Math 302: Existence and uniqueness of solutions

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I won't go into every gritty detail. Rather, the goal is to give you a view of the broad ideas, and some techniques and tricks that are used to reason abstractly about differential equations.

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$$\begin{aligned} y_n' &= -a_{n-1}y_n - \cdots - a_1 y_2 - a_0 y_1 + b \\ y_{n-1}' &= y_n \\ &\vdots \\ y_1' &= y_2 \end{aligned}$$

So solving this system of equations gives us a solution to our original equation.

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So solving this system of equations gives us a solution to our original equation.

We've turned our problem into a new problem. That's kinda like progress!

Solving systems of equations

We turned our nth-order differential equation about an unknown function y into a system of equations. Namely, we have

- n equations
- about n unknown functions $y_1, \ldots, y_n$
- where the ith equation describes y_i' in terms of $y_1, \ldots, y_n$, and x.

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Let's look at this in a bit more generality.

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If we want to think about how to solve this, it makes sense to start with the simplest case, where we have only one unknown function.

Solving $y' = F(x, y)$

- Suppose for now that F is continuous in a rectangle R: $a_0 \leq x \leq a_1$ and $b_0 \leq y \leq b_1$.
- (We'll see later that we actually have to assume a little bit more about F.)
- Fix a point (x_0, y_0) in this rectangle, and suppose we have have the initial condition $y(x_0) = y_0$.

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A trick: Turn the differential equation into an integral equation.

Integrate both sides, starting at x_0, get:

$$y(x) - y(x_0) = \int_{x_0}^{x} F(t, y(t))\,\mathrm{d}t.$$

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- Derivatives are easier to compute than integrals.
- But for theoretical uses, integrals are better behaved.
- Since we want a theoretical result rather than calculations, this makes things easier.

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Spoiler: This stupid idea will work out (with an extra assumption on F). The error gets smaller and smaller, so these Picard approximations converge to the true solution.

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- We turned the first-order differential equation $y' = F(x, y)$ into an integral equation.
- We want to use Picard's method to find a solution to this integral equation.