MATH 218M TECHNICAL CONTENT
FUZZY LOGIC, PART 1

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1. Introduction

In the previous section we explored boolean logic. Also called classical propositional logic, this was a *truth functional*, *bivalent* logic. Truth functional means that the meaning of the connectives is entirely dependent upon the truth value of their inputs; they can be defined with truth tables. Bivalent means that there are only two truth values, true (1) or false (0).

In this section we will study *fuzzy logic*. (You also see this logic or closely related logics under the names *continuous logic* or *Tarski–Łukasiewicz logic*.) This logic, like boolean, is truth functional. However rather than have two values it has infinitely many truth values. This is the intuition behind the name—fuzzy logic is useful to studying things whose truth value is fuzzy.

For example, take the sentence "John is tall". This doesn't admit a simple true-false answer. Surely if John's height is 7 feet we'd say he's tall. But what if he's only 6 feet? 5 feet 11 inches? 5 feet 10 inches? Where is the boundary between tall and not tall? So rather than represent this as a binary true versus false it's more natural to be able to say that John is kinda tall, or that John is tall but Karen is more tall.

In general, fuzzy logic is useful for modeling things that are *vague* or *imprecise*.

2. Truth values and the basic connectives

Fuzzy logic has as its truth values the closed interval $[0, 1]$ consisting of all real numbers between 0 and 1, including the endpoints. These truth values, all infinitely many of them, make up the propositional constants. And these are the values the propositional variables can take on. The most false something can be is 0 and the most true it can be is 1.

As with boolean logic, we want to have connectives which allow us to build up more complicated statements. As with boolean logic, we can define the connectives by describing them as functions—saying what output we get for different possible inputs. In that context, we could give the function by explicitly listing every possible input and the corresponding output. For fuzzy logic, that is not practical, as it would require a truth table with infinitely many rows. Indeed, a fuzzy logic connective is an n-variable function whose inputs and outputs are real numbers in $[0, 1]$.

Instead we will resort to using ideas from arithmetic. We will build on an exercise from the previous section, where we saw how to define the boolean connectives using arithmetic.

Definition 1. Let x and y be fuzzy truth values, meaning x and y are real numbers in $[0, 1]$.

- $x \wedge y$ is defined to be $\min(x, y)$.
- $x \vee y$ is defined to be $\max(x, y)$.
- $\neg x$ is defined to be $1 - x$.

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Exactly as with boolean logic, we can build up *formulas* by combining propositional constants and variables using connectives. We can think of formulas as giving n-variable functions whose inputs and outputs are real numbers in $[0,1]$, repeatedly applying the rules for the connectives. We will write these using standard function notation, e.g. $\varphi(x,y) = t$ to mean that the formula φ with truth values x and y for its variables has truth value t.

We can carry over some of the same definitions as in boolean logic.

Definition 2.

- Two formulas φ and ψ with the same variables $x, y, \ldots$ are *logically equivalent* if $\varphi(x,y,\ldots) = \psi(x,y,\ldots)$ for every $x, y, \ldots$ in $[0,1]$. We write $\varphi \equiv \psi$.
- A formula φ is a *tautology* if $\varphi(x,\ldots) = 1$ for every $x, \ldots$ in $[0,1]$.
- A formula φ is *satisfiable* if $\varphi(x,\ldots) = 1$ for some $x, \ldots in [0,1]$, and otherwise it is *unsatisfiable*.
- For fixed t in $(0,1]$, say φ is *t-satisfiable* if $\varphi(x,\ldots \geq t)$ for some $x, \ldots$ in $[0,1]$, and otherwise it is *t-unsatisfiable*.

Note that satisfiable and 1-satisfiable mean the same thing.

With a notion of equivalence in hand one can readily check that many of the equivalences of boolean logic hold for fuzzy logic. The exercises for week 9 ask you to do just that.

Defining implication is a more delicate affair.

To illustrate what we want, consider the inequality (using a not-yet defined $\Rightarrow$ connective):

$$x \wedge (x \Rightarrow y) \leq y.$$

This equality expresses a fuzzy version of the well-known logical rule of *modus ponens*. This rule says that if you know x is true and you know $x \Rightarrow y$ is true then you know y is true. We want this equality to hold. However, there's we don't want it hold for silly reasons. For example, it'd trivially be true if we defined $x \Rightarrow y$ to always be 0. But that's not a reasonable definition of the IF-THEN connective.

To rule out silly definitions we want that $x \Rightarrow y$ should be the *pointwise largest* binary operation which makes that inequality always hold. What this means is, if $x \Rightarrow^* y$ is some other connective which makes that inequality always hold, then $(x \Rightarrow^* y) \leq (x \Rightarrow y)$. With that goal in mind let's see the definition.

