Mediacy and Independence

Kameryn Julia Williams
they/she

Bard College at Simon's Rock

120 Years of Choice
2024 Jul 12

What does it mean to be infinite?

- X is finite if $|X| < \omega$. Otherwise X is infinite.
- X is infinite iff $|X| \geq n$ for all $n < \omega$.

This isn't circular, because we can define ω by its induction properties.

Dedekind's analysis

Dedekind's analysis

- X is Dedekind-infinite if there is $f : X \to X$ a non-surjective injection.
- X is Dedekind-finite if any injection $f : X \to X$ is a surjection.

Dedekind's analysis

X is Dedekind-infinite iff $\omega \leq |X|$.

- X is Dedekind-infinite if there is $f : X \to X$ a non-surjective injection.
- X is Dedekind-finite if any injection $f : X \to X$ is a surjection.

Dedekind's analysis

- X is Dedekind-infinite if there is $f : X \to X$ a non-surjective injection.
- X is Dedekind-finite if any injection $f : X \to X$ is a surjection.

X is Dedekind-infinite iff $\omega \leq |X|$.

- $(\Leftarrow)$ Push forward the $+1$ function on ω.
- $(\Rightarrow)$ Fix $z \in X \setminus \operatorname{ran} f$. Then the map $n \mapsto f^n(z)$ gives an injection $\omega \to X$.
    - Use fact that f is one-to-one to inductively prove this map is an injection.

Are they the same?

Are they the same?

Yes.

- If X is Dedekind-infinite then $\omega \leq |X|$ so X is infinite.
- If X is infinite, choose for each n an injection $e_n : n \to X$. Inductively glue them together into an injection $e : \omega \to X$.

Are they the same?

Yes.

- If X is Dedekind-infinite then $\omega \leq |X|$ so X is infinite.
- If X is infinite, choose for each n an injection $e_n : n \to X$. Inductively glue them together into an injection $e : \omega \to X$.

Are they the same?

Yes, assuming AC_ω.

- If X is Dedekind-infinite then $\omega \leq |X|$ so X is infinite.
- If X is infinite, choose for each n an injection $e_n : n \to X$. Inductively glue them together into an injection $e : \omega \to X$.

- Dedekind-infinite implies infinite goes through in ZF.
- Infinite implies Dedekind-infinite needs a small fragment of AC.

Are they the same?

Yes, assuming AC_ω.

- If X is Dedekind-infinite then $\omega \leq |X|$ so X is infinite.
- If X is infinite, choose for each n an injection $e_n : n \to X$. Inductively glue them together into an injection $e : \omega \to X$.

- Dedekind-infinite implies infinite goes through in ZF.
- Infinite implies Dedekind-infinite needs a small fragment of AC.

Theorem (Cohen 1963)

It is consistent with ZF *that there exists a Dedekind-finite, infinite set.*

But you can't get a reversal: there's no hope the non-existence of a DFI set implies AC because the former is local while the latter is global.

The first question

Question

Is there a suitable generalization of a Dedekind-finite, infinite set whose nonexistence gives a characterization of AC?

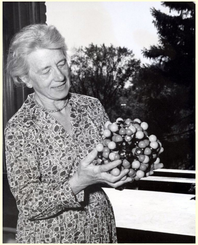
A look back in history

- What we've seen is, under other language, the state of the art for the first decade of AC's life.

A look back in history

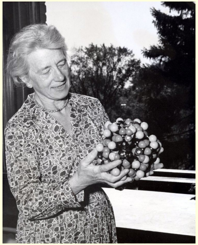
- What we've seen is, under other language, the state of the art for the first decade of AC's life.
- In the late 1910s, Bertrand Russell is a few years after the last volume of *Principia Mathematica*. His time is occupied by legal troubles over his pacifism during World War I and thinking about the foundations of mathematics.
- Working with him are multiple students, including Dorothy Wrinch.
- The next decade (1923) she will publish a paper answering our first question.

Dorothy Wrinch



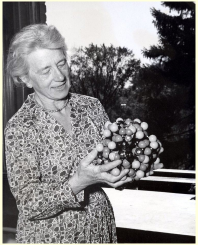
- Born 1894, died 1976.
- Studied logic under Russell, did her doctorate (1921) under applied mathematician John Nicholson.
- Wrote in a range of subjects: logic, pure mathematics, philosophy of science, and mathematical biology.
- Was awarded a Rockefeller Foundation fellowship to support her work in mathematical biology.
- Early career was in the UK, later emigrated to the USA. Latter years of her career were at Smith College (Mass, USA).
- Had the misfortune of being on the losing side of a scientific dispute with Linus Pauling over the structure of proteins.

Wrinch's work in logic



- Part of a group of Russell's students who studied mathematical logic with him.
- Her 1917 essay *Transfinite Types* won Girton College's Gamble Prize.
- (1923) Gave a characterization of AC based on a generalization of Dedekind-finite, infinite sets.
  (Independently rediscovered in 1964 by Lévy.)
- Worked on logic's applications in the philosophy of science.

Wrinch's work in logic



- Part of a group of Russell's students who studied mathematical logic with him.
- Her 1917 essay *Transfinite Types* won Girton College's Gamble Prize.
- (1923) Gave a characterization of AC based on a generalization of Dedekind-finite, infinite sets.
  (Independently rediscovered in 1964 by Lévy.)
- Worked on logic's applications in the philosophy of science.

This one's a stretch:

- She thought her model of protein structure would be a 'theorem', but was later partially vindicated with a consistency result: there are crystals whose molecular structure fit her cyclol model.

