Mediate cardinals

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they/them

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This isn't circular, because we can define ω by its induction properties.

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- X is Dedekind-finite if any injection $f : X \to X$ is a surjection.

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X is Dedekind-infinite iff $\omega \leq |X|$.

- ($\Leftarrow$) Push forward the $+1$ function on ω.
- ($\Rightarrow$) Fix $z \in X \setminus \operatorname{ran} f$. Then the map $n \mapsto f^n(z)$ gives an injection $\omega \to X$.
  - Use fact that f is one-to-one to inductively prove this map is an injection.

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- If X is infinite, choose for each n an injection $e_n : n \to X$. Inductively glue them together into an injection $e : \omega \to X$.

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- Infinite implies Dedekind-infinite needs a small fragment of AC.

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Theorem (Cohen 1963)

It is consistent with ZF *that there exists a Dedekind-finite, infinite set.*

But you can't get a reversal: there's no hope the non-existence of a DFI set implies AC because the former is local while the latter is global.

The first question

Question

Is there a suitable generalization of a Dedekind-finite, infinite set whose nonexistence gives a characterization of AC?

A look back in history

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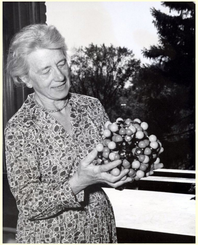
- In the late 1910s, Bertrand Russell is a few years after the last volume of *Principia Mathematica*. His time is occupied by legal troubles over his pacifism during World War I and thinking about the foundations of mathematics.
- Working with him are multiple students, including Dorothy Wrinch.
- The next decade (1923) she will publish a paper answering our first question.

Dorothy Wrinch



- Born 1894, died 1976.
- Studied logic under Russell, did her doctorate (1921) under applied mathematician John Nicholson.
- Wrote in a range of subjects: logic, pure mathematics, philosophy of science, and mathematical biology.
- Was awarded a Rockefeller Foundation fellowship to support her work in mathematical biology.
- Early career was in the UK, later emigrated to the USA. Latter years of her career were at Smith College (Mass, USA).
- Had the misfortune of being on the losing side of a scientific dispute with Linus Pauling over the structure of proteins.

Wrinch's question, and mine

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Question

Can we use modern techniques to prove more precise consistency results?

Cardinals sans choice

Notation:

- $\kappa, \lambda, \ldots$ will be used for well-orderable, infinite cardinals.
- $\mathfrak{p}, \mathfrak{q}, \ldots$ will be used for cardinals in general.
- I'll sometimes use $\mathfrak{p}$ to refer to an arbitrary set of cardinality $\mathfrak{p}$.

- Under AC, every cardinal is well-orderable. We can thus define the cardinals as the initial ordinals.
- Without AC we have to fall back on defining cardinals as equivalence classes.
- Can use Scott's trick to make these sets.

Mediate cardinals

Fix a cardinal $\mathfrak{p}$. Then X is $\mathfrak{p}$-mediate if

- $\mathfrak{q} \leq |X|$ for all $\mathfrak{q} < \mathfrak{p}$;
- $\mathfrak{p} \not\leq |X|$; and
- $|X| \not\leq \mathfrak{p}$.

A $\mathfrak{p}$-mediate cardinal is a cardinal number of a $\mathfrak{p}$-mediate set.

Mediate means $\mathfrak{p}$-mediate for some infinite $\mathfrak{p}$.

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Mediate means $\mathfrak{p}$-mediate for some infinite $\mathfrak{p}$.

- Dedekind-finite infinite $\Leftrightarrow$ $\aleph_0$-mediate.
- ZF proves there are no n-mediate for finite n.

A few facts

Some facts about DFI sets generalize.

