

Mediate cardinals

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they/them

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What does it mean to be infinite?

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This isn't circular, because we can define ω by its induction properties.

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- X is **Dedekind-infinite** if there is $f : X \rightarrow X$ a non-surjective injection.
- X is **Dedekind-finite** if any injection $f : X \rightarrow X$ is a surjection.

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- (\Leftarrow) Push forward the $+1$ function on ω .
- (\Rightarrow) Fix $z \in X \setminus \text{ran } f$. Then the map $n \mapsto f^n(z)$ gives an injection $\omega \rightarrow X$.
 - Use fact that f is one-to-one to inductively prove this map is an injection.

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Yes.

- If X is Dedekind-infinite then $\omega \leq |X|$ so X is infinite.
- If X is infinite, choose for each n an injection $e_n : n \rightarrow X$. Inductively glue them together into an injection $e : \omega \rightarrow X$.

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- Dedekind-infinite implies infinite goes through in ZF.
- Infinite implies Dedekind-infinite needs a small fragment of AC.

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Theorem (Cohen 1963)

It is consistent with ZF that there exists a Dedekind-finite, infinite set.

But you can't get a reversal: there's no hope the non-existence of a DFI set implies AC because the former is **local** while the latter is **global**.

The first question

Question

Is there a suitable generalization of a Dedekind-finite, infinite set whose nonexistence gives a characterization of AC?

A look back in history

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- In the late 1910s, Bertrand Russell is a few years after the last volume of *Principia Mathematica*. His time is occupied by legal troubles over his pacifism during World War I and thinking about the foundations of mathematics.
- Working with him are multiple students, including [Dorothy Wrinch](#).
- The next decade (1923) she will publish a paper answering our first question.

Dorothy Wrinch



- Born 1894, died 1976.
- Studied logic under Russell, did her doctorate (1921) under applied mathematician John Nicholson.
- Wrote in a range of subjects: logic, pure mathematics, philosophy of science, and mathematical biology.
- Was awarded a Rockefeller Foundation fellowship to support her work in mathematical biology.
- Early career was in the UK, later emigrated to the USA. Latter years of her career were at Smith College (Mass, USA).
- Had the misfortune of being on the losing side of a scientific dispute with Linus Pauling over the structure of proteins.

Wrinch's question, and mine

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Question

Can we use modern techniques to prove more precise consistency results?

Cardinals sans choice

Notation:

- κ, λ, \dots will be used for well-orderable, infinite cardinals.
- $\mathfrak{p}, \mathfrak{q}, \dots$ will be used for cardinals in general.
- I'll sometimes use \mathfrak{p} to refer to an arbitrary set of cardinality \mathfrak{p} .
- Under AC, every cardinal is well-orderable. We can thus define the cardinals as the **initial ordinals**.
- Without AC we have to fall back on defining cardinals as equivalence classes.
- Can use Scott's trick to make these sets.

Mediate cardinals

Fix a cardinal p . Then X is **p -mediate** if

- $q \leq |X|$ for all $q < p$;
- $p \not\leq |X|$; and
- $|X| \not\leq p$.

A **p -mediate cardinal** is a cardinal number of a p -mediate set.

Mediate means p -mediate for some infinite p .

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Mediate means p -mediate for some infinite p .

- Dedekind-finite infinite $\Leftrightarrow \aleph_0$ -mediate.
- ZF proves there are no n -mediate for finite n .

A few facts

Some facts about DFI sets generalize.

Fact

Suppose \mathfrak{q} and \mathfrak{r} are \mathfrak{p} -mediate. Then:

- $\mathfrak{q} + \mathfrak{r}$ is \mathfrak{p} -mediate;
- $\mathfrak{q} \cdot \mathfrak{r}$ is \mathfrak{p} -mediate; and
- $2^{2^{\mathfrak{q}}}$ is not \mathfrak{p} -mediate.

