MATH 211: 3/14 WORKSHEET

Trig substitution is helpful for integrals containing one of the three patterns $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$. Transforming to an angle domain lets us use the pythagorean identity to simplify an integrand. Here is a summary of the substitution to use:

Integral contains	Use substitution
$\sqrt{a^2 - x^2}$	$x = a \sin\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec\theta$

Trig substitution produces integrals involving powers of trig functions, for which the formulas from section 7.5 are useful.

Use trig substitution to calculate the following integrals. Some can be solved by other methods. If so, say what the alternative method is.

(1) $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x^2 - 1}}$

(2) $\displaystyle\int \frac{\sqrt{4 + x^2}}{x}\,\mathrm{d}x$

(3) $\displaystyle\int x^2(16 - x^2)^{3/2}\,\mathrm{d}x$

(4) $\displaystyle\int \frac{x}{3 - x^2}\,\mathrm{d}x$

Some of the applications we saw earlier give integrals you can solve using trig substitution.

(1) Use trig substitution to calculate an integral giving the circumference (arc length) of the circle of radius 1.

(2) The curve $y = x^3$, where $0 \le x \le 1$ is rotated around the x-axis.
   (a) Say why the following integral gives the surface area of the resulting surface:
$$\int_0^1 x^3\sqrt{1+9x^4}\,\mathrm{d}x.$$
   (b) Use the substitution $u = 3x^2$ to transform this into the u-domain. To do this, you will need to say what x^3 is terms of u and what $\mathrm{d}x$ is in terms of u and $\mathrm{d}u$.
   (c) Having done this substitution you should have an integral involving $\sqrt{1+u^2}$. Use trig substitution to solve this integral. You should get a number at the end, since you are evaluating a definite integral.

You might wonder why we only use half the trig functions for trig substitution. Can we use the other three? You can, but they don't give you any new power.

(1) Integrate $\displaystyle\int \frac{\sqrt{4+x^2}}{x}\,\mathrm{d}x$ using the substitution $x = 2\cot\theta$. Compare to your solution from the previous page using the substitution $x = 2\tan\theta$.

(2) When could you use the substitution $x = a\cos\theta$? When could you use the substitution $x = a\csc\theta$?

(3) Indeed, the trig functions aren't the only functions you can do this kind of substitution with. For the trig functions, what makes it work is the pythagorean identity. Other functions with a similar identity can also be used.
   (a) The *hyperbolic sine* and *hyperbolic cosine* are defined as:
$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \text{(pronounced “cinch”)}$$
$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \text{(pronounced “cosh”)}$$
   Check that $\frac{\mathrm{d}}{\mathrm{d}x}\sinh x = \cosh x$ and $\frac{\mathrm{d}}{\mathrm{d}x}\cosh x = \sinh x$. Use this to determine their integrals.
   (b) Check the *hyperbolic identity* $\sinh^2 x - \cosh^2 x = 1$.
   (c) Using this identity we can use *hyperbolic substitution.* Integrate the following integral using the substitution $x = a\sinh\theta$.
$$\int \sqrt{a^2+x^2}\,\mathrm{d}x$$
   (d) Integrate the following using the substitution $x = a\cosh\theta$.
$$\int \sqrt{x^2-a^2}\,\mathrm{d}x$$
   (e) Like we did with trig substitution, work out the general pattern for when you can do hyperbolic substitution.