Composite Precipitation Spectrum Usage in GEMINI

Jules van Irsel

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1 Derivation of accelerated bi-Maxwellian differential number flux

In order to implement the impact ionization calculations by Fang et al. (2010), we need the differential (as a function of energy) hemispherical number flux, i.e. electrons/eV/s/cm^2, at the topside of the ionosphere for every lat./lon. pair. To accurately derive this flux for an accelerated population we start with a bi-Maxwellian source at the plasmasheet as is done by Fridman and Lemaire (1980):

$$g_S(v_{\parallel,S}, v_{\perp,S}, \phi)\mathrm{d}^3v = n_{e,S}\left(\frac{m_e}{2\pi}\right)^{3/2}\frac{1}{E_{0,\parallel}^{1/2}E_{0,\perp}}\exp\left[-\frac{m_e v_{\parallel,S}^2}{2E_{0,\parallel}} - \frac{m_e v_{\perp,S}^2}{2E_{0,\perp}}\right]v_{\perp,S}\mathrm{d}v_\parallel \mathrm{d}v_\perp \mathrm{d}\phi. \tag{1}$$

As electrons precipitate down towards the ionosphere they undergo no collisions; their velocities change in two ways only (Knight, 1973; Fridman and Lemaire, 1980; Kaeppler, 2013):

1. The conservation of the first adiabatic invariant, i.e. the mirror force, increases their perpendicular velocity:

   $$v_{\perp,S} = \frac{1}{\sqrt{\beta}}v_{\perp,I}, \tag{2}$$

   where $\beta = B_I/B_S > 1$.

2. The conservation of energy increases the square magnitude speed as they fall through the parallel potential difference, U_0:

   $$v_{\parallel,I}^2 + v_{\perp,I}^2 = v_{\parallel,S}^2 + v_{\perp,S}^2 + \frac{2U_0}{m_e} \implies v_{\parallel,S} = \pm\sqrt{v_{\parallel,I}^2 + v_{\perp,I}^2\frac{\beta-1}{\beta} - \frac{2U_0}{m_e}}. \tag{3}$$

From here, we use Liouville's theorem which tells us that, along a well-defined path through phase space, e.g. $(\mathbf{x},\mathbf{v})_S \to (\mathbf{x},\mathbf{v})_I$, the phase space density is held constant such that

$$g_I(\mathbf{x}_I, \mathbf{v}_I) = g_S(\mathbf{x}_S, \mathbf{v}_S). \tag{4}$$

A good assumption is to say that we may separate spatial and velocity coordinates, $g(\mathbf{x},\mathbf{v}) = n(\mathbf{x})f(\mathbf{v})$, and that locally the densities are constants: $n_I(\mathbf{x}) = n_{e,I}$, $n_S(\mathbf{x}) = n_{e,S}$. This tells us

$$g_I(\mathbf{v}_I) = g_S(\mathbf{v}_S) = g_S(\mathbf{v}_S(\mathbf{v}_I)) \tag{5}$$

such that

$$g_I(v_{\parallel,I}, v_{\perp,I})\mathrm{d}^2v = n_{e,S}\frac{m_e^{3/2}/\sqrt{2\pi}}{E_{0,\parallel}^{1/2}E_{0,\perp}}\exp\left[-\frac{m_e\left(v_{\parallel,I}^2 + v_{\perp,I}^2\frac{\beta-1}{\beta} - \frac{2U_0}{m_e}\right)}{2E_{0,\parallel}} - \frac{m_e v_{\perp,I}^2/\beta}{2E_{0,\perp}}\right]\frac{v_{\perp,I}}{\sqrt{\beta}}\mathrm{d}v_\parallel \mathrm{d}v_\perp \tag{6}$$

where we've integrated over ϕ. The ionospheric density is thus

$$n_{e,I} = n_{e,S} \frac{E_{0,\parallel}\sqrt{\beta}}{E_{0,\parallel} + E_{0,\perp}(\beta - 1)} e^{\frac{U_0}{E_{0,\parallel}}}. \tag{7}$$

