

## MATH 113: 4/23 WORKSHEET

### MODAL LOGIC AND FRAME CONDITIONS

(This material corresponds to chapter 44 of the textbook.)

Two weeks ago we learned about different systems of modal logic. Each extends the rules for TFL by adding new rules for the modal symbols.

- $K$  has all the rules of TFL plus:
  - (Necessitation Rule) If  $\varphi$  is a theorem (that is,  $\vdash \varphi$ ) then  $\Box\varphi$  is also a theorem ( $\vdash \Box\varphi$ ).
  - (Distribution Axiom)  $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$  is always a theorem.
  - (Duality Axioms)  $\neg\Box\varphi \rightarrow \Diamond\neg\varphi$  and  $\neg\Diamond\varphi \rightarrow \Box\neg\varphi$  are always theorems.
- $T$  has all the rules of  $K$  plus:
  - ( $M$  Axiom)  $\Box\varphi \rightarrow \varphi$  is always a theorem.
- $S4$  has all the rules of  $T$  plus:
  - (4 Axiom)  $\Box\varphi \rightarrow \Box\Box\varphi$  is always a theorem.
- $S5$  has all the rules of  $S4$  plus:
  - (5 Axiom)  $\Diamond\varphi \rightarrow \Box\Diamond\varphi$  is always a theorem.

To distinguish entailment in the different systems we put a subscript on the turnstile, e.g.  $\vdash_K$  or  $\vdash_{S4}$ .

These systems correspond to different conditions on frames. For each system, semantic entailment  $\models A$  means “for every frame satisfying such-and-such condition and any interpretation (= assignment of truth values to worlds) on that frame,  $A$  is true at every world”.

**The system  $K$ .**

This system corresponds to frames as a whole.  $\models_K A$  means that if you take any frame and any interpretation on that frame, then  $A$  is true at every world in that frame.

**The system  $T$ .**

This system corresponds to frames whose accessibility relation is *reflexive*.  $\models_T A$  means that if you take any reflexive frame and any interpretation on that frame, then  $A$  is true at every world in that frame.

**The system  $S4$ .**

This system corresponds to frames whose accessibility relation is *reflexive* and *transitive*.  $\models_{S4} A$  means that if you take any reflexive and transitive frame and any interpretation on that frame, then  $A$  is true at every world in that frame.

**The system  $S5$ .**

This system corresponds to frames whose accessibility relation is an *equivalence relation* (i.e. *reflexive*, *transitive*, and *symmetric*).  $\models_{S5} A$  means that if you take any equivalence relation frame and any interpretation on that frame, then  $A$  is true at every world in that frame.

For each of these systems, syntactic entailment is equivalent to semantic entailment:

$$P \models_* C \quad \text{if and only if} \quad P \vdash_* C,$$

where  $*$  is any of the four systems.