Fact

Suppose $\mathfrak{q}$ *and* $\mathfrak{r}$ *are* $\mathfrak{p}$*-mediate. Then:*

- $\mathfrak{q} + \mathfrak{r}$ *is* $\mathfrak{p}$*-mediate;*
- $\mathfrak{q} \cdot \mathfrak{r}$ *is* $\mathfrak{p}$*-mediate; and*
- $2^{2^{\mathfrak{q}}}$ *is not* $\mathfrak{p}$*-mediate.*

Wrinch's theorem

Theorem (Wrinch 1923)

Over ZF, *the following are equivalent.*

1. AC;
2. *There are no mediate cardinals; and*
3. *There are no κ-mediate cardinals for well-ordered κ.*

(Wrinch originally formulated this result in the framework of *Principia Mathematica*.)

Wrinch's theorem, $(1 \Rightarrow 2)$

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Definition

$\mathfrak{m}$ is $\mathfrak{p}$-mediate if

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Prove $(1 \Rightarrow 2)$ by contrapositive.

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Prove $(1 \Rightarrow 2)$ by contrapositive.

- Suppose $\mathfrak{q}$ is $\mathfrak{p}$ mediate. Then $\mathfrak{p}$ and $\mathfrak{q}$ are incomparable, so Cardinal Trichotomy fails.
- (Hartogs 1915) AC iff Cardinal Trichotomy.

Wrinch's theorem, $(3 \Rightarrow 1)$

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$(2 \Rightarrow 3)$ is trivial. Prove $(3 \Rightarrow 1)$ by contrapositive.

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- (Hartogs) For any $\mathfrak{p}$ there is a smallest well-orderable cardinal $\aleph(\mathfrak{p})$ so that $\aleph(\mathfrak{p}) \not\leq \mathfrak{p}$.
- If $\mathfrak{p}$ is not well-orderable then $\mathfrak{p}$ is $\aleph(\mathfrak{p})$-mediate.

Dependent choice

Dependent choice (DC) informally says you can make ω many choices where each choice depends on the previous ones.

- Suppose R is a relation on a set X so that for each $x \in X$ there is $y \in X$ with $x\,R\,y$. Then there is a branch $\langle x_i : i \in \omega \rangle$ through R: for each i have $x_i\;R\;x_{i+1}$.

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DC_κ says:

- Suppose R is a relation on $X^{<\kappa} \times X$ so that for each $s \in X^{<\kappa}$ there is $y \in X$ with $s \, R \, y$.
  Then there is a branch $b = \langle x_i : i < \kappa \rangle$ through R: for each i have $(b \restriction i) \; R \; b_i$.

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Facts:

- AC is equivalent to $\forall \kappa \; \mathsf{DC}_\kappa$.
- $\lambda < \kappa$ implies $\mathsf{DC}_\kappa \Rightarrow \mathsf{DC}_\lambda$.
- $\mathsf{ZF} + \mathsf{DC}_{<\kappa} + \neg \mathsf{DC}_\kappa$ is consistent.
- DC implies AC_ω over ZF, but not vice versa.
- DC is equivalent to "a relation is well-founded iff it has no infinite descending sequence".
- (Solovay) $\mathsf{ZF} + \mathsf{DC} +$ "every set of reals is Lebesgue-measurable" is consistent.

DC and mediate cardinals

Lemma: DC_κ implies there are no κ-mediates.

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- Suppose $\lambda \leq \mathfrak{p}$ for all $\lambda < \kappa$ but $\mathfrak{p} \not\leq \kappa$.
- Consider the collection of all injections $\alpha \to \mathfrak{p}$ for $\alpha < \mathfrak{p}$.
- None of the injections are onto, so you can always extend them to an injection $\alpha + 1 \to \mathfrak{p}$.
- By DC_κ there's a branch, which gives an injection $\kappa \to \mathfrak{p}$.

Refining mediacy

Observation:

- If $\mathfrak{p}$ is κ-mediate and $\lambda > \kappa$ then $\mathfrak{p} + \lambda$ is λ^+-mediate.
- So if you have κ-mediates for one κ you have mediates for larger cardinals.

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$\mathfrak{p}$ is exact κ-mediate if

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- if $Y \subseteq \mathfrak{p}$ has cardinality $< \kappa$ then $\mathfrak{p} \setminus Y$ is κ-mediate.

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Lemma: If $\mathfrak{p}$ is κ-mediate where κ is smallest such that κ-mediates exist, then $\mathfrak{p}$ is exact κ-mediate.