Note that as $U_0 \to 0$ and $E_{0,\parallel} \to E_{0,\perp}$ this gives a familiar $n_{e,I} = n_{e,S}/\sqrt{\beta}$. Now that we have the the velocity distribution function at the ionosphere, we find the differential number flux with $J_{\parallel,I}(\mathbf{v}_I)\mathrm{d}^3v = v_{\parallel,I}\, g_I(\mathbf{v}_I)\mathrm{d}^3v$ and then we perform the following change of coordinates:

$$v_{\parallel,I} = v\cos\theta = \sqrt{2E/m_e}\cos\theta, \tag{8}$$
$$v_{\perp,I} = v\sin\theta = \sqrt{2E/m_e}\sin\theta, \tag{9}$$

with Jacobian determinant, $|J| = 1/m_e$. This gives

$$J_{\parallel,I}(E,\theta)\mathrm{d}E\mathrm{d}\theta = \frac{n_{e,S}}{\sqrt{2\pi m_e}} \frac{1}{E_{0,\parallel}^{1/2} E_{0,\perp}} \frac{\sin 2\theta}{\sqrt{\beta}} E \exp\left[-\frac{E - U_0}{E_{0,\parallel}} - \left(\frac{E}{E_{0,\perp}} - \frac{E}{E_{0,\parallel}}\right)\frac{\sin^2\theta}{\beta}\right]\mathrm{d}E\mathrm{d}\theta. \tag{10}$$

With unit-less parameters $\varepsilon \equiv E/E_{0,\parallel}$, $u_0 \equiv U_0/E_{0,\parallel}$, and $\delta \equiv E_{0,\perp}/E_{0,\parallel}$, we get

$$\frac{\sqrt{m_e E_{0,\parallel}}}{n_{e,S}} J_{\parallel,I}(E,\theta) \equiv j_\parallel(\varepsilon,\theta) = \frac{\sin 2\theta}{\sqrt{2\pi\beta}} \frac{\varepsilon}{\delta} \exp\left[-(\varepsilon - u_0) - \left(\frac{\varepsilon}{\delta} - \varepsilon\right)\frac{\sin^2\theta}{\beta}\right]. \tag{11}$$

We find that the potential drop, u_0, simply scales the differential number flux by factor of e^{u_0}. The dependence on asymmetric source temperature is shown in Figures 1-2 for $\beta = 10$ and $\beta = 1000$.

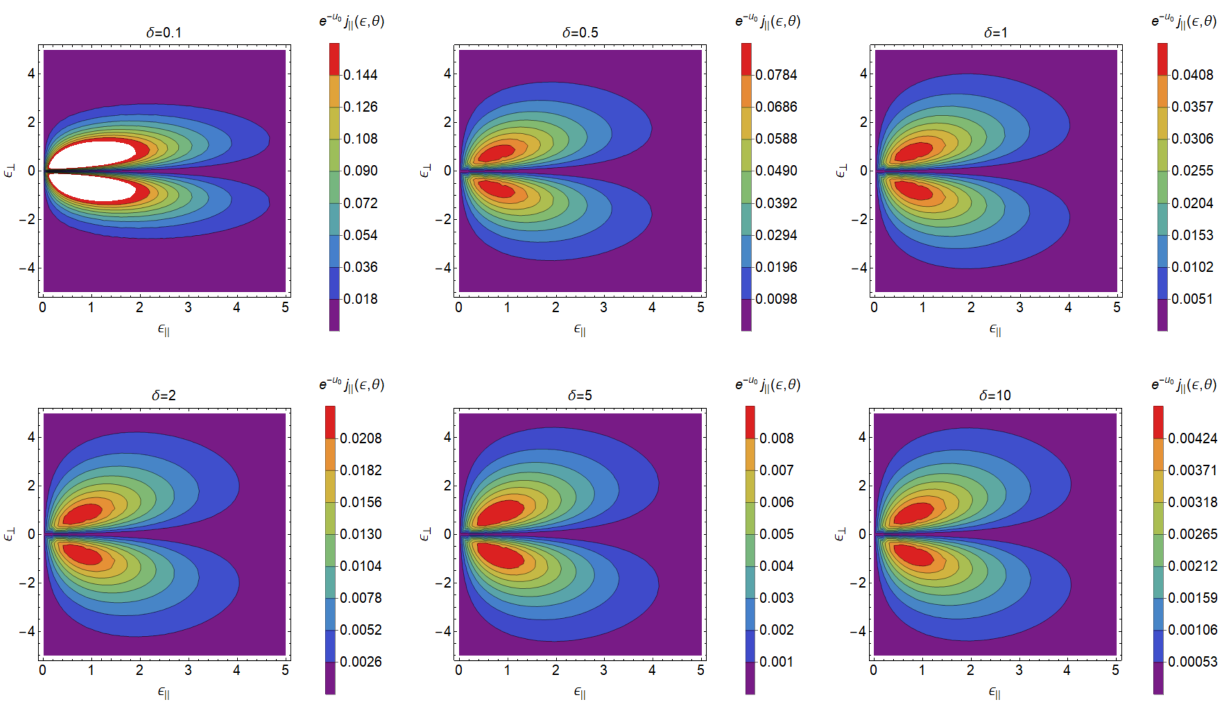

Figure 1: Normalized pitch angle number flux distributions for different values of $\delta \equiv E_{0,\perp}/E_{0,\parallel}$ with $\beta = 10$.

