

MATH 321: HOMEWORK 5
DUE THURSDAY, OCT 8 BY 11:59PM

This homework is focused on writing. Your finished product should be a complete write-up of your proof. I broke down the problem into sub-problems, with two of the subproblems asking you to prove a lemma. A *lemma* is a mathematical result that used as part of proving something else. Mathematicians will state and prove them before stating and proving their theorem. So for example, your write-up for this homework might have the following structure.

Lemma. *The first lemma.*

Proof. The proof of the first lemma. □

Lemma. *The second lemma.*

Proof. The proof of the second lemma. □

Theorem. *The main result. [In this case, \sqrt{p} is irrational for all primes p .]*

Proof. The proof of the main result. □

Problem. Prove that for all prime numbers p , \sqrt{p} is irrational.

Here I break down what you need to show step by step.

Sub-Problem 1. Prove Euclid's Lemma: for integers a, b, n , if $\gcd(a, n) = 1$ and $n \mid ab$, then $n \mid b$. [Hint: by Bézout's identity you know you have x and y so that $ax + ny = 1$. Multiply both sides by b and use that to conclude that $n \mid b$.]

Sub-Problem 2. As a lemma prove that: for all natural numbers p , if n is not a perfect square then for all integers a , if $a^2 \equiv 0 \pmod{p}$ then $a \equiv 0 \pmod{p}$. [Hint 1: prove this innermost if-then by contrapositive: assume $a = qn + r$ for $0 < r < n$, and use this to show that $a^2 \not\equiv 0 \pmod{n}$.] [Hint 2: Following hint 1, you will eventually need to show that $p \nmid r^2$. Do this by contradiction.]

Sub-Problem 3. Using these two lemmata, you can now take the ideas behind showing $\sqrt{2}$ is irrational and port them over to the more general context of \sqrt{p} for prime p .