Math 321: Coda, or the limits of mathematics

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Recapping the semester

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- We learned strategies for proofs.
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To close off this class, I want to have a bit of a class discussion about why this view of what math is misses out on some real complexity.

Alternate logics

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- We can think of rules for proofs as saying that certain moves are valid.
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- Let's see an example.

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- One possible answer: saying a mathematical object exists means saying you can explicitly construct it.
- If you take this view, that affects what mathematical statements are true.

Kőnig's Lemma

Lemma

Any infinite binary tree has an infinite branch.

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But this result is not constructively valid.

Constructivism

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- The upshot of this is that it means using different rules of logic, based on a different interpretation of the meaning of logical symbols.
- For example, to know $P \vee Q$ means you know P is true, or you know Q is true. That is, it's not allowed where you know one of the two is true but you cannot say which.
- In particular, the law of excluded middle $P \vee \neg P$ is not valid in constructivism.

This leads to a different mathematics. We saw that Kőnig's Lemma is one point of difference. There are also differences in, for example, real analysis, the mathematics underlying calculus.

To sum up

If you use different rules of logic—rules which can be well motivated by a different understanding of how logic works—then you can get different mathematical results.

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- We can try to push this down even further, but we could get down to an axiom we disagree about.
- At this point, our disagreement cannot be resolved by mathematics. Maybe I can convince you that my axiom is better than your axiom, or maybe you can convince me to accept your axiom, but this cannot happen just by exhibiting a proof.

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We might agree about the rules of logic and the basic axioms of mathematics, but still disagree.

- The way math works is, there are some intuitive concepts we want to study—shapes, quantity, space, continuity, etc.
- We take these intuitive concepts and formalize them, coming up with precise mathematical definitions that we can then prove theorems about.
- How do we know whether a definition is correct? Is it even sensible to ask such a thing?

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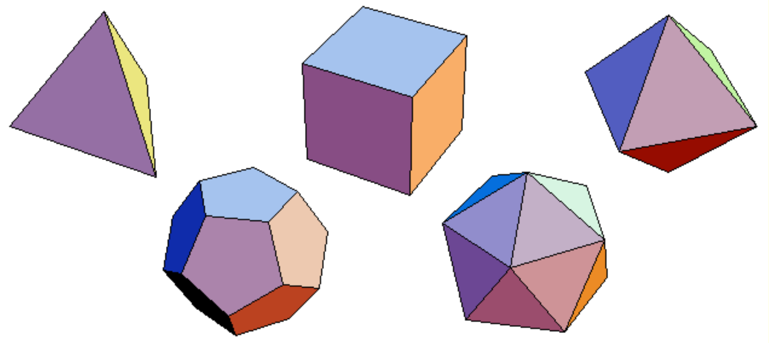
This brings us to the reading: Lakatos's *Proofs and Refutations*.

What is a polyhedron?



Euler's theorem ("theorem"?)

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Consider a polyhedron. Let V be its number of vertices, E be its number of edges, and F be its number of faces. Then,

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Whether this theorem is actually true depends on how you define polyhedron.