MATH655 LECTURE NOTES: PART $\frac{1}{2}$

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In this part of the course we will review some logical results we will need in the remainder.

Mathematical logicians settled upon first-order logic as *the* standard. Like other conventions in mathematics, the history of this is littered with controversy and disagreement. But we will take the usual approach of ignoring all that, barely touching upon alternative logics.

First-order logic has two main flavors of advantages. The first is that it enjoys many nice metalogical properties—compactness, completeness, Löwenheim–Skolem theorems, and so on. The second is that it has proven to be well suited for applications within mathematics. We will see some of these nice properties below. And the in parts 1 through 3 of the course we will get exposed to some applications.

A first-order *structure* or *model* is a set M equipped with some number of constants, functions, and relations over M, where we require all functions and relations to have finite arity. That is, each function is $f : M^n \to M$ and each relation is $R \subseteq M^n$. I will write $(M; c, f, R, \ldots)$ for the structure with universe M and constants, functions, and relations $c, f, R, \ldots$. I will often abuse notation and refer to the structure simply as M, leaving the rest implicit.

Example 1. The ordered ring $(\mathbb{Z}; 0, 1, +, \cdot, <)$ is a first-order structure. Another example would be the structure $(V_\kappa, \in)$ for a cardinal κ.

For many applications, we can assume that our structures only have relations, with no constants nor functions. (For example, we could instead consider $\mathbb{Z}$ where we attached the graphs of $+$ and $\cdot$ as relations, rather than having the functions.) And since we are mainly interested in set theoretic structures, transitive sets equipped with the membership relation, this restriction will be of no harm to us.

Given a structure $(M; R_0, \ldots, R_i, \ldots)$, the *signature* of M is the map s which sends i to the arity of R_i. The *language* of M is the collection of first-order formulae with the usual logical symbols $\exists, \forall, \wedge, \vee, \neg, =$, countably many variable symbols, and nonlogical relational symbols R_i each of arity $s(i)$. Following standard convention, we will tend to not notationally distinguish R_i the relation on M from R_i the symbol. For example, we will use "+" to refer to both the addition operation on $\mathbb{Z}$ and the formal symbol used in formulae. In case I want to emphasize the relation on M, I will write ${R_i}^M$.

To be more precise, the language of M is the collection of formulae recursively defined by the following schema. Here, I use $x, y, z, \ldots$ as metasyntactic variables for logical variables. It is to be understood that, e.g., saying $x = y$ is in the language of M means that corresponding formula is in the language of M for any choice of logical variables.

- $x = y$ is in the language of M;
- $R_i(x_0, \ldots, x_{n-1})$ is in the language of M, where R_i has arity n;
- If φ and ψ are in the language of M, then so are $\neg(\varphi)$, $(\varphi) \wedge (\psi)$, and $(\varphi) \vee (\psi)$.

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- If φ is in the language of M, then so are $\exists x(\varphi)$ and $\forall x(\varphi)$. For this case, we will tacitly assume that x is not a *bound variable* of φ, meaning under the scope of a quantifier in φ.

  The intention here is that quantification is to be over elements of a structure. Contrast this with second-order logic, which also allows quantification over subsets of the structure.

For readability, we will freely drop parentheses where it is unambiguous. E.g. we write $\varphi \wedge \psi \wedge \theta$ rather than $(\varphi) \wedge ((\psi) \wedge (\theta))$ or any other way of putting parentheses in, all of which are logically equivalent.

An important feature of first-order logic is that formulae are finite. There are infinitary logics, allowing infinite conjunctions/disjunctions or infinite quantifier blocks, but we will not talk about them.

Example 2. The language of set theory is built up from the logical symbols and the relational symbol $\in$ of arity 2. As is common for binary relations, we will write this in infix, writing $x \in y$ rather than $\in(x, y)$.

In addition to describing the *syntax* for first-order logic, the rules for the formal languages and symbol manipulation, we must also talk about the *semantics*, the connection between the formal language and structures. If we are being sticklers, then we first must talk about substitution—replacing free variables (= unbound variables) in a formula with elements from a structure. This works exactly as you expect, but I will skip over the picky details. I will simply write φ to refer to a formula, possibly after substitution. If I want to emphasize that a formula has a free variable I will write $\varphi(x)$ or similar. And to emphasize that an element a has been substituted for the free variable I will write $\varphi(a)$.