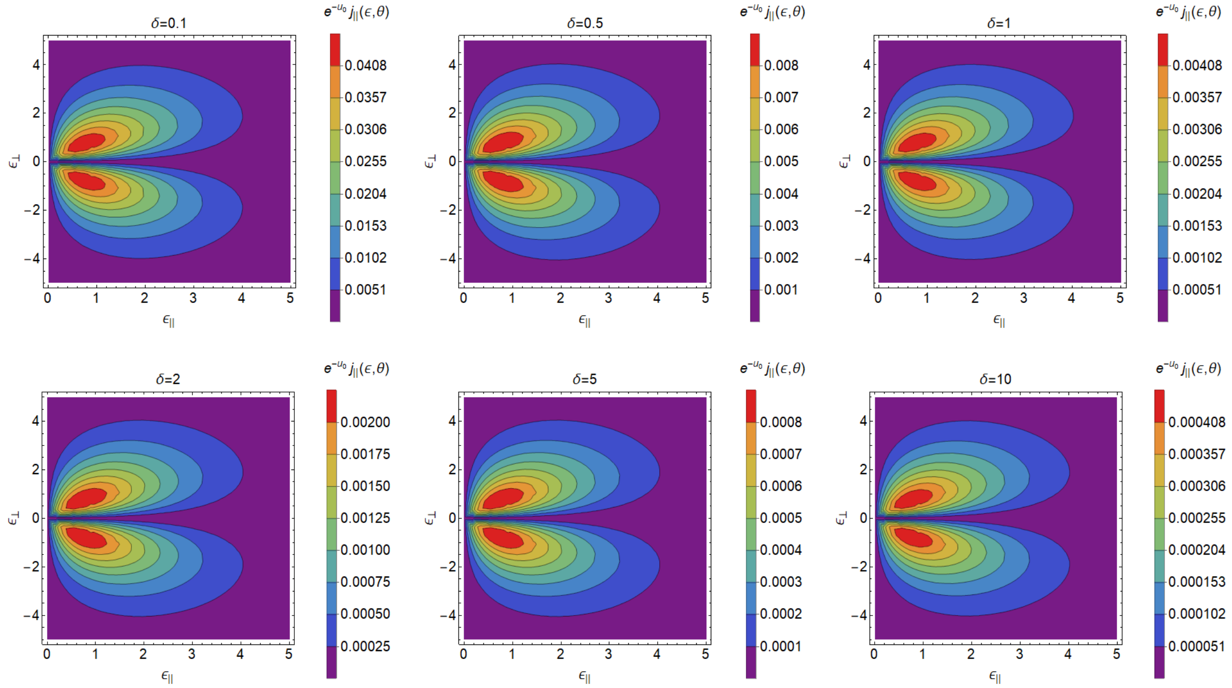

Figure 2: Same as Figure 1 but with a more realistic $\beta = 1000$.

We now integrate[1] over $v_{\|,I} > 0$, i.e. $0 \leqslant \theta \leqslant \pi/2$, and find the hemispherical differential number flux.

$$J_{\|,I}(\varepsilon)\mathrm{d}\varepsilon = n_{e,S}\sqrt{\frac{E_{0,\|}}{2\pi m_e}}\frac{1}{\delta\sqrt{\beta}}G\left(\frac{\delta-1}{\delta\beta}\varepsilon\right)\varepsilon e^{-\varepsilon+u_0}\mathrm{d}\varepsilon \tag{12}$$

where

$$G(x) \equiv \frac{e^x-1}{x}. \tag{13}$$

For similar parallel and perpendicular source temperatures, we have $\delta \sim 1$, and we have $\beta \sim 10^3$ for a plasmasheet source region (Fridman and Lemaire, 1980), where $G(x \ll 1) \to 1 + x/2 + \mathcal{O}(x^2)$ such that

$$J_{\|,I}(\varepsilon)\mathrm{d}\varepsilon \approx n_{e,S}\sqrt{\frac{E_{0,\|}}{2\pi m_e}}\frac{1}{\delta\sqrt{\beta}}\left(1+\frac{\delta-1}{2\delta\beta}\varepsilon\right)e^{-\varepsilon+u_0}\mathrm{d}\varepsilon \tag{14}$$

