MATH 218M TECHNICAL CONTENT
FUZZY LOGIC, PART 1

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1. Introduction

In the previous section we explored boolean logic. Also called classical propositional logic, this was a *truth functional*, *bivalent* logic. Truth functional means that the meaning of the connectives is entirely dependent upon the truth value of their inputs; they can be defined with truth tables. Bivalent means that there are only two truth values, true (1) or false (0).

In this section we will study *fuzzy logic*. (You also see this logic or closely related logics under the names *continuous logic* or *Tarski–Łukasiewicz logic*.) This logic, like boolean, is truth functional. However rather than have two values it has infinitely many truth values. This is the intuition behind the name—fuzzy logic is useful to studying things whose truth value is fuzzy.

For example, take the sentence "John is tall". This doesn't admit a simple true-false answer. Surely if John's height is 7 feet we'd say he's tall. But what if he's only 6 feet? 5 feet 11 inches? 5 feet 10 inches? Where is the boundary between tall and not tall? So rather than represent this as a binary true versus false it's more natural to be able to say that John is kinda tall, or that John is tall but Karen is more tall.

In general, fuzzy logic is useful for modeling things that are *vague* or *imprecise*.

2. Truth values and the basic connectives

Fuzzy logic has as its truth values the closed interval $[0, 1]$ consisting of all real numbers between 0 and 1, including the endpoints. These truth values, all infinitely many of them, make up the propositional constants. And these are the values the propositional variables can take on. The most false something can be is 0 and the most true it can be is 1.

As with boolean logic, we want to have connectives which allow us to build up more complicated statements. As with boolean logic, we can define the connectives by describing them as functions—saying what output we get for different possible inputs. In that context, we could give the function by explicitly listing every possible input and the corresponding output. For fuzzy logic, that is not practical, as it would require a truth table with infinitely many rows. Indeed, a fuzzy logic connective is an n-variable function whose inputs and outputs are real numbers in $[0, 1]$.

Instead we will resort to using ideas from arithmetic. We will build on an exercise from the previous section, where we saw how to define the boolean connectives using arithmetic.

Definition 1. Let x and y be fuzzy truth values, meaning x and y are real numbers in $[0, 1]$.

- $x \wedge y$ is defined to be $\min(x, y)$.
- $x \vee y$ is defined to be $\max(x, y)$.
- $\neg x$ is defined to be $1 - x$.

Date: November 8, 2024.

Exactly as with boolean logic, we can build up *formulas* by combining propositional constants and variables using connectives. We can think of formulas as giving n-variable functions whose inputs and outputs are real numbers in $[0,1]$, repeatedly applying the rules for the connectives. We will write these using standard function notation, e.g. $\varphi(x,y) = t$ to mean that the formula φ with truth values x and y for its variables has truth value t.

We can carry over some of the same definitions as in boolean logic.

Definition 2.

- Two formulas φ and ψ with the same variables $x, y, \ldots$ are *logically equivalent* if $\varphi(x,y,\ldots) = \psi(x,y,\ldots)$ for every $x, y, \ldots$ in $[0,1]$. We write $\varphi \equiv \psi$.
- A formula φ is a *tautology* if $\varphi(x,\ldots) = 1$ for every $x, \ldots$ in $[0,1]$.
- A formula φ is *satisfiable* if $\varphi(x,\ldots) = 1$ for some $x, \ldots$ in $[0,1]$, and otherwise it is *unsatisfiable*.
- For fixed t in $(0,1]$, say φ is t-*satisfiable* if $\varphi(x, \ldots \geq t)$ for some $x, \ldots$ in $[0,1]$, and otherwise it is t-*unsatisfiable*.

Note that satisfiable and 1-satisfiable mean the same thing.

With a notion of equivalence in hand one can readily check that many of the equivalences of boolean logic hold for fuzzy logic. The exercises for week 9 ask you to do just that.

Defining implication is a more delicate affair.