Definition 3 (Tarski)**.** Let $(M; R, \ldots)$ be a structure. Then the satisfaction relation $M \models \varphi(\bar{a})$ between formulae φ in the language of M and tuples $\bar{a}$ of elements from M is recursively defined as follows. Think: "snow is white" is true iff snow is white.

- $M \models a = b$ iff $a = b$;
- $M \models R(\bar{a})$ iff $\bar{a} \in R^M$;
- $M \models \varphi \wedge \psi$ iff $M \models \varphi$ and $M \models \psi$;
- $M \models \varphi \vee \psi$ iff $M \models \varphi$ or $M \models \psi$;
- $M \models \neg\varphi$ iff $M \not\models \varphi$;
- $M \models \exists x\ \varphi(x)$ iff there is $a \in M$ so that $M \models \varphi(a)$; and
- $M \models \forall x\ \varphi(x)$ iff for every $a \in M$ we have $M \models \varphi(a)$.

If T is a *theory in the language of* M—a set of formulae in the language of M—we write $M \models T$ whenever $M \models \varphi$ for each $\varphi \in T$. And we write $T \models \varphi$ if whenever $M \models T$ we have $M \models \varphi$.

I will not say any details about formal proofs, but you should know that you can formally define a proof system for first-order logic. Moreover, the proof system can be defined to be *effective*. If you start out with a computable set of nonlogical symbols then recognizing formulae in the language, checking whether a proof is valid, etc. are all computable procedures. If T is a set of formulae, we write $T \vdash \varphi$ if there is a formal proof of φ using T as axioms.

Lacking an explication of a proof system, we cannot prove the following two results. Nevertheless, they are important and you should be aware of them.

Theorem 4 (Soundness theorem)**.** *If $T \models \varphi$ then $T \vdash \varphi$.*

Theorem 5 (Gödel's completeness theorem)**.** *If $T \vdash \varphi$ then $T \models \varphi$.*

An immediate corollary of these results is that a theory is consistent—i.e. it does not prove a contradiction—iff it has a model.

Another result you should know of but I shan't prove is the compactness theorem. You can immediately derive it from the completeness theorem plus the fact that a proof can only use finitely many axioms. The more model theoretically inclined may prefer to prove it using ultraproducts.

Theorem 6 (Compactness theorem)**.** *A theory T has a model iff every finite $T_0 \subseteq T$ has a model.*

A central notion in this context is that of the elementary submodel.

Definition 7. M is a *submodel* of N, written $M \subseteq N$, if they are structures with the same arity and for each relation symbol R in their language we have $R^M = R^N \cap M^k$, where k is the arity of R. That is, M is the restriction of N to a smaller domain.

M is an *elementary submodel* of N, written $M \prec N$, if $M \subseteq N$ and for all $\bar{a} \in M$ we have $M \models \varphi(\bar{a})$ iff $N \models \varphi(\bar{a})$. In short, M is an elementary substructure of N if they agree on the truth about elements of M.

Exercise 8. Show that $(\mathbb{Q}, <)$, that is the rationals equipped only with their order, is an elementary substructure of $(\mathbb{R}, <)$. Show that $(\mathbb{Q}, +, \cdot, <)$ is *not* an elementary substructure of $(\mathbb{R}, +, \cdot, <)$.

The following theorem gives a nice test for being an elementary substructure.

Theorem 9 (Tarski–Vaught test)**.** *Suppose that M is a first-order structure and $N \subseteq M$ is a substructure of M. Then $N \prec M$ iff the following test is satisfied: for any formula $\varphi(x, \bar{y})$ in the language of M and any $\bar{a} \in N$, if $M \models \exists x\ \varphi(x, \bar{a})$ then such an x is in N.*

Proof. ($\Rightarrow$) Fix $\bar{a} \in N$ and a formula $\varphi(x, \bar{a})$. If $M \models \exists x\ \varphi(x, \bar{a})$ then by elementarity $N \models \exists x\ \varphi(x, \bar{a})$. So pick $b \in N$ so that $N \models \varphi(b, \bar{a})$. Then by elementarity $M \models \varphi(b, \bar{a})$, as desired.

