A MULTILINEAR PROOF OF JACOBI'S FORMULA

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Jacobi's formula gives the determinant of a matrix exponential in terms of the trace:

$$\det(e^{tA}) = e^{t\,\mathrm{tr}(A)}$$

The usual proof of this uses the Laplace expansion of the determinant. Here I'll give a proof that relies on multilinearity of the determinant.

Consider the function $f(t) = \det(e^{tA})$. To show that $f(t) = e^{t\,\mathrm{tr}(A)}$ it suffices to show that f satisfies the differential equation that defines $e^{t\,\mathrm{tr}(A)}$:

$$\begin{aligned} f(0) &= 1 \\ f'(t) &= \mathrm{tr}(A)f(t) \end{aligned}$$

The condition $f(0) = 1$ follows immediately, so it remains to calculate the derivative of f.

To calculate the derivative of a determinant, first consider the derivative of a product:

$$(a_1 \cdot a_2 \cdots a_n)' = (a_1' \cdot a_2 \cdots a_n) + (a_1 \cdot a_2' \cdots a_n) + \cdots + (a_1 \cdot a_2 \cdots a_n')$$

The analogous formula works for any multilinear function, in particular the determinant:

$$\det(a_1|a_2|\cdots|a_n)' = \det(a_1'|a_2|\cdots|a_n) + \det(a_1|a_2'|\cdots|a_n) + \cdots + \det(a_1|a_2|\cdots|a_n')$$

where $\det(a_1|a_2|\cdots|a_n)$ is the determinant of the matrix with columns $a_2, a_2, \cdots, a_n$.

Applying this to f, we get:

$$\begin{aligned} \det(e^{tA})' = {} & \det((e^{tA})_1'|(e^{tA})_2|\cdots|(e^{tA})_n) + \\ & \det((e^{tA})_1|(e^{tA})_2'|\cdots|(e^{tA})_n) + \\ & + \cdots + \\ & \det((e^{tA})_1|(e^{tA})_2|\cdots|(e^{tA})_n') \\ = {} & \det((Ae^{tA})_1|(e^{tA})_2|\cdots|(e^{tA})_n) + \\ & \det((e^{tA})_1|(Ae^{tA})_2|\cdots|(e^{tA})_n) + \\ & + \cdots + \\ & \det((e^{tA})_1|(e^{tA})_2|\cdots|(Ae^{tA})_n) \\ = {} & (\det(A_1|b_2|\cdots|b_n) + \\ & \det(b_1|A_2|\cdots|b_n) + \\ & + \cdots + \\ & \det(b_1|b_2|\cdots|A_n)) \cdot \det(e^{tA}) \\ = {} & \mathrm{tr}(A)\det(e^{tA}) \\ = {} & \mathrm{tr}(A)f(t) \end{aligned}$$

where $b_1, \cdots, b_n$ are the basis vectors. This completes the proof.

A similar argument where we use $1 = \det(A^{-1}) \cdot \det(A)$ shows that

$$\det(A)' = \mathrm{tr}(A^{-1}A')\det(A)$$