MATH655 LECTURE NOTES: PART 1.1 MEASURABLE CARDINALS

KAMERYN J. WILLIAMS

We next turn to the most important large cardinal notion in set theory, namely that of the measurable cardinal. If you want some historical background to their development, which originates with Banach's measure problem, then Chapter 1 of Kanamori's *The Higher Infinite* has a nice discussion. I, however, will not touch on the history.

1. Measurable cardinals

We will need some definitions first.

Definition 1. Let U be an ultrafilter on a set D and let α be a cardinal. Say that U is *α-complete* if given any sequence $\langle A_\xi : \xi < \beta \rangle$ with $\beta < \alpha$ of sets from U, we have that $\bigcap_{\xi<\beta} A_\xi \in U$.

Note that α-complete means the ultrafilter is closed under intersections of sequences of *strictly fewer than* α many sets. This is the typical naming convention in set theory, with α-foo means that foo holds for things smaller than α. If we also want to include things of size α, then we talk about α^+-foo. If we went with the other convention of α-foo meaning $\leq\alpha$-foo, then how would we express $<\lambda$-foo for limit cardinals λ?

Exercise 2. Show that an ultrafilter U is α-complete iff for any $\beta < \alpha$ and any sequence $\langle A_\xi : \xi < \beta \rangle$ of sets so that $\bigcup_{\xi<\beta} A_\xi \in U$, there is $\xi < \beta$ so that $A_\xi \in U$.

For principal ultrafilters, completeness is trivial. So our only interest is in nonprincipal ultrafilters.

Exercise 3. Show that if U is a principal ultrafilter than U is α-complete for every ordinal α.

Observe that if U is a nonprincipal α-complete ultrafilter then every $A \in U$ must have $|A| \geq \alpha$. Otherwise, if $|A| < \alpha$ then A would be the union of $< \alpha$ many singletons. So by the exercise there would be a singleton in U, contradicting the nonprincipality of U.

Observe that every ultrafilter is ω-complete, by a simple induction. And ω_1-completeness is another name for countable completeness.

Exercise 4. Show that there is a correspondence between ω_1-complete ultrafilters on D and $\{0, 1\}$-valued measures on the σ-algebra $\mathcal{P}(D)$.[1]

It is also easy to see that nontrivial ultrafilters have an upper bound on their completeness.

Exercise 5. Suppose that U is a nonprincipal ultrafilter on D.[2] Show that U is not $|D|^+$-complete.

Measurable cardinals are those which admit ultrafilters as complete as possible.

Date: February 6, 2019.

[1] Here "measure" is in the sense of measures from measure theory, i.e. functions from a σ-algebra to $[0, \infty]$, satisfying blah blah. This is not the same as the measure we will soon define.

[2] Recall that an ultrafilter U is principal if there is $d \in D$ so that $A \in U$ iff $d \in A$.

Definition 6 (Ulam)**.** An uncountable cardinal κ is *measurable* if there is a nonprincipal κ-complete ultrafilter on κ. Such an ultrafilter is known as a *measure*.

Note that if we didn't require κ to be uncountable, then ω would satisfy the definition. There's two reasons for this restriction. First, it would be awkward to continually have to say things like "ZFC does not prove there is a measurable cardinal, besides ω". Second, we will soon see an alternative characterization of measurable cardinals which does not hold for ω. Since this alternative characterization is the really important fact about measurable cardinals, we want the definition to exactly pick them out.

But first let us see that measurable cardinals are large cardinals.

Theorem 7 (Ulam and Tarski)**.** *If κ is measurable then κ is inaccessible. So* ZFC *cannot prove the existence of measurable cardinals.*

Proof. Fix U a measure on κ. We must see that κ is regular and strong limit.

(κ is regular) Suppose otherwise. Then $\kappa = \bigcup_{\xi<\alpha} \lambda_\xi$ for $\lambda_\xi < \kappa$ and $\alpha < \kappa$. As we observed above, $\lambda_\xi \notin U$. So $\kappa = \bigcup_{\xi<\alpha} \lambda_\xi \notin U$, by κ-completeness. This contradicts that $\kappa \in U$.

(κ is strong limit) Suppose otherwise that $f : \kappa \to {}^\lambda 2$ is injective for $\lambda < \kappa$. By κ-completeness, for each $\alpha < \lambda$ there $i_\alpha < 2$ so that

$$A_\alpha = \{\xi < \kappa : f(\xi)(\alpha) = i_\alpha\} \in U.$$

Thus $A = \bigcap_{\alpha<\lambda} A_\alpha \in U$. But note that if $\xi \in A$ then $f(\xi)(\alpha) = i_\alpha$ for all $\alpha < \lambda$. Because f is one-to-one, we thereby conclude that A has at most one member, which is impossible. □

This result was first published in 1930. A natural next question is: can the least inaccessible cardinal be measurable? Or must it be that below any measurable cardinal there are inaccessible cardinals? This question would remain open for thirty years, until it was settled by Hanf and Tarski. We will not see their proof, but will instead derive this fact as a corollary of Scott's seminal characterization of measurable cardinals, which is our next topic.

Recall that one use of ultrafilters is to construct ultrapowers. Scott's idea was to use this to take an ultrapower of the universe of sets V. If our ultrafilter is a measure on some uncountable cardinal κ, then the resulting ultrapower is well-founded and set-like. So we can take its transitive collapse, and get an elementary embedding of V into an inner model.

Our next topic to explicate this paragraph.

2. Elementary embeddings of the universe

There is a bit to work out here, to make sure all this makes sense when talking about proper class sized structures.

We start by discussing a bit of notation. To illustrate the reason for the notation: Many objects we refer to by definitions. For example, ω_1 is the least uncountable ordinal, $V_{\omega+\omega}$ is set obtained from iterating the powerset operation $\omega + \omega$ times starting from $\emptyset$, and so on. When we are dealing with different transitive models, there is no guarantee that they agree on these definitions, even if they have some sets in common. We will see striking examples of this later; for example, it's possible to have an inner model M so that the ordinal M thinks is ω_1 is what V sees as a countable ordinal, while the ordinal V thinks is ω_1 is thought by M to be an inaccessible cardinal.

So we need a way to distinguish from which context we are making a definition. The convention here is to use superscripts for this. If we don't write a subscript, we mean the definition is made from V. So ${\omega_1}^M$ denotes the set which M thinks satisfies the definition of ω_1. We can also use parameters in definitions. For example, if κ is some fixed ordinal, then $\mathcal{P}(\kappa)^M$ denotes the set which M thinks is the collection of subsets of κ. Let's think a bit about this example. Since being an ordinal and $x \subseteq y$ are Δ_0, it must be that every element of $\mathcal{P}(\kappa)^M$ is actually a subset of κ. So $\mathcal{P}(\kappa)^M \subseteq \mathcal{P}(\kappa)$.

In general, we can often use absoluteness facts to extract some facts about sets defined in some transitive model. While M may be wrong about, say, what set is ω_1, it is correct that ${\omega_1}^M$ is an ordinal.

As a warmup, let us first consider embeddings with only a limited amount of elementarity. We will specialize to the case of transitive sets and classes.

Definition 8. Let M and N be either transitive sets or transitive classes. and let k be a finite ordinal. Then an embedding $j : M \to N$ is Σ_k-*elementary*, written $j : M \prec_k N$, if for all $a_0, \ldots, a_n \in M$ and all Σ_k formulae $\varphi(x_0, \ldots, x_n)$ we have $M \models \varphi(a_0, \ldots, a_n)$ iff $N \models \varphi(j(a_0), \ldots, j(a_n))$. That is, we only require agreement for truth about the Σ_k formulae, unlike full elementarity where we require agreement for truth about all formulae.

Observe that Σ_k-elementarity immediately implies Π_k elementarity. (Exercise: write the one line argument for this!)

If M and N are sets, then it's clear this notion is actually definable. We have already seen that for set-sized structures the Tarskian satisfaction relation exists, and so we can query the satisfaction relations for M and N to check for agreement among the Σ_k assertions.

On the other hand, it was an exercise for you to prove that for proper class sized structures that the Tarskian satisfaction relation need not exist. (Or to speak explicitly in linguistic terms: our way of speaking: a formula $\varphi(x)$ may define a class-sized structure without there being some other formula $\sigma(x, y)$ defining the satisfaction relation for the structure defined by $\varphi(x)$.) So there is something to be checked here.

First, let us check it makes sense to talk about embeddings between transitive sets. This is fairly straightforward. Suppose $\mu(x)$ defines transitive M,[3] $\nu(x)$ defines transitive N, and $\gamma(x, y)$ defines a function j. Then, we say that j is an embedding from M to N by using the usual definition, but subbing in the formulae in appropriate places. To be precise, this can be formulated as:

$$\begin{gathered}\forall x\ [\mu(x) \Rightarrow \exists! y\ \nu(y) \wedge \gamma(x, y)] \\ \wedge \forall x, x'\ [(\mu(x) \wedge \mu(x') \wedge x \in x') \Rightarrow \forall y, y'\ (\gamma(x, y) \wedge \gamma(x', y') \Rightarrow y \in y')].\end{gathered}$$

Next, it must needs be remarked what we mean by $M \models \varphi$ for transitive proper class M. We cash this out as an abbreviation for asserting φ^M, where φ^M is obtained from φ by restricting all quantifiers to range over M. More formally, φ^M is defined according to the following recursion, where $\mu(x)$ is the formula defining M:

- $(a \in b)^M$ is $a \in b$ and $(a = b)^M$ is $a = b$;
- $(\varphi \wedge \psi)^M$ is $\varphi^M \wedge \psi^M$, and similarly for disjunctions and negations;
- $(\exists x \in a\ \varphi(x))^M$ is $\exists x \in a\ (\varphi(x))^M$, and similarly for bounded universal quantification; and
- $(\exists x \varphi(x))^M$ is $\exists x\ \mu(x) \wedge (\varphi(x))^M$, and similarly for unbounded universal quantification.