To illustrate what we want, consider the inequality (using a not-yet defined $\Rightarrow$ connective):

$$x \wedge (x \Rightarrow y) \leq y.$$

This equality expresses a fuzzy version of the well-known logical rule of *modus ponens*. This rule says that if you know x is true and you know $x \Rightarrow y$ is true then you know y is true. We want this equality to hold. However, there's we don't want it hold for silly reasons. For example, it'd trivially be true if we defined $x \Rightarrow y$ to always be 0. But that's not a reasonable definition of the IF-THEN connective.

To rule out silly definitions we want that $x \Rightarrow y$ should be the *pointwise largest* binary operation which makes that inequality always hold. What this means is, if $x \Rightarrow^* y$ is some other connective which makes that inequality always hold, then $(x \Rightarrow^* y) \leq (x \Rightarrow y)$. With that goal in mind let's see the definition.

Definition 3. $x \Rightarrow y$ is defined to be the largest z so that $x \wedge z \leq y$.

We can give a more concrete characterization of $\Rightarrow$, in case you don't like that one.

Proposition 4. *$x \Rightarrow y$ is 1 if $x \leq y$ and is y otherwise.*

As a temporary definition while we prove this, let $i(x,y)$ denote this value. That is, $i(x,y) = 1$ if $x \leq y$ and $i(x,y) = y$ if $x > y$.

To see this we have to check two things: (1) if $x \wedge z \leq y$ then $z \leq i(x,y)$ and (2) $x \wedge i(x,y) \leq y$. For (1), $x \wedge z \leq y$ means $\min(x,z) \leq y$ which means either $x \leq y$ or $z \leq y$. If $x \leq y$ then $z \leq 1 = i(x,y)$. If $x > y$ then the only option is $z \leq y = i(x,y)$. For (2), we again consider the $x \leq y$ and the $x > y$ cases separately. If $x \leq y$ then $i(x,y) = 1$ and so $x \wedge i(x,y) = x \leq y$. If $x > y$ then $i(x,y) = y$ and so $x \wedge i(x,y) = y \leq y$. □

Proposition 5. *The following is true for any x, y for this definition of $\Rightarrow$:*

$$x \wedge (x \Rightarrow y) \leq y.$$

If $x \leq y$ then $x \Rightarrow y = 1$ and so $x \wedge (x \Rightarrow y) = x \leq y$. If $x > y$ then $x \Rightarrow y = y$ and so $x \wedge (x \Rightarrow y) = y \leq y$. $\square$

Proposition 6. *For all x, y, z, $x \wedge y \leq z$ if and only if $x \leq (y \Rightarrow z)$.*

We have to see two things: (1) if $x \wedge y \leq z$ then $x \leq (y \Rightarrow z)$ and (2) if $x \leq (y \Rightarrow z)$ then $x \wedge y \leq z$.

(1): $x \wedge y \leq z$ means either $x \leq z$ or $y \leq z$. If $x \leq z$ then $x \leq z \leq (y \Rightarrow z)$. If $y \leq z$ then $x \leq 1 = (y \Rightarrow z)$.

(2): We need to see that either $x \leq z$ or $y \leq z$. If $y \leq z$ then we're done. Otherwise, if $y > z$ then $y \Rightarrow z = z$ so $x \leq (y \Rightarrow z)$ simply means $x \leq z$. $\square$

Once you have $\Rightarrow$ defined, the IFF connective $\Leftrightarrow$ can be given by the same definition as in boolean logic:

$$x \Leftrightarrow y \qquad \text{means} \qquad (x \Rightarrow y) \wedge (y \Rightarrow x)$$

Proposition 7. *$x \Leftrightarrow y$ is 1 if $x = y$ and otherwise is $\min(x, y)$.*

Consider the three possible cases for how x and y compare.