Wrinch's theorem

Theorem (Wrinch 1923)

Over ZF, the following are equivalent.

- ① AC;
- ② There are no mediate cardinals; and
- ③ There are no κ -mediate cardinals for well-ordered κ .

(Wrinch originally formulated this result in the framework of *Principia Mathematica*.)

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Definition

m is **p-mEDIATE** if

- $q \leq m$ for all $q < p$;
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Prove $(1 \Rightarrow 2)$ by contrapositive.

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- Suppose q is p mediate. Then p and q are incomparable, so **Cardinal Trichotomy** fails.
- (Hartogs 1915) AC iff Cardinal Trichotomy.

Wrinch's theorem, $(3 \Rightarrow 1)$

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$(2 \Rightarrow 3)$ is trivial. Prove $(3 \Rightarrow 1)$ by contrapositive.

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- (Hartogs) For any p there is a smallest well-orderable cardinal $\aleph(p)$ so that $\aleph(p) \not\leq p$.
- If p is not well-orderable then p is $\aleph(p)$ -mediate.

Dependent choice

Dependent choice (DC) informally says you can make ω many choices where each choice depends on the previous ones.

- Suppose R is a relation on a set X so that for each $x \in X$ there is $y \in X$ with $x R y$. Then there is a **branch** $\langle x_i : i \in \omega \rangle$ through R : for each i have $x_i R x_{i+1}$.

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DC $_{\kappa}$ says:

- Suppose R is a relation on $X^{<\kappa} \times X$ so that for each $s \in X^{<\kappa}$ there is $y \in X$ with $s R y$. Then there is a branch $b = \langle x_i : i < \kappa \rangle$ through R : for each i have $(b \restriction i) R b_i$.

DC $_{<\kappa}$ is DC $_{\lambda}$ for all $\lambda < \kappa$.

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DC_κ says:

- Suppose R is a relation on $X^{<\kappa} \times X$ so that for each $s \in X^{<\kappa}$ there is $y \in X$ with $s R y$. Then there is a branch $b = \langle x_i : i < \kappa \rangle$ through R : for each i have $(b \restriction i) R b_i$.

$DC_{<\kappa}$ is DC_λ for all $\lambda < \kappa$.

Facts:

- AC is equivalent to $\forall \kappa \ DC_\kappa$.
- $\lambda < \kappa$ implies $DC_\kappa \Rightarrow DC_\lambda$.
- $ZF + DC_{<\kappa} + \neg DC_\kappa$ is consistent.
- DC implies AC_ω over ZF, but not vice versa.
- DC is equivalent to “a relation is well-founded iff it has no infinite descending sequence”.
- (Solovay) $ZF + DC +$ “every set of reals is Lebesgue-measurable” is consistent.

DC and mediate cardinals

Lemma: DC_κ implies there are no κ -mediates.

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- Suppose $\lambda \leq \mathfrak{p}$ for all $\lambda < \kappa$ but $\mathfrak{p} \not\leq \kappa$.
- Consider the collection of all injections $\alpha \rightarrow \mathfrak{p}$ for $\alpha < \mathfrak{p}$.
- None of the injections are onto, so you can always extend them to an injection $\alpha + 1 \rightarrow \mathfrak{p}$.
- By DC_κ there's a branch, which gives an injection $\kappa \rightarrow \mathfrak{p}$.

Refining mediacy

Observation:

- If \mathfrak{p} is κ -mediate and $\lambda > \kappa$ then $\mathfrak{p} + \lambda$ is λ^+ -mediate.
- So if you have κ -mediates for one κ you have mediates for larger cardinals.

Definition

\mathfrak{m} is \mathfrak{p} -mediate if

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- if $Y \subseteq \mathfrak{p}$ has cardinality $< \kappa$ then $\mathfrak{p} \setminus Y$ is κ -mediate.

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Lemma: If \mathfrak{p} is κ -mediate where κ is smallest such that κ -mediates exist, then \mathfrak{p} is exact κ -mediate.