Note that since M is assumed to be transitive we don't have to add the restriction to bounded quantifiers, if $a \in M$, meaning $\mu(a)$, then any $x \in a$ must satisfy $\mu(x)$.

So this gives us a way to make sense $M \models \varphi$ for an individual φ when M is a proper class. But the induction for how we make sense of this took place in the meta-theory; there is no uniform way

[3]That is, $\mu(x)$ and $y \in x$ implies $\mu(y)$.

to express $M \models \varphi$ for all formulae φ. (Else we could define the full satisfaction class, contradicting Tarski's theorem on the undefinability of truth.) Next let's see that we can get definable express satisfaction for Σ_k formulae.

Theorem Schema 9. *Fix a transitive class M and a finite number k. Then the relation $M \models \varphi(\bar{a})$ restricted to Σ_k formulae is definable.*

Note that this is a theorem schema, with instances for all transitive classes M and all finite numbers k. The definitions for the satisfaction relation for more and more complex formulae themselves get more and more complex.

Proof. We prove this by induction on k in the meta-theory. Start with $k = 0$. Then we define $M \models \varphi(\bar{a})$ restricted to the Σ_0 formulae as:

$$M \models^0 \varphi(\bar{a}) \Leftrightarrow \exists y \in M \ y \text{ is transitive and } \bar{a} \in y \text{ and } y \models \varphi(\bar{a}).$$

Because Σ_0 properties are absolute between transitive models, $y \models \varphi(\bar{a})$ iff $V \models \varphi(\bar{a})$ iff $\varphi(\bar{a})^M$. Now suppose we have defined $M \models^n \varphi(\bar{a})$ for Σ_k formulae φ. To get the definition for Σ_{k+1} formulae $\exists \bar{x} \ \neg\varphi(\bar{x}, \bar{a})$ where φ is Σ_k, define

$$M \models^{n+1} \exists \bar{x} \ \neg\varphi(\bar{x}, \bar{a}) \Leftrightarrow \exists \bar{x} \ \neg(M \models^n \varphi(\bar{x}, \bar{a})). \qquad \square$$

The upshot of all this is that we can express that $j : M \to N$ is a Σ_k elementary embedding from M to N. (Remark: it is good to keep in mind the formal apparatuses behind all this. But the way to work with these in practice is to just think of the standard satisfaction relation, but remembering that we can only talk about the truth of formulae with a bounded quantifier complexity.)

Definition 10. Let k be a natural number. Let ZFC_k be the theory obtained from ZFC by restricting the Separation and Replacement schemata to Σ_k-formulae. Similarly ZF_k is obtained from ZF.

Observe that, since the axioms of ZFC_k have bounded quantifier depth, we can definably express $M \models \mathsf{ZFC}_k$ for transitive classes M.

It turns out that many basic theorems of ZFC are provable just from ZFC_1. For example, the Mostowski collapse theorem, the theorem that being well-founded is equivalent to the existence of ranking functions, and the theorem that there is a rank function $V \to \mathrm{Ord}$ measuring the rank of a set with respect to $\in$, are all theorems of ZFC_1. I shan't prove these claims I just made; doing so would require a more careful analysis of how we got transfinite recursion from Replacement, carefully counting up how many quantifiers we need for everything, and a close look at what instances of Separation we used.

If you are skeptical of my claim here—a very reasonable attitude for students to have!—you have two options. The first is to do these more careful arguments yourself (or go look in a textbook, if you're that way...). This is a good exercise for really understanding those theorems. The second option is to observe that for each of these theorems, we only used finitely many axioms of ZFC—because proofs are finite. So there is *some* m for which ZFC_m is enough to prove all these theorems. Since in the sequel we will be dealing with transitive classes that satisfy ZFC_k for every k, it's not harmful if you don't actually compute the minimal m which works—but you have to edit the following proposition to replace the 1s with ms. I opted to give the precise bound, namely $m = 1$, but stated without proof.

Proposition 11. *Suppose $j : M \prec_1 N$ is a Σ_1-elementary embedding between transitive classes, where M and N both contain all ordinals.*

(1) *For all ordinals* α, $j(\alpha)$ *is an ordinal and* $j(\alpha) \geq \alpha$.
(2) *Suppose* j *is not the identity, that* $M \models \mathsf{ZF}_1$, *and that* $N \subseteq M$. *Then there is some ordinal* α *so that* $j(\alpha) > \alpha$.
(3) *Suppose* j *is not the identity and that* $M, N \models \mathsf{ZFC}_1$. *Then there is some ordinal* α *so that* $j(\alpha) > \alpha$.
(4) *Fix a natural number* k. *Suppose* $M, N \models \mathsf{ZFC}_k$. *Then,* $j < M \prec_k N$.

Proof. (1) Being an ordinal is a Σ_0 property, so by elementarity j must preserve ordinals. Now prove $j(\alpha) \geq \alpha$ by an easy induction.

(2) Let x be of minimal rank so that $j(x) \neq x$. Such exists because j is not the identity. Observe that $x \subseteq j(x)$, because by minimality of x we have for $y \in x$ that $y = j(y) \in j(x)$. Pick $z \in j(x) \setminus x$. If $\mathrm{rank}(j(x)) = \mathrm{rank}(x)$ then, because $z \in N \subseteq M$ and $\mathrm{rank}\, z < \mathrm{rank}\, x$, we get that $j(z) = z \in j(x)$. But then by elementarity, $z \in x$, a contradiction. So if $\alpha = \mathrm{rank}(x)$ then $j(\alpha) = \mathrm{rank}(j(x)) > \alpha$, as desired.

(3) Suppose towards a contradiction that $j(\alpha) = \alpha$ for all ordinals α. Observe that it is a Σ_0 property to be a set of ordinals, so if x is a set of ordinals then $j(x)$ is a set of ordinals. But since the ordinals are fixed, $j(x) = x$ for all sets of ordinals.

Now fix arbitrary $a \in M$ and let $t = \mathrm{tc}(\{a\})$. By the axiom of choice in M, there is a bijection f between some ordinal γ and t. Now define a relation E on γ as $(\alpha, \beta) \in E$ iff $f(\alpha) \in f(\beta)$, i.e. E is an isomorphic copy of the membership relation restricted to t. Using the Gödel pairing function, which is Σ_0-definable, we can identify E with a set of ordinals. So by the observation of the previous paragraph, $j(E) = E$. Because $N \models \mathsf{ZFC}_1$, in N we can take the Mostowski collapse of $j(E) = E$ to the membership relation restricted to a transitive set $\bar{t}$. Because being the Mostowski collapse of a well-founded extensional relation is a Σ_1 property—there exists a function such that blah blah—V would also think that $\bar{t}$ is the Mostowski collapse of E. But V must agree with M that the transitive collapse is to t, again because this is Σ_1, and so $\bar{t} = t$. It then follows by Σ_1-elementarity of j that $j(t) = t$. But it's now clear that $j(a) = a$, since a is Σ_0 definable from t.

But since a was arbitrary, we get that j is the identity, a contradiction.

(4) We prove this by induction in the metatheory. Suppose we have inductively shown that $j : M \prec_n N$, where $n \geq 1$. And assume that $M, N \models \mathsf{ZFC}_{n+1}$. Let $\exists x\ \varphi(x, y)$ be a Σ_{n+1} formula, where φ is Π_n. Fix $b \in M$. We want to see $M \models \exists x\ \varphi(x, b)$ iff $N \models \exists x\ \varphi(x, j(b))$. For the forward implication, suppose M satisfies this and pick $a \in M$ so that $M \models \varphi(a, b)$. Then by inductive hypothesis $N \models \varphi(j(a), j(b))$, so $N \models \exists x\ \varphi(x, j(b))$.

For the backward implication, suppose N satisfies this and pick $a \in N$ so that $N \models \exists \varphi(a, j(b))$. Now the problem is that a may not be in the range of j, so we cannot directly pull this back. Nevertheless, note that by (1) there is some ordinal $\alpha \in M$ so that $a \in {V_{j(\alpha)}}^N$. And note that being $x = V_\alpha$ is a Π_1 property (Exercise: Do this!), so by Σ_1-elementarity of j we get that ${V_{j(\alpha)}}^N = j({V_\alpha}^M)$. Thus we have shown that $N \models \exists x \in j({V_\alpha}^M)\ \varphi(x, j(b))$. Now we can apply Σ_n-elementarity, which we have by inductive hypothesis, to get $M \models \exists x \in {V_\alpha}^M\ \varphi(x, b)$. But then $M \models \exists x\ \varphi(x, b)$, as desired. $\square$

(1–3) suggest the following definition, which we will see is of critical importance to the theory of elementary embeddings of the universe into an inner model.

