A simple proof of Kirchoff's theorem, and some other combinatorial graph determinants

Jules Jacobs

December 9, 2020

Abstract

Kirchoff's matrix tree theorem states that the number of spanning trees in a graph is $\det L'$, where L' is the Laplacian matrix L of the graph, with any row and column deleted. We give a direct proof of the fact that $\det(xI + L)$ is the generating function of spanning forests with k roots. Our proof does not rely on the Cauchy-Binet formula, yet is arguably simpler than the standard proof, and applies to directed graphs as well.

The proof is based on a lemma that if A is the adjacency matrix of a graph where each vertex has at most one outgoing edge, then $\det(I - A) = 1$ if A is a forest and 0 otherwise. We also generalize this lemma to any graph, in which case $\det(I - xA)^{-1}$ is shown to be the generating function of *upward routes*.

Lastly, we generalize Kirchoff's theorem to a theorem about $\det(A + L)$ where A is the adjacency matrix of a second graph, which reduces to counting spanning forests when $A = xI$ and to the fact that $\det(A)$ counts signed cycle covers when $L = 0$. The all-minors matrix tree theorem also follows as a corollary. For instance, the fact that $\det(L')$ counts spanning forests, where L' is L with any column i and row j deleted, follows by picking A to be the matrix where $A_{ij} = 1$ and zero elsewhere.

1 Introduction

Given finite sets of numbers $S_1, \dots, S_n \subset \mathbb{R}$, we have the identity:

$$\sum_{x_1 \in S_1, \cdots, x_n \in S_n} x_1 x_2 \cdots x_n = \left(\sum_{x_1 \in S_1} x_1 \right) \cdots \left(\sum_{x_n \in S_n} x_n \right)$$

On the left hand side, we get one term for every way of choosing $(x_1, \dots, x_n) \in S_1 \times \cdots \times S_n$. This identity is useful in combinatorics, for instance to show that the coefficient of x^k of $(x + x^2 + \cdots + x^6)^n$ counts the number of ways of obtaining k as the sum of n dice.

Similarly, given finite sets of vectors $S_1, \dots, S_n \subset \mathbb{R}^k$, we have the following identity, by multilinearity of the determinant:

$$\sum_{v_1 \in S_1, \cdots, v_n \in S_n} \det(v_1, \cdots, v_n) = \det\left(\sum_{v_1 \in S_1} v_1, \cdots, \sum_{v_n \in S_n} v_n \right)$$

Where $\det(v_1, \dots, v_n)$ is the determinant of a matrix with those columns. We shall see that Kirchoff's theorem can be obtained from this identity, but we first need some definitions and a lemma.

Definition 1.1. We define the concepts *01-graph*, *forest*, *root*, and *tree*:

- A 01-graph is a directed graph where each vertex has 0 or 1 outgoing edges.
- A *forest* is a 01-graph with no cycles.

- The *roots* are vertices with 0 outgoing edges.
- A *tree* is a forest with one root.

If we start at a vertex v in a 01-graph and keep following the unique outgoing edge, then we either loop in a cycle, or we end up at a vertex with no outgoing edges. Therefore a general 01-graph consists of roots and cycles, plus trees converging onto the roots and cycles.

Definition 1.2. The adjacency matrix A of a graph G is defined as follows, where $e_j \in \mathbb{R}^n$ are the basis vectors:

$$A_i = \sum_{(i \to j) \in G} e_j$$

2 The matrix-tree theorem

We start with a lemma that gives us an indicator function for forests.

Lemma 2.1. *Let A be the adjacency matrix of a 01-graph G, then*

$$\det(I - A) = \begin{cases} 1 & \text{if } G \text{ is a forest} \\ 0 & \text{if } G \text{ has a cycle} \end{cases}$$

Proof. In a 01-graph, the vector $A^i e_k$ follows the path out of k for $i = 0, 1, \ldots$ (note that this path is unique because each vertex has at most one outgoing edge).

