Nonstandard methods versus Nash-Williams

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they/them

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Joint work with Timothy Trujillo (SHSU)

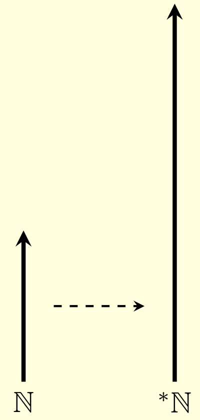
Our project

- Nonstandard methods have been fruitfully applied to prove theorems about combinatorics on $\mathbb{N}$
  - Namedrop: Di Nasso, Goldbring, Jin, Lupini, Tao, ...
- Topological Ramsey theory studies combinatorial topological spaces which generalize Ellentuck space ($\approx$ the space of subsets of $\mathbb{N}$), the familiar setting for ordinary Ramsey theory
- Let's apply nonstandard methods to a more general setting than Ellentuck space
- Starting point: the Nash-Williams theorem for Ellentuck space and its generalization

Nonstandard methods

We can use tools from model theory to prove theorems outside of logic

- Take a structure. For this talk, it will mostly be $\mathbb{N}$
- Take an ultrapower of $\mathbb{N}$ to embed $\mathbb{N}$ into a saturated elementary extension ${}^*\mathbb{N}$
- Exploit the connection $\mathbb{N} \hookrightarrow {}^*\mathbb{N}$ to prove theorems about $\mathbb{N}$



A gentle warmup: the pigeonhole principle

Theorem (Pigeonhole Principle)

If you partition $\mathbb{N}$ into finitely many pieces $X_0, \ldots, X_n$ then one of the pieces is infinite.

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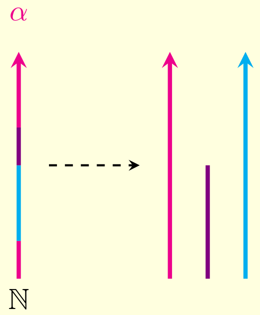
Proof:

- Consider $\alpha \in {}^*\mathbb{N} \setminus \mathbb{N}$
- ${}^*X_0, \ldots, {}^*X_n$ are a partition of ${}^*\mathbb{N}$ (by elementarity)
- So α is in some ${}^*X_i$
- So X_i is infinite (by elementarity)



Iterating the $*$ map

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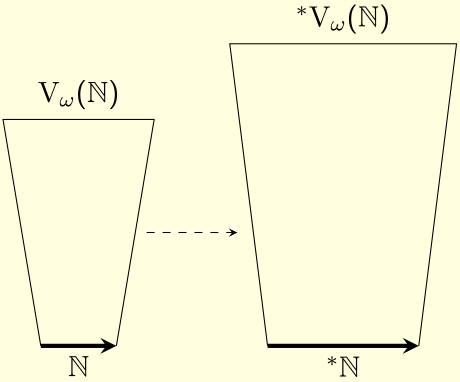
- Actually we embed $\mathrm{V}_\omega(\mathbb{N})$ into a saturated elementary extension
- Then ${}^*\mathrm{V}_\omega(\mathbb{N})$ is a definable class in $\mathrm{V}_\omega(\mathbb{N})$
- So ${}^*\mathbb{N}$ is a set in the domain of the embedding
- We can apply the * map to it and its elements
- If $\alpha \in {}^*\mathbb{N} \setminus \mathbb{N}$ then $\alpha < {}^*\alpha$
- And we can iterate:

$$\mathbb{N} \hookrightarrow {}^*\mathbb{N} \hookrightarrow {}^{*(2)}\mathbb{N} \hookrightarrow \cdots \hookrightarrow {}^{*(k)}\mathbb{N} \hookrightarrow \cdots$$



A slightly less gentle warmup: Ramsey's theorem

Theorem (Ramsey 1930)

Partition $[\mathbb{N}]^k$ into finitely many pieces $X_0, \ldots, X_n$. Then there is infinite $H \subseteq \mathbb{N}$ so that $[H]^k \subseteq X_i$ for some i.