Definition 12. The *critical point* of a nontrivial Σ_1-elementary embedding j, denoted $\mathrm{crit}\, j$, is the least ordinal α so that $j(\alpha) > \alpha$.

As a consequence of (4), if M and N are proper classes containing all the ordinals which satisfy ZFC_k for all natural numbers k, then $j : M \prec_1 N$ implies that j is in fact fully elementary. So to get an adequate formalization of the notion of fully elementary embeddings between proper classes containing all ordinals, we need only to check that we can formalize them satisfying ZFC_k for all k. This is the topic to which we now turn.

Definition 13. M is an *inner model* if M is transitive, $\mathrm{Ord} \subseteq M$, and $M \models \mathsf{ZF}$.

Of course, this concept cannot be directly formalized, since $M \models \mathsf{ZF}$ cannot be directly formalized. Nevertheless, it will turn out that we can in fact get a satisfactory definition of this within ZFC.

We will need a few preliminaries.

Definition 14. Let a be a transitive set. Then $b \subseteq a$ is *definable over* a if there is a formula $\varphi(x, \bar{y})$ in the language of set theory and $\bar{p} \in a$ so that $x \in b$ iff $a \models \varphi(x, \bar{p})$.

Definition 15. Let a be a transitive set. Then $\mathrm{Def}(a) \subseteq \mathcal{P}(a)$ is the collection of all subsets of a definable over a.

Definition 16. Let $\mathrm{Inn}(x)$ be the formula

$$\bigcup x \subseteq x \wedge \forall \alpha \ \mathrm{Def}(x \cap V_\alpha) \subseteq v.$$

We are interested in the case $\mathrm{Inn}(M)$ where M is a proper class. So $\mathrm{Inn}(M)$ is really a longer formula using the definition of M.

Exercise 17. If $\mu(x)$ defines M, explicitly write out $\mathrm{Inn}(M)$ in terms of μ.

Exercise 18. Show that $\mathrm{Inn}(M)$ implies $\mathrm{Ord} \subseteq M$. So $\mathrm{Inn}(M)$ can only hold for proper classes M.

Fact 19 (Schema)**.** *Let M be a class and let φ be an axiom of* ZF*. Then* ZF *proves* $\mathrm{Inn}(M) \Rightarrow \varphi^M$.

Fact 20 (Metatheoretic)**.** *Suppose M is a transitive class containing* Ord *so that φ^M holds for every axiom of* ZF*. Then* $\mathrm{Inn}(M)$.

Taken together, these show that $\mathrm{Inn}(M)$ gives a satisfactory formalization of the concept "M is an inner model". And it's clear that if we can talk about inner models of ZF, we can talk about inner models of ZFC, simply by appending $\wedge \mathsf{AC}^M$ onto the end of $\mathrm{Inn}(M)$. Similarly, we can talk about inner models of ZF plus finitely many extra axioms.

So formally, "M is an inner model" will mean $\mathrm{Inn}(M)$. But we will really think of M as being a transitive model of ZF which contains the ordinals. This is the useful way to think about it, even though it's not directly formalizable.

This completes the last step in our set up. We have now seen that we can formalize all the notions necessary to make sense of the statement "j is an elementary embedding of V into an inner model M". (We formalize this as: $\mathrm{Inn}(M)$ and j is Σ_1-elementary.)

3. Embedding the universe into an inner model

Hereon, we adopt the convention that we only speak of non-identity elementary embeddings between transitive models of set theory (either set or class-sized).

We will also informally quantify over proper classes. Strictly speaking, you should read this as a schema for a definition, similar to the earlier schema for $\mathrm{Inn}(M)$. If someone ever corners you in an alley, puts a gun to your head, and demands that you only quantify over sets, then you can use the concepts from the previous section to satisfy them. But otherwise, it is more convenient to sweep the formal details under the rug, leaving the important ideas to float on the surface of this mixed metaphor. Math is hard enough without going out of our way to make it harder.

Let us now turn to ultrapowers of V. Let U be an ultrafilter on a cardinal κ. As in part $\frac{1}{2}$, we would like to define $\mathrm{Ult}(V,U)$. Taking the same definition as before, the domain of the ultrapower should be the collection of functions $\kappa \to V$, modding out by the equivalence relation $=_U$ defined as

$$f =_U g \Leftrightarrow \{i \in \kappa : f(i) = g(i)\} \in U.$$

But these equivalence classes are themselves proper classes. So we cannot directly talk about them forming the codomain for an embedding, even with our informal treatment of classes. (That would require talking about "hyperclasses", collections of classes.) Fortunately, Dana Scott has a nice trick.

Fact 21 (Scott's trick)**.** *Let $\sim$ be an equivalence relation on a proper class domain. Then we can always pick out set-sized collections of representatives for $\sim$-equivalence classes. Namely, for each $a \in \mathrm{dom} \sim$ the corresponding representatives are:*

$$[a] = \{b \in \mathrm{dom}(\sim) : a \sim b \text{ and } \forall c \in \mathrm{dom}(\sim)\ a \sim c \Rightarrow \operatorname{rank} b \leq \operatorname{rank} c\}.$$

That is, $[a]$ consists of the minimal rank elements $\sim$-equivalent to a.

Exercise 22. Show that $[a]$ is always a set.

Observe that a may not be an element of a. Nevertheless, if $a \sim b$ then $[a] = [b]$.

With Scott's trick in hand we will let the domain for $\mathrm{Ult}(V,U)$ consist of the $[f]$ for $f : \kappa \to V$ under the equivalence relation $=_U$. So the elements of the domain of $\mathrm{Ult}(V,U)$ are sets, and the collection of all of them is a proper class. We must also define the membership relation for $\mathrm{Ult}(V,U)$:

$$[f] \in_U [g] \Leftrightarrow \{i \in \kappa : f(i) \in g(i)\} \in U.$$

In this context, Łoś's theorem is a schema.

Theorem Schema 23 (Łoś)**.** *For any formula $\varphi(x_0,\dots,x_n)$ in the language of set theory and any functions $f_0,\dots,f_n : \kappa \to V$, we have*

$$\mathrm{Ult}(V,U) \models \varphi([f_0],\dots,[f_n]) \Leftrightarrow \{i \in \kappa : \varphi(f_0(i),\dots,f_n(i))\} \in U.$$

This is proven just like the Łoś's theorem for set-sized structures, except that now the induction takes place in the metatheory, rather than the object theory.

The following result is important.

Proposition 24. *U is ω_1-complete iff $\in_U$ is well-founded.*

Proof. ($\Rightarrow$) This is the easier of the two directions. Suppose towards a contradiction that there were a sequence $\langle [f_n] : n \in \omega\rangle$ so that $[f_{n+1}] \in_U [f_n]$ for all n. But then, because

$$\bigcap_{n\in\omega} \{i \in \kappa : f_{n+1}(i) \in f_n(i)\}$$

is in U by ω_1-completeness, we in particular get that it is non empty. But then we have an infinite descending $\in$-chain of sets, which contradicts the axiom of Foundation.

($\Leftarrow$) By contrapositive. Suppose that $X_n \in U$ for each $n < \omega$ and yet $\bigcap_{n\in\omega} X_n \notin U$. Now define functions $g_k : \kappa \to V$ for each $k \in \omega$ as

$$g_k(i) = \begin{cases} n-k & \text{if } n \geq k \text{ and } i \in (\bigcap_{m<n} X_m) \setminus X_n \\ 0 & \text{else} \end{cases}$$

Then, for each $k \in \omega$ we have that $\{i \in \kappa : g_{k+1}(i) \in g_k(i)\}$ contains $\bigcap_{m \le k} X_m \setminus \bigcap_{n \in \omega} X_n$. The former set is in U by finite completeness of U and the latter set is not in U by assumption. But then the difference between the two sets is in U by the complementary property of U. So then the sequence $\langle [g_k] : k \in \omega \rangle$ witnesses that $\in_U$ is ill-founded. $\square$

Proposition 25. *$\in_U$ is set-like.*

Proof. Suppose $[g] \in_U [f]$ and pick $g_0 \in [g]$. Now define $g_1 : \kappa \to V$ as

$$g_1(i) = \begin{cases} g_0(i) & \text{if } g_0(i) \in f(i) \\ 0 & \text{else} \end{cases}$$

Then $g_1 \in [g]$ and $\operatorname{rank}(g_1) \le \operatorname{rank} f$. Thus, by the definition of $[g]$ as consisting of the least-rank examples, we get that $\operatorname{rank}([g]) \le \operatorname{rank}(f) + 1$, and so $\{[g] : [g] \in_U [f]\}$ is a set. $\square$

Because $\in_U$ is well-founded and set-like the Mostowski collapse theorem tells us that it is isomorphic to $\in$ restricted to some transitive proper class. That is, there is

$$\pi_U : (\operatorname{Ult}(V,U), \in_U) \cong (M_U, \in)$$

an isomorphism onto some M_U. And by Łoś's theorem M_U is in fact an inner model. To avoid overly cumbersome notation, we will write $[f]_U$ for $\pi_U([f])$ for $f : \kappa \to V$. Then $M_U = \{[f]_U : f \in {}^\kappa V\}$.

