MATH 455 COMPACTNESS HANDOUT
MARCH 9, 2020

Each of these exercises can be solved using the compactness theorem. Write up a full solution of one of these exercises—one which we did not do together in class!—and return it to me by Friday for up to 2 extra points on Homework 6.

Exercise 1. Recall that the 4 color theorem says that every planar graph is 4-colorable, where a graph is n-colorable if you can color its nodes with n many colors so that adjacent nodes get different colors. Assume the 4 color theorem for finite planar graphs and use it to prove the infnite case of the 4 color theorem. [Hint: all you need about planar graphs for this argument is that any subgraph of a planar graph is planar.]

Exercise 2. Show that any partial order on a set X can be extended to a linear order on X. Here, by $<$ extends to $<^*$ I mean that if $x < y$ then $x <^* y$.

Exercise 3. Let T be an extension of the theory of linear orders in the language whose only symbol is $<$. Show that there is $\mathcal{M} \models T$ so that $\mathcal{Q} = (\mathbb{Q}, <)$ is a substructure of $\mathcal{M}$.

Exercise 4. Show there is no set Φ of axioms, in any language, so that given a structure $\mathcal{X}$ we have $\mathcal{X} \models \Phi$ iff $\mathcal{X}$ is finite.

Exercise 5. Let Eq be the theory of equivalence relations, in the language $\mathcal{L}$ with a single binary relation symbol $\equiv$. Show that there is no set Φ of $\mathcal{L}$-sentences so that given $\mathcal{M} \models \mathsf{Eq}$ we have that $\mathcal{M} \models \Phi$ iff $\mathcal{M}$ has an infinite equivalence class.

Exercise 6. Suppose we have three $\mathcal{L}$-structures $\mathcal{M}$, $\mathcal{M}_0$, and $\mathcal{M}_1$ so that $\mathcal{M} \prec \mathcal{M}_0$ and $\mathcal{M} \prec \mathcal{M}_1$. Show that there is an $\mathcal{L}$-structure $\mathcal{N}$ so that $\mathcal{M}_0 \prec \mathcal{N}$ and $\mathcal{M}_1 \prec \mathcal{N}$.