MATH454 HOMEWORK 11
DUE THURSDAY, NOVEMBER 14

Exercise 1. Let $p \in \omega$ and let U_p be the principal ultrafilter on ω consisting of sets which contain p as an element. Show that the ultrapower of $(\omega, <)$ by U_p is isomorphic to $(\omega, <)$. [Hint: read the discussion in the book starting on page 149.]

Exercise 2. Let X be an infinite set. Show that an ultrafilter U on X is nonprincipal if and only if U extends the Fréchet filter on X.

Exercise 3. Let X be a finite nonempty set. Show that every ultrafilter on X is a principal ultrafilter.

Exercise 4. Let X be a finite set of cardinality ≥ 2. Give an example of a filter on X which is not a principal ultrafilter.

Say that a filter $\mathcal{F}$ is *countably closed* if given a countable sequence $\langle A_i : i \in \omega \rangle$ of elements of $\mathcal{F}$ we have that $\bigcap_{i \in \omega} A_i \in \mathcal{F}$.

Exercise 5. Show that no filter on ω can be countably closed.

In class we only talked about ultrapowers using ultrafilters on ω. But you can also talk about ultrapowers using ultrafilters on larger sets.

Exercise 6 (Reach). Let I be a set and U be an ultrafilter on I. Consider the structure $(\omega, <)$. Define relations $=_U$ and $<_U$ on ${}^I\omega$ as:

$$x =_U y \quad \Leftrightarrow \quad \{i \in I : x(i) = y(i)\} \in U$$
$$x <_U y \quad \Leftrightarrow \quad \{i \in I : x(i) < y(i)\} \in U$$

Show that $=_U$ is an equivalence relation and $<_U$ is a congruence relation modulo $=_U$.

Exercise 7 (Reach). I and U are as in the previous exercise. Let $\mathrm{Ult}((\omega, <), U)$ be the structure with domain $\{[x]_U : x \in {}^I\omega\}$ and order relation $<_U$ be the ultrapower of $(\omega, <)$ by U. Show that $\mathrm{Ult}((\omega, <), U)$ is a linear order.

Exercise 8 (Reach). Let I be an uncountable set and suppose that U is a countably closed ultrafilter on I. Show that $\mathrm{Ult}((\omega, <), U)$ is a well-order.