SOLUTION FOR PROBLEM 3 IN MATH 454 TAKE-HOME FINAL

Name: ______________________________

Problem 3. *Let* $W = \{R \subseteq \omega \times \omega : (\operatorname{dom} R, R)$ *is a well-order*$\}$. *Let* $\preceq$ *be the binary relation on* W *defined as* $R \preceq S$ *if there is an order-embedding* $f : (\operatorname{dom} R, R) \to (\operatorname{dom} S, S)$. *Consider* $W/{\cong}$, *where you quotient out* W *by identifying isomorphic well-orders.* (a) *Show that* $\preceq$ *is a congruence modulo* $\cong$*; that is, show that if* $(\operatorname{dom} R, R) \cong (\operatorname{dom} R', R')$, *and* $(\operatorname{dom} S, S) \cong (\operatorname{dom} S', S')$ *and* $R \preceq S$, *then* $R' \preceq S'$. (b) *Show that* $(W/{\cong}, \prec)$ *is a well-order and determine the ordertype of* $(W/{\cong}, \prec)$, *where* $R \prec S$ *if* $R \preceq S$ *but* $R \ncong S$.

Solution. (a) Let $f : (\operatorname{dom} R, R) \to (\operatorname{dom} R', R')$ and $g : (\operatorname{dom} S, S) \to (\operatorname{dom} S', S')$ be isomorphisms and let $e : (\operatorname{dom} R, R) \to (\operatorname{dom} S, S)$ be an order-embedding. Then $g \circ e \circ f^{-1} : (\operatorname{dom} R', R') \to (\operatorname{dom} S', S')$ is an embedding.

(a) (Alternate Proof) Because R and R' are isomorphic, they are both isomorphic to the same ordinal, call it ρ. And similarly S and S' are both isomorphic to some ordinal σ. Because R order-embeds into S we get that $\rho \leq \sigma$. But because $\rho \leq \sigma$ we get that R' order-embeds into S'.

(b) First let us see that $(W/{\cong}, \prec)$ is a well-order. That $\prec$ is irreflexive is immediate from the definition. That it is transitive is clear from composing embeddings. To see that it is total, take $R, S \in W$. Let ρ be the ordinal isomorphic to R and σ be the ordinal isomorphic to S. Then either $\rho < \sigma$, $\rho = \sigma$, or $\rho > \sigma$. These three possibilities correspond to $R \prec S$, $R \cong S$, or $R \succ S$, showing that $\prec$ is total on the equivalence classes. Finally, we want to see that $\prec$ is well-founded. Suppose toward a contradiction we have a descending sequence

$$R_0 \succ R_1 \succ \cdots \succ R_n \succ \cdots$$

in W. Let ρ_n be the ordinal isomorphic to R_n. Then we have

$$\rho_0 > \rho_1 > \cdots > \rho_n > \cdots,$$

an infinite descending sequence in the ordinals which is impossible.

Now let us see that the ordertype of $(W/{\cong}, \prec)$, call it γ is ω_1. First, let us see that if α is a countable ordinal then $\alpha \leq \gamma$. To see this, note that because α is countable there is $R \in W$ so that R is isomorphic to α, via an isomorphism $f : \alpha \to (\operatorname{dom} R, R)$. Now consider $\beta \leq \alpha$. Look at $S_\beta = R \restriction f[\beta]$. Then S_β is a well-order of a subset of ω, i.e. $S_\beta \in W$. And if $\beta < \beta' \leq \alpha$ then $S_\beta \prec S_{\beta'}$. So we have an isomorphic copy of α inside W, so $\alpha \leq \gamma$.

Next, observe that $\gamma \leq \omega_1$, because every well-order in W is countable. So together we get that $\gamma = \omega_1$. □

Remark. You can do a similar construction to get an explicit well-order of ordertype ω_2, ω_3, and so on. In general, if X is any infinite, well-orderable set then you get a well-order with ordertype $|X|^+$ by considering the collection of well-orders of subsets of X ordered by embeddability.