- If G is forest then $A^n e_k = 0$ for all k, where n is the number of vertices of G. So $A^n = 0$, so all eigenvalues of A are zero, so all eigenvalues of $I - A$ are one, so $\det(I - A) = 1$.
- If G has a cycle consisting of vertices C, take $v = \sum_{i \in C} e_i$. Then $Av = v$, so $(I - A)v = 0$, so $\det(I - A) = 0$.

Thus, for 01-graphs, $\det(I - A)$ indicates whether the graph is a forest or not. □

If using eigenvalues to prove a finitary lemma feels wrong, here is an alternative proof:

Proof. In a 01-graph, the i-th column of A consists of only the diagonal 1 when i is a root, and has one other entry equal to -1 if i is not a root.

We may therefore do a Laplace expansion along the column of a root, which deletes column i and row i from the matrix. On the graph side, this corresponds to deleting root i and all its incoming edges.

Repeating this process, we end up deleting the entire graph if the graph was a forest, in which case the determinant is one[1]. If the graph had a cycle, this we will be left with a matrix that has a 1 and a -1 in each column, so the sum of the rows is 0 so $\det(I - A) = 0$. □

Definition 2.1. The Laplacian matrix L of a graph G is defined as:

$$L_i = \sum_{(i \to j) \in G} (e_i - e_j)$$

Theorem 2.2. *(Kirchoff, Tutte) The determinant* $\det(I + L)$ *gives the number of ways to choose a forest as a subgraph of G.*

[1] The determinant of a 0×0 matrix is 1, but one can also stop deleting when the matrix is 1×1, and use $\det([1]) = 1$.

Proof. The strategy is to first consider all possible ways of choosing a 01-graph as a subgraph of G, and then summing $\det(I-A)$ over those, which counts only the forests. To choose a 01-graph in G, we choose for each vertex i whether we make it a root, or whether we give it an outgoing edge from G. Therefore:

$$\det(I+L) = \det(e_1+L_1,\dots,e_n+L_n) = \sum_{\text{01-graph } A \subseteq G} \det(I-A) = \sum_{\text{forest } A \subseteq G} 1$$

In the second step, we have expanded the determinant by multilinearity in each column

$$e_i + L_i = e_i + \sum_{(i\to j)\in G} (e_i - e_j)$$

In each column we either pick e_i or we pick one of the terms $(e_i - e_j)$ in the sum over the outgoing edges[2]. The former corresponds to picking e_i as a root, and the latter corresponds to picking $i \to j$ as the outgoing edge for vertex i. $\square$

The same proof obtains a weighted version of the theorem. Let A be a matrix of weights with A_{ij} corresponding to edge $i \to j$, and define the Laplacian $L_i = \sum_j A_{ij}(e_i - e_j)$. Let D be a diagonal matrix of weights with D_{ii} corresponding to vertex i as a root. Let the weight of a forest be

$$w(F) = \prod_{i\in\text{roots}(F)} D_{ii} \prod_{(i\to j)\in\text{edges}(F)} A_{ij}$$

Then we have the following weighted version of the previous theorem.

Theorem 2.3. *(Kirchoff, Tutte) The determinant* $\det(D+L)$ *is the weighted sum of all forests on n-vertices:*

$$\det(D+L) = \sum_{\textit{forest } G} w(G)$$

Proof. The i-th column of the matrix is $(D+L)_i = D_{ii}e_i + \sum_j A_{ij}(e_i - e_j)$, so

$$\det(D+L) = \det\left(\begin{array}{c|c|c|c} D_{11}e_1 & D_{22}e_2 & & D_{nn}e_n \\ + & + & & + \\ A_{11}(e_1-e_1) & A_{21}(e_2-e_1) & & A_{n1}(e_n-e_1) \\ + & + & \cdots & + \\ A_{12}(e_1-e_2) & A_{22}(e_2-e_2) & & A_{n2}(e_n-e_2) \\ \vdots & \vdots & & \vdots \\ A_{1n}(e_1-e_n) & A_{2n}(e_2-e_n) & & A_{nn}(e_n-e_n) \end{array}\right) = \sum_{\text{01-graph } G} w(G)\det(I-A_G) = \sum_{\text{forest } G} w(G)$$

