

# A simple proof of the matrix-tree theorem, upward routes, and a matrix-tree-cycle theorem

Jules Jacobs

July 25, 2021

## Abstract

The matrix-tree theorem states that the number of spanning trees in a graph is  $\det L'$ , where  $L'$  is the Laplacian matrix  $L$  of the graph, with any row and column deleted. We give a direct proof of the fact that  $\det(xI + L)$  is the generating function of spanning forests with  $k$  roots. Our proof does not rely on the Cauchy-Binet formula, yet is arguably simpler than the standard proof, and applies to weighted & directed graphs as well.

The proof is based on a lemma that if  $A$  is the adjacency matrix of a graph where each vertex has at most one outgoing edge, then  $\det(I - A) = 1$  if  $A$  is a forest and  $\det(I - A) = 0$  otherwise. We generalize this lemma to any graph, in which case  $\det(I - xA)^{-1}$  is shown to be the generating function of *upward routes*.

Lastly, we generalize the matrix-tree theorem to a theorem about  $\det(A + L)$  where  $A$  is the adjacency matrix of a second graph, which reduces to counting spanning forests when  $A = xI$  but allows some cycles when  $A$  is not diagonal. The special case  $L = 0$  gives that  $\det(A)$  counts signed cycle covers. The all-minors matrix-tree theorem follows as a corollary. For instance, the fact that  $\det(L')$  counts spanning forests, where  $L'$  is  $L$  with any column  $i$  and row  $j$  deleted, follows by picking  $A$  to have  $A_{ij} = 1$  and zero elsewhere.

## 1 Introduction

Given finite sets of numbers  $T_1, \dots, T_n \subset \mathbb{R}$ , we have the identity:

$$\left( \sum_{x_1 \in T_1} x_1 \right) \cdots \left( \sum_{x_n \in T_n} x_n \right) = \sum_{x_1 \in T_1} \cdots \sum_{x_n \in T_n} x_1 \cdots x_n$$

On the right hand side, we get one term for every way of choosing  $(x_1, \dots, x_n) \in T_1 \times \cdots \times T_n$ . Similarly, given finite sets of vectors  $T_1, \dots, T_n \subset \mathbb{R}^k$ , we have the following identity, by multilinearity of the determinant:

$$\det \left( \sum_{v_1 \in T_1} v_1 \mid \cdots \mid \sum_{v_n \in T_n} v_n \right) = \sum_{v_1 \in T_1} \cdots \sum_{v_n \in T_n} \det(v_1 \mid \cdots \mid v_n)$$

Where  $\det(v_1 \mid \dots \mid v_n)$  is the determinant of a matrix with those columns. We shall see that Kirchoff's theorem can be obtained from this identity, but we first need some definitions and a lemma.

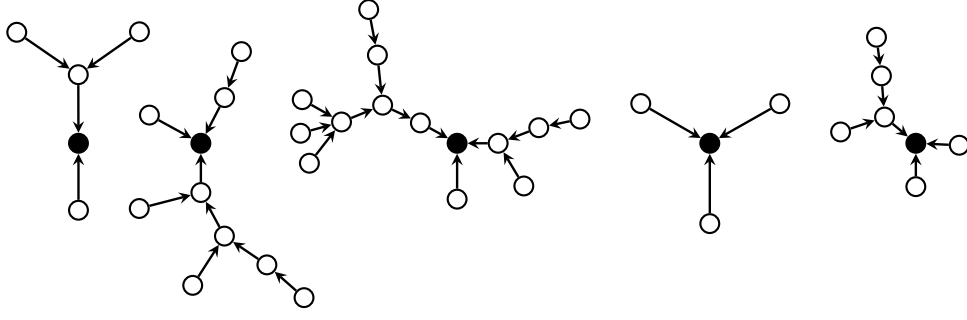


Figure 1: A forest with 5 roots.