Consistency questions

Question

- *Consistently, what can be the smallest κ so that κ -mediates exist?*
- *Consistently, what can be the class of κ for which exact κ -mediates exist?*

Symmetric extensions

Motivating example: Add ω many reals, then forget the order you added them.

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- $\mathbb{P} = \text{Add}(\omega, \omega)$ is the poset. Conditions are finite partial functions $\omega \times \omega \rightarrow 2$.
- Changing the order is permuting the columns in the $\omega \times \omega$ grid.
- Any permutation $\pi : \omega \rightarrow \omega$ generates an automorphism of \mathbb{P} :
$$\pi p(n, i) = p(\pi n, i).$$
- Also generates an automorphism on the \mathbb{P} -names:
$$\pi \dot{x} = \{(\pi p, \pi \dot{y}) : (p, \dot{y}) \in \dot{x}\}$$

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- “Forgetting the order” is restricting to names fixed by a ‘large’ group of automorphisms:
A group H of automorphisms is large if there is finite $e \subseteq \omega$ so that each $\pi \in H$ fixes e pointwise: $H \supseteq \text{fix}(e)$.
- This gives a **normal filter** \mathcal{F} on the lattice of subgroups.
- A name \dot{x} is **\mathcal{F} -symmetric** if $\text{sym}(\dot{x}) = \{\pi : \pi \dot{x} = \dot{x}\} \in \mathcal{F}$.
- The **symmetric extension** consists of the interpretations of all hereditarily symmetric names.

Symmetric extensions, in general

A **symmetric system** is $(\mathbb{P}, G, \mathcal{F})$ so that

- \mathbb{P} is a forcing poset;
- $G \leq \text{Aut}(\mathbb{P})$; and
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A \mathbb{P} -name \dot{x} is symmetric if $\text{sym } x \in \mathcal{F}$.

- (**Symmetry lemma**) $p \Vdash \varphi(\dot{x})$ iff $\pi p \Vdash \varphi(\pi \dot{x})$.

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- (**Symmetry lemma**) $p \Vdash \varphi(\dot{x})$ iff $\pi p \Vdash \varphi(\pi \dot{x})$.

The **symmetric extension** by $(\mathbb{P}, G, \mathcal{F})$ via a generic $g \subseteq \mathbb{P}$:

- Consists of the interpretations of hereditarily symmetric names.
- $V[g/\mathcal{F}] = \{\dot{x}^g : \dot{x} \text{ is hereditarily symmetric}\}$.

$V[g/\mathcal{F}] \models \text{ZF}$, but the point is to make AC fail in a controlled way.

The Cohen symmetric extension

Fix regular κ and assume $\kappa^{<\kappa} = \kappa$.

- $\mathbb{P}_\kappa = \text{Add}(\kappa, \kappa)$;
- $G_\kappa \leq \text{Aut}(\mathbb{P}_\kappa)$ is generated by permutations of κ ;
- $H \in \mathcal{F}_\kappa$ if $\exists e \in [\kappa]^{<\kappa}$ so that $\text{fix}(e) \subseteq H$.

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In $V[g_\kappa/\mathcal{F}_\kappa]$ the set $A = \{c_i : i < \kappa\}$ for Cohen subsets of κ is not well-orderable.

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Facts:

- \mathbb{P}_κ is κ -closed and has the κ^+ -cc.
- \mathcal{F}_κ is κ -complete.

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Facts:

- \mathbb{P}_κ is κ -closed and has the κ^+ -cc.
- \mathcal{F}_κ is κ -complete.

Thus, $(\mathbb{P}_\kappa, G_\kappa, \mathcal{F}_\kappa)$ will preserve $\text{DC}_{<\kappa}$.

In particular, there will be no λ -mediates for $\lambda < \kappa$.

