Math 302: Existence and uniqueness of solutions

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I won't go into every gritty detail. Rather, the goal is to give you a view of the broad ideas, and some techniques and tricks that are used to reason abstractly about differential equations.

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$$y'_{n} = -a_{n-1}y_{n} - \dots - a_{1}y_{2} - a_{0}y_{1} + b$$
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We've turned our problem into a new problem. That's kinda like progress!

We turned our *n*th-order differential equation about an unknown function *y* into a system of equations. Namely, we have

- *n* equations
- about *n* unknown functions y_1, \ldots, y_n
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If we want to think about how to solve this, it makes sense to start with the simplest case, where we have only one unknown function.



- Suppose for now that F is continuous in a rectangle R: a₀ ≤ x ≤ a₁ and b₀ ≤ y ≤ b₁.
- (We'll see later that we actually have to assume a little bit more about *F*.)
- Fix a point (x_0, y_0) in this rectangle, and suppose we have have the initial condition $y(x_0) = y_0$.



A trick: Turn the differential equation into an integral equation.

Integrate both sides, starting at x_0 , get:

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Why?

- Derivatives are easier to compute than integrals.
- But for theoretical uses, integrals are better behaved.
- Since we want a theoretical result rather than calculations, this makes things easier.



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Spoiler: This stupid idea will work out (with an extra assumption on F). The error gets smaller and smaller, so these Picard approximations converge to the true solution.

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has solutions.

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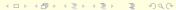
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- We turned the first-order differential equation y' = F(x, y) into an integral equation.
- We want to use Picard's method to find a solution to this integral equation.

