Math 243, Section 14.8: Lagrange Multipliers

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Last time

Last time, we discussed section 14.7 of the textbook. This was about finding maximum and minimum values for functions with two real number inputs.

- We discussed the second derivative test for finding local maximums/minimums.
- We also learned about the extreme value theorem: if a function $f(x, y)$ is continuous on a closed, bounded region R in the plane $\mathbb{R}^2$, then $f(x, y)$ acheives a maximum and a minimum somewhere in R.
  - If the max/min is in the interior of R, then the second derivative test is how to find where they are.
  - If they are on the boundary, we need a new method.

The problem we want to solve

- We have a (nice) function $f(x, y)$ and we are looking at a (nice) curve C in the plane $\mathbb{R}^2$.
- We want to find where the max and min of $f(x, y)$ on C occur.

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- Let's be a a bit more concrete: the curve C is given by an equation, which we can rearrange to be in the form $g(x, y) = k$, where k is a constant.
- We want to find the point(s) (x_M, y_M) so that $g(x_M, y_M) = k$ and $f(x_M, y_M)$ is as big as possible. (And similarly for finding the minimum.)

What's going on, pictorally

Look at the curve $g(x, y) = k$ and the level curves $f(x, y) = c$.

What's going on, symbolically

We reasoned pictorally that we want to find points (x_M, y_M) so that the gradiant vectors for f and g at the points are parallel. In symbols, we want:

$$\nabla f(x_M, y_M) = \lambda \nabla g(x_M, y_M) \qquad (\lambda \neq 0 \text{ a scalar}).$$

We call this constant λ a Lagrange multiplier.

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We can also argue purely symbolically, in case you prefer that style.

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- By Calculus I, $h'(t_M) = 0$. But we can use the chain rule to compute $h'(t_M)$:

$$\begin{aligned}
0 &= h'(t_M) \\
&= f_x(x_M, y_M)x'(t_M) + f_y(x_M, y_M)y'(t_M) \\
&= \nabla f(x_M, y_M) \cdot \vec{r}\,'(t_M)
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- That is, $\nabla f(x_M, y_M)$ and $\vec{r}\,'(t_M)$ are orthogonal.
- But if the curve is also given by the equation $g(x, y) = k$ then $\nabla g(x, y)$ is also orthogonal to $\vec{r}\,'(t)$.
- Thus, $\nabla f(x_M, y_M)$ and $\nabla g(x_M, y_M)$ must be parallel.

The method of Lagrange multipliers

The problem: we want to find the max and min values for $f(x, y)$ on the curve given by $g(x, y) = k$.

1. Find all values x, y, λ so that

$$\nabla f(x, y) = \lambda \nabla g(x, y) \text{ and}$$
$$k = g(x, y).$$

   That is, we want to solve the following simultaneous system of equations:

$$f_x(x, y) = \lambda g_x(x, y), \quad f_y(x, y) = \lambda g_y(x, y), \quad k = g(x, y).$$

2. Evaluate $f(x, y)$ at all the points you found in the previous step. The largest gives you the maximum, the smallest gives you the minimum.

An example

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(x, y)	$f(x, y)$
$(1, 1)$	
$(1, -1)$	
$(-1, 1)$	
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$(1, 1)$	e
$(1, -1)$	$1/e$
$(-1, 1)$	$-e$
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(x, y)	$f(x, y)$
$(1, 1)$	e MAX
$(1, -1)$	$1/e$
$(-1, 1)$	$-e$ MIN
$(-1, -1)$	$-1/e$

Finding extreme values

We can combine the method of Lagrange multipliers with the second derivative test to find extreme values:
Given a (nice) function $f(x, y)$ on a closed, bounded region R, we want to find the maximum and minimum.

1. Use the second derivative test to find all local maxima/minima in the interior of R.
2. Use Lagrange multipliers to find the maximum and minimum on the boundary of R.
3. The very largest value gives the maximum, the very smallest gives the minimum.

> 2 dimensions

The method of Lagrange multipliers also applies in 3 dimensions.
The problem: you want to find the maximum and minimum of the function $f(x, y, z)$ on the surface given by the equation $g(x, y, z) = k$.

1. Find all values x, y, z, λ so that
$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \text{ and}$$
$$k = g(x, y, z).$$
2. Evaluate $f(x, y, z)$ at all these points. The largest is the maximum, the smallest is the minimum.

3 dimensions, 2 surfaces

The problem: you want to find the maximum and minimum of the function $f(x, y, z)$ on the curve which is the boundary of the two surfaces given by $g(x, y, z) = k$ and $h(x, y, z) = \ell$.

1. Find all values x, y, z, λ so that

$$\begin{aligned} \nabla f(x, y, z) &= \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z); \\ k &= g(x, y, z); \text{ and} \\ \ell &= h(x, y, z) \end{aligned}$$

2. Evaluate $f(x, y, z)$ at all these points. The largest is the maximum, the smallest is the minimum.