Embedding classical logic into constructive logic by double negation translation

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Abstract

We will embed classical logic into constructive logic via double negation translations. The explanation presented here is intended to be a bit easier to understand and less mysterious than the explanations you'll usually find.[1]

Constructive logic differs from classical logic in that we do not assume $\neg\neg P \implies P$ for all propositions. Proofs that are constructively valid are also classically valid, but not the other way around if the proof uses this principle or an equivalent one like $P \vee \neg P$. Thus, it may seem that the theorems one can prove with classical logic are a superset of those you can prove with constructive logic, but it turns out that from a more refined perspective, the situation is the other way around.

[Describe here that classical proofs of existence give you a method to construct a witness?]

We can translate every classical theorem and proof to some constructive theorem and proof by means of a *double negation translation*. Thus, classical logic may be viewed as the *subset* of constructive logic that lies in the image of this translation. Classical mathematicians restrict themselves to theorems that lie in that image.

Within constructive logic we say that a proposition P is classical if $\neg\neg P \implies P$. The classical propositions form a subset of the constructive ones. Given any P, the proposition $\neg\neg P$ is always classical, because $\neg\neg(\neg\neg P) \implies \neg\neg P$, even constructively. The double negation $\neg\neg$ acts as a projection of constructive propositions onto classical propositions.

Using this projection, we can define the classical connectives in terms of the constructive ones:

$$
\begin{aligned}
P \wedge^c Q &:= \neg\neg(P \wedge Q) \\
P \vee^c Q &:= \neg\neg(P \vee Q) \\
P \to^c Q &:= \neg\neg(P \to Q) \\
\forall^c x, P(x) &:= \neg\neg\forall x, P(x) \\
\exists^c x, P(x) &:= \neg\neg\exists x, P(x) \\
True^c &:= \neg\neg True \\
False^c &:= \neg\neg False \\
\neg^c P &:= \neg\neg(\neg P)
\end{aligned}
$$

These connectives enjoy all the rules of classical logic, such as $\neg^c(P \wedge^c Q) \iff P \vee^c Q$. Thus, a classical proof of a theorem T can be converted into a constructive proof of theorem T' where T' uses the classical connectives.

Now we can understand the point of view that classical logic is a subset of constructive logic. The classical mathematician can never prove $\exists n \in \mathbb{N}, P(n)$; they only ever prove $\exists^c n \in \mathbb{N}, P(n)$, which is a weaker statement. The former means "I can give you a concrete number n and a proof that $P(n)$ holds", whereas the latter means "If **you** give me a concrete number n and a proof of $\neg P(n)$, then I can derive a contradiction".

[1] For instance, on wikipedia (https://en.wikipedia.org/wiki/Double-negation_translation).

You may think that the difference between those two is not very large. After all, we could simply search for an n satisfying $P(n)$, and this search is guaranteed to terminate if we know $\exists^c n \in \mathbb{N}, P(n)$. This glosses over the fact that to even do the search, we must be able to decide whether $P(n)$ or $\neg P(n)$ for each n.

The situation gets worse if the set we are quantifying over is $\mathbb{R}$ instead of $\mathbb{N}$. The constructivist promises to give you a real number, that is, a concrete description of a Cauchy sequence $r_i \in \mathbb{Q}$ with which you can compute arbitrarily accurate rational approximations, and a proof that $P(r)$ holds. The classical mathematician only promises that they can derive a contradiction if **you** give them such a Cauchy sequence, plus a proof of $\neg P(r)$. Searching over all real numbers for one that works is hopeless, even if we somehow had a magic machine that tells us whether $P(r)$ or $\neg P(r)$ for each particular one.

The definitions of the classical connectives can be simplified. For instance, $\neg\neg True \iff True$, so we might as well define $True^c := True$. The same holds for $False^c$ and $\neg^c$. Another simplication can be had if we assume that *classical connectives are only used on classical propositions*. In that case we can define $P \wedge^c Q := P \wedge Q$, because for classical P and Q, we have $\neg\neg(P \wedge Q) \iff P \wedge Q$. The same is true for $\forall^c x, P(x) := \forall x, P(x)$ and $P \to^c Q := P \to Q$.

The same is *not* true for disjunctions and existentials. We do need to retain the $\neg\neg$ around those, even if the propositions involved are classical. This simplifies the translation to:

$$
\begin{aligned}
P \wedge^c Q &:= P \wedge Q \\
P \vee^c Q &:= \neg\neg(P \vee Q) \\
P \to^c Q &:= P \to Q \\
\forall^c x, P(x) &:= \forall x, P(x) \\
\exists^c x, P(x) &:= \neg\neg\exists x, P(x) \\
True^c &:= True \\
False^c &:= False \\
\neg^c P &:= \neg P
\end{aligned}
$$

This still ensures that a formula involving the classical connectives is a classical proposition, and that all the classical laws hold constructively for these connectives, provided that all the basic propositions are classical. We can always *make* the basic propositions classical by putting a $\neg\neg$ around them. For instance, an equality $x = y$ becomes $\neg\neg(x = y)$.

Since $\neg\neg(P \vee Q) \iff \neg(\neg P \wedge \neg Q)$, we could also have used that in the translation. Various different possibilities of shuffling the $\neg$'s around have different names (Gödel-Gentzen's translation, Komolgorov translation, Kuroda's translation). These translations are all constructively equivalent.

- How to use the translation to (1) import classical theorems and proofs (2) prove constructive propositions that involve negation by doing the translation in reverse. Decidable propositions.

- Expand on Kuroda's translation.