Wrinch's question, and mine

Question

Is there a suitable generalization of a Dedekind-finite, infinite set whose nonexistence gives a characterization of AC?

Question

Can we use modern techniques to prove more precise consistency results?

Cardinals sans choice

Notation:

- $\kappa, \lambda, \ldots$ will be used for well-orderable, infinite cardinals.
- $\mathfrak{p}, \mathfrak{q}, \ldots$ will be used for cardinals in general.
- I'll sometimes use $\mathfrak{p}$ to refer to an arbitrary set of cardinality $\mathfrak{p}$.

- Under AC, every cardinal is well-orderable. We can thus define the cardinals as the initial ordinals.
- Without AC we have to fall back on defining cardinals as equivalence classes.
- Can use Scott's trick to make these sets.

Mediate cardinals

Fix a cardinal $\mathfrak{p}$. Then X is $\mathfrak{p}$-mediate if

- $\mathfrak{q} \leq |X|$ for all $\mathfrak{q} < \mathfrak{p}$;
- $\mathfrak{p} \not\leq |X|$; and
- $|X| \not\leq \mathfrak{p}$.

A $\mathfrak{p}$-mediate cardinal is a cardinal number of a $\mathfrak{p}$-mediate set.

Mediate means $\mathfrak{p}$-mediate for some infinite $\mathfrak{p}$.

Mediate cardinals

Fix a cardinal $\mathfrak{p}$. Then X is $\mathfrak{p}$-mediate if

- $\mathfrak{q} \leq |X|$ for all $\mathfrak{q} < \mathfrak{p}$;
- $\mathfrak{p} \not\leq |X|$; and
- $|X| \not\leq \mathfrak{p}$.

A $\mathfrak{p}$-mediate cardinal is a cardinal number of a $\mathfrak{p}$-mediate set.

Mediate means $\mathfrak{p}$-mediate for some infinite $\mathfrak{p}$.

- Dedekind-finite infinite $\Leftrightarrow$ $\aleph_0$-mediate.
- ZF proves there are no finite degrees of mediacy.

A few facts

Some facts about DFI sets generalize.

Fact

Suppose $\mathfrak{q}$ and $\mathfrak{r}$ are $\mathfrak{p}$-mediate. Then:

- $\mathfrak{q} + \mathfrak{r}$ *is $\mathfrak{p}$-mediate;*
- $\mathfrak{q} \cdot \mathfrak{r}$ *is $\mathfrak{p}$-mediate;*

Suppose $\mathfrak{q}$ is κ-mediate. Then:

- $2^{2^{\mathfrak{q}\cdot\mathfrak{q}}}$ *is not κ-mediate; and*
- *If κ is limit then $2^{2^{\mathfrak{q}}}$ is not κ-mediate.*

Wrinch's theorem

Theorem (Wrinch 1923)

Over ZF, *the following are equivalent.*

1. AC*;*
2. *There are no mediate cardinals; and*
3. *There are no κ-mediate cardinals for well-ordered κ.*

- Wrinch originally formulated this result in the framework of *Principia Mathematica*.
- Lévy (1964) independently rediscovered this result.

Wrinch's theorem, ($1 \Rightarrow 2$)

Theorem (Wrinch 1923)

Over ZF, *the following are equivalent.*

1. AC;
2. *There are no mediate cardinals; and*
3. *There are no κ-mediate cardinals for well-ordered κ.*

Definition

$\mathfrak{m}$ is $\mathfrak{p}$-mediate if

- $\mathfrak{q} \leq \mathfrak{m}$ for all $\mathfrak{q} < \mathfrak{p}$;
- $\mathfrak{p} \not\leq \mathfrak{m}$; and
- $\mathfrak{m} \not\leq \mathfrak{p}$.

Prove ($1 \Rightarrow 2$) by contrapositive.

Wrinch's theorem, $(1 \Rightarrow 2)$

Theorem (Wrinch 1923)

Over ZF, the following are equivalent.

1. AC;
2. *There are no mediate cardinals; and*
3. *There are no κ-mediate cardinals for well-ordered κ.*

Definition

$\mathfrak{m}$ is $\mathfrak{p}$-mediate if

- $\mathfrak{q} \leq \mathfrak{m}$ for all $\mathfrak{q} < \mathfrak{p}$;
- $\mathfrak{p} \not\leq \mathfrak{m}$; and
- $\mathfrak{m} \not\leq \mathfrak{p}$.

Prove $(1 \Rightarrow 2)$ by contrapositive.

- Suppose $\mathfrak{q}$ is $\mathfrak{p}$ mediate. Then $\mathfrak{p}$ and $\mathfrak{q}$ are incomparable, so Cardinal Trichotomy fails.

Wrinch's theorem, $(1 \Rightarrow 2)$

Theorem (Wrinch 1923)

Over ZF, the following are equivalent.

1. AC;
2. *There are no mediate cardinals; and*
3. *There are no κ-mediate cardinals for well-ordered κ.*

Definition

$\mathfrak{m}$ is $\mathfrak{p}$-mediate if

- $\mathfrak{q} \leq \mathfrak{m}$ for all $\mathfrak{q} < \mathfrak{p}$;
- $\mathfrak{p} \not\leq \mathfrak{m}$; and
- $\mathfrak{m} \not\leq \mathfrak{p}$.

Prove $(1 \Rightarrow 2)$ by contrapositive.

- Suppose $\mathfrak{q}$ is $\mathfrak{p}$ mediate. Then $\mathfrak{p}$ and $\mathfrak{q}$ are incomparable, so Cardinal Trichotomy fails.
- (Hartogs 1915) AC iff Cardinal Trichotomy.