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Proof ($k = 3$):

- Consider $\alpha \in {}^*\mathbb{N} \setminus \mathbb{N}$
- Then $\langle \alpha, {}^*\alpha, {}^{*(2)}\alpha \rangle$ is in some ${}^{*(3)}X_i$
- So $\alpha \in {}^*\{a \in \mathbb{N} : \langle a, \alpha, {}^*\alpha \rangle \in {}^{*(2)}X_i\}$.
- So $\{a \in \mathbb{N} : \langle a, \alpha, {}^*\alpha \rangle \in {}^{*(2)}X_i\}$ is infinite
- Let h_0 be the minimum member

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Now induct:

- Already built $H_i = \langle h_0, \ldots, h_i \rangle$
- Inductively, $\alpha \in {}^*\{a \in \mathbb{N} : t^\frown a \in X_i\}$ for each $t \in [H_i]^2$
- And $\alpha \in {}^*\{a \in \mathbb{N} : t^\frown a^\frown \alpha \in {}^*X_i\}$ for each $t \in [H_i]^1$

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- Finitely many, and α is in their nonstandard intersection
- So their standard intersection is infinite
- Pick $h_{i+1} > h_i$ from that intersection

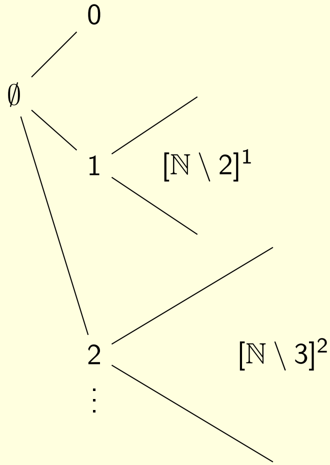
Finally $H = \langle h_i \rangle$ is monochromatic

Generalizing Ramsey to families of sets of nonuniform size

Definition

The Schreier barrier $\mathcal{S}$ consists of all $s \in [\mathbb{N}]^{<\omega}$ so that $|s| = \min s + 1$.

- The first element of s tells you how long s is
- You can think of $\mathcal{S}$ as a tagged amalgamation of (copies of) all $[\mathbb{N}]^k$



A Ramsey property for the Schreier barrier

Theorem (Nash-Williams for $\mathcal{S}$)

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$$\mathcal{S} \upharpoonright H = \{s \in \mathcal{S} : s \subseteq H\}$$

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- For $[\mathbb{N}]^k$ we looked at what piece of the partition contained $\langle \alpha, {}^*\alpha, \ldots, {}^{*(k-1)}\alpha \rangle$
- But now we don't know in advance how long a sequence in $\mathcal{S}$ will be
- Intuitively, we want to look at

$$\langle \alpha, {}^*\alpha, \ldots {}^{*(\alpha)}\alpha \rangle$$

- But this is nonsensical—what would it even mean to iterate * a nonstandard number of times?

A proxy for $\langle \alpha, {}^*\alpha, \ldots {}^{*(\alpha)}\alpha \rangle$

Notation:

- ${}^\star\mathbb{N} = \operatorname{dir}\lim_{k \in \omega} {}^{*(k)}\mathbb{N}$
- For $\beta \in {}^\star\mathbb{N}$, let $k(\beta)$ be the least k so that $\beta \in {}^{*(k)}\mathbb{N}$

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Claim: Fix $\alpha \in {}^*\mathbb{N}$. For any sequence $\langle \beta_i : i \in \omega \rangle$ there is (a non-unique) $\sum_\alpha \beta_i \in {}^*\mathbb{N}$ so that for all $X \subseteq \mathbb{N}$

$$\sum_{i\in\mathbb{N};\,\alpha} \beta_i \in {}^*X \quad \Leftrightarrow \quad \alpha \in {}^*\{i \in \mathbb{N} : \beta_i \in {}^{*(k(\beta_i))}X\}$$

