Mediacy and Independence

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120 Years of Choice
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What does it mean to be infinite?

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- X is finite if $|X| < \omega$. Otherwise X is infinite.
- X is infinite iff $|X| \geq n$ for all $n < \omega$.

This isn't circular, because we can define ω by its induction properties.

- X is Dedekind-infinite if there is $f : X \to X$ a non-surjective injection.
- X is Dedekind-finite if any injection $f : X \to X$ is a surjection.
- X is Dedekind-infinite iff $\omega \leq |X|$.

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- Dedekind-infinite $\Rightarrow$ infinite is true in ZF.
- Infinite $\Rightarrow$ Dedekind-infinite uses AC_ω.

Theorem (Cohen 1963)

It is consistent with ZF *that there exists a Dedekind-finite, infinite set.*

But you can't get a reversal: there's no hope the non-existence of a DFI set implies AC because the former is local while the latter is global.

The first question

Question

Is there a suitable generalization of a Dedekind-finite, infinite set whose nonexistence gives a characterization of AC?

A look back in history

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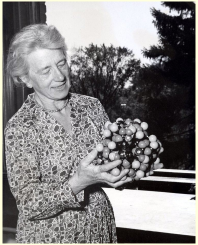
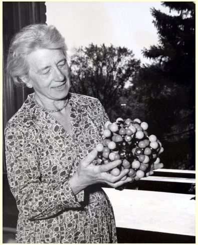
- In the late 1910s, Bertrand Russell is a few years after the last volume of *Principia Mathematica*. His time is occupied by legal troubles over his pacifism during World War I and thinking about the foundations of mathematics.
- Working with him are multiple students, including Dorothy Wrinch.
- The next decade (1923) she will publish a paper answering our first question.

Dorothy Wrinch



- Born 1894, died 1976.
- Studied logic under Russell, did her doctorate (1921) under applied mathematician John Nicholson.
- Wrote in a range of subjects: logic, pure mathematics, philosophy of science, and mathematical biology.
- Was awarded a Rockefeller Foundation fellowship to support her work in mathematical biology.
- Early career was in the UK, later emigrated to the USA. Latter years of her career were at Smith College (Mass, USA).
- Had the misfortune of being on the losing side of a scientific dispute with Linus Pauling over the structure of proteins.

Wrinch's work in logic



- Part of a group of Russell's students who studied mathematical logic with him.
- Her 1917 essay *Transfinite Types* won Girton College's Gamble Prize.
- (1923) Gave a characterization of AC based on a generalization of Dedekind-finite, infinite sets.
- Worked on logic's applications in the philosophy of science.

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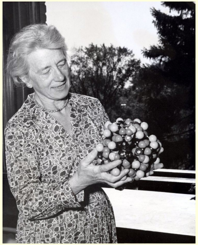

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This one's a stretch:

- She thought her model of protein structure would be a 'theorem', but was later partially vindicated with a consistency result: there are crystals whose molecular structure fit her cyclol model.

Wrinch's question, and mine

Question

Is there a suitable generalization of a Dedekind-finite, infinite set whose nonexistence gives a characterization of AC?

Question

Can we use modern techniques to prove more precise consistency results?

Cardinals sans choice

Notation:

- $\kappa, \lambda, \ldots$ will be used for well-orderable, infinite cardinals.
- $\mathfrak{p}, \mathfrak{q}, \ldots$ will be used for cardinals in general.
- I'll sometimes use $\mathfrak{p}$ to refer to an arbitrary set of cardinality $\mathfrak{p}$.

- Under AC, every cardinal is well-orderable. We can thus define the cardinals as the initial ordinals.
- Without AC we have to fall back on defining cardinals as equivalence classes.
- Can use Scott's trick to make these sets.

Mediate cardinals

Fix a cardinal $\mathfrak{p}$. Then X is $\mathfrak{p}$-mediate if

- $\mathfrak{q} \leq |X|$ for all $\mathfrak{q} < \mathfrak{p}$;
- $\mathfrak{p} \not\leq |X|$; and
- $|X| \not\leq \mathfrak{p}$.

A $\mathfrak{p}$-mediate cardinal is a cardinal number of a $\mathfrak{p}$-mediate set.

Mediate means $\mathfrak{p}$-mediate for some infinite $\mathfrak{p}$.

