THE PRODUCT OF GCD AND LCM

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This is the standard identity for the product of gcd and lcm:

$$\gcd(a,b)\cdot\text{lcm}(a,b) = ab$$

One might wonder whether it holds that $\gcd(a,b,c)\cdot\text{lcm}(a,b,c) = abc$. Unfortunately, it does not; consider $a = b = c = 2$. It does however hold that

$$\gcd(a,b,c)\cdot\text{lcm}(ab,ac,bc) = abc \tag{1}$$

In fact, it also holds that

$$\gcd(ab,ac,bc)\cdot\text{lcm}(a,b,c) = abc$$

To see this, think of a number as a vector of its prime factorisation:

$$2^2\cdot 3^1\cdot 7^2 = (2,1,0,2,0,0,\cdots)$$

On this representation, the gcd corresponds to taking the pointwise minimum, and the lcd the pointwise maximum:

$$\begin{aligned}\gcd((a_1,a_2,\cdots),(b_1,b_2,\cdots),(c_1,c_2,c_3,\cdots)) &= (\min(a_1,b_1,c_1),\min(a_2,b_2,c_2),\cdots)\\ \text{lcm}((a_1,a_2,\cdots),(b_1,b_2,\cdots),(c_1,c_2,c_3,\cdots)) &= (\min(a_1,b_1,c_1),\max(a_2,b_2,c_2),\cdots)\end{aligned}$$

And the product corresponds to the pointwise sum:

$$(a_1,a_2,\cdots)\cdot(b_1,b_2,\cdots)\cdot(c_1,c_2,c_3,\cdots) = (a_1+b_1+c_1,a_2+b_2+c_2,\cdots)$$

Thus, in this representation, equation (1) translates to:

$$\min(a_i,b_i,c_i)+\max(a_i+b_i,a_i+c_i,b_i+c_i) = a_i+b_i+c_i \qquad \text{(for all i)}$$

Now it is easy to see that the identity holds: fix i and assume without loss of generality that $a_i \leqslant b_i \leqslant c_i$, then the minimum reduces do a_i and the maximum to $b_i + c_i$.

We see that more generally, given n numbers instead of 3 numbers:

$$\gcd(\text{k-fold products})\cdot\text{lcm}(\text{(n-k)-fold products}) = \text{product}$$

For $n = 4$, this gives that the following values are all equal to $abcd$.

$$\begin{aligned}
&\gcd(\emptyset)\cdot\text{lcm}(abcd) && (k=0)\\
&\gcd(a,b,c,d)\cdot\text{lcm}(bcd,acd,abd,abc) && (k=1)\\
&\gcd(ab,ab,ad,bc,bd,cd)\cdot\text{lcm}(ab,ab,ad,bc,bd,cd) && (k=2)\\
&\gcd(bcd,acd,abd,abc)\cdot\text{lcm}(a,b,c,d) && (k=3)\\
&\gcd(abcd)\cdot\text{lcm}(\emptyset) && (k=4)
\end{aligned}$$

In fact, if we allow negative powers in the prime factorization, we can see that such identities hold over the positive rationals too, with gcd and lcm suitably extended.