DIVISIBILITY OF MULTINOMIAL COEFFICIENTS

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Together with Ike Mulder we discovered a fun little divisibility property of multinomial coefficients (probably well-known!).

Our starting point is that not only the binomial coefficient $\frac{(a+b)!}{a!b!}$ is a whole number, but also the Catalan numbers $\frac{(2n)!}{n!(n+1)!} = \frac{(2n)!}{n!n!}/(n+1)$ are whole numbers. That $(a+b)!$ is divisible by $a!b!$ is already a minor miracle, but the Catalan numbers show that binomial coefficient $\frac{(2n)!}{n!n!}$ can be divided even further. Our question is: does this generalize to other binomial coefficients?

The answer turns out to be yes: if $\gcd(a, b) = 1$ then $\frac{(a+b-1)!}{a!b!}$ is a whole number. This implies the Catalan divisibility because $\gcd(n, n+1) = 1$.

This generalizes to multinomial coefficients:

Lemma. If $\gcd(a_1, \ldots, a_n) = 1$ then

$$\frac{(a_1 + \cdots + a_n - 1)!}{a_1! \cdots a_n!} \tag{1}$$

is a whole number.

Proof. From the gcd assumption, we have integers $k_1, \ldots, k_n$ such that

$$1 = a_1 k_1 + \cdots + a_n k_n$$

Multiply both sides by (1):

$$\begin{aligned} \frac{(a_1 + \cdots + a_n - 1)!}{a_1! \cdots a_n!} &= \frac{(a_1 + \cdots + a_n - 1)!}{a_1! \cdots a_n!} a_1 k_1 + \cdots + \frac{(a_1 + \cdots + a_n - 1)!}{a_1! \cdots a_n!} a_n k_n \\ &= \frac{(a_1 + \cdots + a_n - 1)!}{(a_1 - 1)! \cdots a_n!} k_1 + \cdots + \frac{(a_1 + \cdots + a_n - 1)!}{a_1! \cdots (a_n - 1)!} k_n \end{aligned}$$

The right hand side is an integer because multinomial coefficients are integers. □

Slightly more generally, the proof shows that $\frac{(a_1 + \cdots + a_n - 1)! \gcd(a_1, \ldots, a_n)}{a_1! \cdots a_n!}$ is always a whole number.