If $x < y$ then $x \Rightarrow y = 1$ and $y \Rightarrow x = x$, so $x \Leftrightarrow y = x$. The same calculation, but with x and y swapped, shows that if $y < x$ then $x \Leftrightarrow y = y$. In case $x = y$, we have that $x \Rightarrow y = y \Rightarrow x = 1$, so $x \Leftrightarrow y = 1$. $\square$

2.1. **Week 9 exercises (due Friday 11/8).**

Exercise 1. Show that all of the following pairs of formulas are equivalent:

- (Associativity of $\wedge$) $(x \wedge y) \wedge z \equiv x \wedge (y \wedge z)$
- (Commutativity of $\wedge$) $x \wedge y \equiv y \wedge x$
- (Associativity of $\vee$) $(x \vee y) \vee z \equiv x \vee (y \vee z)$
- (Commutativity of $\vee$) $x \vee y \equiv y \vee x$

Exercise 2. Check that both DeMorgan's laws hold in fuzzy logic:

- $\neg(x \wedge y) \equiv \neg x \vee \neg y$
- $\neg(x \vee y) \equiv \neg x \wedge \neg y$

Exercise 3. Are the following two formulas equivalent? Say why.

$$x \Rightarrow 0 \qquad \text{and} \qquad \neg x.$$

Exercise 4. In boolean logic the formula $(x \Rightarrow y) \vee (y \Rightarrow x)$ was a tautology. Is it a tautology in fuzzy logic? Justify your answer

Exercise 5. In boolean logic the formulas $x \Rightarrow (y \Rightarrow z)$ and $(x \wedge y) \Rightarrow z$ were logically equivalent. Are they logically equivalent in fuzzy logic? Justify your answer.

3. Other connectives and t-norms

When we defined the connectives we took an arithmetic characterization of the connectives in boolean logic, and asked what if truth values could be intermediate between 0 and 1? However, there's multiple ways we might have done that choice. We defined $P \wedge Q$ as $\min(P, Q)$, but we could've equally well defined it as $P \cdot Q$ and gotten the same truth table.

What happens if we use this instead as a definition of $\wedge$ in the fuzzy realm?

Let's call the $\wedge$ based on min simply as $\wedge$ and the $\wedge$ defined using multiplication as $\wedge_\times$, so we can talk about both with less confusion.

One feature distinguishing $\wedge$ from $\wedge_\times$ is that $\wedge$ is *idempotent* while $\wedge_\times$ is not. That is, $x \wedge x$ is always x while $x \wedge_\times x$ is only x when x is 0 or 1. Why might we prefer one behavior over the other? The non-idempotent AND gives a better model for the idea that if you use a partially true hypothesis multiple times it should lower the level of truth of your conclusion.

Whether or not you find this a model you would want to use, it is mathematically interesting that there is more than one reasonable way to define $\wedge$ in fuzzy logic.

Let's isolate some notions of what would make something a reasonable definition of AND and then we'll check that $\wedge_\times$ satisfies them.

Definition 8. Let $\wedge^*$ be a binary connective in fuzzy logic. That is, $\wedge^*$ is a function with two inputs in $[0, 1]$ and outputs in $[0, 1]$. Say that $\wedge^*$ *seems reasonable as conjunction* if all of the following hold.

- $\wedge^*$ is commutative: $x \wedge^* y = y \wedge^* x$;
- $\wedge^*$ is associative: $x \wedge^* (y \wedge^* z) = (x \wedge^* y) \wedge^* z$;
- $\wedge^*$ is monotonous: if $x_0 \leq x_1$ then $x_0 \wedge^* y \leq x_1 \wedge^*$;
- $\wedge^*$ has 1 as a neutral element: $x \wedge 1 = x$ for any x;
- $\wedge^*$ has 0 has a null element: $x \wedge 0 = 0$ for any x;
- $\wedge^*$ is continuous, in the sense of calculus:

$$\lim_{(a,b)\to(x,y)} a \wedge^* b = x \wedge^* y.$$

Note that elsewhere in mathematics a binary function which satisfies the first five conditions is called a *t-norm* or *triangular norm*. So for fuzzy logic seeming reasonable as conjunction means being a continuous t-norm.

Commutativity and associativity express that order doesn't matter for AND. Monotony expresses that if you increase the truth value of a conjunct this shouldn't lower the truth value of the whole conjunction. Having 1 as a neutral element and 0 as a null element express that 1 is absolute truth and 0 is absolute falsity. Continuity expresses that a small change to the truth values of the inputs should have only a small change to the truth value of the output.

