On axioms for multiverses of set theory

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Set-theoretic multiverses

Set theorists have studied many different multiverses.

- The generic multiverse of a model of set theory.
- S. Friedman's hyperverse of countable transitive models.
- Zermelo's upwardly dynamic conception of set can be seen as a multiverse with worlds V_κ for inaccessible κ.

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One perspective: The axiomatic approach. Write down axioms which a multiverse can satisfy.

- Philosophy: What axioms are true of the real multiverse of sets?
- Mathematics: Given a toy multiverse—a collection of set-sized models of set theory—what axioms are true of it?

Multiverse axioms

Hamkins introduced a series of axioms which describe his view of what the set-theoretic multiverse looks like.

- **Realizability** If M is a world and N is a set- or class-sized model of ZFC in M, then N is a world.
- **Closure Under Forcing** If M is a world and $\mathbb{P}$ is a poset in M then the multiverse contains a forcing extension of M by $\mathbb{P}$.
- **Countability** Every world M is an element of a larger world which thinks M is countable.

Remark

Under suitable consistency assumptions: The collection of countable transitive models of ZFC form a multiverse satisfying Standard Realizability, Closure Under Forcing, and Countability; and
The collection of countable models of ZFC form a multiverse satisfying Realizability, Closure Under Forcing, and Countability.

The well-foundedness mirage axiom

The most provocative of Hamkins's multiverse axioms is his well-foundedness mirage axiom.

- **Well-Foundedness Mirage** If M is a wold there is another world N with $M \in N$ and $N \models M$ is ω-nonstandard. That is, N sees an embedding of ω^M onto a strict initial segment of ω^N.

WFM has profound consequences for the structure of the multiverse, more so than Hamkins's other axioms. It forces every world to be ω-nonstandard, and more.

Recursive saturation

Definition

A structure is recursively saturated if it realizes every finitely consistent computable type.

Recursive saturation is an important concept in the model theory of nonstandard models.

- Every theory with an infinite model has a countable recursively saturated model.
- Every recursively saturated model of set theory is ω-nonstandard.
- The definable ordinals in a recursively saturated model of set theory are bounded.
- Every element of an ω-nonstandard model of set theory is recursively saturated.

Thus, if a multiverse satisfies Hamkins's WFM axiom then every world in the multiverse must be recursively saturated.

A natural model of the Hamkins multiverse axioms

Theorem (Gitman–Hamkins (2010))

The collection of countable, recursively saturated models of set theory form a multiverse satisfying Realizability, Closure Under Forcing, Countability, and Well-Foundedness Mirage.

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Yes.

Gitman, Godziszewsky, Meadows, and I consider two possible weakenings.

Avoiding recursive saturation

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- Can weaken being a Paris model to having cofinally many ordinals definable without parameters.
- Can weaken even further to allowing a fixed parameter in the definitions.

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Every consistent extension of ZF *has a Paris model.*

The first weakening of the WFM axiom

The weak well-foundedness mirage axiom

- **Weak Well-Foundedness Mirage** If M is a world there is another world N with $M \in N$ and $N \models M$ is nonstandard (but possibly ω-standard).

A natural model of the weak WFM axiom

Theorem (Gitman, Godziszewsky, Meadows, W.)

The collection of countable, nonstandard models of set theory form a multiverse satisfying Realizability, Closure Under Forcing, Countability, and Weak Well-Foundedness Mirage.

This multiverse contains many worlds which are not recursively saturated, e.g. ω-standard models and Paris models.

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Theorem (Gitman, Godziszewsky, Meadows, W.)

The collection of countable, nonstandard but ω-standard models of set theory form a multiverse satisfying Standard Realizability, Closure Under Forcing, Countability, and Weak Well-Foundedness Mirage.

No world in this multiverse is recursively saturated.

The second weakening of the WFM axiom

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(Need some set-up first.)

Covering extensions

- $N \supseteq M$ is an end-extension if $b \in^N a \in M$ implies $b \in M$.
- An end-extension $N \supseteq M$ is covering if there exists $m \in N$ so that $a \in^N m$ for all $a \in M$.
- $N \supseteq M$ is a rank-extension if $b \in N \setminus M$ implies $\mathrm{rank}^N\, b > \alpha$ for all $\alpha \in \mathrm{Ord}^M$.

Observe that every rank-extension is a covering end-extension and every elementary end-extension is a rank-extension.

Theorem (Keisler–Morley (1968))

Every countable model of ZFC *has an elementary end-extension.*

The covering multiverse axioms

- **Covering Well-Foundedness Mirage** If M is a world then there is a world N with $(k, \in^k) \in N$ so that k is a covering end-extension of M and $N \models k$ is ω-nonstandard.
- **Covering Countability** If M is a world then there is a world N with $(k, \in^k) \in N$ so that k is a covering end-extension of M and $N \models k$ is countable.

