Math 321: Relations and functions

Kameryn J Williams

University of Hawaiʻi at Mānoa

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These functions and relations are themselves objects of mathematical study.

Relations

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The standard in mathematics is to take the extension of a relation as *the* definition of relations.

- A binary relation on a set A is a subset of A^2, the set of ordered pairs from A.
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(You can talk about relations between more than two objects, but binary relations are used the most.)

Some examples

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- $\equiv \pmod 3$ on $\mathbb{Z}$ (that is, a and b have the same remainder when divided by 3)
  This is called equivalence modulo 3.

Some properties a relation can have

Let $\star$ be a binary relation on A.

- $\star$ is reflexive if $a \star a$ for all $a \in A$.
- $\star$ is symmetric if $a \star b$ implies $b \star a$ for all $a, b \in A$.
- $\star$ is transitive if $a \star b$ and $b \star c$ implies $a \star c$ for all $a, b, c \in A$.

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- (RST) We checked earlier that $=$ on $\mathbb{R}$ has all three properties.
- (RT) We checked earlier that $\leq$ on $\mathbb{R}$ is reflexive and transitive but not symmetric.
- (T) We checked earlier that $<$ on $\mathbb{R}$ is transitive but neither reflexive nor symmetric.
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We have examples for all eight cases, completing the proof of the theorem.

Closures of a relation

Let $\star$ be a relation on a set A. We can add new instances to $\star$ to make it satisfy these properties.

- The reflexive closure of $\star$ is the smallest reflexive relation on A which contains $\star$, i.e. as a subset.
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- What is the reflexive closure of the empty relation on $\mathbb{R}$?

Another way to define closures

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Instead, we have to close off $\dagger$ in a recursive process with infinitely many steps.

- Start with $\dagger_0 = \dagger$.
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- Then the transitive closure of $\dagger$ is

$$\bar{\dagger} = \bigcup_{n=0}^{\infty} \dagger_n = \dagger_0 \cup \dagger_1 \cup \dagger_2 \cup \cdots$$

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  This is the equivalence relation used in the statement of the fundamental theorem of arithmetic: when we proved that any two prime factorizations of n must be the same, what we meant is that the two lists were related in this way.

Equivalence classes and partitions

Let $\sim$ be an equivalence relation on A. Then $\sim$ partitions A into equivalence classes.

- Consider $a \in A$. The $\sim$-equivalence class for a is:

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  - Pairwise disjoint means that if we take $X \neq Y$ from the family, then $X \cap Y = \emptyset$

- Each $[a]$ is nonempty.

  Because $a \in [a]$, because $a \sim a$.
- If $[a] \cap [b] \neq \emptyset$, then $[a] = [b]$.

  Fix $c \in [a] \cap [b]$. Consider $x \in [a]$. Then $a \sim x$, $a \sim c$, and $b \sim c$. By symmetry, $x \sim a$ and $c \sim b$. By transitivity, $x \sim b$, and so $b \sim x$. We've seen that $[a] \subseteq [b]$. An identical argument proves $[b] \subseteq [a]$, establishing their equality.
- The union of the $[a]$ is all of A

  Because $a \in [a]$ for each $a \in A$.

Equivalence classes and partitions

Let $\sim$ be an equivalence relation on A. Then $\sim$ partitions A into equivalence classes.

- Consider $a \in A$. The $\sim$-equivalence class for a is:

$$[a] = [a]_\sim = \{b \in A : a \sim b\}.$$

- A partition of A is a family of nonempty pairwise disjoint subsets of A whose union is all of A.
    - Pairwise disjoint means that if we take $X \neq Y$ from the family, then $X \cap Y = \emptyset$

- Each $[a]$ is nonempty.

  Because $a \in [a]$, because $a \sim a$.
- If $[a] \cap [b] \neq \emptyset$, then $[a] = [b]$.

  Fix $c \in [a] \cap [b]$. Consider $x \in [a]$. Then $a \sim x$, $a \sim c$, and $b \sim c$. By symmetry, $x \sim a$ and $c \sim b$. By transitivity, $x \sim b$, and so $b \sim x$. We've seen that $[a] \subseteq [b]$. An identical argument proves $[b] \subseteq [a]$, establishing their equality.
- The union of the $[a]$ is all of A

  Because $a \in [a]$ for each $a \in A$.

Notation: Mathematicians write $A/\sim$ for the family of $\sim$-equivalence classes.