All these properties are readily checked to hold for $\wedge$. It's not too hard to convince oneself they also hold for $\wedge_\times$.

Proposition 9. $\wedge_\times$ *is a continuous t-norm (i.e. it seems reasonable as conjuction).*

Commutativity, associativity, monotony, neutrality of 1, and nullity of 0 are all well-known properties of multiplication. Continuity is the elementary calculus fact that multiplication is continuous.

Any binary connective which seems reasonable as conjunction allows an analysis of IF-THEN similar to the one we did with $\wedge$. (Indeed, you don't need the full strength of continuity, you only need it to be so-called left-continuous.)

One of the properties of $\Rightarrow$ we looked at was that x, y, z, $x \wedge y \leq z$ if and only if $x \leq (y \Rightarrow z)$. It will turn out that this can be taken as a defining property.

Theorem 10. *Let $\wedge^*$ be a continuous t-norm. Then there is a unique binary connective $\Rightarrow^*$ so that*

$$x \wedge^* y \leq z \qquad \textit{iff} \qquad x \leq (y \Rightarrow^* z).$$

We call $\Rightarrow^*$ the residuum *of $\wedge^*$.*

The idea to define $\Rightarrow^*$ is the same as how we defined $\Rightarrow$ from the max AND. Namely, it turns out

$$x \Rightarrow^* y = \max\{z : x \wedge^* z \leq y\}.$$

For a specific continuous t-norm we can give a concrete formula for the residuum.

Proposition 11. *$x \Rightarrow_\times$, the residuum for $\wedge_\times$, could equivalently have been defined as*

$$x \Rightarrow_\times y = \begin{cases} 1 & \textit{if } x \leq y \\ \frac{y}{x} & \textit{if } x > y \end{cases}$$

This works by following the same idea as when we checked the analogous fact for the max AND. Namely, we need to show two things: (1) $x \wedge_\times (x \Rightarrow_\times y) \leq y$ and (2) if $x \wedge_\times z \leq y$ then $z \leq (x \Rightarrow_\times y)$. Again we break things into two cases based on which case of the definition we are in.

Assume $x \leq y$. Then $x \Rightarrow_\times y = 1$ and so $x \wedge_\times (x \Rightarrow_\times y) = x \leq y$. This shows (1). And (2) is simply the fact that 1 is the maximum truth value so $z \leq (x \Rightarrow_\times y)$ no matter what z is.

Now assume $x > y$. Then $x \Rightarrow_\times y = y/x$. Thus $x \wedge_\times (x \Rightarrow_\times y) = x \cdot y/x = y$. And $y \leq y$ so this shows (1). For (2), take z so that $x \wedge_\times z \leq y$. This says $xz \leq y$ and rearranging gives us $z \leq y/x = (x \Rightarrow_\times y)$. □

The two possible definitions of AND in fuzzy logic we've seen—the maximum or the product—are not the only continuous t-norms. Without giving any calculations, let's mention one more.

Definition 12. The *Łukasiewicz t-norm* $x \wedge_{\text{Ł}} y$ is given by the formula

$$x \wedge_{\text{Ł}} y = \max(x + y - 1, 0).$$

Fact 13. *The residuum of the Łukasiewicz t-norm is*

$$x \Rightarrow_{Ł} y = \min(1, 1 - x + y).$$

You can also define other connectives based on the residuum. In some contexts this is used to get a definition of negation.

Definition 14. Let $\wedge^*$ be a continuous t-norm. The *negation associated to* $\wedge *$ is defined using the residuum $\Rightarrow^*$:

$$\neg^* x = x \Rightarrow^* 0.$$

Fact 15. *The* Łukasiewicz negation *is $\neg_{Ł} x = 1 - x$, the standard negation we used above. The negation associated to both the maximum and product t-norms is called the* Gödel negation*:*

$$\neg_G x = \begin{array}{ll} 1 & \textit{if } x = 0 \\ 0 & \textit{if } x > 0 \end{array}$$

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