Noether's theorem: the simplest case

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Noether's theorem states that conservation laws correspond to symmetries:

- Translation invariance $\iff$ conservation of momuntum
- Rotation invariance $\iff$ convervation of angular momentum
- Time translation invariance $\iff$ conservation of energy

Versions of this correspondence hold very generally throughout physics: in classical mechanics, field theory, relativity, quantum mechanics. In this note I'll explain the simplest case: the classical mechanics of a single particle in a potential $V(x, y)$ in two dimensions.

Newton's laws $ma = F$ describe the motion of this particle:

$$m\ddot{x} = \frac{\partial V}{\partial x} \qquad\qquad m\ddot{y} = \frac{\partial V}{\partial y}$$

We see: if $\frac{\partial V}{\partial x} = 0$, then $m\ddot{x} = 0$ so the momentum $m\dot{x}$ is conserved.

The condition on the function V can be stated in different ways:

- $\frac{\partial V}{\partial x} = 0$
- $V(x + \Delta x, y) = V(x, y)$ for all Δx
- $V(x, y)$ is independent of x
- $V(x, y) = f(y)$ for some function f

We say that V **is invariant under translation in the x direction**, and this **symmetry** of V is connected with the **conservation law** $L(x, \dot{x}, y, \dot{y}) = m\dot{x} =$ constant.

Symmetries of V are not always so obvious. For instance, for a particle in a gravitational well,

$$V(x, y) = -\frac{GMm}{\sqrt{x^2 + y^2}}$$

This potential depends on both variables x, y but it has rotational symmetry. If we write this in polar coordinates (r, θ),

$$V(r, \theta) = -\frac{GMm}{r}$$

and then $V(r, \theta)$ is independent of θ. The question arises: is this symmetry associated with any conservation law? For the moment, take my word for it that these are Newton's laws in polar coordinates:

$$m\ddot{r} = \frac{\partial V}{\partial r} \qquad\qquad \frac{d}{dt}(mr^2\dot{\theta}) = \frac{\partial V}{\partial \theta}$$

We see: if $\frac{\partial V}{\partial \theta} = 0$, then $\frac{d}{dt}(mr^2\dot{\theta}) = 0$, so $L(r, \dot{r}, \theta, \dot{\theta}) = mr^2\dot{\theta}$ is conserved!

If we change coordinates back to x, y we have $mr^2\dot{\theta} = m\dot{y}x - m\dot{x}y$, which is the usual expression for angular momentum. We can verify that this is conserved in the original coordinate system: take the derivative, and substitute the equations of motion whenever $\ddot{x}$ and $\ddot{y}$ appear, and simplify the resulting expression to 0.

The question is: **why did Newton's laws have the form $\frac{d}{dt}[L_1] = \frac{\partial V}{\partial \theta}$** and $\frac{d}{dt}[L_2] = \frac{\partial V}{\partial r}$ and do they have this form in any coordinate system? If they do, then whenever one of the partial derivatives of V is zero, then we have a conservation law.

We can see that Newton's laws have this form in any coordinate system using Lagrangian mechanics.

Noethers idea is that you can describe the symmetries in a different way, so that you do not need to find a coordinate system in which V is independent of one of the the coordinates.

The existence of a coordinate system (α, β) in which $\frac{\partial V}{\partial \alpha} = 0$ corresponds to a **continuous symmetry** of $V(x, y)$.

$$A_\theta(x, y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$$

$$\begin{aligned}
\frac{A_\theta(x,y)}{d\theta} &= (y, -x) \\
V(A_\theta(x,y)) &= V(x,y) \\
DV(A_\theta(x,y)) \cdot A'_\theta &= 0 \\
DV(x,y) \cdot A'_0 &= 0 \\
DV(x,y) \cdot A'_0 &= 0 \\
f(x,y)\ddot{x} + g(x,y)\ddot{y} &= 0 \\
(f(x,y)\dot{x} + g(x,y)\dot{y})' &= \dot{f}(x,y)\dot{x} + f(x,y)\ddot{x} + \dot{g}(x,y)\dot{y} + g(x,y)\ddot{y} \\
&= \dot{f}(x,y)\dot{x} + \dot{g}(x,y)\dot{y} \\
&= (f_1(x,y)\dot{x} + f_2(x,y)\dot{y})\dot{x} + (g_1(x,y)\dot{x} + g_2(x,y)\dot{y})\dot{y}
\end{aligned}$$

HAS TO BE A SYMMETRY OF THE LAGRANGIAN, NOT JUST OF V!

So also ok if it's a symmetry of the kinetic energy and V separately. Not ok, things like $(x, y) \mapsto (x + \theta y, y)$. But ok things like $(x, y) \mapsto (x + \theta, y)$.