Methods of finding electric potential maps to interpolated flow data

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October 25, 2023

This document outlines three choices of fitting an electric potential map to a linearly interpolated, topside plasma flow field, $\mathbf{v} = \mathbf{E} \times \mathbf{B}/B^2$, where $\mathbf{B}$ is the magnetic field and $\mathbf{E}$ is the ionospheric electric field perpendicular to $\mathbf{B}$. This interpolation is sourced from a collection of distributed measurement points, whether in-situ or via some means of replication. The interpolated flow field used in this example is generated by the replication, translation, scaling, and rotation of Isinglass flow data. Its associated interpolated electric field reads as follows:

$$\mathbf{E}(x, y) = -\mathbf{v}(x, y) \times \mathbf{B} = \mathbf{E}_I + \mathbf{E}_S = -\nabla\phi(x, y) + \nabla \times \mathbf{A}(x, y) \tag{1}$$

We want to remove the non-electrostatic part, i.e. find the irrotational electric field, $\mathbf{E}_I$, and remove the solenoidal field, $\mathbf{E}_S$, in a way that best agrees with the interpolated flow field. Three choices of doing so are as follows:

1. **Potential fit**: A least-squares fitting algorithm (MATLAB's `lsqcurvefit` from its *Optimization Toolbox*) that fits a potential map, $\phi(x, y)$, such that

$$\min_\phi \|\nabla \times \mathbf{A}\|_2^2 = \min_\phi \|-\nabla\phi(x, y) - \mathbf{E}(x, y)\|_2^2 = \min_\phi \sum_{i,j} \Big(-(\nabla\phi)_{ij} - \mathbf{E}_{ij}\Big)^2 \tag{2}$$

2. **Averaged path-integrated**: The average of several path-integrated potential maps in an attempt to drown out the non-conservative part of the interpolated field, i.e.

$$\phi_i(x, y) = \int_{x_i}^{x} \mathrm{d}x' \mathbf{E}(x', y_i) \cdot \hat{x} + \int_{y_i}^{y} \mathrm{d}y' \mathbf{E}(x, y') \cdot \hat{y} \tag{3}$$

$$\tag{4}$$

where we take $\phi(x, y) = \langle \phi_i(x, y) \rangle$.

3. **Helmholtz decomposition**: We use the Fourier transform method for Helmholtz decomposition[1][2]. Taking the Fourier transform of Eq. (1) gives us

$$\mathbf{G}(k_x, k_y) = -i\mathbf{k}G_\phi(k_x, k_y) + i\mathbf{k} \times \mathbf{G}_\mathbf{A}(k_x, k_y) \tag{5}$$

where we define $\mathbf{G}(k_x, k_y)$ to be the Fourier transform of $\mathbf{E}$, i.e.

$$\mathbf{E}(x, y) = \mathcal{F}^{-1}\{\mathbf{G}(k_x, k_y)\} = \int_{-\infty}^{+\infty} \mathrm{d}k_x \int_{-\infty}^{+\infty} \mathrm{d}k_y \mathbf{G}(k_x, k_y) e^{i(xk_x + yk_y)} \tag{6}$$

[1] https://en.wikipedia.org/wiki/Helmholtz_decomposition#Derivation_from_the_Fourier_transform

[2] Pers. comm. A. Mule (2023)

Taking the dot product of $\mathbf{k}$ with equation (5) gives us that

$$G_\phi(k_x, k_y) = i\frac{\mathbf{k}\cdot\mathbf{G}(k_x, k_y)}{\|\mathbf{k}\|^2} \tag{7}$$

such that

$$\phi_0(x, y) = \mathcal{F}^{-1}\{G_\phi(x, y)\} = \int_{-\infty}^{+\infty} \mathrm{d}k_x \int_{-\infty}^{+\infty} \mathrm{d}k_y G_\phi(k_x, k_y) e^{i(xk_x + yk_y)} \tag{8}$$

where we've found the potential up to a harmonic function (both irrotational and solenoidal), i.e. $\phi = \phi_0 + f(x, y)$, where $\nabla^2 f = 0$. Our choice of f has the average electric field remain the same such that

$$f(x, y) = \langle -\nabla\phi_0 - \mathbf{E}(x, y)\rangle \cdot \mathbf{r} \tag{9}$$

where $\mathbf{E}$ is that from Eq. (1).