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- Dedekind-finite infinite $\Leftrightarrow$ $\aleph_0$-mediate.
- ZF proves there are no finite degrees of mediacy.

A few facts

Some facts about DFI sets generalize.

Fact

Suppose $\mathfrak{q}$ and $\mathfrak{r}$ are $\mathfrak{p}$-mediate. Then:

- $\mathfrak{q} + \mathfrak{r}$ *is* $\mathfrak{p}$*-mediate;*
- $\mathfrak{q} \cdot \mathfrak{r}$ *is* $\mathfrak{p}$*-mediate;*

Suppose $\mathfrak{q}$ is κ-mediate. Then:

- $2^{2^{\mathfrak{q}\cdot\mathfrak{q}}}$ *is not* κ*-mediate; and*
- *If κ is an aleph-fixed point then $2^{2^{\mathfrak{q}}}$ is not κ-mediate.*

Wrinch's theorem

Theorem (Wrinch 1923)

Over ZF*, the following are equivalent.*

1. AC*;*
2. *There are no mediate cardinals; and*
3. *There are no κ-mediate cardinals for well-ordered κ.*

- Wrinch originally formulated this result in the framework of *Principia Mathematica*.
- Lévy (1964) independently rediscovered this result, under different language.

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Definition

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Prove $(1 \Rightarrow 2)$ by contrapositive.

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- Suppose $\mathfrak{q}$ is $\mathfrak{p}$ mediate. Then $\mathfrak{p}$ and $\mathfrak{q}$ are incomparable, so Cardinal Trichotomy fails.
- (Hartogs 1915) AC iff Cardinal Trichotomy.

Wrinch's theorem, $(3 \Rightarrow 1)$

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$(2 \Rightarrow 3)$ is trivial. Prove $(3 \Rightarrow 1)$ by contrapositive.

- (Hartogs) For any $\mathfrak{p}$ there is a smallest well-orderable cardinal $\aleph(\mathfrak{p})$ so that $\aleph(\mathfrak{p}) \nleq \mathfrak{p}$.
- If $\mathfrak{p}$ is not well-orderable then $\mathfrak{p}$ is $\aleph(\mathfrak{p})$-mediate.

Dependent choice

Dependent choice (DC) informally says you can make ω many choices where each choice depends on the previous ones.

- Suppose R is a relation on a set X so that for each $x \in X$ there is $y \in X$ with $x \, R \, y$. Then there is a branch $\langle x_i : i \in \omega \rangle$ through R: for each i have $x_i \, R \, x_{i+1}$.

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DC_κ says:

- Suppose R is a relation on $X^{<\kappa} \times X$ so that for each $s \in X^{<\kappa}$ there is $y \in X$ with $s \, R \, y$.
  Then there is a branch $b = \langle x_i : i < \kappa \rangle$ through R: for each i have $(b \restriction i) \, R \, b_i$.

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$\mathrm{DC}_{<\kappa}$ is DC_λ for all $\lambda < \kappa$.

Facts:

- AC is equivalent to $\forall\kappa\ \mathrm{DC}_\kappa$.
- $\lambda < \kappa$ implies $\mathrm{DC}_\kappa \Rightarrow \mathrm{DC}_\lambda$.
- $\mathrm{ZF} + \mathrm{DC}_{<\kappa} + \neg\mathrm{DC}_\kappa$ is consistent.
- DC implies AC_ω over ZF, but not vice versa.
- DC is equivalent to "a relation is well-founded iff it has no infinite descending sequence".
- (Solovay) $\mathrm{ZF} + \mathrm{DC} +$ "every set of reals is Lebesgue-measurable" is consistent.

DC and mediate cardinals

Can get a level-by-level version of Wrinch's theorem.

Lemma: DC_κ implies there are no κ-mediates.

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Lemma: DC_κ implies there are no κ-mediates.

Corollary: AC iff for all κ there are no κ-mediates.

- Suppose $\lambda \leq \mathfrak{p}$ for all $\lambda < \kappa$ but $\mathfrak{p} \nleq \kappa$.
- Consider the collection of all injections $\alpha \to \mathfrak{p}$ for $\alpha < \mathfrak{p}$.
- None of the injections are onto, so you can always extend them to an injection $\alpha + 1 \to \mathfrak{p}$.
- By DC_κ there's a branch, which gives an injection $\kappa \to \mathfrak{p}$.