In the middle step we expand the determinant by multilinearity: we sum over all possible ways to pick one term in each column. To choose a 01-graph on n vertices, for each vertex i (column i) we choose whether to make i a root (term $D_{ii}e_i$) or to give i an outgoing edge $i \to j$ (term $A_{ij}(e_i - e_j)$). After taking the weights D_{ii} and A_{ij} out of the determinant, we're left with a sum of $w(G)\det(I-A_G)$ over all 01-graphs G on n vertices, which is equal to the sum of $w(G)$ over all forests, by the lemma. $\square$

[2] We do not further expand $(e_i - e_j)$ into two separate terms.

3 Upwards routes

We now know that when A is the adjacency matrix of a 01-graph, then $\det(I - A) = 1$ if G is a forest and $\det(I - A) = 0$ if G has a cycle. One naturally wonders about the value of $\det(I - A)$ when A is an arbitrary adjacency matrix.

Definition 3.1. Given a directed graph G with an order on the vertices, we define *(strict) upwards loops* and *(strict) upwards routes*:

- An *upwards loop* at vertex i is a path from i to i that does not visit vertices lower than i.
- A *strictly upwards loop* at vertex i is a path from i to i that only visits vertices higher than i (except at the start/endpoint of the path, where it does visit i itself).
- A *(strictly) upwards route* is a choice of (strictly) upwards loop at each vertex.

Let $f_i(x)$ be the generating function of strictly upwards loops of length k at vertex i. Then

$$\bar{f}_i(x) = (1 - f_i(x))^{-1}$$

is the generating function of upwards loops of length k at vertex i, because an upwards loop of length k splits up uniquely into a sequence of strictly upwards loops. Furthermore, the generating functions $f(x)$ and $\bar{f}(x)$ of (strictly) upward routes of k edges are given by:

$$f(x) = \prod_{i=1}^{n} f_i(x) \qquad\qquad \bar{f}(x) = \prod_{i=1}^{n} \bar{f}_i(x)$$

Recall Cramer's rule:

Theorem 3.1. *(Cramer's rule) Let A be a matrix and let $A_{[i,j]}$ be the same with column i and row j deleted, then:*

$$A^{-1}_{ij} = \frac{\det(A_{[i,j]})}{\det(A)}$$

From this, we get the following lemma that allows us to calculate $\det(A)^{-1}$ in terms of entries of inverses of submatrices of A:

Lemma 3.2. *Given an invertible matrix A,*

$$\det(A)^{-1} = \prod_{i=0}^{n-1} (A_{[1\ldots i,1\ldots i]})^{-1}_{11}$$

Where $A_{[1\ldots i,1\ldots i]}$ is the matrix A with the first i rows and columns deleted.

Proof. Cramer's rule implies:

$$\det(A)^{-1} = A^{-1}_{1,1} \cdot \det(A_{[1,1]})^{-1}$$

Continuing this by induction, we get:

$$\begin{aligned}
\det(A)^{-1} &= A^{-1}_{1,1} \cdot \det(A_{[1,1]})^{-1} \\
&= A^{-1}_{1,1} \cdot (A_{[1,1]})^{-1}_{11} \cdot \det(A_{[1..2,1..2]})^{-1} \\
&= \cdots \\
&= A^{-1}_{1,1} \cdot (A_{[1,1]})^{-1}_{11} \cdot (A_{[1..2,1..2]})^{-1}_{1,1} \cdots (A_{[1..n-1,1..n-1]})^{-1}_{1,1} \cdot 1
\end{aligned}$$

□

We apply this lemma to the matrix $I - xA$, to obtain:

Lemma 3.3. *The generating function of upwards routes with k edges is* $\det(I-xA)^{-1}$.