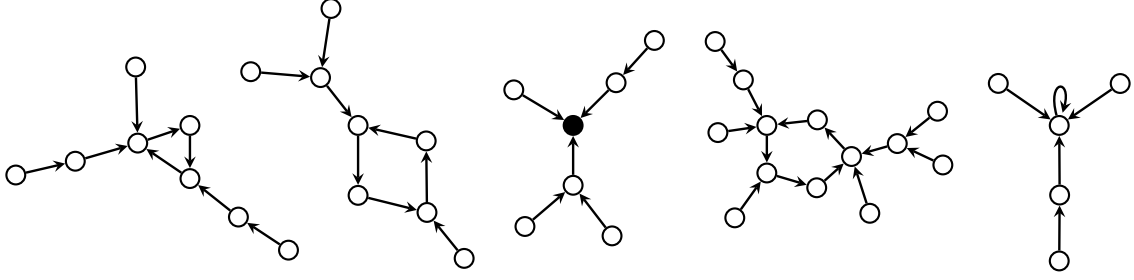


Figure 2: A 01-graph with one root. If we delete the third component, we obtain a 1-graph.

**Definition 1.1.** We define the concepts *1-graph*, *01-graph*, *forest*, *root*, and *tree*:

- A *1-graph* is a directed graph where each vertex has 1 outgoing edge.
- A *01-graph* is a directed graph where each vertex has 0 or 1 outgoing edges.
- A *forest* is a 01-graph with no cycles.
- The *roots* are vertices with 0 outgoing edges.
- A *tree* is a forest with one root.

See [Figure 1](#) and [Figure 2](#) for examples.

If we start at a vertex  $v$  in a 01-graph and keep following the unique outgoing edge, then we either loop in a cycle, or we end up at a vertex with no outgoing edges. Therefore a general 01-graph consists of roots and cycles, plus trees converging onto the roots and cycles. A 1-graph is a 01-graph with no roots, so regardless of where we start, we always end up in a cycle.

## 2 The matrix-tree theorem

We need a lemma that give us an indicator function for forests.

**Lemma 2.1.** Let  $A_G$  be the adjacency matrix of a 1-graph  $G$ , then

$$\det(I - A_G) = \begin{cases} 1 & \text{if } G \text{ is empty} \\ 0 & \text{if } G \text{ is not empty} \end{cases}$$

*Proof.* If  $G$  is empty, we have a  $0 \times 0$  matrix, which has determinant 1. If  $G$  is not empty, then each column of  $I - A_G$  has one +1 diagonal entry and one -1 entry from  $A_G$ , so the sum of the rows is zero, so  $\det(I - A_G) = 0$ . This remains true in the presence of self loops.  $\square$

**Lemma 2.2.** Let  $A_G$  be the adjacency matrix of a 01-graph  $G$ , then

$$\det(I - A_G) = \begin{cases} 1 & \text{if } G \text{ is a forest} \\ 0 & \text{if } G \text{ has a cycle} \end{cases}$$

*Proof.* We calculate the determinant by repeatedly performing Laplace expansion on a column  $i$  that corresponds to a root. The column of a root has a single  $+1$  entry on the diagonal, so performing Laplace expansion along this column deletes column  $i$  and row  $i$ . Row  $i$  contains all the incoming edges of the root. Therefore, this operation corresponds to deleting root  $i$  and all its incoming edges from the graph. Deleting a root may create new roots. Repeating this process of deleting roots, the remaining 1-graph will be empty iff the original graph was a forest. Applying the previous lemma gives the desired result.  $\square$

An alternative shorter proof using eigenvalues:

*Proof.* If  $G$  is a forest then  $A_G$  is nilpotent, so all its eigenvalues are 0, so all the eigenvalues of  $I - A_G$  are 1, so  $\det(I - A_G) = 1$ . If  $G$  has a cycle, then  $A_G$  has 1 as an eigenvalue (take the eigenvector that is 1 on the cycle and 0 elsewhere), so  $I - A_G$  has 0 as an eigenvalue, so  $\det(I - A_G) = 0$ .  $\square$

**We now fix  $n \times n$  matrices  $A$  and  $D$  over some commutative ring:**

- An arbitrary matrix  $A$  of edge weights (with  $A_{ij}$  being the weight of edge  $i \rightarrow j$ )
- A diagonal matrix  $D$  of vertex weights (with  $D_{ii}$  being the weight of vertex  $i$ ).