If we re-cast this in terms of normalized total precipitating energy flux, $q_0 \equiv Q_0/E_{0,\|}$, where

$$q_0 = \int_{u_0}^{\infty} \varepsilon J_{\|,I}(\varepsilon)\mathrm{d}\varepsilon, \tag{15}$$

we get

$$J_{\|,I}(\varepsilon)\mathrm{d}\varepsilon = q_0\frac{1+\chi\varepsilon}{2+6\chi+u_0(2+u_0+(6+u_0(3+u_0))\chi)}\varepsilon e^{-\varepsilon+u_0}\mathrm{d}\varepsilon \tag{16}$$

[1]No need to integrate over the loss cone; this physics has already been taken into account.

with

$$\chi = \frac{\delta - 1}{\delta\beta}.$$

Figure 3 shows the dependence of $J_{||,I}(\varepsilon)/q_0$ on δ for $u_0 = 3$ and $\beta = 10$, 1000. Clearly, in our regime of $\beta \sim 1000$ we may ignore the temperature difference at the source, so if we take the limit of $\delta \to 1$ we get a familiar result:

$$J_{||,I}(E,\theta)\mathrm{d}E\mathrm{d}\theta = \frac{Q_0}{E_0^2 + (E_0 + U_0)^2}\frac{E}{E_0}\exp\left(-\frac{E - U_0}{E_0}\right)\mathrm{d}E\sin 2\theta\mathrm{d}\theta \tag{17}$$

$$J_{||,I}(E)\mathrm{d}E = \frac{Q_0}{E_0^2 + (E_0 + U_0)^2}\frac{E}{E_0}\exp\left(-\frac{E - U_0}{E_0}\right)\mathrm{d}E \tag{18}$$

which is plotted in Figure 4. Figure 5 shows their respective pitch-angle distributions.

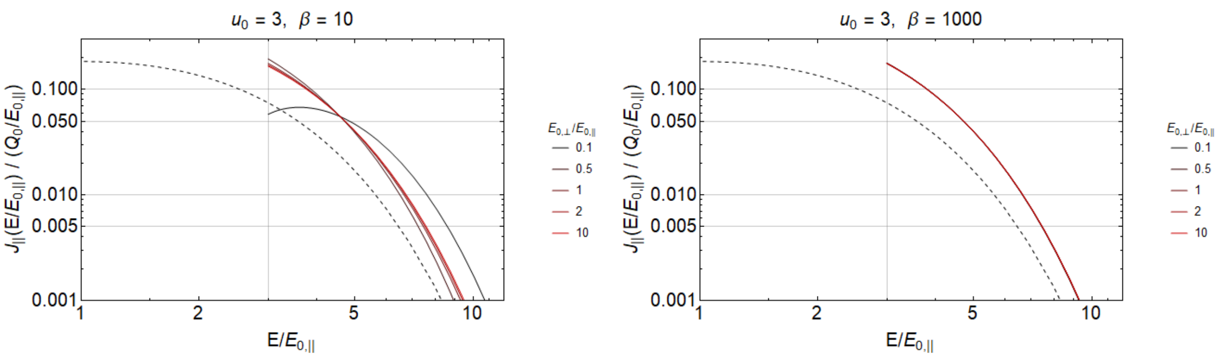

Figure 3: Normalized differential number flux with $u_0 = 3$, and $\beta = 10$ (left) and $\beta = 1000$ (right). Dashed lines represent Maxwellians, i.e. $u_0 = 0$, $\delta = 1$.