Definition 3. $x \Rightarrow y$ is defined to be the largest z so that $x \wedge z \leq y$.

We can give a more concrete characterization of $\Rightarrow$, in case you don't like that one.

Proposition 4. *$x \Rightarrow y$ is 1 if $x \leq y$ and is y otherwise.*

As a temporary definition while we prove this, let $i(x,y)$ denote this value. That is, $i(x,y) = 1$ if $x \leq y$ and $i(x,y) = y$ if $x > y$.

To see this we have to check two things: (1) if $x \wedge z \leq y$ then $z \leq i(x,y)$ and (2) $x \wedge i(x,y) \leq y$. For (1), $x \wedge z \leq y$ means $\min(x,z) \leq y$ which means either $x \leq y$ or $z \leq y$. If $x \leq y$ then $z \leq 1 = i(x,y)$. If $x > y$ then the only option is $z \leq y = i(x,y)$. For (2), we again consider the $x \leq y$ and the $x > y$ cases separately. If $x \leq y$ then $i(x,y) = 1$ and so $x \wedge i(x,y) = x \leq y$. If $x > y$ then $i(x,y) = y$ and so $x \wedge i(x,y) = y \leq y$. □

Proposition 5. *The following is true for any x, y for this definition of $\Rightarrow$:*

$$x \wedge (x \Rightarrow y) \leq y.$$

If $x \leq y$ then $x \Rightarrow y = 1$ and so $x \wedge (x \Rightarrow y) = x \leq y$. If $x > y$ then $x \Rightarrow y = y$ and so $x \wedge (x \Rightarrow y) = y \leq y$. $\square$

Proposition 6. *For all x, y, z, $x \wedge y \leq z$ if and only if $x \leq (y \Rightarrow z)$.*

We have to see two things: (1) if $x \wedge y \leq z$ then $x \leq (y \Rightarrow z)$ and (2) if $x \leq (y \Rightarrow z)$ then $x \wedge y \leq z$.

(1): $x \wedge y \leq z$ means either $x \leq z$ or $y \leq z$. If $x \leq z$ then $x \leq z \leq (y \Rightarrow z)$. If $y \leq z$ then $x \leq 1 = (y \Rightarrow z)$.

(2): We need to see that either $x \leq z$ or $y \leq z$. If $y \leq z$ then we're done. Otherwise, if $y > z$ then $y \Rightarrow z = z$ so $x \leq (y \Rightarrow z)$ simply means $x \leq z$. $\square$

Once you have $\Rightarrow$ defined, the IFF connective $\Leftrightarrow$ can be given by the same definition as in boolean logic:

$$x \Leftrightarrow y \qquad \text{means} \qquad (x \Rightarrow y) \wedge (y \Rightarrow x)$$

Proposition 7. *$x \Leftrightarrow y$ is 1 if $x = y$ and otherwise is $\min(x, y)$.*

Consider the three possible cases for how x and y compare.

If $x < y$ then $x \Rightarrow y = 1$ and $y \Rightarrow x = x$, so $x \Leftrightarrow y = x$. The same calculation, but with x and y swapped, shows that if $y < x$ then $x \Leftrightarrow y = y$. In case $x = y$, we have that $x \Rightarrow y = y \Rightarrow x = 1$, so $x \Leftrightarrow y = 1$. $\square$

2.1. **Week 9 exercises (due Friday 11/8).**

Exercise 1. Show that all of the following pairs of formulas are equivalent:

- (Associativity of $\wedge$) $(x \wedge y) \wedge z \equiv x \wedge (y \wedge z)$
- (Commutativity of $\wedge$) $x \wedge y \equiv y \wedge x$
- (Associativity of $\vee$) $(x \vee y) \vee z \equiv x \vee (y \vee z)$
- (Commutativity of $\vee$) $x \vee y \equiv y \vee x$

Exercise 2. Check that both DeMorgan's laws hold in fuzzy logic:

- $\neg(x \wedge y) \equiv \neg x \vee \neg y$
- $\neg(x \vee y) \equiv \neg x \wedge \neg y$

Exercise 3. Are the following two formulas equivalent? Say why.

$$x \Rightarrow 0 \qquad \text{and} \qquad \neg x.$$

Exercise 4. In boolean logic the formula $(x \Rightarrow y) \vee (y \Rightarrow x)$ was a tautology. Is it a tautology in fuzzy logic? Justify your answer

Exercise 5. In boolean logic the formulas $x \Rightarrow (y \Rightarrow z)$ and $(x \wedge y) \Rightarrow z$ were logically equivalent. Are they logically equivalent in fuzzy logic? Justify your answer.

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