($\Leftarrow$) We prove this by induction on formulae. The atomic case holds because N is a substructure of M. The boolean cases are straightforward. For quantifiers, it suffices to consider only existential quantifiers, since every formula is logically equivalent to a formula without universal quantifiers. So fix $\bar{a} \in N$ and assume that for all $b \in N$ we have that $N \models \varphi(b, \bar{a})$ iff $M \models \varphi(b, \bar{a})$. We want to see $N \models \exists x\ \varphi(x, \bar{a})$ iff $M \models \exists x\ \varphi(x, \bar{a})$. The forward direction is immediate—if $b \in N$ witnesses that $N \models \exists x\ \varphi(x, \bar{a})$ then b also witnesses that $M \models \exists x\ \varphi(x, \bar{a})$. (But notice that we appealed to the inductive hypothesis.) For the backward direction, the Tarski–Vaught test tells us that if there is a witness to $M \models \exists x\ \varphi(x, \bar{a})$, then there is a witness in N. $\square$

Theorem 10 (Downward Löwenheim–Skolem)**.** *Let T be a first-order theory in a language $\mathcal{L}$. Suppose $M \models T$ is infinite and let $X \subseteq M$. Then there is $N \prec M$ so that $X \subseteq N$ and $|N| = \max\{\aleph_0, |X|, |\mathcal{L}|\}$. In particular, if T is in a finite language then there are elementary submodels of M of every infinite cardinality $\leq |M|$.*

Proof. First, let us see that there are *Skolem functions* for M. That is, for each formula $\varphi(x, \bar{y})$ in the language of M there is a partial function s_φ so that if $M \models \exists x\ \varphi(x, \bar{a})$ then if $b = s_\varphi(\bar{a})$ we have $M \models \varphi(b, \bar{a})$. Such functions exist because we can well-order M. Namely, fix a well-order of M and let $s_\varphi(\bar{a})$ be the least, according to that well-order, witness that $M \models \exists x\ \varphi(x, \bar{a})$, if such exists.

Now we can construct N, which we do in ω many stages. Start with $N_0 = X$. Given N_k, let N_{k+1} be the closure of N_k under the Skolem functions for M. That is, $b \in N_{k+1}$ if there are $\bar{a} \in N_k$ and a formula $\varphi(x, \bar{y})$ in the language of M so that $b = s_\varphi(\bar{a})$. Finally, set $N = \bigcup_{k<\omega} N_k$.

Let us see that $N \prec M$. We do this by using the Tarski–Vaught test. So fix $\bar{a} \in N$ and suppose that $M \models \exists x\ \varphi(x, \bar{a})$. Fix k large enough so that $\bar{a} \in N_k$. Then $b = s_\varphi(\bar{a}) \in N_{k+1} \subseteq N$ and $M \models \varphi(b, \bar{a})$. So the Tarski–Vaught test is passed, so $N \prec M$.

Finally, we must see that N has the claimed cardinality. I leave this for you as an exercise in cardinal arithmetic. □

The N we constructed in the above proof is called the *Skolem Hull of* X (with respect to the choice of Skolem functions).

There is also an upward Löwenheim–Skolem theorem, asserting that M has elementary supermodels of larger cardinality. To prove it we will need some new tools. And since these tools will be hella important later, they are worth spending time on.

Definition 11. Let X be a set. A *filter* on X is a set $\mathcal{F} \subseteq \mathcal{P}(X)$ with the following properties.

(1) $X \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$;
(2) If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$; and
(3) If $A \in \mathcal{F}$ and $B \supseteq A$ then $B \in \mathcal{F}$.

The dual notion is that of an ideal.

Definition 12. Let X be a set. An *ideal* on X is a set $\mathcal{I} \subseteq \mathcal{P}(X)$ with the following properties.

(1) $\emptyset \in \mathcal{I}$ and $X \notin \mathcal{I}$;
(2) If $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$; and
(3) If $A \in \mathcal{I}$ and $B \subseteq A$ then $B \in \mathcal{I}$.

Exercise 13. Let X be a set. Show that if $\mathcal{I}$ is an ideal on X then $\{A : X \setminus A \in \mathcal{I}\}$ is a filter on X. And show that if $\mathcal{F}$ is a filter on X then $\{A : X \setminus A \in \mathcal{F}\}$ is an ideal on X. These are known as the *dual* filter/ideal.

We think of filters as giving a notion of largeness for subsets of X, whereas ideals give a notion of smallness for subsets of X.

