MATH655 EXERCISES: POSET COMBINATORICS

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The basic combinatorial objects used in forcing are posets. This exercise set will lead through you through an investigation of how set theorists handle them. Please write up your solutions to these exercises and turn them in to me before we start Part 2. (Exact date pending; I'll update the course website as we get closer.)

A *poset*[1] $(\mathbb{P}, <)$ is a set $\mathbb{P}$ equipped with a transitive, irrelexive relation $<$. As is typical, we will write $p \leq q$ if $p < q$ or $p = q$, and use $>$ and $\geq$ for the reversed relation. We refer to elements of posets as *conditions*, and we think of conditions as containing partial information about some generic object. We will prefer to think of posets as growing downward, with $p \leq q$ meaning that p is a stronger condition than q, containing more information. We will work with posets that have a maximum element, call it $\mathbf{1}$. (If a poset lacks a maximum, it can easily be modified to add a maximum, so this restriction is a harmless convenience.) This condition $\mathbf{1}$ is the one that contains no information.

If $p, q \in \mathbb{P}$, we say that p and q are *compatible*, written $p \parallel q$ if there is $r \in \mathbb{P}$ so that $r \leq p$ and $r \leq q$. Otherwise, if there is no such r, we say that p and q are *incompatible*, written $p \perp q$. The intuition is: if p and q could be partial descriptions of the same object, then they are compatible. But if they are mutually exclusive partial descriptions, then they are incompatible. If for all $p \in \mathbb{P}$ we can strengthen p to two incompatible conditions—that is, there are $q, r \leq p$ so that $q \perp r$—we say that $\mathbb{P}$ is *separable*.

Example 1. Consider the infinite binary tree $\mathbb{P} = {}^{<\omega}2$, ordered by reverse inclusion: $p \leq q$ if $p \supseteq q$. Then $\mathbb{P}$ is a separable poset with maximum element the empty sequence. (To check that it is separable: given any condition p, extend p to $p^\frown 0$ and $p^\frown 1$, which are incompatible.)

Exercise 2. Show that every separable poset is infinite. Show that no linear order is separable.

Exercise 3. Let α, β be ordinals. Then ${}^{<\alpha}\beta$ is the collection of all sequences of ordinals $< \beta$ of length $< \alpha$. We order this tree by reverse inclusion: $p \leq q$ iff $p \supseteq q$. Show that if α is limit and $\beta \geq 2$ then ${}^{<\alpha}\beta$ is a separable poset.

Given a poset $\mathbb{P}$, we can put a topology on it. Namely, the basic open sets are of the form $N_p = \{q \in \mathbb{P} : q \leq p\}$.

Exercise 4. Give a topological characterization of separability.

Several topological properties of a poset are of intertest to us, with openness and density standing out here. The following two exercises have you give combinatorial characterizations for them.

Exercise 5. Show that $A \subseteq \mathbb{P}$ is open iff A is downward-closed—$p \in A$ and $q \leq p$ implies $q \in A$.

Exercise 6. Show that $D \subseteq \mathbb{P}$ is dense iff given any $p \in \mathbb{P}$ there is $q \leq p$ with $q \in D$.

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[1]**P**artially **o**rdered **set**.

A key notion is that of a generic filter over a poset.

Definition 7. $F \subseteq \mathbb{P}$ is a filter if it satisfies the following.

(1) $\mathbf{1} \in F$;
(2) F is upward-closed—$p \in F$ and $q \geq p$ implies $q \in F$; and
(3) F is directed—$p, q \in F$ implies $p \parallel q$.

Exercise 8. Give a topological characterization of when a subset of $\mathbb{P}$ is a filter.

Definition 9. Let $\mathcal{D}$ be a collection of dense subsets of $\mathbb{P}$. Then a filter G is *$\mathcal{D}$-generic* if for every $D \in \mathcal{D}$ there is $p \in G \cap D$. We summarize this situation by saying that G *meets* $\mathcal{D}$. If $\mathcal{D} = \{D\}$ we will abuse terminology a bit and say that G meets D.

If we are concerned about a filter meeting a dense set, we can freely assume without loss that the dense set is open.

Exercise 10. Let $\mathbb{P}$ be a poset and suppose $D \subseteq \mathbb{P}$ is dense. Let $\bar{D}$ be the smallest open set containing D. That is, $\bar{D} = \{p \in \mathbb{P} : \exists q \in D\ p \leq q\}$. Manifestly, $\bar{D}$ is still dense. Show that a filter F meets D iff F meets $\bar{D}$.

The following is the fundamental lemma about generic filters. It is sufficiently important that I will give the proof, rather than leaving it as an exercise.

Lemma 11. *Let $\mathcal{D}$ be a countable collection of dense subsets of $\mathbb{P}$. Then there is a filter G which is $\mathcal{D}$-generic.*

Proof. Enumerate $\mathcal{D}$ as $D_0, \ldots, D_k, \ldots$, where $k \in \omega$. We will define a sequence of increasingly stronger conditions. Start with $p_0 = \mathbf{1}$. Given p_k, pick $p_{k+1} \leq p_k$ which meets D_k. Such exists because D_k is dense.