For $x \in V$, let $c_x : \kappa \to V$ be the constant function which always outputs x. Now define an embedding $j_U : V \to M_U$ as $j(x) = [c_x]_U$. Then j_U is an elementary embedding of V into an inner model. We summarize the situation as

$$j_U : V \prec M_U \cong \operatorname{Ult}(V,U).$$

Let us return now to measurable cardinals so we can see the connection.

Proposition 26. *Suppose that U is a measure on κ with corresponding embedding $j : V \prec M$. Then* $\operatorname{crit} j = \kappa$.

Proof. First, I claim that $j(\alpha) = \alpha$ for all $\alpha < \kappa$. Otherwise, let $\alpha < \kappa$ be the least ordinal which is moved. If $[f]_U = \alpha$ then $\{i < \kappa : f(i) < c_\alpha(i) = \alpha\} \in U$. So by κ-completeness of U there is $\beta < \alpha$ so that $\{i < \kappa : f(i) = \beta\} \in U$. But then $[f]_U = j(\beta) = \beta$, by leastness of α, contradicting that $[f]_U = \alpha$.

Now we want to see that $j(\kappa) > \kappa$. By κ-completeness, no bounded subset of κ is in U, so $\{i \in \kappa : \alpha < i\} \in U$ for all $\alpha < \kappa$. Let $f : \kappa \to \kappa$ be the identity. Then $\alpha = j(\alpha) < [f]_U < j(\kappa)$. So $\kappa \le [f]_U < j(\kappa)$. as desired. $\square$

There is a sort of converse to this result. First, a little lemma will be needed, which I give you as an exercise.

Exercise 27. Show that if F is a κ-complete filter on κ then κ must be regular.

Theorem Schema 28 (Keisler)**.** *Suppose $j : V \prec M$ is an elementary embedding. Then* $\operatorname{crit} j$ *is a measurable cardinal.*

Proof. Let $\kappa = \operatorname{crit} j$. First, observe that $\kappa > \omega$, since every ordinal $\le \omega$ is definable and so they must all be fixed by j. Now define $U \subseteq \mathcal{P}(\kappa)$ as

$$X \in U \Leftrightarrow \kappa \in j(X).$$

I claim that U is a nonprincipal κ-complete ultrafilter on κ. In particular, this implies that κ is regular and hence a cardinal.

First, see that $\kappa \in U$, because $j(\kappa)$ is a cardinal $> \kappa$. Next note that $\{\alpha\} \notin U$ for all $\alpha < \kappa$ because $j(\{\alpha\}) = \{j(\alpha)\} = \{\alpha\}$. That U is as ultrafilter follows from elementarity. (Exercise: check this!)

Finally, let us check U is κ-complete. So fix $\alpha < \kappa$ and suppose $\bar{X} = \langle X_i : i < \alpha \rangle$ is a sequence of sets from U. Clearly, $\kappa \in \bigcap_{i<\alpha} j(\bar{X}(i))$. But since $j(i) = i$ for all $i < \alpha$ we have that $j(\bar{X}(i)) = j(\bar{X})(j(i)) = j(\bar{X})(i)$. So

$$\kappa \in \bigcap \{j(\bar{X}(i)) : i < \alpha\} = \bigcap \{j(\bar{X})(i) : i < j(\alpha)\} = j\left(\bigcap \{\bar{X}(i) : i < \alpha\}\right).$$

So $\bigcap_{i<\alpha} X_i \in U$, as desired. □

This is the characterization of measurable cardinals that has been important in the development of set theory: κ is measurable iff κ is the critical point of an embedding of the universe into an inner model.

Observe that this implies that an ultrafilter U on an uncountable cardinal κ being just countably complete already has large cardinal strength. If U is such an ultrafilter, then by an earlier proposition $\in_U$ is well-founded. And observe that our earlier argument that $\in_U$ is set-like goes through regardless of how complete U is. So by taking a collapse we then get a nontrivial elementary embedding $j : V \prec M$. The critical point of this embedding is a measurable cardinal.

We can also use this characterization of measurables to prove that below any measurable cardinal there are inaccessible cardinals.

Proposition 29. *If κ is measurable then there is $\lambda < \kappa$ which is inaccessible.*

Proof. We already saw the Ulam–Tarski theorem that κ must be inaccessible. Now let $j : V \prec M$ have critical point κ. Note that being an inaccessible cardinal is a Π_1-property. (Exercise: Check this!) So $M \models \kappa$ is inaccessible. So $M \models \exists \lambda < j(\kappa)$ so that λ is inaccessible. By elementarity, we then get that $\exists \lambda < \kappa$ so that $V \models \lambda$ is inaccessible. □

Exercise 30. Fix a natural number n. Show that if κ is measurable then there are at least n many inaccessibles $< \kappa$.

This proposition reveals the power of embeddings of the universe. Once we had the machinery set up, in the matter of a few lines we settled a question which had been open from 1930 to the early 1960s! (But note that this argument through embeddings was not the first proof that below any measurable cardinal is an inaccessible cardinal.)

A natural question is to ask is how far we can push this. To go as far as we can will need new technology, which will come in a moment. Before seeing this new technology, let us take a moment to better understand this inner model M.

Proposition 31. *Suppose $j : V \prec M$ with critical point κ is the elementary embedding coming from a measure U on κ. Then the following:*

1. *For all $x \in V_\kappa$ we have $j(x) = x$, and so ${V_\kappa}^M = V_\kappa$.*
2. *For all $x \subseteq V_\kappa$, we have $j(x) \cap V_\kappa = x$, so ${V_{\kappa+1}}^M = V_{\kappa+1}$ and $(\kappa^+)^M = \kappa^+$.*
3. $2^\kappa \leq (2^\kappa)^M < j(\kappa) < (2^\kappa)^+$.
4. *If λ is a strong limit cardinal whose cofinality is not κ, then $j(\lambda) = \lambda$. In particular, j fixes a definable proper class of ordinals.*
5. *M is closed under sequences of length $\leq \kappa$, but not closed under sequences of length κ^+. That is, if $\langle a_i : i \in \kappa \rangle$ is a sequence of sets from M, then $\langle a_i \rangle \in M$, but there is no guarantee of this if the sequence has length κ^+.*

(6) $U \notin M$.

Proof. (1) Take x of minimal rank so that $j(x) \neq x$. It must be that $\operatorname{rank}(j(x)) \neq \operatorname{rank}(x)$, as all y of rank $< \operatorname{rank}(x)$ are fixed. But $\operatorname{rank}(x)$ is an ordinal and κ is the least ordinal moved by j, so x must have rank κ.

(2) Let $x \subseteq V_\kappa$. Suppose $y \in j(x)$ where $y \in V_\kappa$. Then, since $y \in \operatorname{ran} j$, by elementarity we have $j^{-1}(y) = y \in x$. So $j(x) \cap V_\kappa = x$. To now see that $(\kappa^+)^M = \kappa^+$, note first that being a cardinal is a Π_1 property, and so downward absolute. Thus κ^+ is a cardinal in M and so $(\kappa^+)^M \leq \kappa^+$. So we only have to see that $(\kappa^+)^M \not< \kappa^+$. To see this, observe that every well-order of κ is a subset of V_κ. So M has the same well-orders of κ as V does. So M sees that every ordinal $< \kappa^+$ has cardinality κ, as desired.

(3) We just saw that $\mathcal{P}(\kappa)^M = \mathcal{P}(\kappa)$. If $f : \mathcal{P}(\kappa) \to \lambda$ is a bijection in M, then by absoluteness f is also a bijection in V. So $(2^\kappa)^M$ cannot be less than 2^κ. (But note that there's no guarantee that $(2^\kappa)^M = 2^\kappa$, as it could be that M has fewer bijections than V.) To see that $(2^\kappa)^M < j(\kappa)$, note that M thinks $j(\kappa)$ is inaccessible, so M thinks every cardinal $\lambda < j(\kappa)$ has $2^\lambda < j(\kappa)$. Finally, note that $j(\kappa) = \{[f]_U : f \in {}^\kappa\kappa\}$, which has cardinality 2^κ. So $j(\kappa) < (2^\kappa)^+$.

(4) We already know that $j(\lambda) = \lambda$ for all $\lambda < \kappa$. So consider $\lambda > \kappa$ a strong limit cardinal of cofinality $\neq \kappa$. Since $\lambda \leq j(\lambda)$, we only have to see that $j(\lambda) \leq \lambda$. For that it is enough to see that $[f]_U < j(\lambda)$ implies $[f]_U < \lambda$. So assume we have such f. By modifying f on a set not in U, we may assume without loss that $f(i) < \lambda$ for all $i < \kappa$. We now have two cases to consider, depending upon whether $\operatorname{cof}(\lambda)$ is greater than or less than κ. If $\operatorname{cof}(\lambda) < \kappa$ then by κ-completeness of U we can find $\alpha < \lambda$ so that $\{i < \kappa : f(i) < \alpha\} \; inU$. If $\operatorname{cof}(\lambda) > \kappa$ then if $\alpha = \sup(\operatorname{ran} f)$ we get that $\{i < \kappa : f(i) < \alpha\} \in U$. Either way, we get that

$$[f]_U \leq j(\alpha) = \{[g]_U : g \in {}^\kappa\alpha\} < \lambda.$$

The last inequality here is because λ is strong limit.

