Euler product

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Abstract

Start with the formula for df in terms of dx & dy:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

This really means:

$$df = \left.\frac{\partial f}{\partial x}\right|_y dx + \left.\frac{\partial f}{\partial y}\right|_x dy$$

Wedge both sides with dy:

$$df \wedge dy = \left.\frac{\partial f}{\partial x}\right|_y dx \wedge dy + \left.\frac{\partial f}{\partial y}\right|_x dy \wedge dy = \left.\frac{\partial f}{\partial x}\right|_y dx \wedge dy$$

Therefore:

$$\left.\frac{\partial f}{\partial x}\right|_y = \frac{df \wedge dy}{dx \wedge dy}$$

Application: Euler product formula.

$$\begin{aligned}
\left.\frac{\partial u}{\partial v}\right|_w \cdot \left.\frac{\partial v}{\partial w}\right|_u \cdot \left.\frac{\partial w}{\partial u}\right|_v &= \frac{du \wedge dw}{dv \wedge dw} \cdot \frac{dv \wedge du}{dw \wedge du} \cdot \frac{dw \wedge dv}{du \wedge dv} \\
&= \frac{du \wedge dw}{dw \wedge du} \cdot \frac{dv \wedge du}{du \wedge dv} \cdot \frac{dw \wedge dv}{dv \wedge dw} \\
&= (-1) \cdot (-1) \cdot (-1) \\
&= -1
\end{aligned}$$

Cramer's rule using the wedge product [Joo].

Let A be a 2×2 matrix and $b \in \mathbb{R}^2$. System of equations:

$$A \begin{pmatrix} x \\ y \end{pmatrix} = b$$

That is,

$$A_1 x + A_2 y = b$$

Wedge both sides with A_2:

$$A_1 \wedge A_2 x + A_2 \wedge A_2 y = b \wedge A_2$$

So

$$x = \frac{b \wedge A_2}{A_1 \wedge A_2}$$

$$\begin{aligned} b &= xA_1 + yA_2 && \implies \text{wedge with } A_2 \\ df &= \left.\frac{\partial f}{\partial x}\right|_y dx + \left.\frac{\partial f}{\partial y}\right|_x dy && \implies \text{wedge with } dy \end{aligned}$$

$$\left.\frac{\partial a}{\partial b}\right|_c \cdot \left.\frac{\partial c}{\partial d}\right|_b \cdot \left.\frac{\partial b}{\partial a}\right|_d \cdot \left.\frac{\partial d}{\partial c}\right|_a = 1$$

References

[Joo] Peeter Joot. Cramer's rule. URL: http://peeterjoot.com/archives/geometric-algebra/ga_wiki_cramers.pdf.