The Σ_1-definable universal finite sequence

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Set Theory, Model Theory, and their Philosophy

2018 Dec 15

This is joint work with Joel David Hamkins and Philip Welch.

Multiversism versus universism in set theory

- The universist: *The* universe of sets is uniquely determined.
- The multiversist: There are many universes of sets, and every universe is contained inside a bigger, better universe.

Set-theoretic multiverses

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- A more radical multiversism: Hamkins's multiverse axioms, including the Well-foundedness Mirage axiom—every universe is seen to be ill-founded from some larger universe.

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- A more radical multiversism: Hamkins's multiverse axioms, including the Well-foundedness Mirage axiom—every universe is seen to be ill-founded from some larger universe.
  - The multiverse consisting of the countable, recursively saturated models of set theory satisfy Hamkins's axioms. (Gitman and Hamkins)

Potentialism as a general framework

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- Warning! There is no guarantee that a potentialist system be linearly ordered or even directed. There are branching potentialist systems which have incompatible extensions.

The modal logic of potentialism

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- This gives an easy lower bound. The real work is in getting upper bounds.

Buttons and switches (Hamkins and Löwe)

We can analyze the modal validities of a potentialist system using control statements.

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- If there are arbitrarily large families of independent buttons and switches, then the modal validities are contained within S4.2, which is S4 plus the axiom $\Diamond\Box p \Rightarrow \Box\Diamond p$.

Potentialism in set theory

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  - Height potentialist.

- Forcing potentialism: worlds are forcing extensions of M.
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- Countable transitive model potentialism.
  - Height and width potentialist.

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  - S4.3 is S4 plus

$$(\Diamond\varphi \wedge \Diamond\psi) \Rightarrow (\varphi \wedge \Diamond\psi) \vee (\Diamond\varphi \wedge \psi).$$

- Forcing potentialism: worlds are forcing extensions of M.
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  - Modal validities are S4.2. (Hamkins and Löwe)
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Flavors of potentialism

S5	maximality principles
S4.3	linear
S4.2	directed
S4	incompatible branching

The Σ_2-definable universal finite set for rank-extensions

$N \supseteq M$ is a rank-extension of M if every $b \in N \setminus M$ has rank in $N \setminus M$.

Theorem (Hamkins and Woodin)

There is a Σ_2 definition for a finite set $\{b_0, \ldots, b_n\}$ with the following properties.

1. ZFC *proves that the definition defines a finite set.*
2. *In any transitive model of* ZFC *the set is empty.*
3. *If $M \models$ ZFC is countable, has s as its universal finite set, and $t \in M$ is a finite set extending s, then there is $N \models$ ZFC a rank-extension of M which has t as its universal finite set.*

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Theorem (Hamkins and Woodin)

The modal validities of rank-extensional set theoretic potentialism are precisely S4.

End-extensions versus rank-extensions

- $N \supseteq M$ is an end-extension of M if $a \in b$ in N and b in M implies a is in M.
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- But not all end-extensions are rank-extensions. For example, if M is an inner model of N then N end-extends M.

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- But not all end-extensions are rank-extensions. For example, if M is an inner model of N then N end-extends M.
- Key fact: the assertions which are preserved in arbitrary end-extensions are the Σ_1 assertions.

$$\exists y \quad \underbrace{\varphi(x, y)}_{\text{quantifiers bounded}}$$

The Σ_1-definable universal finite sequence for end-extensions

Let ZF^+ be a computably enumerable extension of ZF.

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4. *Indeed, it suffices in* (3) *that M has an inner model W of* ZF^+ *satisfying such.*

The definition of the universal finite sequence—Process A

Intended for ω-nonstandard models. A different process is used for ω-standard models.

- Proceed in stages to produce $b_0, b_1, \ldots, b_n$, using auxiliary information: countable ordinals $\alpha_0 < \alpha_1 < \cdots < \alpha_n$ and natural numbers $k_0 > k_1 > \cdots > k_n$.

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- At stage n: Are there $\alpha > \alpha_{n-1}$, $k < k_{n-1}$, and $b \in L_\alpha$ so that L_α has no end-extension to N satisfying the first k axioms of ZF^+ plus "process A succeeds at stage n and defines b"? If so, stage n is successful and set (b_n, α_n, k_n) to be the L-least triple of such (b, α, k).

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- This self-reference is allowed by the Gödel–Carnap fixed-point lemma.
- Claim: The map $n \mapsto (b_n, \alpha_n, k_n)$ is Σ_1-definable.
- Claim: Each k_i must be nonstandard.
- Claim: There are only finitely many successful stages.

Seeing that Process A has the plus one extension property

Consider countable M in which the universal sequence is $b_0, \ldots, b_{n-1}$ and take any $b \in M$ and nonstandard $k < k_{n-1}$.

- Because stage n is unsuccessful in M this means L^M thinks every countable set can be end-extended to a model of the first k axioms of ZF^+ in which the universal sequence is $b_0, \ldots, b_{n-1}, b$.