Exercise 14. Let X be an infinite set. The *Fréchet filter* on X is $\mathcal{F} = \{Y \subseteq X : X \setminus Y \text{ is finite}\}$. Show that the Fréchet filter is indeed a filter.

We are especially intetrested in certain filters, known as ultrafilters.

Definition 15. A filter $\mathcal{U}$ on X is called an *ultrafilter* if it has the property that for every $A \subseteq X$ either $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$. An ultrafilter $\mathcal{U}$ is *principal* if there is $x \in X$ so that $A \in \mathcal{U}$ iff $x \in A$. Otherwise, $\mathcal{U}$ is *nonprincipal*.

Lemma 16. *The ultrafilters are precisely the maximal filters. That is, if X is a set and $\mathcal{F}$ is a filter on X, then $\mathcal{F}$ is an ultrafilter iff there is no filter $\mathcal{G}$ on X with $\mathcal{F} \subsetneq \mathcal{G}$.*

Proof. ($\Rightarrow$) Assume that $\mathcal{F}$ is an ultrafilter. If there were a filter $\mathcal{G} \supsetneq \mathcal{F}$ on X, then there would be $A \in \mathcal{G} \setminus \mathcal{F}$. But since $A \notin \mathcal{F}$, we would have that $X \setminus A \in \mathcal{F}$. Then $A \cap (X \setminus A) = \emptyset \in \mathcal{G}$, contradicting that $\mathcal{G}$ is a filter.

($\Leftarrow$) By contrapositive. Suppose that $\mathcal{F}$ is not an ultrafilter. Then there is $A \subseteq X$ so that neither A nor $X \setminus A$ is in $\mathcal{F}$. It must be that at least one of these two sets is not in the ideal dual to $\mathcal{F}$; otherwise, without loss of generality assume that this holds for A. Now define

$$\mathcal{G} = \{A \cap F : F \in \mathcal{F}\} \cup \mathcal{F}.$$

It's clear that $\mathcal{G}$ is closed under superset and intersection. And because A is not in the ideal dual to $\mathcal{F}$ we get that $\emptyset \notin \mathcal{G}$. So $\mathcal{G}$ is a filter extending $\mathcal{F}$, and we are done. □

Theorem 17 (Tarski)**.** *Let $\mathcal{F}$ be a filter on X. Then there is an ultrafilter $\mathcal{U}$ on X with $\mathcal{F} \subseteq \mathcal{U}$.*

Proof. By the lemma it is enough to show that $\mathcal{F}$ extends to a maximal filter. This can be proved by a Zorn's lemma argument. (Exercise: do it!) □

Exercise 18. Fix a regular cardinal κ. Say that $C \subseteq \kappa$ is *closed* if for every $\alpha < \kappa$ we have $\sup \alpha \cap C \in C$. And C is *unbounded* if for all $\alpha < \kappa$ we have $\beta > \alpha$ in C. Subsets of κ which are both closed and unbounded are known as *clubs* (for **cl**osed + **unb**ounded). Show that the club subsets of κ form a filter.

Exercise 19. Again fix a regular cardinal κ. Say that $S \subseteq \kappa$ is *stationary* if $S \cap C \neq \emptyset$ for every club $C \subseteq \kappa$. If S is not stationary, then it is called *nonstationary*. Show that the nonstationary subsets of κ form an ideal which is dual to the club filter.

Using ultrafilters we can take the ultraproduct of a structure. Indeed, the machinery works just for filters, but using ultrafilters gives especially nice properties.

Definition 20. Let $\mathcal{F}$ be a filter on a cardinal κ and fix a first-order structure $(M; R, \ldots)$, where $R, \ldots$ are relations on M. We can then talk about the reduced product $M^\kappa/\mathcal{F}$, which is defined as follows.

Define the equivalence relation $=_\mathcal{F}$ on functions $\kappa \to M$ as $f =_\mathcal{F} g$ if $\{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in \mathcal{F}$. Then ${}^\kappa M/{=_\mathcal{F}}$ will be the domain for $M^\kappa/\mathcal{F}$. The relations of $M^\kappa/\mathcal{F}$ are then defined as follows.