(5) Suppose that $[f_i]_U \in M$ for each $i < \kappa$. We want to find $g : \kappa \to V$ so that $[g]_U = \{[f_i]_U : i < \kappa\}$. Fix $h : \kappa \to \kappa$ so that $[h]_U = \kappa$. Now define $g : \kappa \to V$ to be the function so that $g(\xi)$ is the function with domain $h(\xi)$ satisfying that $(g(\xi))(i) = f_i(\xi)$. Then, by Łoś, $[g]_u$ is a function with domain $[h]_U = \kappa$ and for each $i < \kappa$ we have $[g]_U(i) = [f_i]_U$, as desired.

To show M is not closed under κ^+-sequences, it is enough to find an example. I claim that $j''\kappa^+ \notin M$. This is clearly enough, as $j(\alpha) \in \mathrm{Ord} \subseteq M$ for each ordinal α. First, observe that $j''\kappa^+$ is a cofinal subset of $j(\kappa^+)$: if $[f]_U < j(\kappa^+)$ then, without loss, $f(i) < \kappa^+$ for all $i < \kappa$ and then $[f]_U < j(\sup(\operatorname{ran} f)) < j(\kappa^+)$. Now simply note that $j''\kappa^+$ has ordertype $\kappa^+ < j(\kappa^+)$ and so $j''\kappa^+ \in M$ would contradict that M thinks, by elementarity, that $j(\kappa^+)$ is regular.

(6) Suppose toward a contradiction that $U \in M$. Then M would be able to reconstruct the map $f \mapsto [f]_U$ for $f \in M$. We saw in (2) that M and V have the same functions $\kappa \to \kappa$, as they are all subsets of V_κ. So we get that the map sending $f : \kappa \to \kappa$ to $[f]_U$ is in M. But then $j(\kappa) = \{[f]_U : f \in {}^\kappa\kappa\} < ((2^\kappa)^+)^M$, contradicting that $j(\kappa)$ is inaccessible in M ∎

(3), (5), and (6) all imply that $M \neq V$. In particular, this shows that there can be no ultrapower embedding $V \prec V$.

Let's see that this finer analysis of M implies that the GCH cannot first fail at a measurable cardinal.

Corollary 32. *Let κ be measurable. If $2^\lambda = \lambda^+$ for all $\lambda < \kappa$ then $2^\kappa = \kappa^+$.*

Proof. Let $j : V \prec M$ have critical point κ. Then, by elementarity, $(2^\kappa)^M = (\kappa^+)^M$. But then by (2) and (3) of the proposition we then get

$$2^\kappa \leq (2^\kappa)^M = (\kappa^+)^M = \kappa^+. \qquad \square$$

4. Normal ultrafilters

We have seen that measurable cardinals give rise to a reflection phenomenon: we can reflect some properties of measurable κ down to smaller cardinals. For example, from κ being inaccessible we were able to conclude that there are inaccessible cardinals $< \kappa$. In this section we will study a combinatorial property of filters which allows us to exploit this reflection to a strong degree.

Definition 33. Let $\langle X_i : i < \kappa \rangle$ be a sequence of subsets of κ. Then the *diagonal intersection* of the sequence is

$$\mathop{\triangle}_{i<\kappa} X_i = \left\{ \alpha < \kappa : \alpha \in \bigcap_{i<\alpha} X_i \right\}.$$

Definition 34. Let F be a filter on κ. Then F is *normal* if F is closed under diagonal intersections: for any sequence $\langle X_i : i < \kappa \rangle$ of sets from F we have that $\triangle_{i<\kappa} X_i \in F$.

Proposition 35. *Suppose that F is a filter on κ which contains every final segment $[\alpha, \kappa) = \{i < \kappa : i \geq \alpha\}$. Then F being normal implies F is κ-complete.*

Proof. Let $\langle X_i : i < \alpha \rangle$ be a sequence of sets in F of length $\alpha < \kappa$. Now set $\bar{X}_i = X_i$ if $i < \alpha$ and $\bar{X}_i = \kappa$ if $i \geq \alpha$. By normality, we get that $\triangle_{i<\kappa} \bar{X}_i \in F$. Now note that if $\xi \geq \alpha$ then $\xi \in \triangle_{i<\kappa} \bar{X}_i$ iff $\xi \in \bigcap_{i<\alpha} X_i$. So

$$\left(\mathop{\triangle}_{i<\kappa} \bar{X}_i \right) \cap [\alpha, \kappa) \subseteq \bigcap_{i<\alpha} X_i.$$

Because F is closed under superset we conclude $\bigcup_{i<\alpha} X_i \in F$. $\square$

Corollary 36. *If F is a normal filter on κ which contains every final segment then κ is regular and uncountable.*

Proof. To see that κ is regular, suppose otherwise that $\langle \alpha_i : i < \lambda < \kappa \rangle$ were cofinal in κ. Then $\bigcap_{i<\lambda} [\alpha_i, \kappa) = \emptyset \in F$, a contradiction.

To see that κ is uncountable, note that the diagonal intersection of $\langle \{k \in \omega : k \geq i+1\} : i \in \omega \rangle$ is empty. $\square$

Observe that, by the proposition, if U is a normal ultrafilter on a κ then U is a measure. So having normal ultrafilters is beyond ZFC in strength. The following exercise shows that if we merely ask for a normal filter, then we do have many examples just from ZFC.

Exercise 37. The club filter on κ consists of all club subsets of κ. Show that if κ is an uncountable regular cardinal then the club filter is normal.

The following proposition gives a useful characterization of normality. First let us make a couple definitions: (1) Let $X \subseteq \kappa$ and say that a (partial) function $f : X \to \kappa$ is *regressive* if $f(i) < i$ for all $i \in X$. And (2): if F is a filter over κ then $X \subseteq \kappa$ is *F-stationary* if $X \cap A \neq \emptyset$ for all $A \in F$. (So a stationary set is stationary with respect to the club filter.)

Exercise 38. Suppose F is a filter on κ and X is non-F-stationary, meaning that X is not F-stationary. Show that $\kappa \setminus X \in F$. That is, the non-F-stationary sets form an ideal dual to F.

Proposition 39. *A filter F on κ in normal iff for any F-stationary X and regressive $f : X \to \kappa$ there is a $\alpha < \kappa$ so that $\{i < \kappa : f(i) = \alpha\}$ is F-stationary.*

Proof. ($\Rightarrow$) Suppose that for all $\alpha < \kappa$ we have that $X_i = \{i < \kappa : f(i) = \alpha\}$ is not F-stationary. By the exercise, we then get that $\kappa \setminus X_i \in F$. Now see that

$$\alpha \in \mathop{\triangle}_{i<\kappa} (\kappa \setminus X_i) \Leftrightarrow \alpha \in \bigcap_{i<\alpha} (\kappa \setminus X_i) = \kappa \setminus f^{-1}{''}[0, \alpha).$$

But since f is regressive on X, if $\alpha \in X$ we have that $\alpha \in f^{-1}{''}[0, \alpha)$. So $X \cap \triangle_{i<\kappa}(\kappa \setminus X_i) = \emptyset$, contradicting that X is F-stationary.

($\Leftarrow$) Suppose we had $\langle X_i : i < \kappa \rangle$ a sequence of sets from F but $\triangle_{i<\kappa} X_i \notin F$. Then $X = \kappa \setminus \triangle_{i<\kappa} X_i$ is F-stationary. Now define $f : X \to \kappa$ as $f(i)$ is the minimum α so that $i \notin X_\alpha$. By the definition of X we have that f is regressive. But note that $X_\alpha \cap \{i < \kappa : f(i) = \alpha\} = \emptyset$ by the definition of f. This contradicts that f must be constant on an F-stationary set. □

Exercise 40. Let F be a normal filter on κ. Show that F contains every club subset of κ. (Hint: fix a club $C \subseteq \kappa$ and consider the function f on $\kappa \setminus (C \cup \{0\})$ defined as $f(i) = \sup(C \cap i)$. Show that f is regressive and thereby conclude that C not being in F leads to a contradiction.)

We turn now to normal ultrafilters. By an earlier comment, normal ultrafilters are automatically measures. First let us see some alternative characterizations of normality for measures.

Proposition 41. *Let U be a measure on κ. Then the following are equivalent.*

- *U is normal;*
- *If $f : \kappa \to \kappa$ and $\{i < \kappa : f(i) < i\} \in U$, then there is $\alpha < \kappa$ so that $\{i < \kappa : f(i) = \alpha\} \in U$; and*
- *If* id *is the identity function on κ then $[\mathrm{id}]_U = \kappa$.*

Proof. ($1 \Leftrightarrow 2$) Exercise. (Hint: use the earlier characterization of normality.)