Proof. Apply the preceding lemma:

$$\det(I-xA)^{-1} = \prod_{i=0}^{n-1}((I-xA)_{[1\ldots i,1\ldots i]})^{-1}_{11}$$

Thus, for each i we first obtain a subgraph by deleting vertices with lower number than i, and then $((I-xA)_{[1\ldots i,1\ldots i]})^{-1}_{11}$ is the generating function of loops from vertex i to i in the resulting graph. Thus, in terms of the original graph, these are loops that do not visit vertices with lower number than i. Multiplying this over each vertex i in the original graph, we obtain the result. □

Corollary 3.3.1. The number of (strictly) upwards routes of k edges does not depend on the order of the vertices.

Proof. If we permute the order of the vertices by a permutation P, the generating function

$$\det(I-xPAP^{-1}) = \det(P(I-xA)P^{-1}) = \det(I-xA)$$

stays the same. □

Lemma 3.4. *For an arbitrary adjacency matrix A,*

$$\det(I-xA) = \prod_{i=1}^{n}(1-f_i(x))$$

Where $f_i(x)$ is the generating function of strictly upwards loops at vertex i.

Proof. We have the following relationship between the generating functions:

$$\det(I-xA) = \bar{f}(x)^{-1} = \prod_{i=1}^{n}\bar{f}_i(x)^{-1} = \prod_{i=1}^{n}(1-f_i(x))$$

□

This is kind of interesting, because $\det(I-xA)$ is a polynomial, whereas $f_i(x)$ is a power series, so many terms cancel on the right hand side. The original lemma follows as a corollary, which gives us a third proof of the lemma:

Corollary 3.4.1. For a 01-graph, $\det(I-A)=1$ if G is a forest, and 0 if G has a cycle.

Proof. If G is a forest, then it has no strictly upward loops, so $f_i(x)=0$, so $\det(I-xA)=1$. If G has a cycle, let i be the lowerst vertex on the cycle. Then $f_i(x)=x^k$ where k is the length of the cycle. Now substitute $x=1$ to obtain $\det(I-A)=0$. □

It goes without saying that weighted versions of the preceding lemmas hold too.

4 Kirchoff's theorem with cycles

Let G_A be a graph with adjacency matrix A and let G_L be a graph with Laplacian L. We shall generalize Kirchoff's theorem from $\det(I+L)$ to $\det(A+L)$. In order to do this we need to define 1-graphs.

Definition 4.1. A 1-graph is a directed graph where each vertex has exactly one outgoing edge.

Thus, at each vertex we can continue following a unique path indefinitely. In a finite graph that path must eventually cycle. So a general 1-graph looks like a bunch of disjoint cycles and a bunch of trees converging onto those cycles.

In our 1-graphs, some edges will be selected from G_A and some will be selected from G_L. We define the weight function:

$$w(F) = \begin{cases} \dots & \dots \\ 0 & \text{otherwise} \end{cases}$$

Theorem 4.1. *Kirchoff's theorem with cycles.*

$$\det(A + L) = \sum_{\text{1-graph } F \subseteq (G_A + G_L)} w(F)$$

Proof. ... Main idea: generalize the lemma to account for cycles. Each time we Laplace expand a column with one entry (from G_A), which is now not necessarily in diagonal position, we obtain a sign. □

Bunch of corollaries:

Corollary 4.1.1. $\det(A)$ is the number of signed cycle covers.

Corollary 4.1.2. $\det(I + L)$ is the number of spanning forests.

Corollary 4.1.3. $\det(L_{[i,j]})$ spanning trees, for all i, j.

Corollary 4.1.4. All-minor matrix tree theorem.

Corollary 4.1.5. Undirected matrix tree theorem.