MATH 210: 10-21 WORKSHEET
3.7 DERIVATIVES AND THE SHAPE OF A GRAPH

The first and second derivative give a lot of information about the shape of the graph of a function.

- $f(x)$ is *increasing* on an interval $I \iff f'(x) > 0$ on all interior points of I;
- $f(x)$ is *decreasing* on an interval $I \iff f'(x) < 0$ on all interior points of I;
- $f(x)$ is *constant* on an interval $I \iff f'(x) = 0$ on all interior points of I;
- $f(x)$ is *concave up* on an interval $I \iff f''(x) > 0$ on all interior points of I;
- $f(x)$ is *concave down* on an interval $I \iff f''(x) < 0$ on all interior points of I;
- $f(x)$ is *linear* on an interval $I \iff f''(x) = 0$ on all interior points of I;

These have the following meanings:

- $f(x)$ is *increasing* on an interval I means the graph goes up as you go to the right: if $a < b$ then $f(a) < f(b)$;
- $f(x)$ is *decreasing* on an interval I means the graph goes down as you go to the right: if $a < b$ then $f(a) > f(b)$;
- $f(x)$ is *constant* on an interval I means the graph doesn't change height as you go to the right: for any a and b we have $f(a) = f(b)$;
- $f(x)$ is *concave up* on an interval I means the graph curves upward: if you draw the straight line from the point $(a, f(a))$ to the point $(b, f(b))$ then the line is above the graph;
- $f(x)$ is *concave down* on an interval I means the graph curves downward: if you draw the straight line from the point $(a, f(a))$ to the point $(b, f(b))$ then the line is below the graph;
- $f(x)$ is *linear* on an interval I means the graph doesn't curve either way: if you draw the straight line from the point $(a, f(a))$ to the point $(b, f(b))$ then the line overlaps the graph.

Points where a graph changes between increasing and decreasing or between concave up and concave down are interesting.

- A *local maximum* is where the graph changes from increasing to decreasing. At a local maximum $f'(x)$ is either 0 or undefined. If $f''(x)$ is defined at a local maximum then it must be negative.
- A *local minimum* is where the graph changes from decreasing to increasing. At a local minimum $f'(x)$ is either 0 or undefined. If $f''(x)$ is defined at a local minimum then it must be positive.
- A *inflection point* is where the graph changes between concave up and concave down. At an inflection point $f''(x)$ is either 0 or undefined.

The upshot of all this is that if you produce a *sign diagram* for $f(x)$, $f'(x)$, and $f''(x)$ then you have a lot of information about what the graph of $f(x)$ looks like.

(1) Calculate the first and second derivatives of $a(x) = x^3 - x$. Then create sign diagrams for $a(x)$, $a'(x)$, and $a''(x)$. Use these sign diagrams to say on which intervals $a(x)$ is increasing, decreasing, concave up, and concave down. Then use this information to sketch a graph of $a(x)$. Compare your work to what a graphing calculator gives, and also use the graphing calculator to view the graphs of $a'(x)$ and $a''(x)$.
(2) Do the same for $b(x) = (x^2 - x)e^x$.
(3) Do the same for $c(x) = \arctan(x)$.
(4) Do the same for $d(x) = \frac{1}{1+x^2}$.