Wrinch's theorem, $(3 \Rightarrow 1)$

Theorem (Wrinch 1923)

Over ZF, *the following are equivalent.*

1. AC*;*
2. *There are no mediate cardinals; and*
3. *There are no κ-mediate cardinals for well-ordered κ.*

Definition

$\mathfrak{m}$ is $\mathfrak{p}$-mediate if

- $\mathfrak{q} \leq \mathfrak{m}$ for all $\mathfrak{q} < \mathfrak{p}$;
- $\mathfrak{p} \not\leq \mathfrak{m}$; and
- $\mathfrak{m} \not\leq \mathfrak{p}$.

$(2 \Rightarrow 3)$ is trivial. Prove $(3 \Rightarrow 1)$ by contrapositive.

Wrinch's theorem, $(3 \Rightarrow 1)$

Theorem (Wrinch 1923)

Over ZF, *the following are equivalent.*

1. AC;
2. *There are no mediate cardinals; and*
3. *There are no κ-mediate cardinals for well-ordered κ.*

Definition

$\mathfrak{m}$ is $\mathfrak{p}$-mediate if

- $\mathfrak{q} \leq \mathfrak{m}$ for all $\mathfrak{q} < \mathfrak{p}$;
- $\mathfrak{p} \not\leq \mathfrak{m}$; and
- $\mathfrak{m} \not\leq \mathfrak{p}$.

$(2 \Rightarrow 3)$ is trivial. Prove $(3 \Rightarrow 1)$ by contrapositive.

- (Hartogs) For any $\mathfrak{p}$ there is a smallest well-orderable cardinal $\aleph(\mathfrak{p})$ so that $\aleph(\mathfrak{p}) \not\leq \mathfrak{p}$.
- If $\mathfrak{p}$ is not well-orderable then $\mathfrak{p}$ is $\aleph(\mathfrak{p})$-mediate.

Dependent choice

Dependent choice (DC) informally says you can make ω many choices where each choice depends on the previous ones.

- Suppose R is a relation on a set X so that for each $x \in X$ there is $y \in X$ with $x \mathrel{R} y$. Then there is a branch $\langle x_i : i \in \omega \rangle$ through R: for each i have $x_i \mathrel{R} x_{i+1}$.

Dependent choice

Dependent choice (DC) informally says you can make ω many choices where each choice depends on the previous ones.

- Suppose R is a relation on a set X so that for each $x \in X$ there is $y \in X$ with $x \mathrel{R} y$. Then there is a branch $\langle x_i : i \in \omega \rangle$ through R: for each i have $x_i \mathrel{R} x_{i+1}$.

DC_κ says:

- Suppose R is a relation on $X^{<\kappa} \times X$ so that for each $s \in X^{<\kappa}$ there is $y \in X$ with $s \mathrel{R} y$.
  Then there is a branch $b = \langle x_i : i < \kappa \rangle$ through R: for each i have $(b \restriction i) \mathrel{R} b_i$.

$\mathrm{DC}_{<\kappa}$ is DC_λ for all $\lambda < \kappa$.

Dependent choice

Dependent choice (DC) informally says you can make ω many choices where each choice depends on the previous ones.

- Suppose R is a relation on a set X so that for each $x \in X$ there is $y \in X$ with $x \, R \, y$. Then there is a branch $\langle x_i : i \in \omega \rangle$ through R: for each i have $x_i \, R \, x_{i+1}$.

DC_κ says:

- Suppose R is a relation on $X^{<\kappa} \times X$ so that for each $s \in X^{<\kappa}$ there is $y \in X$ with $s \, R \, y$.
  Then there is a branch $b = \langle x_i : i < \kappa \rangle$ through R: for each i have $(b \upharpoonright i) \, R \, b_i$.

$\mathsf{DC}_{<\kappa}$ is DC_λ for all $\lambda < \kappa$.

Facts:

- AC is equivalent to $\forall \kappa \; \mathsf{DC}_\kappa$.
- $\lambda < \kappa$ implies $\mathsf{DC}_\kappa \Rightarrow \mathsf{DC}_\lambda$.
- $\mathsf{ZF} + \mathsf{DC}_{<\kappa} + \neg \mathsf{DC}_\kappa$ is consistent.
- DC implies AC_ω over ZF, but not vice versa.
- DC is equivalent to "a relation is well-founded iff it has no infinite descending sequence".
- (Solovay) $\mathsf{ZF} + \mathsf{DC} +$ "every set of reals is Lebesgue-measurable" is consistent.

DC and mediate cardinals

Can get a level-by-level version of Wrinch's theorem.

Lemma: DC_κ implies there are no κ-mediates.

Corollary: AC iff for all κ there are no κ-mediates.

DC and mediate cardinals

Can get a level-by-level version of Wrinch's theorem.

Lemma: DC_κ implies there are no κ-mediates.

Corollary: AC iff for all κ there are no κ-mediates.

- Suppose $\lambda \leq \mathfrak{p}$ for all $\lambda < \kappa$ but $\mathfrak{p} \not\leq \kappa$.
- Consider the collection of all injections $\alpha \to \mathfrak{p}$ for $\alpha < \mathfrak{p}$.
- None of the injections are onto, so you can always extend them to an injection $\alpha + 1 \to \mathfrak{p}$.
- By DC_κ there's a branch, which gives an injection $\kappa \to \mathfrak{p}$.

An alternate proof of the connection to DC_κ

W_κ asserts that every set has cardinality comparable to κ: either κ injects into X or X injects into κ.

- DC_κ implies W_κ. (See Chapter 8 of Jech's monograph.)
- W_κ implies there are no κ-mediates.

