Reducing separation logic with conjunction, disjunction, and star to boolean-quantified linear integer inequalities

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We start with an entailment with conjunction, disjunction, and star, for example:

$$(P \vee Q) \star (Q \wedge P) \vdash (P \wedge Q) \star (Q \vee P)$$

The first step is to rewrite conjunction and disjunction in terms of star and universal and existential quantification over $\{0,1\}$, using the following equivalences:

$$P \wedge Q \equiv \forall b \in \{0,1\}, P_b \star Q_{1-b}$$
$$P \vee Q \equiv \exists b \in \{0,1\}, P_b \star Q_{1-b}$$

where

$$P_n := \underbrace{P \star P \star \cdots \star P}_{n \text{ times}}$$

For the example we end up with:

$$(\exists a, P_a \star Q_{1-a}) \star (\forall b, Q_b \star P_{1-b}) \vdash (\forall c, P_c \star Q_{1-c}) \star (\forall d, Q_d \star P_{1-d})$$

We pull all the quantifiers to the outermost level, and group the atomic propositions:

$$(\exists a, \forall b, P_{a+1-b} \star Q_{1-a+b}) \vdash (\forall c, \exists d, P_{c+1-d} \star Q_{1-c+d})$$

We bring the quantifiers outside of the entailment:

$$\forall a, \exists b, \forall c, \exists d, (P_{a+1-b} \star Q_{1-a+b} \vdash P_{c+1-d} \star Q_{1-c+d})$$

Finally, we transform the entailment to inequalities (or equalities, for a linear separation logic):

$$\forall a, \exists b, \forall c, \exists d, a+1-b \geq c+1-d \wedge 1-a+b \geq 1-c+d$$

This is a statement that we could solve with a SMT solver. Note that the inequalities are in fact pseudo-boolean constraints, so we can also use a quantified pseudo boolean solver.[1]

[1] Whether this is a good way to prove separation logic entailments is another question :) And depends on the solver.