**Definition 2.1.** The Laplacian matrix  $L$  is defined as having columns:

$$L_i = \sum_j A_{ij}(e_i - e_j)$$

**Definition 2.2.** The weight of a forest  $G$  is:

$$w(G) = \prod_{i \in \text{roots}(G)} D_{ii} \prod_{(i \rightarrow j) \in \text{edges}(G)} A_{ij}$$

We're ready to state Kirchoff's theorem for multiple-root forests in weighted directed graphs.

**Theorem 2.3** (Kirchoff, Tutte). *The determinant  $\det(D + L)$  is the weight-sum of all forests on  $n$  vertices:*

$$\det(D + L) = \sum_{\text{forest } G} w(G)$$

*Proof.* The  $i$ -th column of the matrix is  $(D + L)_i = D_{ii}e_i + \sum_j A_{ij}(e_i - e_j)$ , so

$$\det(D + L) = \det \left( \begin{array}{c|c|c} \begin{matrix} D_{11}e_1 \\ + \\ A_{11}(e_1 - e_1) \\ + \\ A_{12}(e_1 - e_2) \\ \vdots \\ A_{1n}(e_1 - e_n) \end{matrix} & \begin{matrix} D_{22}e_2 \\ + \\ A_{21}(e_2 - e_1) \\ + \\ A_{22}(e_2 - e_2) \\ \vdots \\ A_{2n}(e_2 - e_n) \end{matrix} & \cdots & \begin{matrix} D_{nn}e_n \\ + \\ A_{n1}(e_n - e_1) \\ + \\ A_{n2}(e_n - e_2) \\ \vdots \\ A_{nn}(e_n - e_n) \end{matrix} \end{array} \right) = \sum_{\text{01-graph } G} w(G) \det(I - A_G) = \sum_{\text{forest } G} w(G)$$

The first step is by the definition of  $L$ . In the middle step we expand the determinant by multilinearity: we sum over all possible ways to pick one term of each column. To choose a 01-graph on  $n$  vertices, is to choose for each vertex  $i$  (column  $i$ ) whether to make  $i$  a root (term  $D_{ii}e_i$ ) or to give  $i$  an outgoing edge  $i \rightarrow j$  (term  $A_{ij}(e_i - e_j)$ ). Then we take the weights  $D_{ii}$  and  $A_{ij}$  out of the determinant, and we're left with  $w(G) \det(I - A_G)$ , where  $A_G$  is the adjacency matrix of the chosen 01-graph. The final step is applying the lemma.  $\square$

The classic form of Kirchhoff's matrix tree theorem gives us a way to count the number of spanning trees of an *undirected* and *unweighted* graph  $G$ . It is a special case of [Theorem 2.3](#), as follows.

We interpret the undirected unweighted graph as a directed weighted graph where we insert two edges  $i \rightarrow j$  and  $j \rightarrow i$  of weight 1 for each undirected edge  $i - j$ . This makes the matrix  $A$  the adjacency matrix of  $G$ :

$$A_{ij} = \begin{cases} 1 & \text{if } \{i, j\} \in G \\ 0 & \text{otherwise} \end{cases}$$

The Laplacian  $L$  is still given by [definition 2.1](#).

**Corollary 2.3.1** (Classic form of Kirchhoff's theorem). Let  $L'$  be the Laplacian matrix  $L$  with the first row and column deleted. Then  $\det(L')$  is the number of spanning trees of  $G$ .