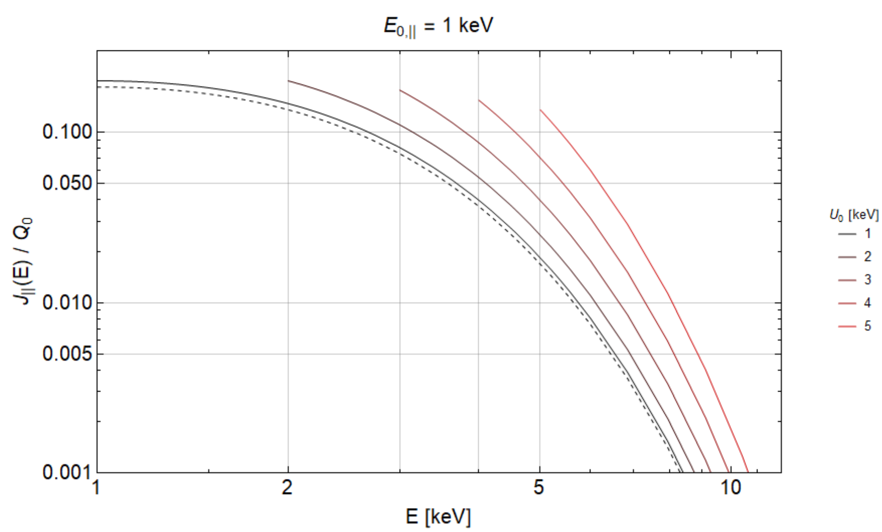

Figure 4: Differential number flux for an accelerated Maxwellian at different potential drops. Again, the dashed line is the non-accelerated Maxwellian.

These results have been concatenated from knowledge obtained in publications by Medicus (1961); Evans (1974); Fridman and Lemaire (1980); Strickland et al. (1989); Kaeppler (2013) as

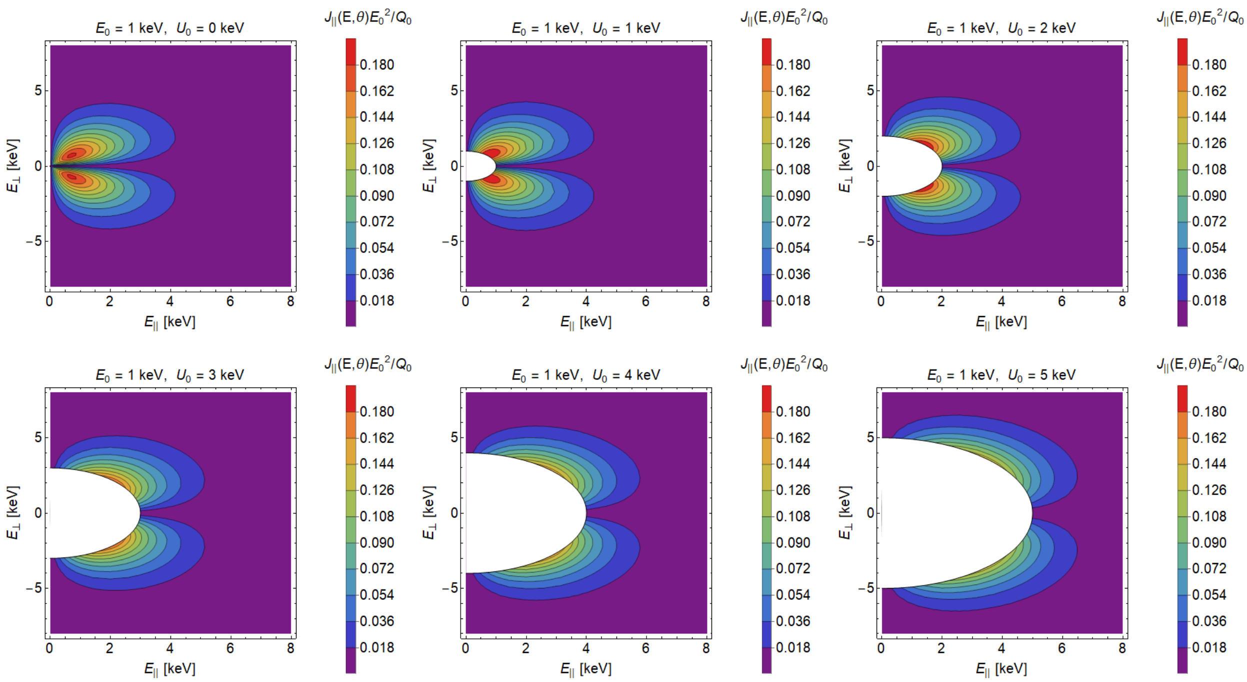

Figure 5: Pitch-angle distributions for accelerated Maxwellians for different acceleration potentials.

well as in depth discussions had with Stephen Kaeppler, Yi-Hsin Liu, Alex Mule, Meghan Burleigh, Matthew Zettergren, and Kristina Lynch. Thanks!

References

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