

A MAGIC DETERMINANT FORMULA FOR SYMMETRIC POLYNOMIALS OF EIGENVALUES

Jules Jacobs

$$\sum_i p_i \lambda_1^{i_1} \lambda_2^{i_2} \cdots \lambda_n^{i_n} = \sum_i p_i \det(A_1^{i_1} | A_2^{i_2} | \cdots | A_n^{i_n})$$

TRACE AND DETERMINANT

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \quad (\text{e.g. integer matrix})$$

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A: Fundamental theorem of symmetric polynomials: all symmetric ones.

$$p(\lambda_1, \lambda_2, \lambda_3) = p(\lambda_2, \lambda_1, \lambda_3) = \cdots = p(\lambda_3, \lambda_2, \lambda_1)$$

THE MAGIC METHOD

$$p(\lambda_1, \lambda_2, \lambda_3) = \lambda_1 \lambda_2^4 \lambda_3^4 + \lambda_1^4 \lambda_2 \lambda_3^4 + \lambda_1^4 \lambda_2^4 \lambda_3$$

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A_2^4 = the second column of A^4

$$\cdots + a \lambda_1^{i_1} \cdot \lambda_2^{i_2} \cdots \lambda_n^{i_n} + \cdots$$

\Downarrow

$$\cdots + a \det(A_1^{i_1} | A_2^{i_2} | \cdots | A_n^{i_n}) + \cdots$$

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$$\begin{aligned}\lambda_1 + \lambda_2 + \lambda_3 &= \lambda_1 \lambda_2^0 \lambda_3^0 + \lambda_1^0 \lambda_2 \lambda_3^0 + \lambda_1^0 \lambda_2^0 \lambda_3 \\ &= \det(A_1|A_2^0|A_3^0) + \det(A_1^0|A_2|A_3^0) + \det(A_1^0|A_2^0|A_3)\end{aligned}$$

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PROVE OR DISPROVE

$$\sum_i p_i \lambda_1^{i_1} \lambda_2^{i_2} \cdots \lambda_n^{i_n} = \sum_i p_i \det(A_1^{i_1} | A_2^{i_2} | \cdots | A_n^{i_n})$$

No peeking! <https://julesjacobs.com/pdf/sympoly.pdf>