An alternate proof of the connection to DC_κ

W_κ asserts that every set has cardinality comparable to κ: either κ injects into X or X injects into κ.

- DC_κ implies W_κ. (See Chapter 8 of Jech's monograph.)
- W_κ implies there are no κ-mediates.

Note: It's known that $\mathrm{ZF} + W_\kappa + \neg\mathrm{DC}_\kappa$ is consistent, so the nonexistence of κ-mediates cannot imply DC_κ.

Refining mediacy

Observation:

- If $\mathfrak{p}$ is κ-mediate and $\lambda > \kappa$ then $\mathfrak{p} + \lambda$ is λ^+-mediate.
- So if there is a degree of mediacy then every larger successor cardinal is also a degree of mediacy.

Definition

$\mathfrak{m}$ is $\mathfrak{p}$-mediate if

- $\mathfrak{q} \leq \mathfrak{m}$ for all $\mathfrak{q} < \mathfrak{p}$;
- $\mathfrak{p} \not\leq \mathfrak{m}$; and
- $\mathfrak{m} \not\leq \mathfrak{p}$.

Refining mediacy

Observation:

- If $\mathfrak{p}$ is κ-mediate and $\lambda > \kappa$ then $\mathfrak{p} + \lambda$ is λ^+-mediate.
- So if there is a degree of mediacy then every larger successor cardinal is also a degree of mediacy.

Definition

$\mathfrak{m}$ is $\mathfrak{p}$-mediate if

- $\mathfrak{q} \leq \mathfrak{m}$ for all $\mathfrak{q} < \mathfrak{p}$;
- $\mathfrak{p} \not\leq \mathfrak{m}$; and
- $\mathfrak{m} \not\leq \mathfrak{p}$.

$\mathfrak{p}$ is exact κ-mediate if

- $\mathfrak{p}$ is κ-mediate and
- if $Y \subseteq \mathfrak{p}$ has cardinality $< \kappa$ then $\mathfrak{p} \setminus Y$ is κ-mediate.

Refining mediacy

Observation:

- If $\mathfrak{p}$ is κ-mediate and $\lambda > \kappa$ then $\mathfrak{p} + \lambda$ is λ^+-mediate.
- So if there is a degree of mediacy then every larger successor cardinal is also a degree of mediacy.

Definition

$\mathfrak{m}$ is $\mathfrak{p}$-mediate if

- $\mathfrak{q} \leq \mathfrak{m}$ for all $\mathfrak{q} < \mathfrak{p}$;
- $\mathfrak{p} \nleq \mathfrak{m}$; and
- $\mathfrak{m} \nleq \mathfrak{p}$.

$\mathfrak{p}$ is exact κ-mediate if

- $\mathfrak{p}$ is κ-mediate and
- if $Y \subseteq \mathfrak{p}$ has cardinality $< \kappa$ then $\mathfrak{p} \setminus Y$ is κ-mediate.

Lemma: If $\mathfrak{p}$ is κ-mediate where κ is smallest such that κ-mediates exist, then $\mathfrak{p}$ is exact κ-mediate.

Consistency questions

Question

- *Consistently, what can be the smallest degree of mediacy?*
- *Consistently, what can be the class of degrees of exact mediacy?*

Symmetric extensions

Motivating example (Cohen's first model):
Add ω many reals, then forget the order you added them.

Motivating example (Cohen's first model):
Add ω many reals, then forget the order you added them.

- $\mathbb{P} = \mathrm{Add}(\omega, \omega)$ is the poset. Conditions are finite partial functions $\omega \times \omega \to 2$.
- Changing the order is permuting the columns in the $\omega \times \omega$ grid.
- Any permutation ϖ of ω generates an automorphism of $\mathbb{P}$: $\varpi p(n, i) = p(\varpi n, i)$.
- Also generates an automorphism on the $\mathbb{P}$-names: $\varpi\sigma = \{(\varpi\tau, \varpi p) : (\tau, p) \in \sigma\}$

Symmetric extensions

Motivating example (Cohen's first model):
Add ω many reals, then forget the order you added them.

- $\mathbb{P} = \mathrm{Add}(\omega, \omega)$ is the poset. Conditions are finite partial functions $\omega \times \omega \to 2$.
- Changing the order is permuting the columns in the $\omega \times \omega$ grid.
- Any permutation ϖ of ω generates an automorphism of $\mathbb{P}$:
  $\varpi p(n, i) = p(\varpi n, i)$.
- Also generates an automorphism on the $\mathbb{P}$-names:
  $\varpi\sigma = \{(\varpi\tau, \varpi p) : (\tau, p) \in \sigma\}$

- "Forgetting the order" is restricting to names fixed by a 'large' group of automorphisms:
  A group H of automorphisms is large if there is finite $e \subseteq \omega$ so that each $\varpi \in H$ fixes e pointwise: $H \supseteq \mathrm{fix}(e)$.
- This gives a normal filter $\mathcal{F}$ on the lattice of subgroups.
- A name σ is $\mathcal{F}$-symmetric if $\mathrm{sym}(\sigma) = \{\varpi : \varpi\sigma = \sigma\} \in \mathcal{F}$.
- The symmetric extension consists of the interpretations of all hereditarily symmetric names.

Symmetric extensions, in general

A symmetric system is $(\mathbb{P}, G, \mathcal{F})$ so that

- $\mathbb{P}$ is a forcing poset;
- $G \leq \mathrm{Aut}(\mathbb{P})$; and
- $\mathcal{F}$ is a normal filter on the lattice of subgroups of G.