*Proof.* Set the vertex weight  $D_{11} = 1$  for vertex 1 and  $D_{ii} = 0$  for the other vertices. Then on the one hand,  $\det(D + L) = \det(L')$ , and on the other hand,  $\det(D + L)$  is the number of directed trees with 1 as the root, by [theorem 2.3](#). Such trees are in bijective correspondence with undirected spanning trees of  $G$ , because we can turn a tree rooted at 1 into an undirected spanning tree by forgetting the direction of the edges, and we can turn an undirected spanning tree into a tree rooted at 1 by directing all the edges toward 1.  $\square$

More generally, [theorem 2.3](#) allows us to count the number of forests with a given number of roots. For our commutative ring we pick polynomials in  $x$  and set  $D = xI$ . [Theorem 2.3](#) gives that the (slightly modified) characteristic polynomial  $\det(xI + A)$  of the adjacency matrix  $A$  is the generating polynomial of forests with  $k$  roots.

It is essential that we consider directed forests. The correspondence between undirected spanning trees and directed spanning trees rooted at 1 fails to work as smoothly for  $k > 1$ . Thus it could be argued that Kirchhoff's theorem is really a theorem about directed forests. The directed version was Tutte's contribution to the theorem.

### 3 Upward routes

We now know that when  $A$  is the adjacency matrix of a 01-graph, then  $\det(I - A) = 1$  if  $G$  is a forest and  $\det(I - A) = 0$  if  $G$  has a cycle. One naturally wonders about the value of  $\det(I - A)$  when  $A$  is an arbitrary adjacency matrix. The goal of this section is a combinatorial interpretation of  $\det(I - xA)$  and  $\det(I - xA)^{-1}$  for an arbitrary adjacency matrix, which generalizes the lemma to arbitrary graphs.

**Definition 3.1.** Given a directed graph  $G$  with an order on the vertices, we define *(strictly) upward loops* and *(strictly) upward routes*:

- An *upward loop* at vertex  $i$  is a walk from  $i$  to  $i$  that does not visit vertices lower than  $i$ .
- A *strictly upward loop* at vertex  $i$  is a walk from  $i$  to  $i$  that only visits vertices higher than  $i$  (except at the start/endpoint of the walk, where it does visit  $i$  itself).
- A *(strictly) upward route* is a choice of (strictly) upward loop at each vertex.

Let  $f_i(x)$  be the generating function of strictly upward loops of length  $k$  at vertex  $i$ . Then

$$f_i^*(x) = (1 - f_i(x))^{-1}$$

is the generating function of upward loops of length  $k$  at vertex  $i$ , because an upward loop splits uniquely into a sequence of strictly upward loops. Furthermore, the generating functions  $f(x)$  and  $f^*(x)$  of (strictly) upward routes of  $k$  edges are given by:

$$f(x) = \prod_{i=1}^n f_i(x) \qquad f^*(x) = \prod_{i=1}^n f_i^*(x)$$

Recall Cramer's rule:

**Theorem 3.1.** (Cramer's rule) Let  $A$  be an invertible matrix and let  $A_{[i,j]}$  be the same with column  $i$  and row  $j$  deleted, then:

$$(A^{-1})_{ij} = \frac{\det(A_{[i,j]})}{\det(A)}$$

From this, we get the following lemma that allows us to calculate  $\det(A)^{-1}$  in terms of entries of inverses of submatrices of  $A$ :

**Lemma 3.2.** Given a matrix  $A$ , provided both sides are defined,

$$\det(A)^{-1} = \prod_{i=0}^{n-1} (A_{[1\dots i, 1\dots i]})_{11}^{-1}$$

Where  $A_{[1\dots i, 1\dots i]}$  is the matrix  $A$  with the first  $i$  rows and columns deleted.

