MATH 321: HOMEWORK 5
DUE THURSDAY, OCT 8 BY 11:59PM

This homework is focused on writing. Your finished product should be a complete write-up of your proof. I broke down the problem into sub-problems, with two of the subproblems asking you to prove a lemma. A *lemma* is a mathematical result that used as part of proving something else. Mathematicians will state and prove them before stating and proving their theorem. So for example, your write-up for this homework might have the following structure.

> **Lemma.** *The first lemma.*
>
> *Proof.* The proof of the first lemma. □
>
> **Lemma.** *The second lemma.*
>
> *Proof.* The proof of the second lemma. □
>
> **Theorem.** *The main result. [In this case, $\sqrt{p}$ is irrational for all primes p.]*
>
> *Proof.* The proof of the main result. □

Problem. Prove that for all prime numbers p, $\sqrt{p}$ is irrational.

Here I break down what you need to show step by step.

Sub-Problem 1. Prove Euclid's Lemma: for integers a, b, n, if $\gcd(a, n) = 1$ and $n \mid ab$, then $n \mid b$. [Hint: by Bézout's identity you know you have x and y so that $ax + ny = 1$. Multiply both sides by b and use that to conclude that $n \mid b$.]

Sub-Problem 2. As a lemma prove that: for all natural numbers p, if n is not a perfect square then for all integers a, if $a^2 \equiv 0 \mod p$ then $a \equiv 0 \mod p$. [Hint 1: prove this innermost if-then by contrapositive: assume $a = qp + r$ for $0 < r < p$, and use this to show that $a^2 \not\equiv 0 \mod p$.] [Hint 2: Following hint 1, you will eventually need to show that $p \nmid r^2$. Do this by contradiction.]

Sub-Problem 3. Using these two lemmata, you can now take the ideas behind showing $\sqrt{2}$ is irrational and port them over to the more general context of $\sqrt{p}$ for prime p.