Consistency questions

Question

- *Consistently, what can be the smallest* κ *so that* κ*-mediates exist?*
- *Consistently, what can be the class of* κ *for which exact* κ*-mediates exist?*

Symmetric extensions

Motivating example: Add ω many reals, then forget the order you added them.

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- $\mathbb{P} = \mathrm{Add}(\omega, \omega)$ is the poset. Conditions are finite partial functions $\omega \times \omega \to 2$.
- Changing the order is permuting the columns in the $\omega \times \omega$ grid.
- Any permutation $\pi : \omega \to \omega$ generates an automorphism of $\mathbb{P}$: $\pi p(n, i) = p(\pi n, i)$.
- Also generates an automorphism on the $\mathbb{P}$-names: $\pi \dot{x} = \{(\pi p, \pi \dot{y}) : (p, \dot{y}) \in \dot{x}\}$

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- "Forgetting the order" is restricting to names fixed by a 'large' group of automorphisms:
  A group H of automorphisms is large if there is finite $e \subseteq \omega$ so that each $\pi \in H$ fixes e pointwise: $H \supseteq \mathrm{fix}(e)$.
- This gives a normal filter $\mathcal{F}$ on the lattice of subgroups.
- A name $\dot{x}$ is $\mathcal{F}$-symmetric if $\mathrm{sym}(\dot{x}) = \{\pi : \pi \dot{x} = \dot{x}\} \in \mathcal{F}$.
- The symmetric extension consists of the interpretations of all hereditarily symmetric names.

Symmetric extensions, in general

A symmetric system is $(\mathbb{P}, G, \mathcal{F})$ so that

- $\mathbb{P}$ is a forcing poset;
- $G \leq \mathrm{Aut}(\mathbb{P})$; and
- $\mathcal{F}$ is a normal filter on the lattice of subgroups of G.

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A $\mathbb{P}$-name $\dot{x}$ is symmetric if $\operatorname{sym} x \in \mathcal{F}$.

- (Symmetry lemma) $p \Vdash \varphi(\dot{x})$ iff $\pi p \Vdash \varphi(\pi \dot{x})$.

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- (Symmetry lemma) $p \Vdash \varphi(\dot{x})$ iff $\pi p \Vdash \varphi(\pi\dot{x})$.

The symmetric extension by $(\mathbb{P}, G, \mathcal{F})$ via a generic $g \subseteq \mathbb{P}$:

- Consists of the interpretations of hereditarily symmetric names.
- $\mathrm{V}[g/\mathcal{F}] = \{\dot{x}^g : \dot{x}$ is hereditarily symmetric$\}$.

$\mathrm{V}[g/\mathcal{F}] \models \mathrm{ZF}$, but the point is to make AC fail in a controlled way.

The Cohen symmetric extension

Fix regular κ and assume $\kappa^{<\kappa} = \kappa$.

- $\mathbb{P}_\kappa = \mathrm{Add}(\kappa, \kappa)$;
- $G_\kappa \leq \mathrm{Aut}(\mathbb{P}_\kappa)$ is generated by permutations of κ;
- $H \in \mathcal{F}_\kappa$ if $\exists e \in [\kappa]^{<\kappa}$ so that $\mathrm{fix}(e) \subseteq H$.

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In $\mathrm{V}[g_\kappa/\mathcal{F}_\kappa]$ the set $A = \{c_i : i < \kappa\}$ for Cohen subsets of κ is not well-orderable.

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Facts:

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- $\mathcal{F}_\kappa$ is κ-complete.

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Facts:

- $\mathbb{P}_\kappa$ is κ-closed and has the κ^+-cc.
- $\mathcal{F}_\kappa$ is κ-complete.

Thus, $(\mathbb{P}_\kappa, G_\kappa, \mathcal{F}_\kappa)$ will preserve $\mathrm{DC}_{<\kappa}$.

In particular, there will be no λ-mediates for $\lambda < \kappa$.