□

Exercise 42. Let W be a measure on κ and fix $f : \kappa \to \kappa$ so that $[f]_W = \kappa$. Show that

$$U = \{X \subseteq \kappa : f^{-1}{''}X \in W\}$$

is a normal measure on κ.

Exercise 43. Let $j : V \prec M$ have critical point κ. Now define, as in the proof of Keisler's theorem above, a measure U on κ as

$$X \in U \Leftrightarrow \kappa \in j(X).$$

Show that U is normal.

Call this U the *normal measure derived from the embedding j*.

These exercises show that if κ is measurable then there is a normal measure on κ. Indeed, the canonical measures coming from the important characterization of measurability—κ is measurable iff κ is the critical point of an embedding $j : V \prec M$—must be normal.

Normality lets us extract extra information about the ultrapower. This proposition gives a first look at that.

Proposition 44. *Suppose U is a normal measure on κ, with corresponding embedding $j_U : V \prec M_U$. Then $M_U = \{j_U(f)(\kappa) : f \in {}^\kappa V\}$. In other words, κ is a* seed *which generates the entire ultrapower M_U.*

Proof. Fix $x = [f]_U \in M_U$, where $f : \kappa \to V$ is some function in V. Then,

$$j_U(f)(\kappa) = j_U(f)([\mathrm{id}]_U) = [f]_U,$$

because $\{i < \kappa : f(\mathrm{id}(i)) = f(i)\} = \kappa \in U$. □

We can also see that M_U is minimal in a natural sense among the embeddings $j : V \prec M$.

Proposition 45. *Suppose U is a normal measure on κ, with corresponding embedding $j_U : V \prec M_U$. Suppose $j : V \prec M$ is some other embedding with critical point κ, and that U is the normal measure derived from j. Then, there is an elementary embedding $k : M_U \prec M$ so that $k \circ j_U = j$. Moreover, if $j = j_W$ for some normal measure W then $U = W$ and so k is the identity.*

Proof. Define $k : M_U \to M$ as $k([f]_U) = j(f)(\kappa)$. Now fix a formula $\varphi(x_0, \ldots, x_n)$. Then,

$$\begin{aligned} M_U \models \varphi([f_0]_U, \ldots, [f_n]_U) &\Leftrightarrow \{i < \kappa : \varphi(f_0(i), \ldots, f_n(i))\} \in U \\ &\Leftrightarrow \kappa \in j(\{i < \kappa : \varphi(f_0(i), \ldots, f_n(i))\}) \\ &\Leftrightarrow M \models \varphi(j(f_0)(\kappa), \ldots, j(f_n)(\kappa)). \end{aligned}$$

So k is an elementary embedding, and by inspection $k \circ j_U = j$.

Finally, suppose $j = j_W$ for W a normal measure. But then, for $X \subseteq \kappa$, we have

$$\begin{aligned} X \in U &\Leftrightarrow \kappa \in j_W(X) \\ &\Leftrightarrow [\mathrm{id}]_W \in [c_X]_W \\ &\Leftrightarrow \{i < \kappa : \mathrm{id}(i) \in c_X(i)\} \in W \\ &\Leftrightarrow X \in W. \end{aligned}$$

□

One might be tempted to try to conclude from this proposition that normal measures on κ must be unique. Note, however, that the argument that $U = W$ used that U was the measure derived from j. As we saw before, the measure derived from $j : V \prec M$ cannot be in M. But it's conceivable that we could have a normal measure in M itself, in which case it would not be the same as the normal measure derived from j. We will later see that this can in fact happen, though it exceeds the existence of a measurable cardinal in consistency strength. This gives us a way of transcending measurable cardinals in strength, similar to how 2-inaccessibles transcended 1-inaccessibles in strength.

But that is a peak ahead. For now, we are still focused on using normal measures to extract reflection properties on κ.

Proposition 46. *Suppose U is a normal measure on κ and $S \subseteq \kappa$ is stationary. Then, $\{\alpha < \kappa : S \cap \alpha$ is stationary in $\alpha\} \in U$. In particular, if $X \in U$ then $\{\alpha < \kappa : X \cap \alpha$ is stationary in $\alpha\} \in U$.*

Proof. It follows from an earlier proposition that $j_U(S) \cap \kappa = S$. And since ${V_{\kappa+1}}^{M_U} = V_{\kappa+1}$, $j_U(S) \cap \kappa$ is stationary in κ in M_U. Unwrapping definitions a bit, $\{\alpha < \kappa : c_S(\alpha) \cap \mathrm{id}(\alpha)$ is stationary$\} \in U$, so $\{\alpha < \kappa : S \cap \alpha$ is stationary$\} \in U$. □

Proposition 47. *If U is a normal measure on κ then $\{\alpha < \kappa : \alpha$ is inaccessible$\} \in U$.*

As a slogan: "the inaccessibility of κ reflects". So κ is the κth inaccessible cardinal.

Proof. We first see that $M_U \models \kappa$ is inaccessible. We already saw this earlier by observing that inaccessibility is downward absolute. But let's use an alternative argument: whether κ is inaccessible is witnessed by $V_{\kappa+1}$. And ${V_{\kappa+1}}^{M_U} = V_{\kappa+1}$, so M_U has all the information needed to check that κ is inaccessible.

Thus, if $S \subseteq \kappa$ is the collection of inaccessible cardinals $< \kappa$, then $\kappa \in j_U(S)$. We thereby can can conclude that $S \in U$. □

This argument can be generalized. Any large cardinal property of κ that is witnessed inside $V_{\kappa+1}$ will reflect. So, for example, there are κ many Mahlo cardinals below κ. (Mahlo cardinals were defined in an exercise.)

But there are limits to this reflection.

Proposition 48. *Let κ be measurable. Then there is a normal measure U on κ so that $\{\alpha < \kappa : \alpha$ is not measurable$\} \in U$.*

Proof. Let $A = \{\alpha < \kappa : \alpha \text{ is measurable}\}$. Suppose U is a normal measure on κ. If $A \notin U$, then we are done, So suppose we are in the other case. We proceed inductively, supposing that we have U_α a normal measure on α so that $A \cap \alpha \notin U_\alpha$. Now define $W \subseteq \mathcal{P}(\kappa)$ as

$$X \in W \Leftrightarrow \{\alpha \in A : X \cap \alpha \in U_\alpha\} \in U.$$

Exercise: show that W is a normal measure and that $A \notin W$. □

5. Seed theory

In this section we shall study seeds, which provide us with a way of handling normal measures and ultrapowers. The germ of seed theory is in the fact, proven earlier, that if U is the normal measure derived from $j : V \prec M$ then κ generates an elementary substructure of M, namely $M_U = \{j_U(f)(\kappa) : f \in {}^\kappa V\}$.

Definition 49. Suppose $j : V \prec M$ is an elementary embedding, and D is some set. Then $a \in j(D)$ is a *seed* for the ultrafilter $U_a \subseteq \mathcal{P}(D)$ defined as $X \in U$ iff $a \in j(X)$. (Strictly speaking, we should also tag U_a with j, but we will leave it implicit when j is understood to be fixed.) We say that the seed *a generates U_a via j*. If $b = j(f)(a)$ for some ground model function f we say that *a generates b (via j)*. If every member of M is generated by a then we say *a generates all of M*.

Lemma 50 (Seed lemma)**.** *An elementary embedding $j : V \prec M$ is an ultrapower embedding iff there is a seed a which generates all of M. In this case, if U is the measure generated by a then $[f]_U = j(f)(a)$.*

Proof. Suppose $j = j_U$ where U is a normal measure on D. Let $a = [\mathrm{id}]_U \in j(D)$, where $\mathrm{id} : D \to D$ is the identity function. Now fix a function $f : D \to V$ and let c_f be the corresponding constant function. Then

$$j(f)(a) = [c_f]_U([\mathrm{id}]_U) = [\langle f(i) : i \in D\rangle]_U = [f]_U.$$

So a generates all of M.

For the other direction, suppose that the seed a generates all of M via functions $f : D \to V$. Now define $U \subseteq \mathcal{P}(D)$ as $X \in U$ iff $a \in j(X)$. Similar to before, we get that U is a κ-complete measure on D, where $\kappa = \operatorname{crit} j$. Now we get an ultrapower embedding $j_U : V \to M_U$. Define $\pi : M_U \to M$ as $\pi([f]_U) = j(f)(a)$. First, observe this is well-defined, not depending on the choice of f:

$$\begin{aligned} f =_U g &\Leftrightarrow \{i \in D : f(i) = g(i)\} \in U \\ &\Leftrightarrow a \in j(\{i \in D : f(i) = g(i)\}) \\ &\Leftrightarrow j(f)(a) = j(g)(a). \end{aligned}$$

Similarly, π is a $\in$-homomorphism. More, π is onto, since a generates M. So M is an isomorphism. And since the only isomorphism of $\in$ restricted to an inner model is the identity, we get that π is the identity, so $M = M_U$. Finally, note that

$$j_U(x) = [c_x]_U = \pi([c_x]_U) = j(c_x)(a)\ j(x),$$

so $j = j_U$. So $j : V \prec M$ is the ultrapower by U. □

The critical point itself is a seed which generates the target model, provided the embedding is via an ultrapower with a normal measure.