Symmetric extensions and DC

Lemma: Let κ be regular and $\lambda < \kappa$. If \mathbb{P} is κ -closed and \mathcal{F} is κ -complete then $(\mathbb{P}, G, \mathcal{F})$ preserves DC_λ .

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- Consider appropriate $R \subseteq X^{<\lambda} \times X$ in $V[g/\mathcal{F}]$. We need a branch through R in $V[g/\mathcal{F}]$.
- By κ -closure λ remains a cardinal in $V[g]$.
- In $V[g]$, by DC_λ there is a branch $b = \langle x_i : i < \lambda \rangle$.
- Each x_i comes from a symmetric name \dot{x}_i .
- By κ -completeness $H = \bigwedge_{i < \lambda} \text{sym}(\dot{x}_i)$ is in \mathcal{F} .
- Can get a name \dot{b} for b with $\text{sym}(\dot{b}) \supseteq H$.
- So the branch b is in $V[g/\mathcal{F}]$.

The smallest mediate can be anything

Theorem (W.)

Suppose $\kappa = \kappa^{<\kappa}$ is regular. In the symmetric extension by $(\mathbb{P}_\kappa, G_\kappa, \mathcal{F}_\kappa)$:

- $\text{DC}_{<\kappa}$;
- κ is least so that there is a κ -mediate cardinal; and
- There is an exact λ -mediate iff $\lambda = \kappa$.

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We've already seen $\text{DC}_{<\kappa}$ and so there are no λ -mediates for $\lambda < \kappa$.

Claim: Let A be the set of the Cohen subsets of κ added by \mathbb{P}_κ . Then $V[g/\mathcal{F}_\kappa] \models A$ is κ -mediate.

Like getting DFI set in $(\mathbb{P}_\omega, G_\omega, \mathcal{F}_\omega)$.

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- $\lambda < \kappa$ injects by κ -closure of \mathbb{P}_κ and κ -completeness of \mathcal{F}_κ
- $|A| \not\leq \kappa$ because A can't be well-ordered.
- $\kappa \not\leq |A|$:

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- $|A| \not\leq \kappa$ because A can't be well-ordered.
- $\kappa \not\leq |A|$:
 - Suppose \dot{f} is hereditarily symmetric, $\text{sym}(\dot{f}) \supseteq \text{fix}(e)$, and $p \Vdash \dot{f} : \kappa \rightarrow A$ is one-to-one.
 - Extend p to q deciding $\dot{f}(\alpha) = c_i$ for some $\alpha \neq i$ both $\notin e$.
 - Find π fixing $e \cup \{i\}$, moving α , and $q \parallel \pi q$.
 - So $q \cup \pi q \Vdash \dot{f}$ is not one-to-one. Contradiction.

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- There is an exact λ -mediate iff $\lambda = \kappa$.

Lemma: If X is exact λ -mediate for $\lambda > \kappa$ in $V[g/\mathcal{F}_\kappa]$, then $V[g] \models |X| = \lambda$.

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Lemma: If X is exact λ -mediate for $\lambda > \kappa$ in $V[g/\mathcal{F}_\kappa]$, then $V[g] \models |X| = \lambda$.

Work in $V[g]$:

- Consider the tree of hereditarily symmetric names for injections $\alpha \rightarrow X$ for $\alpha < \lambda$.
- Claim implies the tree has a branch.
- Branch has size $\lambda > \kappa$ and $|\mathcal{F}| = \kappa$, so λ many names \dot{f}_α on the branch have the same $\text{sym}(\dot{f}_\alpha)$.
- Can build a branch b so every injection on branch has same $\text{sym}(\dot{f}_\alpha)$.
- Then b has a hereditarily symmetric name.

Thus $V[g/\mathcal{F}_\kappa] \models \lambda \leq |X|$. Contradiction.

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- Karagila has a framework for iterations of symmetric extensions.
- It's complicated, with scary group theoretic objects like **wreath products**.

Doing it more than once

When a set theorist can do something once, she wants to do it more than once. With forcing, she accomplishes this using **products** or **iterations**.