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- Our proxy for $\langle \alpha, {}^*\alpha, \ldots, {}^{*(\alpha)}\alpha \rangle$ is then

$$\sigma(\alpha) = \sum_{i\in\mathbb{N};\,\alpha} \langle \alpha, \ldots, {}^{*(i)}\alpha \rangle$$

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$\mathcal{S} \restriction H = \{s \in \mathcal{S} : s \subseteq H\}$

$s_k = \langle \alpha, \ldots {}^{*(k)}\alpha \rangle$ approximate $\sigma(\alpha)$

Proof:

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Further generalization: fronts

$\mathcal{F} \subseteq [\mathbb{N}]^{<\omega}$ is a front if

- (antichain or Nash-Williams property)
  $s \not\sqsubseteq t$ for $s \neq t$ from $\mathcal{F}$
- (density)
  For any infinite $b \subseteq \mathbb{N}$ there is $s \sqsubseteq b$ from $\mathcal{F}$

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Examples:

- $[\mathbb{N}]^k$ for any k
- The Schreier barrier $\mathcal{S}$

Ramsey properties for fronts

To prove a Ramsey property for $[\mathbb{N}]^k$ and $\mathcal{S}$ we had an idea of what a generic nonstandard member looked like, based on how the front was built up

- $\langle \alpha, \ldots, {}^{*(k-1)}\alpha \rangle$ for $[\mathbb{N}]^k$
- $\sigma(\alpha)$, a proxy for $\langle \alpha, \ldots, {}^{*(\alpha)}\alpha \rangle$ for $\mathcal{S}$

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If we want to do the same for an arbitrary front $\mathcal{F}$ we need to understand how $\mathcal{F}$ is built up

Trees of fronts

For $\mathcal{F}$ a front, set

$$T(\mathcal{F}) = \{t \in [\mathbb{N}]^{<\omega} : t \sqsubseteq s \text{ for some } s \in \mathcal{F}\}$$

Then $T(\mathcal{F})$ is a tree and $\mathcal{F}$ are the leaves

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Claim: $T(\mathcal{F})$ is well-founded

- If b were an infinite branch through $T(\mathcal{F})$ it'd extend some $s \in \mathcal{F}$ by density
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We can think of $\mathcal{F}$ as built up by induction on $T(\mathcal{F})$

- For $s \in \mathcal{F}$, set $\mathcal{F}_s = \{s\}$
- For $s \in T(\mathcal{F}) \setminus \mathcal{F}$, set $\mathcal{F}_s = \bigcup_{t \in \operatorname{succ} s} \mathcal{F}_t$
- Here $\operatorname{succ} s$ is the set of successors of s in $T(\mathcal{F})$
- Observe that $\mathcal{F}_s$ is a front on $[\mathbb{N}]^{<\omega} \restriction s$

Finally $\mathcal{F} = \mathcal{F}_\emptyset$

An example: the Schreier barrier

$$\mathcal{S} = \{s \in [\mathbb{N}]^{<\omega} : |s| = \min s + 1\}$$

What is $\mathcal{S}_s$ for subsequences s of $\langle 2, 7, 9 \rangle$?

$$\begin{aligned} T(\mathcal{S}) &= \{t \in [\mathbb{N}]^{<\omega} : t \sqsubseteq s \text{ for some } s \in \mathcal{S}\} \\ &= \{t \in [\mathbb{N}]^{<\omega} : |s| \leq \min s + 1\} \end{aligned}$$

$$\begin{aligned} \mathcal{S}_s &= \{s\} && \text{if } s \in \mathcal{S} \\ &= \bigcup_{t \in \operatorname{succ} s} \mathcal{S}_s && \text{if } s \in T(\mathcal{S}) \setminus \mathcal{S} \end{aligned}$$

