Math 321: Induction, part II

Kameryn J Williams

University of Hawaiʻi at Mānoa

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Mathematical objects

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So let's talk about how to do that.

Cartesian products

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For example, $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) : x, y \in \mathbb{R}\}$ is the Cartesian plane.

Properties of Cartesian products

1. If A has m elements and B has n elements then $A \times B$ has mn elements.
2. In general, $A \times B \neq B \times A$.
3. $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
4. $A \times (B \cap C) = (A \times B) \cap (A \times C)$.
5. $A \times \emptyset = \emptyset \times A = \emptyset$.

Beyond two

You can also do Cartesian products with more than two coordinates, for example $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ is three-dimensional Euclidean space. If you have, say, four coordinates, then instead of ordered pairs (a, b) you need ordered quadruples (a, b, c, d). But the idea is the same, and we will mainly be concerned with the binary case.

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- So we can represent $<$ as the set of all pairs (x, y) for which $x < y$.
- That is, we represent $<$ as a certain subset of the Cartesian product $\mathbb{R} \times \mathbb{R}$.
- This perspective on relations is extensional—based only on what elements make the relation true—rather than intensional—based just on how it is defined.

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- $< = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x < y\}$.
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When talking about relations abstractly, we will need to give them a name. We will usually use a letter, saving symbols like $<$, $\subseteq$, $\in$, $|$, etc. for specific relations.

So $a\,R\,b$ just means that $(a, b) \in R$ for some relation R.

Thinking pictorally about relations

Properties of relations

Let R be a relation from A to B.

- The domain of R is $\operatorname{dom} R = \{a \in A : \exists b \in B \; a \, R \, b\}$
- The range of R is $\operatorname{ran} R = \{b \in B : \exists a \in A \; a \, R \, b\}$.
- The inverse of R is the relation $R^{-1} = \{(b, a) \in B \times A : (a, b) \in R\}$.
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These should remind you of the definitions for functions. And indeed the same facts for functions also hold true for relations, e.g. $R \circ (S \circ T) = (R \circ S) \circ T$.

Examples

Let's look at the relations $<$, $|$, and $\equiv \mod n$ on $\mathbb{N}$.

More properties of relations

Let R be a relation on A.

- R is reflexive if $a\,R\,a$ for all $a \in A$.
- R is symmetric if $a\,R\,b$ implies $b\,R\,a$ for all $a, b \in A$.
- R is transitive if $(a\,R\,b$ and $b\,R\,c)$ implies $a\,R\,c$ for all $a, b, c \in A$.

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