Math 321: Induction

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Previously in Math 321

- We've been learning strategies for proofs.
- So far, the strategies have been based on the logical structure of the statements being used.
- On Tuesday we finally got to our last bit of logical notation: the disjunction $\vee$.
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- It might seem so, but it's wrong.

Mathematical Induction

- Our previous proof strategies were all based on logical content. There was nothing specifically mathematical about them, and the same proof strategies can be applied to formal logic outside of math.
- The technique of $\mathfrak{Mathematical\ Induction}$ is based on the structure of certain mathematical objects and how they are built up. You could call it the first properly mathematical proof strategy we'll look at.

A different meaning for induction

- You have probably heard of induction in the sense of inductive reasoning, as opposed to deductive reasoning like we do in mathematics.
- For example, we say that the sun will rise tomorrow morning because we've observed lots of mornings of the sun rising, but have no observations of it not rising. So it seems like it'll continue to rise.
- But we don't have a *proof* that the sun will rise tomorrow, just generalization from our experiences. I'd say we have good reason to think the sun will rise tomorrow, but it doesn't meet the standard of evidence used in mathematics.

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- Mathematical induction is not this sense of induction. Mathematical induction is a fully formal proof technique, much like the previous ones we've learned.

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For example, logical formulae.

- We start with atomic formulae, like $x^2 < 2$ or $1 + 1 = 2$ or $A \subseteq B$. These are the basic building blocks.
- And corresponding to each bit of logical notation is a rule for building up more formulae:
    - If φ and ψ are formulae, then so are $\varphi \wedge \psi$, $\varphi \vee \psi$, and $\neg\varphi$.
    - If φ is a formula and x is a variable, then $\exists x\ \varphi(x)$ and $\forall x\ \varphi(x)$ are also formulae.

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For another example, many functions can be described this way.

- Consider the factorial function $f(n) = n!$ on the natural numbers.
- This object, a function, is built up by saying how it behaves at 0, then saying how it behaves at $n > 0$ based upon what happened below n.
- Namely, the rules are:
  - $0! = 1$;
  - $(n+1)! = (n+1) \cdot n!$.

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Before we talk about the general structure, let's see a specific example.

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(Recursive step). We need to prove that if the proposition holds for $n!$ then it holds for $(n+1)!$. To prove this if-then statement, assume that $n!$ is divisible by all positive integers $k \leq n$. Then $(n+1)! = (n+1) \cdot n!$ is divisible by all positive integers $k \leq n$, since it's a multiple of $n!$. But also $(n+1)!$ is divisible by $n+1$. So $(n+1)!$ is divisible by all positive integers $k \leq n+1$.

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And now we are done.

Why are we done?

Why was checking just those two steps enough to prove the result for *all* $n > 1$? Let's see what's going on by looking at an example.

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In general: since any $n > 1$ is obtained from 1 by adding 1 finitely many times, we get the result for n by starting with the base case 1 and applying the recursive step $n - 1$ times.

A general template for mathematical induction

A template to prove $\forall x \in A\ P(x)$ about a collection A of objects x built up by rules like this:

1. Prove $P(x)$ holds for all base cases. These are the starting points that aren't built up from simpler cases.
   - For logical formulae, the base case is the atomic formulae. For the factorial function, the base case is $n = 0$.
2. For each rule of the form "if $x, y, \ldots \in A$ then $z \in A$" prove: if $P(x)$, $P(y)$, and so on, then $P(z)$.
   - For logical formulae, you would prove: if $P(\varphi)$ and $P(\psi)$ then $P(\varphi \wedge \psi)$, and similar for the other rules.
   - For the factorial function, there's only one rule to worry about here. Prove: if $P(n!)$ then $P((n+1)!)$.

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(Recursive steps) If φ is equivalent to φ^*, then clearly $\neg\varphi$ is equivalent to $\neg\varphi^*$, which has no $\forall$s nor $\wedge$s. And a similar process works for the $\vee$ and $\exists$ cases. So we only need to check the $\wedge$ and $\forall$ cases.

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($\forall$) Suppose φ is equivalent to φ^*. Then $\forall x\ \varphi(x)$ is equivalent to $\forall x\ \varphi^*(x)$ which is equivalent to $\neg\exists x\ \neg\varphi^*(x)$. So we can get rid of the new $\forall$.

The most important case of mathematical induction

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I'm talking about the natural numbers $\mathbb{N}$.

Induction on $\mathbb{N}$

We can describe $\mathbb{N}$ by a building up process:

- (Base case) 0 is a natural number.
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From this way of describing $\mathbb{N}$ we get a method for proving facts about natural numbers by induction. To prove $\forall n \in \mathbb{N}\ P(n)$:

1. Prove $P(0)$. This is usually, though not always, easy.
2. Prove: if $P(n)$ then $P(n + 1)$.
3. Conclude: $P(n)$ holds for all natural numbers n.

Example 3

Proposition

For all natural numbers n, we have $0 + 1 + 2 + \cdots + n = n(n+1)/2$.

You can prove this directly—try this at home!—but let's use induction.

(Base case) The left-hand side and right-hand side of the equation are clearly both 0 when $n = 0$.

(Recursive step) Suppose $0 + 1 + \cdots + n = n(n+1)/2$. Let's add $n + 1$ to both sides:

$$\begin{aligned} 0 + 1 + \cdots + n + (n+1) &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2}. \end{aligned}$$

This is precisely what we wanted to show, so now we are done.