We now use this sequence of conditions to define G. Namely, set $G = \{q \in \mathbb{P} : \exists k\ q \geq p_k\}$. I claim that G is a $\mathcal{D}$-generic filter. First check it's a filter: Clearly $\mathbf{1} \in G$ and G is upward closed. And G is directed because if $q, r \in G$ then $q, r \geq p_k$ for large enough k, witnessing that they are compatible. Finally, we want to check genericity. To this end, fix $D \in \mathcal{D}$. Then $D = D_k$ for some k. Then $p_k \in G \cap D_k$, as desired. $\square$

This argument can be pushed to give generics for larger collections of dense sets, provided our poset satisfies an additional property.

Definition 12. Let κ be an infinite cardinal. Say that $\mathbb{P}$ is *κ-closed* if given any decreasing sequence $\langle p_i : i < \alpha < \kappa \rangle$ of conditions of length $< \kappa$—that is, $p_i \geq p_j$ for $i < j$—there is a condition p below each p_i.

Exercise 13. Show that if $\mathbb{P}$ is κ^+-closed and $\mathcal{D}$ is a collection of dense subsets of $\mathbb{P}$ of cardinality $\leq \kappa$ then there is a filter G which is $\mathcal{D}$-generic.

Exercise 14. Let $\kappa \geq \omega$ and $\lambda \geq 2$ be cardinals, where κ is regular. Show that $\mathbb{P} = {}^{<\kappa}\lambda$ is κ-closed.

Exercise 15. Let $\kappa \geq \omega$ be regular. Find a separable poset which is κ-closed but not κ^+-closed.

On the other hand, there is an upper limit for genericity, for non-boring posets.

Exercise 16. Let $\mathbb{P}$ be a separable poset and let $\mathcal{D}$ be the collection of all dense subsets of $\mathbb{P}$. Show that there is no $\mathcal{D}$-generic filter. (Hint: Suppose $F \subseteq \mathbb{P}$ is a filter and consider $D = \mathbb{P} \setminus F$. Start by showing that D is dense.)

Exercise 17. Show that if $\mathbb{P}$ is a nonseparable poset then there is $p \in \mathbb{P}$ so that $N_p = \{q \in \mathbb{P} : q \leq p\}$ consists of pairwise-compatible conditions. Conclude that if $\mathbb{P}$ is nonseparable then there is always a $\mathcal{D}$-generic filter where $\mathcal{D}$ is the collection of all dense subsets of $\mathbb{P}$.

Another object of interest to us is that of the antichain.

Definition 18. Let $\mathbb{P}$ be a poset. Then $A \subseteq \mathbb{P}$ is an *antichain* if $p, q \in A$ implies $p \perp q$.

Exercise 19. Show that every poset $\mathbb{P}$ admits maximal antichains. That is, there is $A \subseteq \mathbb{P}$ an antichain so that there is no $B \supsetneq A$ an antichain.

One reason antichains are important is that generic filters can only select at most one element from an antichain.

Exercise 20. Suppose $A \subseteq \mathbb{P}$ is an antichain and G is a $\mathcal{D}$-generic filter for $\mathbb{P}$, where $\mathcal{D}$ is a nonempty collection of dense subsets of $\mathbb{P}$. Show that G contains at most 1 element of A.

Definition 21. Let $\kappa \geq \omega$ be a cardinal. Then $\mathbb{P}$ has the *κ-chain condition*, abbreviated *κ-cc*, if every anti-chain in $\mathbb{P}$ has cardinality $< \kappa$.[2] Another name for the ω_1-chain condition is the *countable chain condition*, or *ccc*.

Exercise 22. Fix $\kappa \geq \omega$. Find a separable poset which has the κ-cc but not the κ^+-cc.

Finally, we end this exercise set with a taste of how generic filters relate to models of set theory.

Let M be a transitive model of ZFC. Let $\mathbb{P} \in M$ be a poset. Say that a filter $G \subseteq \mathbb{P}$ is *M-generic* if G is $\mathcal{D}_M$-generic where $\mathcal{D}_M$ is the collection of all dense subsets of $\mathbb{P}$ which are in M.

Exercise 23. Show that M-generics exist whenever M is a countable.

Exercise 24. Suppose $\mathbb{P}$ is separable. Show that if $G \subseteq \mathbb{P}$ is M-generic then $G \notin M$.

Exercise 25. Suppose $A \in M$ is an antichain of $\mathbb{P}$ so that $M \models A$ is a maximal antichain. Suppose G is M-generic. Show that $G \cap A$ has exactly one element.

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[2]Yes, the name is bad. It really should be the κ-antichain condition. But the κ-cc name has enough inertia that it would be silly to try to change it.