MATH 211: 3/28 WORKSHEET

A series
$$\sum_{n=1}^{\infty} a_n$$
converges if and only if its sequence of partial sums converges. Unrolling the definition of convergence of a sequence, this says
$$\begin{aligned}\sum_{n=1}^{\infty} a_n &= \operatorname{st}\left(\sum_{n=1}^{N} a_n\right) \qquad (N \text{ is any positive infinite hyperinteger}) \\ &= \lim_{b\to\infty} \sum_{n=1}^{b} a_n\end{aligned}$$
If this standard part/limit is undefined, then we say the series diverges.

Analogy: You can think of series as being like discrete integrals: $\sum_{n=1}^{\infty} a_n$ behaves a lot like $\int_a^b f(x)\,\mathrm{d}x$.

Algebra Rules. Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, and c is a constant.

- (Sum rule) $\displaystyle\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$
- (Difference rule) $\displaystyle\sum_{n=1}^{\infty}(a_n s b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$
- (Constant rule) $\displaystyle\sum_{n=1}^{\infty} c\, a_n = c\sum_{n=1}^{\infty} a_n$
- (Inequality rule) If each $a_n \le b_n$ then $\displaystyle\sum_{n=1}^{\infty} a_n \le \sum_{n=1}^{\infty} b_n$.

Theorem. A series $\sum_{n=1}^{\infty} a_n$ converges if and only if any of its *tail series* $\sum_{n=k}^{\infty} a_n$ converge (k is any positive integer).

Corollary. Convergence of a series is not affected by adding, removing, or changing finitely many terms.

Everything on this page holds if the series starts at an index besides $n = 1$.

Calculate the value of the following series. [Hint: recall from Wednesday the formula for the sum of a geometric series.]

(1) $1 + \frac{1}{3} + \frac{1}{9} + \cdots + \frac{1}{3^n} + \cdots$
(2) $\sum_{n=3}^{\infty} \frac{1}{5^n}$
(3) $2 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \cdots$
(4) $-3 + 7 + 1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{10^n}$

Prove the theorem and corollary on the previous page by following these steps.

(1) Find the value by which the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=k}^{\infty} a_n$ differ.
(2) Assuming $\sum_{n=1}^{\infty} a_n = S$, what value does $\sum_{n=k}^{\infty} a_n$ converge to?
(3) Assuming $\sum_{n=k}^{\infty} a_n = S$, what value does $\sum_{n=1}^{\infty} a_n$ converge to? (Note the swapped starting indices!)
(4) Why does this give the theorem?
(5) For the corollary, explain why if a series $\sum_{n=1}^{\infty} b_n$ is obtained from $\sum_{n=1}^{\infty} a_n$ by adding, removing, or changing finitely many values then there are k and ℓ so that $a_{k+i} = b_{\ell+i}$ for all $i \geq 0$.
(6) Explain why this means $\sum_{n=k}^{\infty} a_n = \sum_{n=\ell}^{\infty} b_n$, and why that means the two series have the same convergence property.

You can check the algebra rules by hand.

(1) Check the sum rule by letting $\langle S_n \rangle$ and $\langle T_n \rangle$ be the sequences of partial sums of the a_n and b_n series, then calculating the sequence of partial sums of the $a_n + b_n$ series. Note which algebra rule for standard parts/limits you used.
(2) Show that the assumption the two series converge is necessary for the sum rule by giving an example of series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ which each diverge yet $\sum_{n=1}^{\infty} a_n + b_n$ converges.
(3) Do a similar process to check the difference rule.
(4) Do a similar process to check the constant rule.
(5) Do a similar process to check the inequality rule.