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- $\mathcal{S} = \mathcal{S}_\emptyset = \{a^\frown t : t \in [\mathbb{N} \setminus (a + 1)]^a\}$

The Nash-Williams theorem for Ellentuck space

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Proof sketch: Fix $\alpha \in {}^*\mathbb{N} \setminus \mathbb{N}$

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The idea is, inductively build up $\sigma_\emptyset = \sigma_\emptyset(\alpha)$ to play a similar role as $\sigma(\alpha)$ did for $\mathcal{S}$:

- For $s \in \mathcal{F}$, set $\sigma_s = \sigma_s(\alpha)$ to be $\langle \alpha \rangle$
- For $s \in T(\mathcal{F}) \setminus \mathcal{F}$, set $\sigma_s = \sigma_s(\alpha)$ to be $\sum_{t \in \mathrm{succ}\, s;\, \alpha} \sigma_t(\alpha)$

Recall:

$$\sum_{t \in \mathrm{succ}\, s;\, \alpha} \sigma_t \in {}^*X \quad \Leftrightarrow \quad s^\frown \alpha \in {}^*\{a \in \mathbb{N} : \sigma_{s^\frown a} \in {}^*X\}$$

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- $\sigma_\emptyset(\alpha)$ is in some ${}^*X_i$
- Pick h_0 to be the minimum element of $\{a \in \mathbb{N} : \sigma_a \in {}^*X_i\}$
- Then inductively pick $h_{i+1} > h_i$ using that α is in ${}^*\{a \in \mathbb{N} : \sigma_{t^\frown a} \in {}^*X_i\}$ for each subset t of the i-th partial solution H_i

Finally $H = \langle h_i \rangle$ is monochromatic

Abstract Ramsey spaces

Ellentuck space $\mathcal{E}$ has multiple components

- The points are elements of $[\mathbb{N}]^\omega$
- You can associate to each point its k-th finite approximation in $[\mathbb{N}]^k$
- There is a partial order $\subseteq$ on points

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- There is a partial order $\subseteq$ on points

And $\mathcal{E}$ has some nice properties

(A.1) Sequencing: points behave like infinite sequences

(A.2) Finitization: you can port the partial order $\subseteq$ to the finite approximations, and each approximation has a finite number of predecessors

(A.3) Amalgamation: [this one's more technical]

(A.4) Pigeonhole: as it says in the name

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- You can associate to each point its k-th finite approximation in $[\mathbb{N}]^k$
- There is a partial order $\subseteq$ on points

And $\mathcal{E}$ has some nice properties

(A.1) Sequencing: points behave like infinite sequences

(A.2) Finitization: you can port the partial order $\subseteq$ to the finite approximations, and each approximation has a finite number of predecessors

(A.3) Amalgamation: [this one's more technical]

(A.4) Pigeonhole: as it says in the name

A Ramsey space is a tuple $(\mathcal{R}, \mathcal{AR}, \leq, r)$ satisfying (A.1–4) where $\mathcal{R}$ are the points, $r : \mathcal{R} \times \omega \to \mathcal{AR}$ is the finite approximation map, and $\leq$ is the partial order

The topological in topological Ramsey theory

The Ellentuck topology on $\mathcal{R}$ is generated by basic open sets

$$[a, X] = \{Y \in \mathcal{R} : Y \leq X \text{ and } \exists k \; r_k(Y) = a\}.$$

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If $\mathcal{R}$ is closed as a subspace of the product topology on $\mathcal{AR}$, it's quite nice

- $\mathcal{X} \subseteq \mathcal{R}$ is Ramsey if you can refine any basic open set be either contained in or disjoint from $\mathcal{X}$
- $\mathcal{X} \subseteq \mathcal{R}$ is Ramsey null if it is Ramsey and you can always refine to be disjoint from $\mathcal{X}$

- If $\mathcal{R}$ is closed, any Baire subset is Ramsey and any meager subset is Ramsey null
- Indeed any Souslin-measurable or Borel subset is Ramsey

The abstract Nash-Williams theorem

Theorem (Abstract Nash-Williams)

Suppose $\mathcal{R}$ is closed. Then any front on the finite approximations $\mathcal{AR}$ satisfies a Ramsey partition property.