An alternate proof of the connection to DC_κ

W_κ asserts that every set has cardinality comparable to κ: either κ injects into X or X injects into κ.

- DC_κ implies W_κ. (See Chapter 8 of Jech's monograph.)
- W_κ implies there are no κ-mediates.

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Note: It's known that $\mathsf{ZF} + W_\kappa + \neg\mathsf{DC}_\kappa$ is consistent, so the nonexistence of κ-mediates cannot imply DC_κ.

Refining mediacy

Observation:

- If $\mathfrak{p}$ is κ-mediate and $\lambda > \kappa$ then $\mathfrak{p} + \lambda$ is λ^+-mediate.
- So if there is a degree of mediacy then every larger successor cardinal is also a degree of mediacy.

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Lemma: If $\mathfrak{p}$ is κ-mediate where κ is smallest such that κ-mediates exist, then $\mathfrak{p}$ is exact κ-mediate.

Consistency questions

Question

- *Consistently, what can be the smallest degree of mediacy?*
- *Consistently, what can be the class of degrees of exact mediacy?*

Symmetric extensions

Motivating example (Cohen's first model):
Add ω many reals, then forget the order you added them.

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- $\mathbb{P} = \mathrm{Add}(\omega, \omega)$ is the poset. Conditions are finite partial functions $\omega \times \omega \to 2$.
- Changing the order is permuting the columns in the $\omega \times \omega$ grid.
- Any permutation ϖ of ω generates an automorphism of $\mathbb{P}$:
  $\varpi p(n, i) = p(\varpi n, i)$.
- Also generates an automorphism on the $\mathbb{P}$-names:
  $\varpi\sigma = \{(\varpi\tau, \varpi p) : (\tau, p) \in \sigma\}$

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- "Forgetting the order" is restricting to names fixed by a 'large' group of automorphisms:
  A group H of automorphisms is large if there is finite $e \subseteq \omega$ so that each $\varpi \in H$ fixes e pointwise: $H \supseteq \mathrm{fix}(e)$.
- This gives a normal filter $\mathcal{F}$ on the lattice of subgroups.
- A name σ is $\mathcal{F}$-symmetric if $\mathrm{sym}(\sigma) = \{\varpi : \varpi\sigma = \sigma\} \in \mathcal{F}$.
- The symmetric extension consists of the interpretations of all hereditarily symmetric names.

Symmetric extensions, in general

A symmetric system is $(\mathbb{P}, G, \mathcal{F})$ so that

- $\mathbb{P}$ is a forcing poset;
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A $\mathbb{P}$-name σ is symmetric if $\operatorname{sym}(\sigma) \in \mathcal{F}$.

- (Symmetry lemma) $p \Vdash \varphi(\sigma)$ iff $\varpi p \Vdash \varphi(\varpi\sigma)$.

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The symmetric extension by $(\mathbb{P}, G, \mathcal{F})$ via a generic $g \subseteq \mathbb{P}$:

- Consists of the interpretations of hereditarily symmetric names.
- $\mathrm{V}[g/\mathcal{F}] = \{\sigma^g : \sigma \text{ is } \mathcal{F}\text{-HS}\}$.

$\mathrm{V}[g/\mathcal{F}] \models \mathsf{ZF}$, but the point is to make AC fail in a controlled way.

The general Cohen symmetric extension

Fix regular κ and assume $\kappa^{<\kappa} = \kappa$.

- $\mathbb{P}_\kappa = \mathrm{Add}(\kappa, \kappa)$;
- $G_\kappa \leq \mathrm{Aut}(\mathbb{P}_\kappa)$ is generated by permutations of κ;
- $H \in \mathcal{F}_\kappa$ if $\exists e \in [\kappa]^{<\kappa}$ so that $\mathrm{fix}(e) \subseteq H$.

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Facts:

- $\mathbb{P}_\kappa$ is κ-closed and has the κ^+-cc.
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In particular, there will be no λ-mediates for $\lambda < \kappa$.

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Note: The cardinal arithmetic assumption gives the smallest possible chain condition and ensures $\mathcal{F}_\kappa$ has the smallest possible basis. These will be used to tightly control the degrees of exact mediacy.

Symmetric extensions and dependent choice

Lemma: Let κ be regular and $\lambda < \kappa$. If $\mathbb{P}$ is κ-closed and $\mathcal{F}$ is κ-complete then $(\mathbb{P}, G, \mathcal{F})$ preserves DC_λ.