Symmetric extensions and DC

Lemma: Let κ be regular and $\lambda < \kappa$. If $\mathbb{P}$ is κ-closed and $\mathcal{F}$ is κ-complete then $(\mathbb{P}, G, \mathcal{F})$ preserves DC_λ.

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- Consider appropriate $R \subseteq X^{<\lambda} \times X$ in $\mathrm{V}[g/\mathcal{F}]$. We need a branch through R in $\mathrm{V}[g/\mathcal{F}]$.
- By κ-closure λ remains a cardinal in $\mathrm{V}[g]$.
- In $\mathrm{V}[g]$, by DC_λ there is a branch $b = \langle x_i : i < \lambda \rangle$.
- Each x_i comes from a symmetric name $\dot{x}_i$.
- By κ-completeness $H = \bigwedge_{i<\lambda} \mathrm{sym}(\dot{x}_i)$ is in $\mathcal{F}$.
- Can get a name $\dot{b}$ for b with $\mathrm{sym}(\dot{b}) \supseteq H$.
- So the branch b is in $\mathrm{V}[g/\mathcal{F}]$.

The smallest mediate can be anything

Theorem (W.)

Suppose $\kappa = \kappa^{<\kappa}$ is regular. In the symmetric extension by $(\mathbb{P}_\kappa, G_\kappa, \mathcal{F}_\kappa)$:

- $\mathsf{DC}_{<\kappa}$;
- *κ is least so that there is a κ-mediate cardinal; and*
- *There is an exact λ-mediate iff $\lambda = \kappa$.*

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We've already seen $\mathsf{DC}_{<\kappa}$ and so there are no λ-mediates for $\lambda < \kappa$.

Claim: Let A be the set of the Cohen subsets of κ added by $\mathbb{P}_\kappa$. Then $\mathrm{V}[g/\mathcal{F}_\kappa] \models A$ is κ-mediate.

Like getting DFI set in $(\mathbb{P}_\omega, G_\omega, \mathcal{F}_\omega)$.

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- $\lambda < \kappa$ injects by κ-closure of $\mathbb{P}_\kappa$ and κ-completeness of $\mathcal{F}_\kappa$
- $|A| \not\leq \kappa$ because A can't be well-ordered.
- $\kappa \not\leq |A|$:

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- $|A| \nleq \kappa$ because A can't be well-ordered.
- $\kappa \nleq |A|$:
  - Suppose $\dot{f}$ is hereditarily symmetric, $\mathrm{sym}(\dot{f}) \supseteq \mathrm{fix}(e)$, and $p \Vdash \dot{f} : \kappa \to A$ is one-to-one.
  - Extend p to q deciding $\dot{f}(\alpha) = c_i$ for some $\alpha \neq i$ both $\notin e$.
  - Find π fixing $e \cup \{i\}$, moving α, and $q \parallel \pi q$.
  - So $q \cup \pi q \Vdash \dot{f}$ is not one-to-one. Contradiction.

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Suppose $\kappa = \kappa^{<\kappa}$ is regular. Then $(\mathbb{P}_\kappa, G_\kappa, \mathcal{F}_\kappa)$ forces that

- *κ is least so that there is a κ-mediate cardinal.*
- $\mathsf{DC}_{<\kappa}$.
- *There is an exact λ-mediate iff $\lambda = \kappa$.*

Lemma: If X is exact λ-mediate for $\lambda > \kappa$ in $\mathrm{V}[g/\mathcal{F}_\kappa]$, then $\mathrm{V}[g] \models \lambda \leq |X|$.

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- *κ is least so that there is a κ-mediate cardinal.*
- $\mathrm{DC}_{<\kappa}$.
- *There is an exact λ-mediate iff $\lambda = \kappa$.*

Lemma: If X is exact λ-mediate for $\lambda > \kappa$ in $\mathrm{V}[g/\mathcal{F}_\kappa]$, then $\mathrm{V}[g] \models \lambda \leq |X|$.