- For each relation R on M, define $R_\mathcal{F}$ on ${}^\kappa M$ as
$$\bar{f} \in R_\mathcal{F} \Leftrightarrow \{\alpha < \kappa : (f_0(\alpha), \ldots, f_k(\alpha)) \in R\} \in \mathcal{F}.$$

In case $\mathcal{F}$ is an ultrafilter we call the reduced product $M^\kappa/\mathcal{F}$ the *ultrapower of M by $\mathcal{F}$*. It is sometimes written $M^\kappa/\mathcal{F} = \mathrm{Ult}(M, \mathcal{F})$.

The following exercise shows we can consider $R_\mathcal{F}$ as being a relation on $M^\kappa/\mathcal{F}$, and so $(M^\kappa/\mathcal{F}; (R_0)_\mathcal{F}, \ldots)$ is a structure in the same language as M.

Exercise 21. Show that $R_\mathcal{F}$ is a congruence with respect to $=_\mathcal{F}$. That is, if $\bar{f} \in R_\mathcal{F}$ and $f_i =_\mathcal{F} g_i$ for each i then $\bar{g}_i \in R_\mathcal{F}$.

Observe that M canonically embeds into $M^\kappa/\mathcal{F}$ via the embedding $x \mapsto c_x$, where c is the constant function $c_x(\alpha) = x$. If $\mathcal{F}$ is an ultrafilter, then this embedding is elementary.

Theorem 22 (Łoś[1])**.** *Suppose $\mathcal{U}$ is an ultrafilter on κ and let M be a first-order structure, where we assume M only has relations attached. Then,*
$$M^\kappa/\mathcal{U} \models \varphi([f_0]_\mathcal{U}, \ldots, [f_n]_\mathcal{U}) \Leftrightarrow \{\alpha < \kappa : M \models \varphi(f_0(\alpha), \ldots, f_n(\alpha))\} \in \mathcal{U}.$$
In particular, M elementarily embeds into $M^\kappa/\mathcal{U}$, via the map $x \mapsto c_x$.

Proof. For convenience, set $N = M^\kappa/\mathcal{U}$. We prove this by induction on formulae.

(Atomic) This was handled by the above exercise.

(Conjunction) Assume the inductive hypothesis for φ and ψ. The forward direction of the implication follows from $\mathcal{U}$ being closed under intersection, and the backward direction of the implication follows from $\mathcal{U}$ being closed under superset.

[1] "Łoś" is approximately pronounced as "Wash". (Please don't tell any Poles about my awful pronounciation.)

(Negation) Assume the inductive hypothesis for φ. Then

$$\begin{aligned} N \models \neg\varphi &\Leftrightarrow N \not\models \varphi \\ &\Leftrightarrow \{\alpha < \kappa : M \models \varphi\} \notin \mathcal{U} \\ &\Leftrightarrow \{\alpha < \kappa : M \not\models \varphi\} \in \mathcal{U} \\ &\Leftrightarrow \{\alpha < \kappa : M \models \neg\varphi\} \in \mathcal{U}. \end{aligned}$$

Note that this uses the complement property of ultrafilters.

(Existential quantifier) Assume the inductive hypothesis for $\varphi(x, y)$ for all substitutions into the variable x and for a fixed substitution g into y. (The case with multiple parameters works similarly.) We want to see that

$$N \models \exists x\ \varphi(x, [g]_\mathcal{U}) \Leftrightarrow \{\alpha < \kappa : M \models \exists x\ \varphi(x, g(\alpha)\} \in \mathcal{U}.$$

For the forward direction, assume that $N \models \exists x\ \varphi(x, [g]_\mathcal{U})$. Then, there is $[f]_\mathcal{U}$ so that $N \models \varphi([f]_\mathcal{U}, [g]_\mathcal{U})$. So by inductive hypothesis we have that

$$\{\alpha < \kappa : M \models \varphi(f(\alpha), g(\alpha))\} \in \mathcal{U}$$

But this set is clearly a subset of $\{\alpha < \kappa : M \models \exists x\ \varphi(x, g(\alpha))\}$, which must therefore be in $\mathcal{U}$.

For the backward direction, assume $A = \{\alpha < \kappa : M \models \exists x\ \varphi(x, g(\alpha))\} \in \mathcal{U}$. For $\alpha \in A$, pick $x_\alpha \in M$ so that $M \models \varphi(x_\alpha, g(\alpha)$. Now define $f : \kappa \to M$ as $f(\alpha) = x_\alpha$ for $\alpha \in A$ and arbitrarily for $\alpha \notin A$. Then,

$$\{\alpha < \kappa : M \models \varphi(f(\alpha), g(\alpha))\} \in \mathcal{U}.$$

So by inductive hypothesis $N \models \varphi([f]_\mathcal{U}, [g]_\mathcal{U})$, and so $N \models \exists x\ \varphi(x, [g]_\mathcal{U})$. □

Observe that for the backward direction of the quantifier step we used the axiom of choice. It is known that Łoś's theorem is not provable from ZF.