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- By the Keisler–Morley theorem, let M^+ be an elementary end extension of M and fix an ordinal θ in $M^+ \setminus M$. Consider $M^+[G]$ the forcing extension in which V_θ is collapsed to be countable.

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- By the Keisler–Morley theorem, let M^+ be an elementary end extension of M and fix an ordinal θ in $M^+ \setminus M$. Consider $M^+[G]$ the forcing extension in which V_θ is collapsed to be countable.
- So V_θ^M has in $M^+[G]$ an end-extension N in which the universal sequence is $b_0, \ldots, b_{n-1}, b$. But N is also an end-extension of M.

Process B—for ω-standard models

- Again go in stages: produce $b_0, b_1, \ldots, b_n$ using auxiliary information countable ordinals $\alpha_0 < \alpha_1 < \cdots < \alpha_n$ and countable ordinals $\lambda_0 > \lambda_1 > \cdots > \lambda_n$.

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Can merge Processes A and B into a single Process C which works for all models. $\square$

The Barwise extension theorem

The Barwise extension theorem can be derived as a corollary of our theorem.

Theorem (Barwise)

Every countable model of ZF *end-extends to a model of* $\mathrm{ZFC} + V = L$.

The universal sequence for L-extensions

Corollary (Hamkins, Welch, W.)

There is a Σ_1 definition for a finite sequence $b_0, b_1, \ldots, b_n$ with the following properties.

1. $\mathsf{ZF} + V = L$ *proves that the sequence is finite.*
2. *In any standard model of* $\mathsf{ZF} + V = L$ *the sequence is empty.*
3. *Let M be a countable model of* $\mathsf{ZF} + V = L$ *which defines the sequence as s. Then if t in M is any finite sequence extending s, there is an L-extension N of M in which the universal sequence is t.*

Railyard labelings

- A tree is a partial order T so that $\{s \in T : s \leq t\}$ is well-ordered for every $t \in T$. A pre-tree is a pre-order which quotients to a tree.
- A railyard labeling of a pre-tree T is an assignment ρ_t of statements to nodes $t \in T$ so that each structure satisfies exactly one ρ_t and $\Diamond \rho_s$ holds iff $t \leq_T s$.

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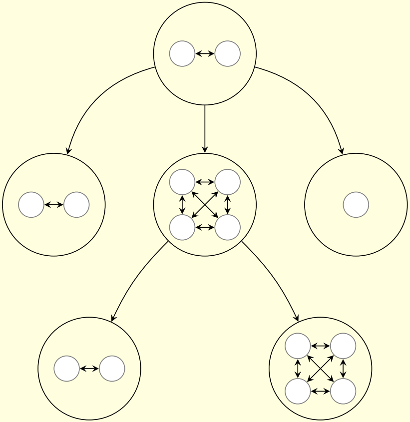
- If there are railyard labelings for every finite pre-tree, then the modal validities for the corresponding potentialist system are contained within S4.

The universal finite sequence and railyard labelings



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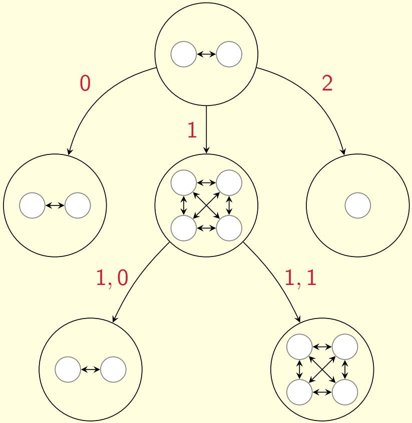
- Step 1: the subsequence $\langle n_i \rangle$ of finite ordinals from the universal finite sequence tell you how to descend the tree to determine your cluster. If B is the branching of the current node, then $n_i \bmod B$ tells you where to go.



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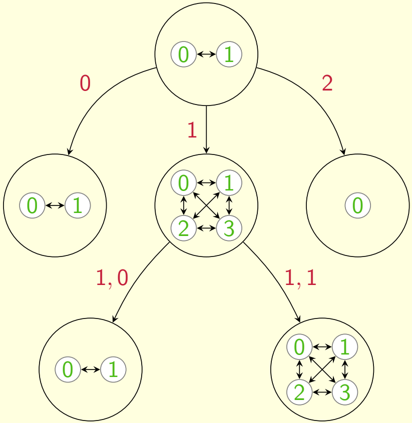
- Step 2: the final infinite ordinal $\lambda + m$ on the sequence tells you where in your cluster you are. If K is the size of the cluster, then $m \mod K$ identifies your node in the cluster. (If no infinite ordinals are on the sequence, default to 0.)



The modal validities of end-extensional set theoretic potentialism

Theorem (Hamkins, Welch, W.)

Consider the potentialist system consisting of countable models of ZF^+ *ordered by end-extension.*

1. *For any world* M*, the modal validities, allowing for a single parameter for the length of the universal finite sequence, are precisely* S4.
2. *For any* ω*-standard world* M*, the modal validities, allowing no parameters, are precisely* S4.

Thank you!