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- I'd like to say our nonstandard proof of the Nash-Williams theorem extends directly to the full abstract Nash-Williams theorem
- But we need the space to be amenable to nonstandard methods
- And we don't (yet?) have a proof that this applies to every nontrivial Ramsey space

What we do have for the abstract Nash-Williams theorem

Under an extra assumption the nonstandard proof goes through.

Theorem (Partial abstract Nash-Williams)

Consider a front $\mathcal{F}$ on $\mathcal{AR}$. Suppose

- *$\mathcal{AR}$ is infinitely branching everywhere; and*
- *There is a filter $\mathcal{C}$ on $\mathcal{R}$ so that for each $s \in T(\mathcal{F}) \setminus \mathcal{F}$ the restriction of $\operatorname{succ} s$ to $\mathcal{C}$ is a nonprincipal ultrafilter on $\operatorname{succ} s$.*

Then $\mathcal{F}$ satisfies a Ramsey partition property.

- $(\mathcal{R}, \leq)$ is a poset, so the usual definition of filter applies to $\mathcal{C}$
- $\operatorname{succ} s \restriction X = \{t \in \operatorname{succ} s : \exists k \; t \leq_{\text{fin}} r_k(X)\}$
- $\operatorname{succ} s \restriction \mathcal{C} = \{\operatorname{succ} s \restriction X : X \in \mathcal{C}\}$

Positive examples

Any Ramsey space which can be thought of as its $(k+1)$-th approximations coming from k-th approximations by concatenating sequences from (cofinite subsets of) a countable alphabet will admit such a filter

- Ellentuck space
    - Restrict any nonprincipal ultrafilter on $\mathcal{P}(\mathbb{N})$ to the infinite subsets to get $\mathcal{C}$

Positive examples

Any Ramsey space which can be thought of as its $(k+1)$-th approximations coming from k-th approximations by concatenating sequences from (cofinite subsets of) a countable alphabet will admit such a filter

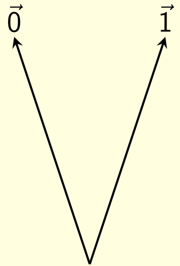
- Ellentuck space
  - Restrict any nonprincipal ultrafilter on $\mathcal{P}(\mathbb{N})$ to the infinite subsets to get $\mathcal{C}$
- The Milliken space of block sequences
- The Hales–Jewett space of variable words
- The space $\mathcal{E}_\omega(\mathbb{N})$ of equivalence relations on $\mathbb{N}$ with infinite quotients

A silly negative example

The V space

- $\mathcal{V}$ has two points, the constant 0 sequence and the constant 1 sequence
- Finite approximations are finite constant 0 or 1 sequences

- Trivially, any front on $\mathcal{AV}$ satisfies a Ramsey partition property
- But $\mathcal{V}$ doesn't satisfy the filter property!



Continuing work

Question

Suppose you have a nontrivial[a] *Ramsey space* $(\mathcal{R}, \mathcal{AR}, \leq, r)$ *and a front* $\mathcal{F}$ *on* $\mathcal{AR}$. *Then there is a filter* $\mathcal{C}$ *on* $\mathcal{R}$ *so that for each* $s \in T(\mathcal{F}) \setminus \mathcal{F}$ *the restriction of* $\operatorname{succ} s$ *to* $\mathcal{C}$ *is a nonprincipal ultrafilter on* $\operatorname{succ} s$.

[a] What should this mean?

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- The abstract Nash-Williams theorem isn't the only theorem in abstract Ramsey theory
- What other results are amenable to nonstandard methods?

Thank you!