Corollary 23 (Upward Löwenheim–Skolem)**.** *Let M be a first-order structure, and let $\kappa > |M|$. Then there is N with $|N| = \kappa$ so that $M \prec N$.*

Proof. Exercise. (Hint: By the downward Löwenheim–Skolem theorem, it is enough to find N with $|N| \geq \kappa$ so that $M \prec N$. Show that $N = M^\kappa/\mathcal{U}$ works.) □

1. Exercises

To get a better understanding of the properties of first-order logic, let's contrast it with second-order logic.

In first-order logic, only quantification over the domain of discourse is allowed. For example, if you want to talk about arithmetic in a first-order context you are only allowed to quantify over numbers, not over sets of numbers. In second-order logic, you are allowed to quantify over subsets of the domain of discourse. (Equivalently, over relations/functions on the domain of discourse.) For example, for second-order arithmetic you are allowed to quantify over both numbers and sets of numbers.[2]

Exercise 24. Formulate axioms in second-order logic for ordered fields with the least upper bound property.[3] Show that, up to isomorphism, there is a unique structure satisfying these axioms. Observe that this unique structure is uncountable. Conclude that second-order logic does not satisfy the Löwenheim–Skolem theorems.

Exercise 25. Formulate axioms in second-order logic for discretely ordered semi-rings which satisfy induction. Show that, up to isomorphism, there is a unique structure satisfying these axioms. Conclude that second-order logic does not satisfy the upward Löwenheim–Skolem theorem.

If you have a bit of a background in logic, also do the following exercise.

Exercise 26. Using one of the categoricity results from the earlier exercises, show that there is no effective proof system for second-order logic which admits a completeness theorem.

Next let us consider equivalences between formulae.

Definition 27. Let T be a theory and consider formulae φ and ψ in the language of T which have no free variables. Then φ and ψ are said to be *T-equivalent* or *equivalent modulo T* if $T \models \varphi \Leftrightarrow \psi$. And φ and ψ are *logically equivalent* if $\emptyset \models \varphi \Leftrightarrow \psi$.

Exercise 28. Show that φ and ψ are T-equivalent iff for any $M \models T$, we have $M \models \varphi$ iff $M \models \psi$.

Exercise 29. Formulate a definition of T-equivalence for formulae $\varphi(x)$ and $\psi(x)$ with free variables.

Try out the following exercise if you have some background in computability theory.

Exercise 30 (Hilbert's *Entscheidungsproblem* is undecidable). Say that φ (without free variables) is *logically valid* if $\emptyset \models \varphi$. Show that no Turing machine decides the set of logically valid formulae (in, say, a finite language).

Exercise 31. Use the completeness and soundness theorems to prove the *deduction theorem*: Let T be a theory. Then $T \vdash \varphi \Rightarrow \psi$ iff $T \cup \{\varphi\} \vdash \psi$. (Hint: by completeness + soundness it is enough to show the analogous thing with $\vdash$ replaced by $\models$. You can also prove this syntactically, but that would require first formulating a formal proof system.)

An important notion in model theory is that of a complete theory.

Definition 32. A consistent theory T is *complete* if for every φ in the language of T, either $\varphi \in T$ or $\neg\varphi \in T$.

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[2] To be 100% accurate, I should mention that there are other semantics for second-order logic, which restrict which sets you are allowed to quantify over. What I'm talking about here is second-order logic with full semantics.

[3] The least upper bound property asserts that if S is a subset of the field and there is $x \geq y$ for all $y \in S$, then there is a smallest x.

We require T to be consistent so as to avoid the trivial theory consisting of all formulae.

Exercise 33. Show that every consistent theory can be extended to a complete theory.

Exercise 34 (Łoś–Vaught test). Say that a theory T is *κ-categorical* if any any two models of T of cardinality κ are isomorphic. Show that if a consistent theory T in a finite language has only infinite models and is κ-categorical for some $\kappa \geq \omega$, then T is complete.

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