

The Σ_1 -definable universal finite sequence

Kameryn J Williams

University of Hawai'i at Mānoa

Cantor Meets Robinson

Set Theory, Model Theory, and their Philosophy

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Multiversism versus universism in set theory

- **The universalist:** *The* universe of sets is uniquely determined.
- **The multiversist:** There are many universes of sets, and every universe is contained inside a bigger, better universe.

Set-theoretic multiverses

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- The generic multiverse: move to new universes by forcing or by going to grounds.
- The toy multiverse of countable transitive models.
- A more radical multiversism: Hamkins's multiverse axioms, including the [Well-foundedness Mirage axiom](#)—every universe is seen to be ill-founded from some larger universe.

Potentialism as a general framework

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- **Warning!** There is no guarantee that a potentialist system be **linearly ordered** or even **directed**. There are **branching** potentialist systems which have incompatible extensions.

The modal logic of potentialism

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- The modal theory **S4** is valid in every potentialist system, where S4 has axioms

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- This gives an easy lower bound. The real work is in getting upper bounds.

Buttons and switches (Hamkins and Löwe)

We can analyze the modal validities of a potentialist system using **control statements**.

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- If there are arbitrarily large families of independent switches, then the modal validities are contained within S5, which is S4 plus the axiom $\Diamond\Box p \Rightarrow p$.
- If there are arbitrarily large families of independent buttons and switches, then the modal validities are contained within S4.2, which is S4 plus the axiom $\Diamond\Box p \Rightarrow \Box\Diamond p$.

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- Forcing potentialism: worlds are forcing extensions of M .
 - Modal validities are S4.2. (Hamkins and Löwe)
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The Σ_2 -definable universal finite set for rank-extensions

$N \supseteq M$ is a **rank-extension** of M if every $b \in N \setminus M$ has rank in $N \setminus M$.

Theorem (Hamkins and Woodin)

There is a Σ_2 definition for a finite set $\{b_0, \dots, b_n\}$ with the following properties.

- ① *ZFC proves that the definition defines a finite set.*
- ② *In any transitive model of ZFC the set is empty.*
- ③ *If $M \models \text{ZFC}$ is countable, has s as its universal finite set, and $t \in M$ is a finite set extending s , then there is $N \models \text{ZFC}$ a rank-extension of M which has t as its universal finite set.*

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Theorem (Hamkins and Woodin)

The modal validities of rank-extensional set theoretic potentialism are precisely S4.

End-extensions versus rank-extensions

- $N \supseteq M$ is an **end-extension** of M if $a \in b$ in N and b in M implies a is in M .
- Elementary end-extensions are always rank-extensions.
- But not all end-extensions are rank-extensions. For example, if M is an inner model of N then N end-extends M .

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- But not all end-extensions are rank-extensions. For example, if M is an inner model of N then N end-extends M .
- Key fact: the assertions which are preserved in arbitrary end-extensions are the Σ_1 assertions.

$$\exists y \underbrace{\varphi(x, y)}_{\text{quantifiers bounded}}$$

The Σ_1 -definable universal finite sequence for end-extensions

Let ZF^+ be a computably enumerable extension of Zermelo–Fraenkel set theory ZF .

Theorem (Hamkins, Welch, W.)

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- 1 ZF^+ proves that the sequence is finite.
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- 3 Let M be a countable model of ZF^+ which defines the sequence as s . Then if t in M is any finite sequence extending s , there is an end-extension N of M in which the universal sequence is t .

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- ❹ Indeed, it suffices in (3) that M has an inner model W of ZF^+ satisfying such.

The definition of the universal finite sequence—Process A

Intended for ω -nonstandard models. A different process is used for ω -standard models.

- Proceed in stages to produce b_0, b_1, \dots, b_n , using auxiliary information: countable ordinals $\alpha_0 < \alpha_1 < \dots < \alpha_n$ and natural numbers $k_0 > k_1 > \dots > k_n$.

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- At stage n : Are there $\alpha > \alpha_{n-1}$, $k < k_{n-1}$, and $b \in L_{\alpha_n}$ so that L_{α_n} has **no** end-extension to N satisfying the first k axioms of ZF plus “process A succeeds at stage n and defines b ”? If so, stage n is successful and set (b_n, α_n, k_n) to be the L -least triple of such (b, α, k) .

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- This self-reference is allowed by the **Gödel–Carnap fixed-point lemma**.
- Claim: The map $n \mapsto (b_n, \alpha_n, k_n)$ is Σ_1 -definable.
- Claim: Each k_i must be nonstandard.

Seeing that Process A has the plus one extension property

Consider countable M in which the universal sequence is b_0, \dots, b_{n-1} and take any $b \in M$ and nonstandard $k < k_{n-1}$.

- Because stage n is unsuccessful in M this means L^M thinks every countable set can be end-extended to a model of the first k axioms of ZF in which the universal sequence is b_0, \dots, b_{n-1}, b .

