Math 321: Countable and uncountable sets

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- For all $b \in B$ there is a unique $a \in A$ so that $f(a) = b$.

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No.

Definitions for cardinality

Last time we saw two definitions about cardinality, letting us compare the sizes of sets.

- $|A| = |B|$ means that there is a bijection $f : A \to B$.
- $|A| \leq |B|$ means that there is a one-to-one function $f : A \to B$.
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- The lesson: if you want to show there is a bijection between A and B it is enough to find one-to-one functions $A \to B$ and $B \to A$.

An example

The intervals $(0, 1)$ and $[0, 1]$ have the same cardinality.

Countable and uncountable sets

- Say that a set A is countable if $|A| \leq |\mathbb{N}|$. That is, A is countable if there is a one-to-one function $f : A \to \mathbb{N}$.
- Every finite set is countable.
- We say countably infinite to distinguish countable, infinite sets from finite sets.
- If A is not countable we call it uncountable.

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(base case) Since A is nonempty, we simply pick any element of A to assign to be $f(0)$.

(induction step) We have already defined $f(0), \ldots, f(n)$. It cannot be that $A = \{f(0), \ldots, f(n)\}$, as if that were the case we would have that A is finite and hence countable, whereas we know A is uncountable. In other words, $A \setminus \{f(0), \ldots, f(n)\}$ is nonempty. So pick some element of this set to assign to be $f(n+1)$.

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We can always continue, so we have a one-to-one function $f : \mathbb{N} \to A$. $\square$

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We have to show there is no one-to-one function $f : \mathbb{R} \to \mathbb{N}$, which we do by contradiction.

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Theorem (Cantor)

$\mathcal{P}(\mathbb{N})$ is uncountable. More generally, if A is any set then $|A| < |\mathcal{P}(A)|$.

Proof.

We can see that $|A| \leq |\mathcal{P}(A)|$ by looking at the one-to-one function $s : A \to \mathcal{P}(A)$ defined as $s(a) = \{a\}$. So we just have to see that there is no bijection $f : A \to \mathcal{P}(A)$, which we do by contradiction.

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Suppose $f : A \to \mathcal{P}(A)$ is a bijection. Consider $D = \{a \in A : a \notin f(a)\}$, a subset of A. Since f is a bijection, there is $d \in A$ so that $f(d) = D$. Let's now consider two cases.

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(Case 1: $d \in D$) By definition of D, we get that $d \notin f(d) = D$, a contradiction.

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(Case 1: $d \in D$) By definition of D, we get that $d \notin f(d) = D$, a contradiction.

(Case 2: $d \notin D$) By definition of D, we get that $d \in f(d) = D$, a contradiction.

Either way we get a contradiction, so there can be no such bijection f. $\square$

Some equinumerosities for uncountable sets

The following sets all have the same cardinality:

- $\mathbb{R}$;
- Any nondegenerate interval (a, b), $[a, b]$, $(a, b]$, or $[b, a)$;
- $\mathcal{P}(\mathbb{N})$.

Cardinalities of infinite sets

- Sets are linearly ordered by cardinality: for two sets A and B, either $|A| < |B|$, $|A| = |B|$, or $|A| > |B|$.
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- We use $\aleph_0$ (the Hebrew letter aleph) for the smallest cardinality of an infinite set. That is, $\aleph_0 = |\mathbb{N}|$.
- And $\aleph_{n+1}$ is the smallest cardinality $> \aleph_n$. And we can continue this upward transfinitely, beyond just the finite indices.
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- Given a set A, we write $2^{|A|}$ for $|\mathcal{P}(A)|$.

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- You should've gotten the answer $|B|^{|A|}$.
- $2^{|A|}$ is the cardinality of the set of functions from A to a two-element set.
- Think: a subset $X \subseteq A$ is really a function mapping each element of A to either `yes` or `no`.

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- There are extra axioms you can add which do settle the continuum hypothesis, some saying yes and others saying no. But as yet no axiom has been proposed which settles CH and which the experts accept.
- But I won't be able to talk about such in this class, since that is a graduate-level topic.