MATH 321: 10-20 IN-CLASS WORK

We continue our investigation of proof strategies. Today we focus on the logical connective $\vee$.

If $P \vee Q$ is a known, then you have a slight problem: you know that either P or Q must be true, but you don't know which! So what are you to do? The answer: do two proofs. You don't know which of P or Q is the true one, so you have to consider both cases. That is, you use the known of P to prove your goal and then separate you use the known of Q to prove your goal.

Often, this $P \vee Q$ known appears in the middle of your proof, and in the guise $P \vee \neg P$. This is a tautology, so you always know it's true. For example, if you are proving something about an integer n then you may find yourself needing to split into cases based upon whether n is even or odd.

What about if $P \vee Q$ is your goal? Often, you will want to split your proof into cases. For example, to show that $x^2 \equiv 0 \mod 4$ or $x^2 \equiv 1 \mod 4$ for all integers x you want to split into the even versus odd cases. But in general, it may not be so easy to figure out what cases to consider.

Fortunately you have another option for how to approach this goal. Namely, take as an extra known $\neg P$ and try to prove Q. (Or swap the role of the two propositions here.) Let me give two explanations why this is a valid proof strategy.

The first is based on how we use $\vee$ as a known. Since $P \vee \neg P$ is always true, we take it as a known. So we want to use this known to do a proof by cases. In case P is true, it's immediate that $P \vee Q$ is also true. So the only work is to assume $\neg P$ and try to prove Q.

The second uses that $P \vee Q$ is equivalent to $\neg P \to Q$. So your strategy for proving an implication says to assume $\neg P$ and try to prove Q.

I also want to mention how statements with $\vee$ work with proofs by contradiction. Suppose you want to prove $P \vee Q$ by contradiction. That is, you take as a known $\neg(P \vee Q)$ and try to derive a contradiction. By DeMorgan, this is saying you get as a known $\neg P \wedge \neg Q$.

Exercise 1. *Prove that, for any two integers a and b, if ab is even then either a is even or b is even.*

Exercise 2. *Consider the following attempt at a proof:*

Theorem. *For all real numbers x if $|x-1| < 2$ then $-1 < x < 3$.*

Proof. Suppose $|x-1| < 2$. We consider two cases.

Case 1 ($x - 1 \geq 0$). Then $|x-1| = x - 1$. So $x - 1 < 2$ so $x < 3$.

Case 2 ($x - 1 < 0$). Then $|x-1| = 1 - x$. So $1 - x < 2$ so $-x < 1$ so $x > -1$.

Since we have proven both $-1 < x$ and $x < 3$, we conclude $-1 < x < 3$. $\square$

Identify the error in this proof, and then correct it to give a valid proof of this theorem.

Exercise 3. *Suppose A, B, and C are sets. Prove that $A \cup C \subseteq B \cup C$ if and only if $A \setminus C \subseteq B \setminus C$.*