Tarski's fixed point theorem

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Let L be a partially ordered set with order $(\leq)$. Given a subset $A \subseteq L$ we say that $x \in L$ is the infimum of A if $x \leq A$ (meaning $x \leq a$ for all $a \in A$) and for any $x' \leq A$ we have $x' \leq x$. The infimum of A is necessarily unique, so if all subsets have infima then we can denote them $\inf A$. Dually, we have the concept of supremum, $\sup A$. We henceforth assume that L is a complete lattice, which means that it has all infima and suprema.

Examples:

- The interval $[0,1] \subset \mathbb{R}$, because for any $A \subseteq [0,1]$, we have $\inf A \in [0,1]$ and $\sup A \in [0,1]$.
- The subsets 2^X of a set X, ordered by $(\subseteq)$ with $\inf A = \bigcup A$ and $\sup A = \bigcap A$.

Denote $0 = \inf L$ and $1 = \sup L$.

We say that a function $f : L \to L'$ is monotone if $x \leq y \implies f(x) \leq f(y)$. This also means that if $x \geq y$ then $f(x) \geq f(y)$. Thus, f is monotone if we can apply it on both sides of an inequality.

We focus on the case $L = L'$ so that $f : L \to L$. In this situation we can repeatedly applying f to x to obtain x, $f(x)$, $f(f(x))$, and so on.

We say that an element $x \in L$ is an upmover if $f(x) \geq x$ and a downmover if $f(x) \leq x$:

$$D = \{x \in L \mid f(x) \leq x\} \tag{1}$$
$$U = \{x \in L \mid f(x) \geq x\} \tag{2}$$

If x is an upmover then $f(x)$ is also an upmover: apply f on both sides of the inequality $f(x) \geq x$. Similarly, if $x \in D$ then $f(x) \in D$. Thus, upmovers keep moving up, and downmovers keep moving down, if we apply f repeatedly.

Note that not all elements need to be upmovers or downmovers; it may be the case that $f(x)$ is neither $\geq x$ nor $\leq x$. Nevertheless, U and D are not empty, since $0 \in U$ and $1 \in D$.

Lemma 1. *If $x \in D$ then $f(x) \in D$.*

Proof. If $x \in U$ then $f(x) \leq x$. Apply f to both sides to obtain $f(f(x)) \leq f(x)$. Therefore $f(x)$ is a downmover. □

Lemma 2. *If $A \subseteq D$ then* $\inf A \in D$.

Proof. To show $\inf A \in D$ we have to show $f(\inf A) \leq \inf A$, but that's the case if $f(\inf A) \leq a$ for all $a \in A$. Since $a \in A \subseteq D$ we have $f(a) \leq a$. So it's sufficient to show $f(\inf A) \leq f(a)$, so it's sufficient if $\inf A \leq a$, which is true.

$$\inf A \in D \impliedby \text{(definition of } D) \tag{3}$$
$$f(\inf A) \leq \inf A \impliedby \text{(property of inf)} \tag{4}$$
$$\forall a \in A, f(\inf A) \leq a \impliedby \text{(because } a \in D \text{ means } f(a) \leq a) \tag{5}$$
$$\forall a \in A, f(\inf A) \leq f(a) \impliedby \text{(monotonicity of } f) \tag{6}$$
$$\forall a \in A, \inf A \leq a \impliedby \text{(property of inf)} \tag{7}$$

□

Lemma 3. $f(\inf D) = \inf D$

Proof. Because $\inf D \in D$ we have $f(\inf D) \leq \inf D$. Because $f(\inf D) \in D$ we have $\inf D \leq f(\inf D)$. $\square$

Proof idea:

1. Prove that inf-complete $\implies$ complete.
2. Prove that D is inf-complete $\implies$ complete.
3. Dually U is complete.
4. Fixed points $F = D \cap U$, the above two properties imply F is complete.

Potential problem: the infimum depends on the surrounding set, so we should really have $\inf_F A$ and $\inf_L A$ and $\inf_D A$ and $\inf_U A$.

Lemma 4. *Let $L' \subseteq L$, and $A \subseteq L'$. Then the following are possible:*

1. *One, or both, of $\inf_L A$ and $\inf_{L'} A$ fail to exist.*
2. *Both $\inf_L A$ and $\inf_{L'} A$ exist, and $\inf_L A \leq \inf_{L'} A$.*

It's possible that $\inf_L A < \inf_{L'} A$, or $\inf_L A = \inf_{L'} A \iff \inf_L A \in L'$. In particular, if $\inf_L A \in L'$ then $\inf_{L'} A$ exists and they're equal.

Lemma 5. *If a lattice L is inf-complete, then L is sup-complete.*

Proof. Let $A \subseteq L$. $\inf\{x \in L \mid x \geq A\} = \sup A$. $\square$