Proposition 51. *An embedding $j : V \prec M$ with critical point κ is the ultrapower by a normal measure on κ iff κ is a seed which generates all of M.*

Proof. Exercise! (Hint: look at the arguments from the previous section.) □

We can also talk about the model generated by a collection of seeds.

Definition 52. Let A be a set and κ be a cardinal. Then ${}^{<\kappa}A$ is the collection of sequences of elements of A with length $< \kappa$. In particular, ${}^{<\omega}A$ is the collection of finite sequences from A.

Definition 53. Let $j : V \prec M$ and let $S \subseteq j(D)$ be nonempty. Then the *seed hull* of S via j in M is

$$X_S = \{j(f)(s) : s \in {}^{<\omega}S \text{ and } f \in {}^{{}^{<\omega}D}V\}$$

If $S = \{a\}$, we write X_a instead of $X_{\{a\}}$, risking ambiguity which will not arise in practice. Note that in this case if $a \in j(D)$ it suffices to only consider functions $f : D \to V$, as we did before.

Note that we can restrict f to a smaller domain, so long as $s \in \operatorname{dom}(j(f))$ stays true and so $j(f)(s)$ is unaffected. So the definition of the seed hull does not depend on the choice of D.

This lemma illustrates the importance of seed hulls.

Lemma 54. *If S is a collection of seeds, then $X_S \prec M$.*

Proof. We use the Tarski-Vaught test. For notational simplicity, we will just verify it in the single parameter case. That is, we suppose $M \models \exists x\ \varphi(x, j(f)(s))$ where $s \in [S]^{<\omega}$. We want to find a witness in X_S.

Toward that goal, for each $t \in \operatorname{dom} f$ let $g(t)$ be x so that $\varphi(x, f(t))$, if such exists, otherwise pick x arbitrarily to be ω_{17}. (Note that we can do this because $\operatorname{dom} f$ is a set and so there is V_α so that V_α is closed under witnesses for $\varphi(x, f(t))$ with $t \in \operatorname{dom} f$. So we can use a choice function on V_α to pick the xs.) Because M has a witness for $\varphi(x, j(f)(s))$ it follows that $M \models \varphi(j(g)(s), j(f)(s))$, by the definition of g. So we have found the desired witness. □

The way to think of X_S is as the Skolem hull of $S \cup \operatorname{ran} j$. Our argument was to show that the function $j(f)$ include Skolem functions.

Note that if we had multiple parameters and S had multiple seeds, then there would be the possibility that we used different seeds for different parameters. So we would have to put them together. This is why we need to use ${}^{\omega}S$ in the definition of the seed hull, not just S.

Now let $\pi_S : X_S \cong M_S$ be the Mostowski collapse of X_S. Because $\operatorname{ran} j \subseteq X_S$ we can then define $j_S = \pi_S \circ j$ and then $j = {\pi_S}^{-1} \circ j_S$.

Set $S_0 = \pi_S{}''S$. Then,

$$\begin{aligned} M_S &= \{\pi_S(j(f)(s)) : f \in V \text{ and } s \in {}^{<\omega}S\} \\ &= \{j_S(f)(\pi(s)) : f \in V \text{ and } s \in {}^{<\omega}S\} \\ &= \{j_S(f)(t) : f \in V \text{ and } t \in {}^{<\omega}S_0\}. \end{aligned}$$

That is, the embedding $j_S : V \prec M_S$ is generated by the seeds of S_0. We will call this the *factor embedding* induced by the seeds of S via j. In particular, if $S = \{a\}$ then $j_a : V \prec M_a$ is the ultrapower by the measure induced by a.

Exercise 55. Suppose $j : V \prec M$ and the seed a generates a measure U via j. Show that the factor embedding $j_a : V \prec M_a$ is precisely the ultrapower by U.

The following lemma generalizes the fact, proven earlier, than if κ is the critical point of $j : V \prec M$ then M is closed under κ-sequences.

Lemma 56. *Suppose $j : V \prec M$ is an ultrapower embedding and λ is an ordinal. Then ${}^\lambda M \subseteq M$ iff $j''\lambda \in M$.*

Proof. ($\Rightarrow$) Immediate.

($\Leftarrow$) By the seed lemma, because M is an ultrapower embedding there is a seed a which generates all of M. Now suppose that $\langle z_\alpha : \alpha < \lambda\rangle \in {}^\lambda M$. Because a generates all of M for each $\alpha < \lambda$ there is a function f_α so that $j(f_\alpha)(a) = z_\alpha$. Now, restrict the sequence $j(\langle f_\alpha : \alpha < \lambda\rangle)$ to the coordinates in $j''\lambda \in M$. This gives us λ many coordinates, so by re-indexing we may see that $\langle j(f_\alpha) : \alpha < \lambda\rangle \in M$. But since $a \in M$ we may evaluate these functions pointwise to get that $\langle j(f_\alpha)(a) : \alpha < \lambda\rangle = \langle z_\alpha : \alpha < \lambda\rangle \in M$, as desired. $\square$

This fact will reappear when we turn to supercompact cardinals.

Lemma 57 (Unique seed lemma)**.** *If $j : V \prec M$ is the ultrapower by a measure U, then $[\mathrm{id}]_U$ is the unique seed for U via j.*

Proof. First, we must see that $[\mathrm{id}]_U$ is a seed for U via j:

$$\begin{aligned} [\mathrm{id}]_U \in j(X) &\Leftrightarrow [\mathrm{id}]_U \in [c_X]_U \\ &\Leftrightarrow \{\alpha : \alpha \in X\} \in U \\ &\Leftrightarrow X \in U. \end{aligned}$$

Next, suppose $[f]_U$ is a seed for U via j. Then,

$$\begin{aligned} X \in U &\Leftrightarrow [f]_U \in j(X) \\ &\Leftrightarrow [f]_U \in [c_X]_U \\ &\Leftrightarrow \{\alpha : f(\alpha) \in c_X(\alpha)\} \in U \\ &\Leftrightarrow \{\alpha : f(\alpha) \in X\} \in U \\ &\Leftrightarrow \{\alpha : \alpha \in f^{-1}{}''X\} \in U. \end{aligned}$$

That is, we have seen that U is closed under taking preimages by f. Let us use this to see that $f =_U \mathrm{id}$.

Think of f, which is a set of ordered pairs, as a directed graph on its domain D. Let A select one element from each connected component of this graph. In then follows that

$$D = \bigcup_{n,k<\omega} (f^{-n} \circ f^k)'' A.$$

U is an ultrapower on D, so by countable completeness of U we get that $(f^{-n} \circ f^k)'' A \in U$ for some $n, k < \omega$. Because U is closed under taking preimages by f, we then get that $B = (f^k)'' A \in U$. Observe that B picks at most one element from each connected component of the graph. But since $f^{-1\prime\prime} B \in U$, there must be $b \in B \cap f^{-1\prime\prime} B$. Now since $b, f(b) \in B$ and they are in the same connected component, it must be that $b = f(b)$. So f is the identity on $B \cap f^{-1\prime\prime} B \in U$, so $f =_U \mathrm{id}$, as desired. □

The unique seed lemma shows that there is not too much going on when dealing with an ultrapower by a measure. Then there is a unique seed which generates the measure, namely the image of the identity function under the embedding. The more interesting case is when we have a collection of many seeds, a topic to which we now turn.

First, however, we need to talk about direct limits of directed systems of embeddings.

5.1. **Direct limits of directed systems.** Let $(P, <)$ be a partially ordered set. P is *directed* if given any $p, q \in P$ there is $r \in P$ so that $p \leq r$ and $q \leq r$. A *directed system* of embeddings indexed by a directed poset P is a collection of structures M_p for $p \in P$ with embedding $\pi_{p,q} : M_p \prec M_q$ for $p \leq q$ in P, where $\pi_{p,p}$ is always the identity and if $p \leq q \leq r$ then $\pi_{p,r} = \pi_{q,r} \circ \pi_{p,q}$. Given a directed system we can form its *direct limit*:

$$M_\infty = \operatorname{dir}\lim_P \langle M_p; \pi_{p,q} : p \leq q \in P \rangle .$$

The domain of M_∞ is the equivalence class of the collection of pairs (p, x) with $p \in P$ and $x \in M_p$, under the following equivalence relation:

$$(p, x) \sim (q, y) \Leftrightarrow \exists r \in P \; p, q \leq r \text{ and } \pi_{p,r}(x) = \pi_{q,r}(y).$$

The M_p then canonically embed into M_∞ via the map $\pi_{p,\infty} : M_p \to M_\infty$ defined as $\pi_{p,\infty}(x) = (p, x)/\sim$.

One can define the relations on M_∞ similarly. Let me only do the case of the relation $\in$, since that is the main relation of interest to us:

$$(p, x) \in^{M_\infty} (q, y) \Leftrightarrow \exists r \in p \; p, q \leq r \text{ and } \pi_{p,r}(x) \in^{M_r} \pi_{q,r}(y).$$

The following exercises establish some basic facts about direct limits.

Exercise 58. Show that $\in^{M_\infty}$ is a congruence modulo $\sim$.