Symmetric extensions, in general

A symmetric system is $(\mathbb{P}, G, \mathcal{F})$ so that

- $\mathbb{P}$ is a forcing poset;
- $G \leq \mathrm{Aut}(\mathbb{P})$; and
- $\mathcal{F}$ is a normal filter on the lattice of subgroups of G.

A $\mathbb{P}$-name σ is symmetric if $\mathrm{sym}(\sigma) \in \mathcal{F}$.

- (Symmetry lemma) $p \Vdash \varphi(\sigma)$ iff $\varpi p \Vdash \varphi(\varpi\sigma)$.

Symmetric extensions, in general

A symmetric system is $(\mathbb{P}, G, \mathcal{F})$ so that

- $\mathbb{P}$ is a forcing poset;
- $G \leq \mathrm{Aut}(\mathbb{P})$; and
- $\mathcal{F}$ is a normal filter on the lattice of subgroups of G.

A $\mathbb{P}$-name σ is symmetric if $\mathrm{sym}(\sigma) \in \mathcal{F}$.

- (Symmetry lemma) $p \Vdash \varphi(\sigma)$ iff $\varpi p \Vdash \varphi(\varpi\sigma)$.

The symmetric extension by $(\mathbb{P}, G, \mathcal{F})$ via a generic $g \subseteq \mathbb{P}$:

- Consists of the interpretations of hereditarily symmetric names.
- $\mathrm{V}[g/\mathcal{F}] = \{\sigma^g : \sigma \text{ is } \mathcal{F}\text{-HS}\}$.

$\mathrm{V}[g/\mathcal{F}] \models$ ZF, but the point is to make AC fail in a controlled way.

The general Cohen symmetric extension

Fix regular κ and assume $\kappa^{<\kappa} = \kappa$.

- $\mathbb{P}_\kappa = \mathrm{Add}(\kappa, \kappa)$;
- $G_\kappa \leq \mathrm{Aut}(\mathbb{P}_\kappa)$ is generated by permutations of κ;
- $H \in \mathcal{F}_\kappa$ if $\exists e \in [\kappa]^{<\kappa}$ so that $\mathrm{fix}(e) \subseteq H$.

The general Cohen symmetric extension

Fix regular κ and assume $\kappa^{<\kappa} = \kappa$.

- $\mathbb{P}_\kappa = \mathrm{Add}(\kappa, \kappa)$;
- $G_\kappa \leq \mathrm{Aut}(\mathbb{P}_\kappa)$ is generated by permutations of κ;
- $H \in \mathcal{F}_\kappa$ if $\exists e \in [\kappa]^{<\kappa}$ so that $\mathrm{fix}(e) \subseteq H$.

In $\mathrm{V}[g/\mathcal{F}_\kappa]$ the set $A = \{c_i : i < \kappa\}$ of Cohen subsets of κ is not well-orderable.

The general Cohen symmetric extension

Fix regular κ and assume $\kappa^{<\kappa} = \kappa$.

- $\mathbb{P}_\kappa = \mathrm{Add}(\kappa, \kappa)$;
- $G_\kappa \leq \mathrm{Aut}(\mathbb{P}_\kappa)$ is generated by permutations of κ;
- $H \in \mathcal{F}_\kappa$ if $\exists e \in [\kappa]^{<\kappa}$ so that $\mathrm{fix}(e) \subseteq H$.

In $\mathrm{V}[g/\mathcal{F}_\kappa]$ the set $A = \{c_i : i < \kappa\}$ of Cohen subsets of κ is not well-orderable.

Facts:

- $\mathbb{P}_\kappa$ is κ-closed and has the κ^+-cc.
- $\mathcal{F}_\kappa$ is κ-complete and is generated by a basis of size κ.

The general Cohen symmetric extension

Fix regular κ and assume $\kappa^{<\kappa} = \kappa$.

- $\mathbb{P}_\kappa = \mathrm{Add}(\kappa, \kappa)$;
- $G_\kappa \leq \mathrm{Aut}(\mathbb{P}_\kappa)$ is generated by permutations of κ;
- $H \in \mathcal{F}_\kappa$ if $\exists e \in [\kappa]^{<\kappa}$ so that $\mathrm{fix}(e) \subseteq H$.

In $\mathrm{V}[g/\mathcal{F}_\kappa]$ the set $A = \{c_i : i < \kappa\}$ of Cohen subsets of κ is not well-orderable.

Facts:

- $\mathbb{P}_\kappa$ is κ-closed and has the κ^+-cc.
- $\mathcal{F}_\kappa$ is κ-complete and is generated by a basis of size κ.

Thus, $(\mathbb{P}_\kappa, G_\kappa, \mathcal{F}_\kappa)$ will preserve $\mathrm{DC}_{<\kappa}$.

In particular, there will be no λ-mediates for $\lambda < \kappa$.

The general Cohen symmetric extension

Fix regular κ and assume $\kappa^{<\kappa} = \kappa$.

- $\mathbb{P}_\kappa = \mathrm{Add}(\kappa, \kappa)$;
- $G_\kappa \leq \mathrm{Aut}(\mathbb{P}_\kappa)$ is generated by permutations of κ;
- $H \in \mathcal{F}_\kappa$ if $\exists e \in [\kappa]^{<\kappa}$ so that $\mathrm{fix}(e) \subseteq H$.

In $\mathrm{V}[g/\mathcal{F}_\kappa]$ the set $A = \{c_i : i < \kappa\}$ of Cohen subsets of κ is not well-orderable.

Facts:

- $\mathbb{P}_\kappa$ is κ-closed and has the κ^+-cc.
- $\mathcal{F}_\kappa$ is κ-complete and is generated by a basis of size κ.