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- Consider appropriate $R \subseteq X^{<\lambda} \times X$ in $\mathrm{V}[g/\mathcal{F}]$. We need a branch through R in $\mathrm{V}[g/\mathcal{F}]$.
- By κ-closure λ remains a cardinal in $\mathrm{V}[g]$.
- In $\mathrm{V}[g]$, by DC_λ there is a branch $b = \langle x_i : i < \lambda \rangle$.
- Each x_i comes from a symmetric name $\dot{x}_i$.
- By κ-completeness $H = \bigwedge_{i<\lambda} \mathrm{sym}(\dot{x}_i)$ is in $\mathcal{F}$.
- Can get a name $\dot{b}$ for b with $\mathrm{sym}(\dot{b}) \supseteq H$.
- So the branch b is in $\mathrm{V}[g/\mathcal{F}]$.

The smallest mediate can be anything

Theorem (Lévy (1964); W.)

Suppose $\kappa = \kappa^{<\kappa}$ is regular. In the symmetric extension by $(\mathbb{P}_\kappa, G_\kappa, \mathcal{F}_\kappa)$:

- $\mathsf{DC}_{<\kappa}$;
- *κ is the smallest degree of mediacy; and*
- *κ is the only degree of exact mediacy.*

(Lévy proved the first two in ZFA*.)*

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We've already seen $\mathrm{DC}_{<\kappa}$.

Claim: Let A be the set of the Cohen subsets of κ added by $\mathbb{P}_\kappa$. Then $\mathrm{V}[g/\mathcal{F}_\kappa] \models A$ is κ-mediate.

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- $\lambda < \kappa$ injects by κ-closure of $\mathbb{P}_\kappa$ and κ-completeness of $\mathcal{F}_\kappa$
- $|A| \not\leq \kappa$ because A can't be well-ordered.
- $\kappa \not\leq |A|$:

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- $|A| \nleq \kappa$ because A can't be well-ordered.
- $\kappa \nleq |A|$:
  - Suppose $\dot{f}$ is hereditarily symmetric, $\mathrm{sym}(\dot{f}) \supseteq \mathrm{fix}(e)$, and $p \Vdash \dot{f} : \kappa \to A$ is one-to-one.
  - Extend p to q deciding $\dot{f}(\alpha) = c_i$ for some $\alpha \neq i$ both $\notin e$.
  - Find ϖ fixing $e \cup \{i\}$, moving α, and $q \parallel \varpi q$.
  - So $q \cup \varpi q \Vdash \dot{f}$ is not one-to-one. Contradiction.

The smallest mediate can be anything

Theorem (Lévy (1964); W.)

Suppose $\kappa = \kappa^{<\kappa}$ is regular. In the symmetric extension by $(\mathbb{P}_\kappa, G_\kappa, \mathcal{F}_\kappa)$:

- $\mathsf{DC}_{<\kappa}$;
- *κ is the smallest degree of mediacy; and*
- *κ is the only degree of exact mediacy.*

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Suppose $\mathrm{V}[g/\mathcal{F}_\kappa] \models X$ is exact λ-mediate for $\lambda > \kappa$.

- First use exactness plus the chain condition to argue that $\mathrm{V}[g]$ has an injection $\lambda \to X$.
  Compare: if A is the set of Cohen generics then $A \cup \mu$ is μ^+-mediate in the symmetric extension but has cardinality μ in $\mathrm{V}[g]$.
- Then use the chain condition plus $\mathcal{F}_\kappa$ having a basis of size $< \lambda$ to get an injection $\lambda \to X$ in $\mathrm{V}[g/\mathcal{F}_\kappa]$.
- Contradiction.

Doing it more than once

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- Karagila and others have worked on iterations of symmetric extensions.
- It's complicated, with scary group theoretic objects like wreath products.
- We are lucky and can get away with products, where the details are significantly less technical.

Products of symmetric extensions

Suppose $(\mathbb{P}, G, \mathcal{F})$ and $(\mathbb{Q}, H, \mathcal{E})$ are symmetric systems. Can define their product $(\mathbb{P}, G, \mathcal{F}) \times (\mathbb{Q}, H, \mathcal{E})$:

- $\mathbb{P} \times \mathbb{Q}$ is usual product of posets;
- $G \times H$ is generated by (ϖ, ϱ) with $\varpi \in G$, $\varrho \in H$; and
- $\mathcal{F} \times \mathcal{E}$ is generated by $G_0 \times H_0$ for $G_0 \in \mathcal{F}$ and $H_0 \in \mathcal{E}$.

