Math 302: Series methods

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$$y'' + y' + y = x^3$$

we guessed that a particular solution looks like

$$A + Bx + Cx^2 + Dx^3$$

Then we solve for the values of the coefficients.

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That is, we want to see what we can figure out if we represent the solution as a power series.

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- If $0 < R < \infty$ the series converges if $|x - p| < R$. At the end points $x = p \pm R$ it may either converge or diverge
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You can use convergence tests like you learned in Calc II to figure out the radius of convergence of a given power series.

From power series to functions

$$\sum_{n=0}^{\infty} a_n x^n$$

If the interval of convergence of this power series is nontrivial (i.e. $R > 0$), then the power series defines a continuous function on the interval $p - R < x < p + R$:

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$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{k=0}^{\infty} (k+1) x_{k+1} x^k$$

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This works centered at p, not just centered at 0.

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So we can write:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

for the power series for $f(x)$. We call this its Taylor series.

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$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

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Let's remember the Taylor series centered at 0 for some important functions:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\cos x = 0 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \pm \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \pm \cdots$$

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Let's plug these into the equation:

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So the solution is $y = a_1x$, where a_1 is an arbitrary constant.

An existence theorem

$$y^{(n)} + \cdots + a_1(x)y' + a_0(x) = b(x)$$

If the coefficient functions $a_i(x)$ and the function $b(x)$ are all analytic on the same interval centered on p, then there is a unique solution satisfying the initial conditions

$$y(p) = v_0, \quad y'(p) = v_1, \quad \cdots, \quad y^{(n-1)}(p) = v_{n-1}$$

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In short, this result tells us that if all the parts of the equation are analytic functions, then it's valid to guess that the solution is given by a power series and use that to determine the solution.

In particular, if the functions are all polynomials, exponential functions, sine/cosine, or combinations thereof, then we get a solution which is valid for all of $\mathbb{R}$.

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Solving for the first few terms:

$$y = 1 + x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24} + \cdots$$

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where a_0 and a_1 are arbitrary constants. This doesn't give us a nice way to write the solution in terms of elementary functions, but why should we expect to always be able to do so?

Nonetheless, we could use this to compute an approximation, using more terms for more precision.