Thus, $(\mathbb{P}_\kappa, G_\kappa, \mathcal{F}_\kappa)$ will preserve $\mathrm{DC}_{<\kappa}$.

In particular, there will be no λ-mediates for $\lambda < \kappa$.

Note: The cardinal arithmetic assumption gives the smallest possible chain condition and ensures $\mathcal{F}_\kappa$ has the smallest possible basis. These will be used to tightly control the degrees of exact mediacy.

Symmetric extensions and dependent choice

Lemma: Let κ be regular and $\lambda < \kappa$. If $\mathbb{P}$ is κ-closed and $\mathcal{F}$ is κ-complete then $(\mathbb{P}, G, \mathcal{F})$ preserves DC_λ.

Symmetric extensions and dependent choice

Lemma: Let κ be regular and $\lambda < \kappa$. If $\mathbb{P}$ is κ-closed and $\mathcal{F}$ is κ-complete then $(\mathbb{P}, G, \mathcal{F})$ preserves DC_λ.

- Consider appropriate $R \subseteq X^{<\lambda} \times X$ in $\mathrm{V}[g/\mathcal{F}]$. We need a branch through R in $\mathrm{V}[g/\mathcal{F}]$.
- By κ-closure λ remains a cardinal in $\mathrm{V}[g]$.
- In $\mathrm{V}[g]$, by DC_λ there is a branch $b = \langle x_i : i < \lambda \rangle$.
- Each x_i comes from a symmetric name $\dot{x}_i$.
- By κ-completeness $H = \bigwedge_{i<\lambda} \operatorname{sym}(\dot{x}_i)$ is in $\mathcal{F}$.
- Can get a name $\dot{b}$ for b with $\operatorname{sym}(\dot{b}) \supseteq H$.
- So the branch b is in $\mathrm{V}[g/\mathcal{F}]$.

The smallest mediate can be anything

Theorem (Lévy (1964); W.)

Suppose $\kappa = \kappa^{<\kappa}$ is regular. In the symmetric extension by $(\mathbb{P}_\kappa, G_\kappa, \mathcal{F}_\kappa)$:

- $\mathsf{DC}_{<\kappa}$;
- *κ is the smallest degree of mediacy; and*
- *κ is the only degree of exact mediacy.*

(Lévy proved the first two in ZFA*.)*

The smallest mediate can be anything

Theorem (Lévy (1964); W.)

Suppose $\kappa = \kappa^{<\kappa}$ is regular. In the symmetric extension by $(\mathbb{P}_\kappa, G_\kappa, \mathcal{F}_\kappa)$:

- $\mathrm{DC}_{<\kappa}$;
- *κ is the smallest degree of mediacy; and*
- *κ is the only degree of exact mediacy.*

(Lévy proved the first two in ZFA.)

We've already seen $\mathrm{DC}_{<\kappa}$.

Claim: Let A be the set of the Cohen subsets of κ added by $\mathbb{P}_\kappa$. Then $\mathrm{V}[g/\mathcal{F}_\kappa] \models A$ is κ-mediate.

Like getting a DFI set in $(\mathbb{P}_\omega, G_\omega, \mathcal{F}_\omega)$.

The smallest mediate can be anything

Theorem (Lévy (1964); W.)

Suppose $\kappa = \kappa^{<\kappa}$ is regular. In the symmetric extension by $(\mathbb{P}_\kappa, G_\kappa, \mathcal{F}_\kappa)$:

- $\mathrm{DC}_{<\kappa}$;
- *κ is the smallest degree of mediacy; and*
- *κ is the only degree of exact mediacy.*

(Lévy proved the first two in ZFA.)

We've already seen $\mathrm{DC}_{<\kappa}$.

Claim: Let A be the set of the Cohen subsets of κ added by $\mathbb{P}_\kappa$. Then $\mathrm{V}[g/\mathcal{F}_\kappa] \models A$ is κ-mediate.

Like getting a DFI set in $(\mathbb{P}_\omega, G_\omega, \mathcal{F}_\omega)$.

- $\lambda < \kappa$ injects by κ-closure of $\mathbb{P}_\kappa$ and κ-completeness of $\mathcal{F}_\kappa$
- $|A| \not\leq \kappa$ because A can't be well-ordered.
- $\kappa \not\leq |A|$:

The smallest mediate can be anything

Theorem (Lévy (1964); W.)

Suppose $\kappa = \kappa^{<\kappa}$ is regular. In the symmetric extension by $(\mathbb{P}_\kappa, G_\kappa, \mathcal{F}_\kappa)$:

- $\mathrm{DC}_{<\kappa}$;
- *κ is the smallest degree of mediacy; and*
- *κ is the only degree of exact mediacy.*

(Lévy proved the first two in ZFA.)

We've already seen $\mathrm{DC}_{<\kappa}$.

Claim: Let A be the set of the Cohen subsets of κ added by $\mathbb{P}_\kappa$. Then $\mathrm{V}[g/\mathcal{F}_\kappa] \models A$ is κ-mediate.

Like getting a DFI set in $(\mathbb{P}_\omega, G_\omega, \mathcal{F}_\omega)$.

- $\lambda < \kappa$ injects by κ-closure of $\mathbb{P}_\kappa$ and κ-completeness of $\mathcal{F}_\kappa$
- $|A| \not\leq \kappa$ because A can't be well-ordered.
- $\kappa \not\leq |A|$:
  - Suppose $\dot{f}$ is hereditarily symmetric, $\mathrm{sym}(\dot{f}) \supseteq \mathrm{fix}(e)$, and $p \Vdash \dot{f} : \kappa \to A$ is one-to-one.
  - Extend p to q deciding $\dot{f}(\alpha) = c_i$ for some $\alpha \neq i$ both $\notin e$.
  - Find ϖ fixing $e \cup \{i\}$, moving α, and $q \parallel \varpi q$.
  - So $q \cup \varpi q \Vdash \dot{f}$ is not one-to-one. Contradiction.

