MATH 321: 10-1 IN-CLASS WORK

We continue to look at how the logical structure of a mathematical statement informs how you can use it in proofs. Today we focus on quantifiers.

How do you use $\exists x\ P(x)$ as a known? This formula asserts that there is an object for which P is true, but it gives no further information. So to use it, you get to know that there is an object a so that $P(a)$, but you have no further information about a. (Of course, a has to be a new object; if you already have an object you're calling a you need to use a different name.) This rule is called *existential instantiation*

How do you prove $\exists x\ P(x)$ when it's a goal? You want to find an object a so that $P(a)$.

How do you use $\forall x\ P(x)$ as a known? This formula asserts that P is true for any object in your domain of discourse. So to use it, whenever you have an appropriate object a you immeditely know $P(a)$. Usually, when you use a universally quantified statement as a known, it's in the form $\forall x \in A\ P(x)$ or $\forall x\ (P(x) \to Q(x))$. In either of these forms, the objects this statement gives useful info about is restricted. In the first case, if you have an object $a \in A$ then you can conclude $P(a)$. In the second case, if you have an object a for which you know $P(a)$ then you can conclude $Q(a)$.

How do you prove $\forall x P(x)$ when it's a goal? You need to prove that P is true of any object. So you assume you have an object a about which you know nothing, and then you prove $P(a)$. It may seem like you are getting very little info to use, and you're right to think that! However, usually you do get some information. Again, when you encounter statements like this they are usually in the form $\forall x \in A\ P(x)$ or $\forall x\ (P(x) \to Q(x))$. In the first case, you get to assume one thing about your object a, namely that $a \in A$. In the second case, you get to assume that $P(a)$ holds.

Exercise 1. *Prove that if $a \mid b$ and $a \mid c$ then $a \mid (b + c)$. Here, a, b, c are all integers.*

Exercise 2. *Prove that $x^2 \geq 0$ for all real numbers x.*

Many interesting mathematical statements are of the form $\forall x \in A\ \exists y \in B\ P(x, y)$. The way to prove such a statement is to assume you have an object a about which you only know $a \in A$ and try to find an object $b \in B$ so that $P(a, b)$.

Exercise 3. *Prove that for all real numbers $x \neq 1$ there is a real number y so that $x = \frac{y-1}{y+1}$.*