Exercise 59. Show that the direct limit satisfies the following universal property: Let N be a structure in the same language of the directed system $\langle M_p; \pi_{p,q} : p \leq q \in P \rangle$. Suppose N has the property that there are embeddings $\sigma_p : M_p \to N$ which factors through the $\pi_{p,q}$. That is, for each $p \leq q$ in P we have that $\sigma_p = \sigma_q \circ \pi_{p,q}$. Then, there is a unique embedding $\rho : M_\infty \to N$ so that each σ_p factors through ρ: for each $p \in P$ we have $\sigma_p = \rho \circ \pi_{p,\infty}$. (Hint: start by drawing commutative diagrams for all this![4])

[4]Fun fact! The best thing about assigning something as an exercise is not having to TEX up the commutative diagram yourself!

Exercise 60. Show that if each $\pi_{p,q}$ is an elementary embedding then each $\pi_{p,\infty}$ is an elementary embedding.

Exercise 61. Show that if P has a maximum m—that is, $p \leq m$ for all $p \in P$—then $M_\infty \cong M_m$.

The following exercise illustrates the importance of the choice of embeddings when taking a direct limit.

Exercise 62. Show that every countably infinite linear order is a direct limit of the finite linear orders. That is: Let $(L, <)$ be a countably infinite linear order. And let K_n be the linear order with n elements. Show that $(L, <) = \operatorname{dir\,lim}_\omega \langle K_n; \pi_{n,m} : n < m \in \omega \rangle$ for some choice of embeddings $\pi_{n,m} : K_n \to K_m$.

A particular special case of interest is taking direct limits where the underlying poset is a limit ordinal.

Let us apply these direct limit ideas to seed hulls. A bit of notation first. Suppose we have a directed system $\langle M_p : \pi_{p,q} : p \leq q \in P \rangle$ and another structure N in the same language. Say that there are maps $\sigma_p : N \to M_p$ for each $p \in P$ which factor through the $\pi_{p,q}$. Then we say $\sigma : N \to M_\infty$ is the direct limit of the directed system. The intended usage is when each model in the directed system is the image of a common ground model and we want to know what the directed system does to this ground model.

Lemma 63. *Suppose $j : V \prec M$ is generated by the seeds in S. Then $j : V \prec M$ is the direct limit of of the directed system of ultrapowers by the measures U_a for $a \in {}^{<\omega}S$.*

Proof. First let us check that this is really a directed system, clarifying what is meant. For $a \in {}^{<\omega}S$ we can form the measure U_a induced by a along with the corresponding factor embedding $j_a : V \prec M_a$, where M_a is the Mostowski collapse of the seed hull X_a. Given two sequences $a, b \in {}^{<\omega}S$ say that $a \leq b$ if the elements of a appear along increasing indices in b. In other words, $a \leq b$ if a can be obtained from b by deleting some entries and then collapsing down the domain. Note that if $a \leq b$ then $a \in X_b$, because a is definable from b. It easily follows (Exercise: Check it!) that $X_a \prec X_b$ and so j_b factors through j_a through a map $k_{a,b} : M_a \to M_b$.

Exercise 64. Suppose $a \leq b \leq c$ are all in ${}^{<\omega}S$. Show that $k_{a,c} = k_{b,c} \circ k_{a,b}$.

Exercise 65. Suppose that $k_a : M_a \prec M$ is the inverse of the Mostowski collapse map $X_a \cong M_a$, for each $a \in {}^{<\omega}S$. Show that if $a \leq b$ then $k_a = k_b \circ k_{a,b}$.

So to finish the proof we have only to see that M is the smallest model which all maps factor through. But this follows immediately from the fact that M is generated by the seeds in S: Every element of M has the form $j(f)(a)$ for some $a \in {}^{<\omega}S$. And since $j_a = \pi \circ j$, where $\pi : X_a \to M_a$ is the collapse map as we saw earlier, we get that every element of M is in the image of k_a for some a. $\square$

The point is, j can be recovered from the indexed system of measures U_a for $a \in {}^{<\omega}S$, since from this indexed system we can construct the M_a and the embeddings j_a and $k_{a,b}$. And then we recover M by taking the direct limit of this system (and hitting it with the Mostowski collapse). This is similar to how the single measure U contains all the information necessary for the ultrapower embedding $j_U : V \prec M_U$, except it works for embeddings that don't arise as an ultrapower.

Whenever $S \subseteq j(D)$ we can recover the measures U_a from the set

$$E = \{(a, X) : X \subseteq {}^{<\omega}D \text{ and } a \in j(X) \cap {}^{<\omega}S\}.$$

This E is sometimes called an *extender*. More, if in fact $S \subseteq j(\kappa)$ then all we need to recover this information is $j \upharpoonright \mathcal{P}(\kappa)$. In something of an abuse of notation, this $j \upharpoonright \mathcal{P}(\kappa)$ is sometimes referred to as the extender.

Some embeddings arising from large cardinals cannot be ultrapower embeddings but can be captured as extender embeddings. Or, to phrase things from a certain perspective, recalling the formalization issues with directly saying "there is an elementary embedding $j : V \prec M$ with such-and-such": some large cardinal notions are defined by the existence of certain kinds of extenders, for example *strong* cardinals or *superstrong* cardinals.

Project Idea 66. Investigate large cardinals with extender characterizations.

6. Mitchell order

Let us step back from general seed theory and return to measurable cardinals.

Definition 67 (Mitchell Order)**.** Let κ be a measurable cardinal and let U, W be measures on κ. Then $U \lhd W$ if $U \in M_W$.

An earlier proposition shows that $U \not\lhd U$.

Exercise 68. Show that $\lhd$ is a partial order on the measures on κ.

Proposition 69. *Let U and W be measures on κ. Then $U \lhd W$ implies $j_U(\kappa) < j_W(\kappa)$.*

Proof. Consider the embedding $j_W : V \prec M_W$. Observe that $j_U(\kappa)$ is the ordertype of the equivalence classes $[f]_U$ with respect to $\in_U$ for $f : \kappa \to \kappa$. But all of these functions are in M_W, because ${V_{\kappa+1}}^{M_W} = V_{\kappa+1}$ and since $U \in M_W$ we get that M_W can compute the equivalence classes. This implies that M_W correct computes the value of $j_U(\kappa)$. Moreover, M_W thinks that $j_U(\kappa) < (2^\kappa)^+$, because there are 2^κ many functions $\kappa \to \kappa$. And since $M_W \models j_W(\kappa)$ is measurable, it thinks that $j_U(\kappa) < (2^\kappa)^+ < j_W(\kappa)$, as desired. □

Corollary 70. *The Mitchell order $\lhd$ on a measurable cardinal κ is well-founded.*

Proof. Because a $\lhd$-descending sequence of measures produces a descending sequence of ordinals. □

We are most interested in looking at the Mitchell order on the normal measures. Given a cardinal κ, let $o(\kappa)$ be the rank of $\lhd$ restricted to the normal measures on κ. So $o(\kappa) = 0$ means κ is not measurable, $o(\kappa) = 1$ means κ is measurable but there is no normal measure on κ which concentrates on the measurable cardinals.[5] And $o(\kappa) \geq 2$ means there is a normal measure on κ which concentrates on the measurable cardinals.

We immediately get that these form a hierarchy in consistency strength. If $o(\kappa = 2)$ then $V_\kappa \models \mathsf{ZFC}$ + there is a proper class of measurable cardinals. And in general, if $o(\kappa) = \alpha$ and $\beta < \alpha$ then the cardinals λ with $o(\lambda) = \beta$ are unbounded below κ.

There is an upper bound to $o(\kappa)$. Normal measures on κ are collections of subsets of κ, so there are at most 2^{2^κ} normal measures. So $o(\kappa) \leq 2^{2^\kappa}$. Working a small amount, we can get a slightly improved upper bound.

Given a normal measure U on κ, let $o(U)$ denote the $\lhd$-rank of U. That is, $o(U)$ is the rank of $\lhd$ restricted to the normal measures W with $W \lhd U$. So, for example, if U does not concentrate on measurable cardinals then $o(U) = 0$.

[5] In general, we say a measure on κ *concentrates* on $X \subseteq \kappa$ if X is a member of the measure.

Lemma 71. *Let o be the sequence $\{o(\alpha) : \alpha < \kappa\}$. If U is a normal measure on κ then $o(U) = [o]_U$.*

Proof. It follows from the definition of o that $[o]_U = o(\kappa)^{M_U}$, where $j_U : V \prec M_U$ is the embedding by U. Next, see that $\lhd$ is absolute to M_U when applied to normal measures $W, W' \lhd U$; this is because M_U contains all functions $\kappa \to \kappa$ and so can reconstruct the ultrapower by $W \lhd U$ and thereby check whether $W' \in M_W$. Thus, we conclude that $o(U)^V = o(\kappa)^M$, completing the argument. □

As a consequence of this lemma plus the fact that $(2^\kappa)^{M_U} < (2^\kappa)^+$, we conclude that $|o(U)| \leq 2^\kappa$. So $o(\kappa) \leq (2^\kappa)^+$.

(Kameryn J. Williams) University of Hawaiʻi at Mānoa, Department of Mathematics, 2565 McCarthy Mall, Keller 401A, Honolulu, HI 96822, USA

E-mail address: kamerynw@hawaii.edu

URL: http://kamerynjw.net