The smallest mediate can be anything

Theorem (Lévy (1964); W.)

Suppose $\kappa = \kappa^{<\kappa}$ is regular. In the symmetric extension by $(\mathbb{P}_\kappa, G_\kappa, \mathcal{F}_\kappa)$:

- $\mathsf{DC}_{<\kappa}$*;*
- *κ is the smallest degree of mediacy; and*
- *κ is the only degree of exact mediacy.*

(Lévy proved the first two in ZFA*.)*

This is the only place the cardinal arithmetic assumption is used.

The smallest mediate can be anything

Theorem (Lévy (1964); W.)

Suppose $\kappa = \kappa^{<\kappa}$ is regular. In the symmetric extension by $(\mathbb{P}_\kappa, G_\kappa, \mathcal{F}_\kappa)$:

- $\mathsf{DC}_{<\kappa}$;
- *κ is the smallest degree of mediacy; and*
- *κ is the only degree of exact mediacy.*

(Lévy proved the first two in ZFA*.)*

This is the only place the cardinal arithmetic assumption is used.

Suppose $\mathrm{V}[g/\mathcal{F}_\kappa] \models X$ is exact λ-mediate for $\lambda > \kappa$.

- First use exactness plus the chain condition to argue that $\mathrm{V}[g]$ has an injection $\lambda \to X$.
  Compare: if A is the set of Cohen generics then $A \cup \mu$ is μ^+-mediate in the symmetric extension but has cardinality μ in $\mathrm{V}[g]$.
- Then use the chain condition plus $\mathcal{F}_\kappa$ having a basis of size $< \lambda$ to get an injection $\lambda \to X$ in $\mathrm{V}[g/\mathcal{F}_\kappa]$.
- Contradiction.

Doing it more than once

When a set theorist can do something once, she wants to do it more than once. With forcing, she accomplishes this using products or iterations.

Doing it more than once

When a set theorist can do something once, she wants to do it more than once. With forcing, she accomplishes this using products or iterations.

- Karagila and others have worked on iterations of symmetric extensions.
- It's complicated, with scary group theoretic objects like wreath products.

Doing it more than once

When a set theorist can do something once, she wants to do it more than once. With forcing, she accomplishes this using products or iterations.

- Karagila and others have worked on iterations of symmetric extensions.
- It's complicated, with scary group theoretic objects like wreath products.
- We are lucky and can get away with products, where the details are significantly less technical.

Products of symmetric extensions

Suppose $(\mathbb{P}, G, \mathcal{F})$ and $(\mathbb{Q}, H, \mathcal{E})$ are symmetric systems. Can define their product $(\mathbb{P}, G, \mathcal{F}) \times (\mathbb{Q}, H, \mathcal{E})$:

- $\mathbb{P} \times \mathbb{Q}$ is usual product of posets;
- $G \times H$ is generated by (ϖ, ϱ) with $\varpi \in G$, $\varrho \in H$; and
- $\mathcal{F} \times \mathcal{E}$ is generated by $G_0 \times H_0$ for $G_0 \in \mathcal{F}$ and $H_0 \in \mathcal{E}$.

Products of symmetric extensions

Suppose $(\mathbb{P}, G, \mathcal{F})$ and $(\mathbb{Q}, H, \mathcal{E})$ are symmetric systems. Can define their product $(\mathbb{P}, G, \mathcal{F}) \times (\mathbb{Q}, H, \mathcal{E})$:

- $\mathbb{P} \times \mathbb{Q}$ is usual product of posets;
- $G \times H$ is generated by (ϖ, ϱ) with $\varpi \in G$, $\varrho \in H$; and
- $\mathcal{F} \times \mathcal{E}$ is generated by $G_0 \times H_0$ for $G_0 \in \mathcal{F}$ and $H_0 \in \mathcal{E}$.

Like with forcing, we have a product lemma stating that the extension by the product is the same as the two-step extensions, in either order.

Products of symmetric extensions

Suppose $(\mathbb{P}, G, \mathcal{F})$ and $(\mathbb{Q}, H, \mathcal{E})$ are symmetric systems. Can define their product $(\mathbb{P}, G, \mathcal{F}) \times (\mathbb{Q}, H, \mathcal{E})$:

- $\mathbb{P} \times \mathbb{Q}$ is usual product of posets;
- $G \times H$ is generated by (ϖ, ϱ) with $\varpi \in G$, $\varrho \in H$; and
- $\mathcal{F} \times \mathcal{E}$ is generated by $G_0 \times H_0$ for $G_0 \in \mathcal{F}$ and $H_0 \in \mathcal{E}$.

Like with forcing, we have a product lemma stating that the extension by the product is the same as the two-step extensions, in either order.

Can also do this for infinite products, with a notion of support.

- Suppose $(\mathbb{P}_\kappa, G_\kappa, \mathcal{F}_\kappa)$ are symmetric systems for $\kappa \in M$.
- Then there is a product $\prod_{\kappa \in M} (\mathbb{P}_\kappa, G_\kappa, \mathcal{F}_\kappa)$ with that support.
- Again we get a product lemma stating we can split the full extension into two-step extensions.

Note: We do not allow permuting the multiplicands.

Refining earlier ideas

In a two-step symmetric extension, the intermediate step won't satisfy AC. So we need to look more carefully at our assumptions.