*Proof.* Cramer's rule implies:

$$\det(A)^{-1} = A_{1,1}^{-1} \cdot \det(A_{[1,1]})^{-1}$$

Continuing this by induction, we get:

$$\begin{aligned} \det(A)^{-1} &= A_{1,1}^{-1} \cdot \det(A_{[1,1]})^{-1} \\ &= A_{1,1}^{-1} \cdot (A_{[1,1]})_{11}^{-1} \cdot \det(A_{[1..2, 1..2]})^{-1} \\ &= \dots \\ &= A_{1,1}^{-1} \cdot (A_{[1,1]})_{11}^{-1} \cdot (A_{[1..2, 1..2]})_{1,1}^{-1} \cdots (A_{[1..n-1, 1..n-1]})_{1,1}^{-1} \cdot 1 \end{aligned}$$

□

**Lemma 3.3.** The generating function of upward loops at vertex  $i$  is  $((I - xA)_{[1\dots i-1, 1\dots i-1]})_{11}^{-1}$ .

*Proof.* Given an adjacency matrix  $B$ ,  $(I - xB)^{-1} = I + xB + x^2B^2 + \dots$  is the matrix generating function of walks, so  $(I - xB)_{11}^{-1}$  is the generating function of loops from vertex 1 to 1. To obtain upward loops at vertex  $i$  in  $A$  we take  $B = (I - xA)_{[1\dots i-1, 1\dots i-1]}$  with the first  $i - 1$  rows and columns deleted. □

We combine these lemmas to obtain:

**Theorem 3.4.** The generating function of upward routes with  $k$  edges is  $\det(I - xA)^{-1}$ .

*Proof.* Combine the preceding two lemmas with  $f^*(x) = \prod_{i=1}^n f_i^*(x)$ . □

**Corollary 3.4.1.** The number of upward routes of  $k$  edges does not depend on the order of the vertices.

*Proof.* If we permute the order of the vertices by a permutation  $P$ , the generating function

$$\det(I - xPAP^{-1}) = \det(P(I - xA)P^{-1}) = \det(I - xA)$$

stays the same. A bijective proof is left as an exercise. □

**Corollary 3.4.2.** For an arbitrary adjacency matrix  $A$ ,

$$\det(I - xA) = \prod_{i=1}^n (1 - f_i(x))$$

Where  $f_i(x)$  is the generating function of strictly upward loops at vertex  $i$ .

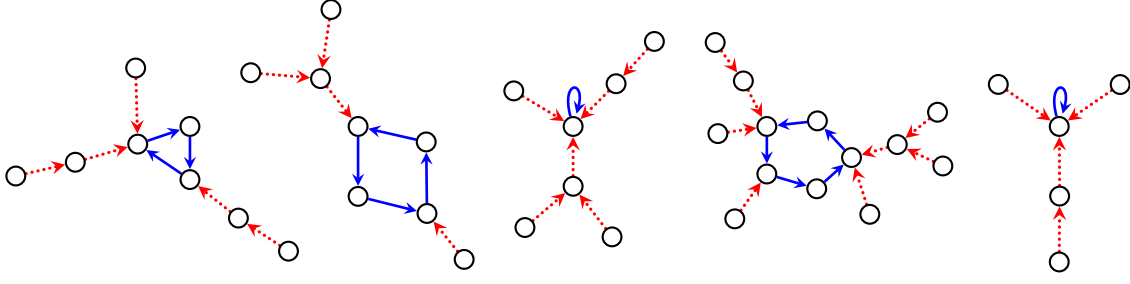


Figure 3: A tree-cyclic 1-graph. T-edges are red dotted edges, and C-edges are solid blue edges.

*Proof.* Use the relationship between the generating functions:

$$\det(I - xA) = f^*(x)^{-1} = \prod_{i=1}^n f_i^*(x)^{-1} = \prod_{i=1}^n (1 - f_i(x))$$

□

Note that  $\det(I - xA)$  is a polynomial, even though  $f_i(x)$  is a power series. If we take the coefficient of  $x^k$  of [Corollary 3.4.2](#), then modulo sign conventions we obtain precisely Theorems 1&2 from [\[Rot01\]](#) about clow sequences (clow sequences are equivalent to strictly upward routes). [Corollary 3.4.2](#) is equivalent to Theorem 2 from [\[Rot01\]](#), but is stated directly in terms of the polynomials, rather than in terms of their coefficients.