Along with the interpolated flow field (column 1), these three options are shown in Figure 1 (columns 2-4). The bottom row shows the divergence of the original interpolated flow field, the difference between the interpolated and potential fit's northward flow, and the difference between the potential fit and the averaged path-integrated and Helmholtz decomposed northward flows. Note the divergence panel indicates the location of rotational, interpretable as Alfvénic, signatures. The pros and cons with each method are as follows:

Method	**Pros**	**Cons**
Potential fit	Easiest to justify being the "best" fit.	Can take up to 5 hours on Discovery with 128×256 cells.
Averaged path-integrated	Easy to explain and faster than Potential fit.	Assumption of random divergence is poor and accumulates systematic error.
Helmholtz decomposition	Fastest of the three and it matches closest to Potential fit.	This method is still left up to a choice of a harmonic function, $f(x, y)$.

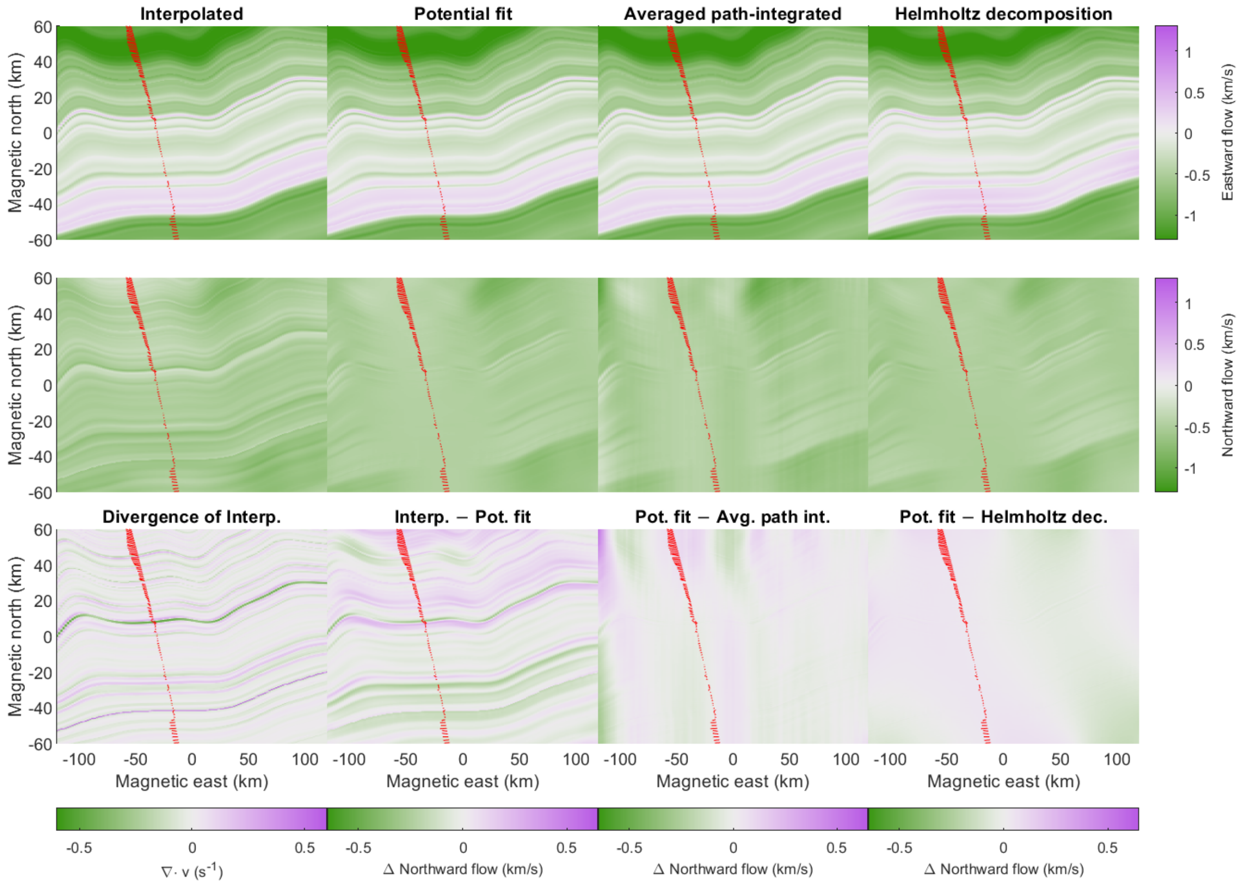

Figure 1: **Top row**: Eastward flow resulting from the interpolation and the three different potential finding methods. **Middle row**: Same as the top row but the northward flow. **Bottom row**: Divergence of the interpolated field and the difference between a) the potential fit and the interpolated field, b) the averaged path-integrated and the potential fit, and c) the Helmholtz decomposition and the potential fit. The original background removed version of the Isinglass flow data is shown in red.