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- By the [Keisler–Morley theorem](#), let M^+ be an elementary end extension of M and fix an ordinal θ in $M^+ \setminus M$. Consider $M^+[G]$ the forcing extension in which V_θ is collapsed to be countable.

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- By the [Keisler–Morley theorem](#), let M^+ be an elementary end extension of M and fix an ordinal θ in $M^+ \setminus M$. Consider $M^+[G]$ the forcing extension in which V_θ is collapsed to be countable.
- So V_θ^M has in $M^+[G]$ an end-extension N in which the universal sequence is b_0, \dots, b_{n-1}, b . But N is also an end-extension of M .

Process B —for ω -standard models

- Again go in stages: produce b_0, b_1, \dots, b_n using auxiliary information countable ordinals $\alpha_0 < \alpha_1 < \dots < \alpha_n$ and countable ordinals $\lambda_0 > \lambda_1 > \dots > \lambda_n$.

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Can merge Processes A and B into a single Process C which works for all models. □

The Barwise extension theorem

The Barwise extension theorem can be derived as a corollary of our theorem.

Theorem (Barwise)

Every countable model of ZF end-extends to a model of $ZFC + V = L$.

The universal sequence for L -extensions

Corollary (Hamkins, Welch, W.)

There is a Σ_1 definition for a finite sequence b_0, b_1, \dots, b_n with the following properties.

- ① $ZF + V = L$ proves that the sequence is finite.
- ② In any standard model of $ZF + V = L$ the sequence is finite.
- ③ Let M be a countable model of $ZF + V = L$ which defines the sequence as s . Then if t in M is any finite sequence extending s , there is an L -extension N of M in which the universal sequence is t .

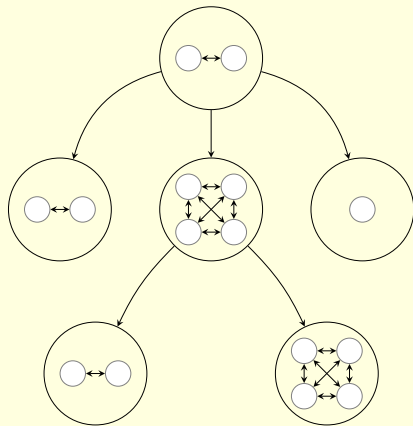
Railyard labelings

- A **tree** is a partial order T so that $\{s \in T : s \leq t\}$ is well-ordered for every $t \in T$. A **pre-tree** is a pre-order which quotients to a tree.
- A **railyard labeling** of a pre-tree T is an assignment ρ_t of statements to nodes $t \in T$ so that each structure satisfies exactly one ρ_t and $\diamond \rho_s$ holds iff $t \leq_T s$.

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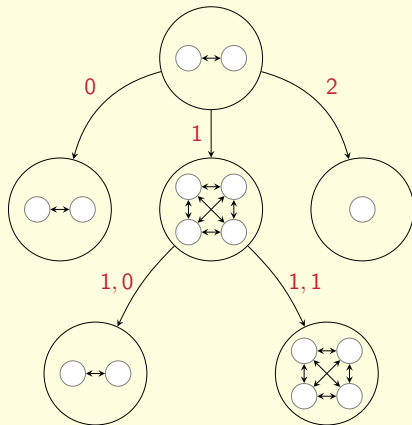
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- If there are railyard labelings for every finite pre-tree, then the modal validities for the corresponding potentialist system are contained within S4.

The universal finite sequence and railyard labelings



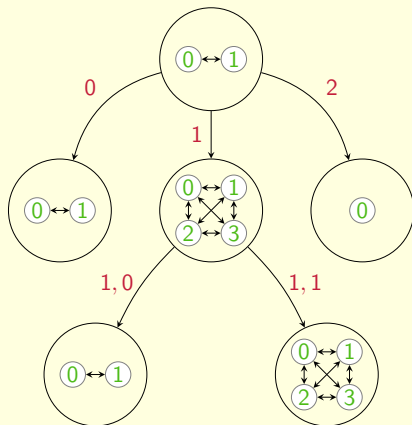
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- Step 1: the subsequence $\langle n_i \rangle$ of finite ordinals from the universal finite sequence tell you how to descend the tree to determine your cluster. If B is the branching of the current node, then $n_i \bmod B$ tells you where to go.



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- Step 2: the final infinite ordinal $\lambda + m$ on the sequence tells you where in your cluster you are. If K is the size of the cluster, then $m \bmod K$ identifies your node in the cluster. (If no infinite ordinals are on the sequence, default to 0.)



The modal validities of end-extensional set theoretic potentialism

Theorem (Hamkins, Welch, W.)

Consider the potentialist system consisting of countable models of ZF^+ ordered by end-extension.

- ① *For any world M , the modal validities, allowing for a single parameter for the length of the universal finite sequence, are precisely S4.*
- ② *For any ω -standard world M , the modal validities, allowing no parameters, are precisely S4.*

Thank you!