Fundamentals of Probabilistic Data Mining

Chapter II - Erratum on class exercise

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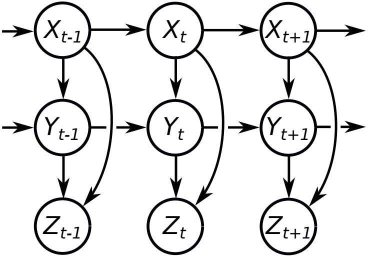
Ensimag/Inria

The model and the question

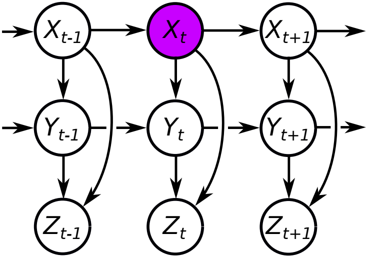
Given the following model,



the question in class was weather or not $\{X_{t-1}\}$ was D-separated from $\{X_{t+1}\}$ by X_t. I wrongly answered that is was not, when it actually is.

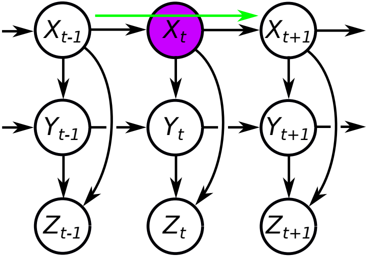
In this erratum we will see why, both graphically and algebraically.

Graphical answer



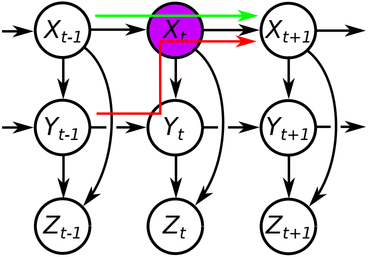
Let us consider all paths from $\{X_{t-1}\}$ to $\{X_{t+1}\}$.

Graphical answer



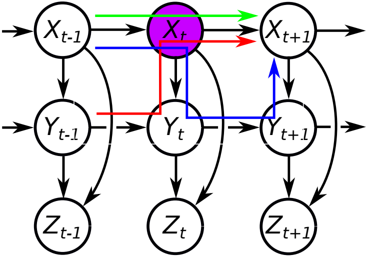
Let us consider all paths from $\{X_{t-1}\}$ to $\{X_{t+1}\}$.
Blocked by $\{X_t\}$ because head-to-tail at X_t.

Graphical answer



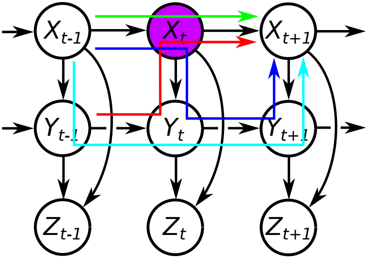
Let us consider all paths from $\{X_{t-1}\}$ to $\{X_{t+1}\}$.
Blocked by $\{X_t\}$ because head-to-tail at X_t.
Blocked by $\{X_t\}$ because tail-to-tail at X_t.

Graphical answer



Let us consider all paths from $\{X_{t-1}\}$ to $\{X_{t+1}\}$.
Blocked by $\{X_t\}$ because head-to-tail at X_t.
Blocked by $\{X_t\}$ because tail-to-tail at X_t.
Blocked by $\{X_t\}$ because head-to-tail at X_t.

Graphical answer



Let us consider all paths from $\{X_{t-1}\}$ to $\{X_{t+1}\}$.

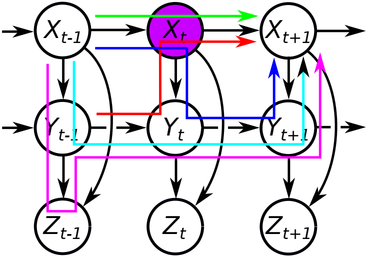
Blocked by $\{X_t\}$ because head-to-tail at X_t.

Blocked by $\{X_t\}$ because tail-to-tail at X_t.

Blocked by $\{X_t\}$ because head-to-tail at X_t.

Blocked by $\{X_t\}$ because head-to-head at Y_{t+1} (and Y_{t+1} is not in $\{X_t\}$).

Graphical answer



Let us consider all paths from $\{X_{t-1}\}$ to $\{X_{t+1}\}$.
Blocked by $\{X_t\}$ because head-to-tail at X_t.
Blocked by $\{X_t\}$ because tail-to-tail at X_t.
Blocked by $\{X_t\}$ because head-to-tail at X_t.
Blocked by $\{X_t\}$ because head-to-head at Y_{t+1} (and Y_{t+1} is not in $\{X_t\}$).
Blocked by $\{X_t\}$ because head-to-head at Z_{t-1} (and Z_{t-1} is not in $\{X_t\}$).
The same rule applies to any path going through any "Z".

Algebraic answer

The model writes:

$$p(x_{1:T}, y_{1:T}, z_{1:T}) = p(z_1|y_1, x_1)p(y_1|x_1)p(x_1)\times \prod_{t\geq 2} p(z_t|y_t, x_t)p(y_t|y_{t-1}, x_t)p(x_t|x_{t-1})$$

We seek for $p(x_{t+1}, x_{t-1}|x_t)$ and use the Bayes theorem for that

$$p(x_{t+1}, x_{t-1}|x_t) = \frac{p(x_{t+1}, x_{t-1}, x_t)}{p(x_t)}.$$

Now we "just" need to compute $p(x_{t+1}, x_{t-1}, x_t)$.

Albegraic answer (II)

Let's compute by marginalizing (I skipped the "d"):

$$p(x_{t+1}, x_{t-1}, x_t) = \int_{x_{1:t-2}, x_{t+2:T}, y_{1:T}, z_{1:T}} p(z_1|y_1, x_1)p(y_1|x_1)p(x_1) \times \prod_{t\geq 2} p(z_t|y_t, x_t)p(y_t|y_{t-1}, x_t)p(x_t|x_{t-1})$$

The z's integrate to one. Similarly for the y's.

$$p(x_{t+1}, x_{t-1}, x_t) = \int_{x_{1:t-2}, x_{t+2:T}} p(x_1) \prod_{t\geq 2} p(x_t|x_{t-1})$$

The x's from $t+2$ to T integrate to one as well, and the previous ones, I can write them as:

$$p(x_{t+1}, x_{t-1}, x_t) = p(x_{t+1}|x_t)p(x_t|x_{t-1}) \int_{x_{1:t-2}} p(x_1, \ldots, x_{t-1})$$

Albegraic answer (III)

The integral is easy to write:

$$p(x_{t+1}, x_{t-1}, x_t) = p(x_{t+1}|x_t)p(x_t|x_{t-1})p(x_t)$$

Going back to Bayes:

$$p(x_{t+1}, x_{t-1}|x_t) = \frac{p(x_{t+1}, x_{t-1}, x_t)}{p(x_t)} = \frac{p(x_{t+1}|x_t)p(x_t|x_{t-1})p(x_{t-1})}{p(x_t)}$$

which is:

$$p(x_{t+1}, x_{t-1}|x_t) = p(x_{t+1}|x_t)p(x_{t-1}|x_t),$$

which obviously aligns with the graphical result.