Math 321: The theory of games

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An example

Let's play the game Twenty-One. There's two players, who take turns counting up to twenty-one, starting at one. On each turn you can say the next one, two, or three numbers, no fewer and no more. The winner is whomever says twenty-one.

Twenty-One

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Here's the strategy: you want to end your turn on 1, 5, 9, 13, 17, or ultimately 21. If you end your turn on one of these, then your opponent can only add 1, 2, or 3 to it, allowing to respond by ending on the next number on the list. Since you can end on 1 for your first turn, this means you can put yourself in a winning position and then win by ensuring you never leave this position. □

A generalized example

Let's generalize. Instead of counting up by up to 3 with a goal of 21, we could make the step size and goal any natural numbers. Let's call the game with goal n and step size s as $G(n, s)$, so Twenty-One was $G(21, 3)$.

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The winning startegy, in either case, is the same: you want to end your turn on the numbers with the same remainder r as n divided by $s + 1$.

$$r, s + 1 + r, 2s + 2 + r, \dots, n$$

Because these are spaced out by exactly $s + 1$, no matter what move your opponent plays you can respond to stay in your winning position.

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If n is a multiple of $s + 1$, this remainder is 0, so the first player cannot end their first turn in a winning position, so it is the second player who can force to be in a winning position. Otherwise, the remainder is $1 \leq r \leq s$, so the first player can get in a winning position on the first move. $\square$

Buckets of fish

Recall the buckets of fish game we talked about earlier as part of an example of an inductive proof. There are finitely many buckets arranged in a row, and each starts with some finite number of fish. Each turn, a player removes a fish from one bucket and puts as many new fish as they like in any of the buckets to its left. The winner is whomever takes the last fish.

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We saw that any game of buckets of fish must eventually end, but must it be that one of the two players has a winning strategy? Or could it be that for every game each player has a shot at winning no matter how their opponent plays?

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Proof.

Observe that after you take the last fish, all buckets have 0 fish, an even number for each.

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Proof.

Observe that after you take the last fish, all buckets have 0 fish, an even number for each. Next, notice that if there's a bucket with an odd number of fish, then you can take one fish from the right-most odd bucket, making it even, and add fish to more left odd buckets to make them odd. So you can get back to the winning position. On the other hand, when your opponent faces an all even setup, because they have to take only one fish from a bucket, they make it odd, keeping them out of the winning position.

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Can we generalize?

- In all the examples we've seen so far, it was always the case that someone has a winning strategy. (And the textbook has more examples of games with winning strategies.) Can we generalize this? What sort of general statement can we make about when a game has a winning strategy?

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- (Games with > 2 players) Consider the game with three players A, B, and C, which has one inning: Player A decides which of B or C wins, and then the game ends. No one has a winning strategy here.
- (Games with draws) In tic-tac-toe any player can force a draw, so there is no winning strategy. Or a simpler example: the game where each player does nothing and then it ends in a draw.

Can we generalize?

- (Games with randomness) In general, there's no way to handle this. You can investigate strategies that have high probability of winning, and this is an area of ongoing mathematical investigation. but that's taking us away from the question of a guaranteed winning strategy—and also it gets really hard fast—so let's not consider these.

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- (Games with > 2 players) These are also hopeless for a general theory, so let's exclude these two.
- (Games with draws) Here, maybe we can amend things to say that either a player has a winning strategy or everyone can force a draw.