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Like with forcing, we have a product lemma stating that the extension by the product is the same as the two-step extensions, in either order.

Can also do this for infinite products, with a notion of support.

- Suppose $(\mathbb{P}_\kappa, G_\kappa, \mathcal{F}_\kappa)$ are symmetric systems for $\kappa \in M$.
- Then there is a product $\prod_{\kappa \in M} (\mathbb{P}_\kappa, G_\kappa, \mathcal{F}_\kappa)$ with that support.
- Again we get a product lemma stating we can split the full extension into two-step extensions.

Note: We do not allow permuting the multiplicands.

Refining earlier ideas

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Suppose $\lambda < \kappa$ are regular.

- $(\mathsf{ZF} + \mathsf{DC}_\kappa)$ If $\mathbb{P}$ is κ-closed and $\mathcal{F}$ is κ-complete then $(\mathbb{P}, G, \mathcal{F})$ preserves DC_λ.
- $(\mathsf{ZF} + \mathsf{DC}_\kappa)$ Suppose $\mathbb{P}$ has the λ^+-cc and $\mathcal{F}$ is generated by a basis of size $\leq \lambda$. Then $\mathrm{V}[g/\mathcal{F}] \models$ there are no exact κ-mediates.

Preserving mediacy

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I couldn't quite see how to make the argument work for mediacy, but with a slight strengthening it goes through.

Lemma: $(\mathsf{ZF} + \mathsf{DC}_\kappa)$ If X is exact κ-mediate$^{+\varepsilon}$ then X remains exact κ-mediate in an extension by a forcing with the κ-cc.

The basic one-step construction gives exact mediacy$^{+\varepsilon}$.

The "$+\varepsilon$" is strengthening a $\not\leq$ to $\not\leq^*$ in the definition:

- X is κ-mediate$^{+\varepsilon}$ if
  - $\lambda \leq |X|$ for all $\lambda < \kappa$;
  - $\kappa \not\leq |X|$; and
  - $|X| \not\leq^* \kappa$: there is no surjection $\kappa \to X$.
- Define exact κ-mediate$^{+\varepsilon}$ analogously to exact mediacy: X remains κ-mediate$^{+\varepsilon}$ after removing a small set.

The pattern of the exact mediates

Theorem (W.)

Assume GCH *and fix a class M of regular cardinals. Do the Easton support product of the $(\mathbb{P}_\kappa, G_\kappa, \mathcal{F}_\kappa)$ for $\kappa \in M$. In the symmetric extension, M is exactly the class of regular degrees of exact mediacy.*

Note: This symmetric extension preserves inaccessibles, and probably more large cardinals.

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Sketch:

- $\mathbb{P}_{>\alpha}$ is α-closed and $\mathcal{F}_{>\alpha}$ is α-complete.
- $\mathbb{P}_{<\alpha}$ has the α^+-cc and $\mathcal{F}_{<\alpha}$ is generated by a basis of cardinality $\leq \alpha$.
- In $\mathrm{V}[g_{>\alpha}/\mathcal{F}_{>\alpha}]$: DC_α is true. So there are no α-mediates.
- In $\mathrm{V}[g_{>\alpha}/\mathcal{F}_{>\alpha}][g_{<\alpha}/\mathcal{F}_{<\alpha}]$: there are no exact α-mediates.
- So the only way there could be an exact α-mediate is if it was added by $(\mathbb{P}_\alpha, G_\alpha, \mathcal{F}_\alpha)$ for $\alpha \in M$.
- But we already know that adds an exact mediate, and that's preserved.

Where does Wrinch fit into 120 years of choice?

- Hartogs (1915) is the root of a tree of research about the possibilities for the ordering of cardinals.
- This includes recent work by people in this room.
- It is a subtree of the tree with root Zermelo (1904).
- Wrinch (1923) should be seen as an early precursor to post-Cohen work in this tree.

Some questions

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- What's up with singular cardinals?
- What if we don't make such strong cardinal arithmetic assumptions?
- What happens if AC fails badly in the ground model?

Thank you!

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