- Karagila has a framework for iterations of symmetric extensions.
- It's complicated, with scary group theoretic objects like **wreath products**.
- We are lucky and can get away with products, where the details are significantly less technical.

Products of symmetric extensions

Suppose $(\mathbb{P}, G, \mathcal{F})$ and $(\mathbb{Q}, H, \mathcal{E})$ are symmetric systems. Can define their product

$(\mathbb{P}, G, \mathcal{F}) \times (\mathbb{Q}, H, \mathcal{E})$:

- $\mathbb{P} \times \mathbb{Q}$ is usual product of posets;
- $G \times H$ is generated by (π, ρ) with $\pi \in G$, $\rho \in H$; and
- $\mathcal{F} \times \mathcal{E}$ is generated by $G_0 \times H_0$ for $G_0 \in \mathcal{F}$ and $H_0 \in \mathcal{E}$.

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Can also do this for infinite products, with a notion of support.

- Suppose $(\mathbb{P}_\kappa, G_\kappa, \mathcal{F}_\kappa)$ are symmetric systems for $\kappa \in M$.
- Then there is a product $\prod_{\kappa \in M} (\mathbb{P}_\kappa, G_\kappa, \mathcal{F}_\kappa)$ with that support.
- Again we get a product lemma stating we can split the full extension into two-step extensions.

Refining earlier ideas

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Suppose $\lambda < \kappa < \mu$ are regular.

- $(\text{ZF} + \text{DC}_\kappa)$ If \mathbb{P} is κ -closed and \mathcal{F} is κ -complete then $(\mathbb{P}, G, \mathcal{F})$ preserves DC_λ .
- $(\text{ZF} + \text{DC}_\kappa)$ Suppose \mathbb{P} has the κ^+ -cc and \mathcal{F} is generated by a basis of size $\leq \kappa$. Then $V[g/\mathcal{F}] \models$ there are no exact μ -mediates.

The pattern of the exact mediates

Theorem (W.)

Assume GCH and fix a class M of regular cardinals. Do the Easton support product of the $(\mathbb{P}_\kappa, G_\kappa, \mathcal{F}_\kappa)$ for $\kappa \in M$. In the symmetric extension, there is an exact α -mediate iff $\alpha \in M$.

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Sketch:

- $\mathbb{P}_{>\alpha}$ is α -closed and $\mathcal{F}_{>\alpha}$ is α -complete.
- $\mathbb{P}_{<\alpha}$ has the α^+ -cc and $\mathcal{F}_{<\alpha}$ is generated by a basis of cardinality $< \alpha$.
- In $V[g_{>\alpha}/\mathcal{F}_{>\alpha}]$: DC_α is true. So there are no α -mediates.
- In $V[g_{>\alpha}/\mathcal{F}_{>\alpha}][g_{<\alpha}/\mathcal{F}_{<\alpha}]$: there are no exact α -mediates.
- So the only way there could be an exact α -mediate is if it was added by $(\mathbb{P}_\alpha, G_\alpha, \mathcal{F}_\alpha)$ which is nontrivial iff $\alpha \in M$.
- But we know that adds an exact mediate when $\alpha \in M$.

Open questions

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- What's up with singular cardinals?
- What if we don't make such strong cardinal arithmetic assumptions?
- What happens if AC fails badly in the ground model?

Thank you!

- Dorothy Wrinch, “On mediate cardinals”, *American Journal of Mathematics*, Vol. 45, No. 2. (1923). pp. 87–92.
DOI: 10.2307/2370490.
- Kameryn J. Williams, “Mediate cardinals: old and new”, *In preparation*.
- Landon D.C. Elkind, “I like her very much—she has very good brains.”: Dorothy Wrinch’s Influence on Bertrand Russell, In: *Bertrand Russell, Feminism, and Women Philosophers in his Circle*, eds. Elkind & Klein. (2024). pp. 259–297.