◃
    blocks[B].mid −= 1
```




```
procedure Split(B, A: B_di → ℕ)
    ▹ Cumulative counts of sub-block sizes ◃
    (start, mid, end) := blocks[B]
    D[0.. max_i A[i] + 1] := 0
    D[0] := mid − start
    for j ∈ B_di do
        D[A[j]] += 1
    i_max = argmax_i D[i]
    for i ∈ 1..|D| do
        D[i] += D[i − 1]
    ▹ Re-order the states by A-value ◃
    dirty := copy(loc2state[mid..end])
    for i ∈ reverse(0..|A|) do
        D[A[i]] −= 1
        j := start + D[A[i]]
        loc2state[j] := dirty[i]
        state2loc[loc2state[i]] := j
    D[0] −= mid − start
    ▹ Create blocks and assign IDs ◃
    D.add(end − start)
    old_block_count := |blocks|
    for i ∈ 0..|D| − 1 do
        j0 := start + D[i]
        j1 := start + D[i + 1]
        if i = i_max then
            blocks[B] = (j0, j1, j1)
        else
            blocks.add(j0, j1, j1)
            idx := |blocks| − 1
            block_of[loc2state[j0..j1]] := idx
    return old_block_count..|blocks|
```

Algorithm 6 Optimized Partition Refinement for all Set functors

```
procedure PartRefSetFun(C, sig, pred)          ▹ i.e. for the implementation of c: C → FC
    Create a new refinable partition structure p: C → ℕ
    Init p to have one block of all states, and all states marked dirty.
    while there is a block B with a dirty state do
        Compute the arrays                                  } Compute signatures,
            (sigs_di: B_di → 2*) := (x ↦ sig(x, p))         } in total O(m log n) calls
            (sigs_cl: B_cl1 → 2*) := (x ↦ sig(x, p))        } to the coalgebra
        A: B_di → ℕ := Renumber'(sigs_di, sigs_cl)          } Split B according to
        B⃗_new := Split(B, A)                                 } signatures in O(|B_di|)
        for every B' ∈ B⃗_new and s ∈ B' do                   } Mark dirty all states with
            for every s' ∈ pred(s) do                       } a successor in a new block
                MarkDirty(s')                               } in total time O(m log n)
    return the partition p
procedure Renumber'(p: B_di → 2*, q : B_cl1 → 2*)
    (A: B_di → ℕ) := Renumber(p)
    if there are d ∈ B_di, c ∈ B_cl1 with p(d) = q(c) then  } Ensure that A(d) = 0 for all dirty
        Swap the values 0 and A(d) in the array A.          } states d that have the same
    return A                                                } signature as the clean states.
```

$\implies$ **signature recomputations dominate**

Comparison

	CoPaR	*DCPR*	*Boa*
Complexity	$O(m \log n)$	$O(\phi_F \cdot n^2)$	$O(\phi_F \cdot m \log n)$
Generality	Zippable	Coalg	Coalg
Language	Haskell	Haskell	Rust

benchmark		time (s)			memory (GB)		
type	**n/10^3**	***CoPaR***	***DCPR***	***Boa***	***CoPaR***	***DCPR***	***Boa***
fms	**1639**	232	84	1.12	16	$0.51_{\times 32}$	0.19
	4459	–	406	4.47	-	$1.69_{\times 32}$	0.58
wlan	**607**	105	855	0.28	16	$0.15_{\times 32}$	0.04
	1632	–	2960	0.79	-	$3.79_{\times 32}$	0.09
wta$_W$	**152**	566	79	0.74	16	$0.64_{\times 32}$	0.08
	944	–	675	11.96	-	$6.79_{\times 32}$	1.23
wta$_Z$	**156**	438	82	0.48	16	$0.68_{\times 32}$	0.09
	1008	–	645	16.75	-	$5.64_{\times 32}$	1.33
wta$_2$	**155**	449	160	0.81	16	$0.62_{\times 32}$	0.08
	1300	–	1377	23.35	-	$7.09_{\times 32}$	1.65

Comparison

	mCRL2	*Boa*
Complexity	$O(m \log n)$	$O(\phi_F \cdot m \log n)$
Generality	LTS+	Coalg
Language	C++	Rust

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benchmark		time (s)		memory (GB)	
type	**n/10³**	***mCRL2***	***Boa***	***mCRL2***	***Boa***
	2417	13.9	1.4	1.78	0.25
cwi	**7839**	214.2	15.8	5.78	0.81
	33950	282.2	31.5	16.62	2.78
	6021	33.8	3.1	2.12	0.52
vasy	**11027**	51.6	6.1	2.77	0.62
	12324	56.9	7.0	3.10	0.73

For ***mCRL2***, we pick its best algorithm and self-reported time.
For ***Boa***, we report wall-clock time.

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ϕ_F *is cheap!*

Don't incrementalize, just recompute:

- Saves memory
- Saves random reads
- Saves iterations*

Conclusion

Minimization can be **simple**, **generic**, and **fast**

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Future

- Other notions of equivalence (*e.g.*, branching)
- Specialization by monomorphisation
- Integration into Storm (with Sebastian Junges)
- Minimize *your* automata!