Work in $\mathrm{V}[g]$:

- Consider the tree of hereditarily symmetric names for injections $\alpha \to X$ for $\alpha < \lambda$.
- Lemma implies the tree has a branch.
- But why should the branch be in $\mathrm{V}[g/\mathcal{F}_\kappa]$?
- Branch has size $\lambda > \kappa$ and $|\mathcal{F}| = \kappa$, so λ many names $\dot{f}_\alpha$ on the branch have the same $\mathrm{sym}(\dot{f}_\alpha)$.
- Can build a branch b so every injection on branch has same $\mathrm{sym}(\dot{f}_\alpha)$.
- Then b has a hereditarily symmetric name.

Thus $\mathrm{V}[g/\mathcal{F}_\kappa] \models \lambda \leq |X|$. Contradiction.

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- Karagila has a framework for iterations of symmetric extensions.
- It's complicated, with scary group theoretic objects like wreath products.
- We are lucky and can get away with products, where the details are significantly less technical.

Products of symmetric extensions

Suppose $(\mathbb{P}, G, \mathcal{F})$ and $(\mathbb{Q}, H, \mathcal{E})$ are symmetric systems. Can define their product $(\mathbb{P}, G, \mathcal{F}) \times (\mathbb{Q}, H, \mathcal{E})$:

- $\mathbb{P} \times \mathbb{Q}$ is usual product of posets;
- $G \times H$ is generated by (π, ρ) with $\pi \in G$, $\rho \in H$; and
- $\mathcal{F} \times \mathcal{E}$ is generated by $G_0 \times H_0$ for $G_0 \in \mathcal{F}$ and $H_0 \in \mathcal{E}$.

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Like with forcing, we have a product lemma stating that the extension by the product is the same as the two-step extensions, in either order.

Can also do this for infinite products, with a notion of support.

- Suppose $(\mathbb{P}_\kappa, G_\kappa, \mathcal{F}_\kappa)$ are symmetric systems for $\kappa \in M$.
- Then there is a product $\prod_{\kappa \in M}(\mathbb{P}_\kappa, G_\kappa, \mathcal{F}_\kappa)$ with that support.
- Again we get a product lemma stating we can split the full extension into two-step extensions.

Refining earlier ideas

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Suppose $\lambda < \kappa$ are regular.

- ($\mathsf{ZF} + \mathsf{DC}_\kappa$) If $\mathbb{P}$ is κ-closed and $\mathcal{F}$ is κ-complete then $(\mathbb{P}, G, \mathcal{F})$ preserves DC_λ.
- ($\mathsf{ZF} + \mathsf{DC}_\kappa$) Suppose $\mathbb{P}$ has the λ^+-cc and $\mathcal{F}$ is generated by a basis of size $\leq \lambda$. Then $\mathrm{V}[g/\mathcal{F}] \models$ there are no exact κ-mediates.

The pattern of the exact mediates

Theorem (W.)

Assume GCH *and fix a class M of regular cardinals. Do the Easton support product of the $(\mathbb{P}_\kappa, G_\kappa, \mathcal{F}_\kappa)$ for $\kappa \in M$. In the symmetric extension, there is an exact α-mediate iff $\alpha \in M$.*

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Sketch:

- $\mathbb{P}_{>\alpha}$ is α-closed and $\mathcal{F}_{>\alpha}$ is α-complete.
- $\mathbb{P}_{<\alpha}$ has the α^+-cc and $\mathcal{F}_{<\alpha}$ is generated by a basis of cardinality $\leq \alpha$.
- In $\mathrm{V}[g_{>\alpha}/\mathcal{F}_{>\alpha}]$: DC_α is true. So there are no α-mediates.
- In $\mathrm{V}[g_{>\alpha}/\mathcal{F}_{>\alpha}][g_{<\alpha}/\mathcal{F}_{<\alpha}]$: there are no exact α-mediates.
- So the only way there could be an exact α-mediate is if it was added by $(\mathbb{P}_\alpha, G_\alpha, \mathcal{F}_\alpha)$ for $\alpha \in M$.
- But we already know that adds an exact mediate.

Open questions

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- What's up with singular cardinals?
- What if we don't make such strong cardinal arithmetic assumptions?
- What happens if AC fails badly in the ground model?

Thank you!

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