Refining earlier ideas

In a two-step symmetric extension, the intermediate step won't satisfy AC. So we need to look more carefully at our assumptions.

Suppose $\lambda < \kappa$ are regular.

- ($\mathsf{ZF} + \mathsf{DC}_\kappa$) If $\mathbb{P}$ is κ-closed and $\mathcal{F}$ is κ-complete then $(\mathbb{P}, G, \mathcal{F})$ preserves DC_λ.
- ($\mathsf{ZF} + \mathsf{DC}_\kappa$) Suppose $\mathbb{P}$ has the λ^+-cc and $\mathcal{F}$ is generated by a basis of size $\leq \lambda$. Then $\mathrm{V}[g/\mathcal{F}] \models$ there are no exact κ-mediates.

Preserving mediacy

To make the product analysis work we need to know we don't kill mediacy in a further extension.

Preserving mediacy

To make the product analysis work we need to know we don't kill mediacy in a further extension.

I couldn't quite see how to make the argument work for mediacy

Preserving mediacy

To make the product analysis work we need to know we don't kill mediacy in a further extension.

I couldn't quite see how to make the argument work for mediacy, but with a slight strengthening it goes through.

Lemma: $(\mathsf{ZF} + \mathsf{DC}_\kappa)$ If X is exact κ-mediate$^{+\varepsilon}$ then X remains exact κ-mediate in an extension by a forcing with the κ-cc.

The basic one-step construction gives exact mediacy$^{+\varepsilon}$.

The "$+\varepsilon$" is strengthening a $\not\leq$ to $\not\leq^*$ in the definition:

- X is κ-mediate$^{+\varepsilon}$ if
  - $\lambda \leq |X|$ for all $\lambda < \kappa$;
  - $\kappa \not\leq^* |X|$: there is no surjection $\kappa \to X$;
  - $|X| \not\leq \kappa$.
- Define exact κ-mediate$^{+\varepsilon}$ analogously to exact mediacy: X remains κ-mediate$^{+\varepsilon}$ after removing a small set.

The pattern of the exact mediates

Theorem (W.)

Assume GCH *and fix a class M of regular cardinals. Do the Easton support product of the $(\mathbb{P}_\kappa, G_\kappa, \mathcal{F}_\kappa)$ for $\kappa \in M$. In the symmetric extension, M is exactly the class of regular degrees of exact mediacy.*

Note: This symmetric extension preserves inaccessibles, and probably more large cardinals.

The pattern of the exact mediates

Theorem (W.)

Assume GCH *and fix a class M of regular cardinals. Do the Easton support product of the $(\mathbb{P}_\kappa, G_\kappa, \mathcal{F}_\kappa)$ for $\kappa \in M$. In the symmetric extension, M is exactly the class of regular degrees of exact mediacy.*

Note: This symmetric extension preserves inaccessibles, and probably more large cardinals.

Sketch:

- $\mathbb{P}_{>\alpha}$ is α-closed and $\mathcal{F}_{>\alpha}$ is α-complete.
- $\mathbb{P}_{<\alpha}$ has the α^+-cc and $\mathcal{F}_{<\alpha}$ is generated by a basis of cardinality $\leq \alpha$.
- In $\mathrm{V}[g_{>\alpha}/\mathcal{F}_{>\alpha}]$: DC_α is true. So there are no α-mediates.
- In $\mathrm{V}[g_{>\alpha}/\mathcal{F}_{>\alpha}][g_{<\alpha}/\mathcal{F}_{<\alpha}]$: there are no exact α-mediates.
- So the only way there could be an exact α-mediate is if it was added by $(\mathbb{P}_\alpha, G_\alpha, \mathcal{F}_\alpha)$ for $\alpha \in M$.
- But we already know that adds an exact mediate, and that's preserved.

Putting Wrinch into the historical context

Lévy (1964) defined a principle $H(\kappa)$ which is essentially "there are no κ-mediates" and comments that its meaning is

- there are no sets whose Hartogs number is κ except the obvious ones.

Putting Wrinch into the historical context

Lévy (1964) defined a principle $H(\kappa)$ which is essentially "there are no κ-mediates" and comments that its meaning is

- there are no sets whose Hartogs number is κ except the obvious ones.

Following Lévy, others have worked further on understanding what the possibilities are for Hartogs numbers and related concepts. This includes recent work by people in this room—e.g. Karagila and Ryan-Smith.

Hartogs (1915) is the root of this tree of research.

Although her focus differed, Wrinch (1923) should be seen as an early precursor to post-Cohen work in this tree.

Some questions

Some questions

- What's up with singular cardinals?
- What if we don't make such strong cardinal arithmetic assumptions?
- What happens if AC fails badly in the ground model?

Thank you!

- Dorothy Wrinch, "On mediate cardinals", *American Journal of Mathematics*, Vol. 45, No. 2. (1923). pp. 87–92.
  DOI: 10.2307/2370490.
- Kameryn Julia Williams, "Mediate cardinals: old and new", *In preparation*.
- Azrael Lévy, "The interdependence of certain consequences of the axiom of choice", *Fundamenta Mathematicae*, Vol. 54, No. 2. (1964). pp. 135–157.
  DOI: 10.4064/fm-54-2-135-157
- Landon D.C. Elkind, "I like her very much—she has very good brains.": Dorothy Wrinch's Influence on Bertrand Russell, In: *Bertrand Russell, Feminism, and Women Philosophers in his Circle*, eds. Elkind & Klein. (2024). pp. 259–297.
  URL: link.springer.com/chapter/10.1007/978-3-031-33026-1_10