The main lemma follows as a corollary, which gives us a third proof of the lemma:

**Corollary 3.4.3.** Let  $A_G$  be the adjacency matrix of a 01-graph  $G$ , then

$$\det(I - A_G) = \begin{cases} 1 & \text{if } G \text{ is a forest} \\ 0 & \text{if } G \text{ has a cycle} \end{cases}$$

*Proof.* If  $G$  is a forest, then it has no strictly upward loops, so  $f_i(x) = 0$ , so  $\det(I - xA_G) = 1$ . If  $G$  has a cycle, let  $i$  be the lowest vertex on the cycle. Then  $f_i(x) = x^k$  where  $k$  is the length of the cycle. Now substitute  $x = 1$  to obtain  $\det(I - A_G) = 0$ . □

## 4 The matrix-tree-cycle theorem

We shall now generalize the matrix-tree theorem about  $\det(D + L)$  to the *matrix-tree-cycle theorem* by allowing  $D$  to be a general matrix. This theorem will express  $\det(D + L)$  as a sum over 1-graphs with each edge labeled either with T (for tree) or with C (for cycle) but not both

**Definition 4.1.** A TC-labeled 1-graph is called *tree-cyclic* if the T-edges form a forest, and each C-edge is part of a cycle.

[Figure 3](#) contains an example tree-cyclic graph.

**Definition 4.2.** The sign of a TC-labeled 1-graph  $G$  is:

$$(-1)^G = (-1)^{\# \text{cycles} + \# \text{C-edges}}$$

where  $\# \text{cycles}$  is the number of cycles of  $G$  and  $\# \text{C-edges}$  is the total number of C-labeled edges in  $G$ .

We associate a matrix  $M_G$  with a TC-labeled 1-graph  $G$ . This matrix will play the role that  $I - A_G$  played in the matrix-tree theorem.

**Definition 4.3.** The matrix  $M_G$  is defined as having columns:

$$(M_G)_i = \begin{cases} e_j & \text{if the outgoing edge } i \rightarrow j \text{ has label C} \\ e_i - e_j & \text{if the outgoing edge } i \rightarrow j \text{ has label T} \end{cases}$$

The following lemma generalizes the main lemma of the matrix-tree theorem.

**Lemma 4.1.** *The determinant of  $M_G$  is given by:*

$$\det(M_G) = \begin{cases} (-1)^G & \text{if } G \text{ is tree-cyclic} \\ 0 & \text{otherwise} \end{cases}$$

*Proof.* Calculate the determinant in steps:

1. Perform Laplace expansion on vertices with no predecessors. This makes the determinant zero if the successor is a C-edge, and deletes the corresponding vertex if the successor is a T-edge.
2. We are now left with a disjoint set of cycles. If there is a cycle consisting solely of T-edges, the determinant is zero because the corresponding rows sum to zero.
3. We are now left with a disjoint set of cycles where each cycle has at least one C-edge. Use the C-edges to turn the T-edges into C-edges by row operations. The determinant obtains a  $-1$  sign for each such switch.
4. We are now left with a disjoint set of cycles where each cycle consists solely of C-edges. In other words, a permutation matrix. The determinant of a permutation matrix is  $(-1)^{\#\text{cycles} + \#\text{edges}}$

If there was a C-edge not part of a cycle we have obtained 0 in step 1, and if there was a cycle among T-edges we have obtained 0 in step 2. For the remaining graphs, we have obtained a  $-1$  sign for each T-edge in the cycles, so together with  $(-1)^{\#\text{cycles} + \#\text{edges}}$  we are left with  $(-1)^{\#\text{cycles} + \#\text{C-edges}} = (-1)^G$ .  $\square$

**We now fix  $n \times n$  matrices  $A$  and  $D$  over some ring:**

- An arbitrary matrix  $A$  of weights for T-edges (with  $A_{ij}$  being the weight of edge  $i \xrightarrow{T} j$ )
- An arbitrary matrix  $D$  of weights for C-edges (with  $D_{ij}$  being the weight of edge  $i \xrightarrow{C} j$ ).
- As before, the Laplacian  $L$  is given by  $L_i = \sum_j A_{ij}(e_i - e_j)$ .