Building a covering axiom multiverse. Step 1: The trellis

$$
\begin{tikzcd}
 & & & M^{\omega}_{e-1} & M^{\omega}_{e} & M^{\omega}_{e+1} & \\
\vdots & & & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \\
M_0^{n+1} \arrow[u] \arrow[r, "U_0"] & {} & \cdots \arrow[r, "U_{e-2}"] & M^{n+1}_{e-1} \arrow[u] \arrow[r, "U_{e-1}"] & M^{n+1}_{e} \arrow[u] \arrow[r, "U_e"] & M^{n+1}_{e+1} \arrow[u] \arrow[r, "U_{e+1}"] & \cdots \\
M_0^{n} \arrow[u] \arrow[r, "U_0"] & {} & \cdots \arrow[r, "U_{e-2}"] & M^{n}_{e-1} \arrow[u] \arrow[r, "U_{e-1}"] & M^{n}_{e} \arrow[u] \arrow[r, "U_e"] & M^{n}_{e+1} \arrow[u] \arrow[r, "U_{e+1}"] & \cdots \\
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\end{tikzcd}
$$

$Z \subseteq \omega^M$, a $\mathbb{Z}$-block

trellis

$M = M_0^0$ is a countable and ω-nonstandard Paris model, $U = U_0 \in M$ is an ultrafilter on ω^M.

Vertical arrows are elementary end-extensions. We can ensure cofinally many ordinals are definable from a fixed parameter.

Horizontal arrows are ultrapowers, iterating the ultrapower of U_0 along ω^M.

Each world in the trellis is ω-nonstandard but not recursively saturated.

Step 2: Grow the multiverse $\mathcal{C}(M)$ on the trellis

- First, add in enough forcing extensions.
  - More precisely: For each $e \in Z$, for each Ord-cc forcing $\mathbb{P} \subseteq M_e^\omega$, for each $G \subseteq \mathbb{P}$ generic over M_e^ω, for each $n \in \omega$: place $M_e^n[G \cap M_e^n]$ into $\mathcal{C}(M)$.

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- Finally, close under set-like realizability.
  - More precisely: For each $M_e^n[G \cap M_e^n]$, if N is a definable over this world model of ZFC so that $M_e^n[G \cap M_e^n]$ thinks N is set-like, then place N into $\mathcal{C}(M)$.

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(Need to restrict to Ord-cc forcings and set-like models to make later arguments work. Use Ord-cc-ness to get that $G \cap M_e^n$ is generic over M_e^n. And use the restriction to set-like models to get covering extensions by moving up the trellis.)

Our main theorem

M is a countable, ω-nonstandard Paris model,
$\mathcal{C}(M)$ is the covering multiverse grown from M.

Theorem (Gitman, Godziszewsky, Meadows, W.)

$\mathcal{C}(M)$ is a multiverse satisfying Set-Like Realizability, Closure Under Ord-cc Forcing, Covering Countability, and Covering Well-Foundedness Mirage.

A partial sketch of the proof

(Worlds in the trellis have Ord-cc forcing extensions)

Take M_e^n and $\mathbb{P} \subseteq M_e^n$ an Ord-cc forcing. Let $\mathbb{P}^+ \subseteq M_e^\omega$ be the forcing defined by the same formula. Then if $G \subseteq \mathbb{P}^+$ is generic over M_e^ω then G meets every antichain, each of which is a set in M_e^ω by Ord-cc-ness. So G meets every antichain in M_e^n, so $G \cap M_e^n$ is generic over M_e^n. So $\mathcal{C}(M)$ contains a forcing extension of M_e^n by $\mathbb{P}$.

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(Covering Countability for worlds in the trellis)
Take M_e^n and look at a forcing extension of M_e^{n+1} which collapses V_α to be countable where α is above M_e^n. Then $(k, \in^k) = (V_\alpha, \in) \in M_e^{n+1}[G]$ witnesses Covering Countability for M_e^n.

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(Covering WFM for worlds in the trellis)
Take M_e^n, with M_e^{n+1} as an elementary end-extension. But M_e^{n+1} is a definable, set-like class in M_{e-1}^{n+1}, which sees that M_e^{n+1} is ω-nonstandard. Cut off M_e^{n+1} at an ordinal above M_e^n to get $(k, \in^k) \in M_{e-1}^{n+1}$ witnessing Covering WFM for M_e^n.

Open questions

Question

Can we get a multiverse for the covering axioms which satisfies Closure Under (Tame) Class Forcing and full Realizability? That is, can we drop the restrictions to Closure Under Ord-cc Forcing and Set-Like Realizability?

Question

Is there a natural model of the covering multiverse axioms?

Thank you!