**Definition 4.4.** The weight of a tree-cyclic 1-graph  $G$  is:

$$w(G) = \prod_{\text{C-edge } (i \rightarrow j) \in G} D_{ij} \prod_{\text{T-edge } (i \rightarrow j) \in G} A_{ij}$$

We're ready to state the matrix-tree-cycle theorem.

**Theorem 4.2.** *The determinant  $\det(D + L)$  is the signed weight-sum of tree-cyclic graphs:*

$$\det(D + L) = \sum_{\text{tree-cyclic } G} (-1)^G w(G)$$

*Proof.* The  $i$ -th column of the matrix is  $(D + L)_i = \sum_j D_{ij}e_j + \sum_j A_{ij}(e_i - e_j)$ , so

$$\det(D + L) = \det \left( \begin{array}{c|c|c|c} D_{11}e_1 & D_{21}e_1 & & D_{n1}e_1 \\ + & + & & + \\ \vdots & \vdots & & \vdots \\ + & + & & + \\ D_{1n}e_n & D_{2n}e_n & \cdots & D_{nn}e_n \\ + & + & & + \\ A_{11}(e_1 - e_n) & A_{21}(e_2 - e_1) & & A_{n1}(e_n - e_1) \\ + & + & & + \\ \vdots & \vdots & & \vdots \\ + & + & & + \\ A_{1n}(e_1 - e_n) & A_{2n}(e_2 - e_n) & & A_{nn}(e_n - e_n) \end{array} \right) = \sum_{\text{TC-1-graph } G} w(G) \det(M_G) = \sum_{\text{tree-cyclic } G} (-1)^G w(G)$$

In the middle step we have again expanded the determinant by multilinearity: we sum over all possible ways to pick one term of each column. To choose a TC-labeled 1-graph on  $n$  vertices, is to choose for each vertex  $i$  (column  $i$ ) an outgoing edge  $i \rightarrow j$  and a label C (term  $D_{ij}e_j$ ) or T (term  $A_{ij}(e_i - e_j)$ ) for this edge. The final step is applying the lemma.  $\square$

The matrix-tree theorem is recovered by taking  $D$  to be diagonal. In that case, the number of cycles and the number of C-edges is equal, so  $(-1)^{\#\text{cycles} + \#\text{C-edges}} = 1$  and there are no signs involved. Each C-labeled self-loop corresponds to a root.

If we take  $A = 0$  we get the fact that  $\det(D)$  is the sum of signed cycle covers. A cycle cover of  $n$  vertices is a choice of edges such that each vertex has precisely one incoming and one outgoing edge, and its sign is the sign of the permutation that this graph depicts.

By taking  $D_{ij} = 1$  for some  $i, j$  and zero elsewhere, we also ensure that the sign  $(-1)^G = 1$ , because in this case we have one C-edge and one cycle. Thus,  $\det(D + L)$  will be the number of spanning trees rooted at  $i$ . The choice of  $j$  does not matter. By developing the determinant and using  $\det(L) = 0$  we see that this is equal to  $\det(L')$  where  $L'$  is obtained from  $L$  by deleting row  $i$  and column  $j$ .

By taking more off-diagonal entries of  $D$  to be nonzero, and by taking the corresponding rows and columns to be zero in  $A$  (and thus in  $L$ ), we obtain the all-minors matrix-tree theorem.

**Acknowledgements** Thanks to Darij Grinberg for pointing out mistakes and suggesting improvements, and for informing me about the relationship of [Corollary 3.4.2](#) to Theorems 1&2 of [\[Rot01\]](#).

## References

[Rot01] Gunter Rote. *Division-Free Algorithms for the Determinant and the Pfaffian: Algebraic and Combinatorial Approaches*, pages 119–135. Springer Berlin Heidelberg, 2001.