Mediacy and Independence

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120 Years of Choice
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What does it mean to be infinite?

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- X is finite if $|X| < \omega$. Otherwise X is infinite.
- X is infinite iff $|X| \geq n$ for all $n < \omega$.

This isn't circular, because we can define ω by its induction properties.

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- X is Dedekind-finite if any injection
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- ($\Leftarrow$) Push forward the $+1$ function on ω.
- ($\Rightarrow$) Fix $z \in X \setminus \operatorname{ran} f$. Then the map $n \mapsto f^n(z)$ gives an injection $\omega \to X$.
  - Use fact that f is one-to-one to inductively prove this map is an injection.

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- Infinite implies Dedekind-infinite needs a small fragment of AC.

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Theorem (Cohen 1963)

It is consistent with ZF *that there exists a Dedekind-finite, infinite set.*

But you can't get a reversal: there's no hope the non-existence of a DFI set implies AC because the former is local while the latter is global.

Question

Is there a suitable generalization of a Dedekind-finite, infinite set whose nonexistence gives a characterization of AC?

A look back in history

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- In the late 1910s, Bertrand Russell is a few years after the last volume of *Principia Mathematica*. His time is occupied by legal troubles over his pacifism during World War I and thinking about the foundations of mathematics.

- Working with him are multiple students, including Dorothy Wrinch.

- The next decade (1923) she will publish a paper answering our first question.

Dorothy Wrinch

- Born 1894, died 1976.
- Studied logic under Russell, did her doctorate (1921) under applied mathematician John Nicholson.
- Wrote in a range of subjects: logic, pure mathematics, philosophy of science, and mathematical biology.
- Was awarded a Rockefeller Foundation fellowship to support her work in mathematical biology.
- Early career was in the UK, later emigrated to the USA. Latter years of her career were at Smith College (Mass, USA).
- Had the misfortune of being on the losing side of a scientific dispute with Linus Pauling over the structure of proteins.

Wrinch's question, and mine

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Question

Can we use modern techniques to prove more precise consistency results?

Cardinals sans choice

Notation:

- $\kappa, \lambda, \ldots$ will be used for well-orderable, infinite cardinals.
- $\mathfrak{p}, \mathfrak{q}, \ldots$ will be used for cardinals in general.
- I'll sometimes use $\mathfrak{p}$ to refer to an arbitrary set of cardinality $\mathfrak{p}$.

- Under AC, every cardinal is well-orderable. We can thus define the cardinals as the initial ordinals.
- Without AC we have to fall back on defining cardinals as equivalence classes.
- Can use Scott's trick to make these sets.

Mediate cardinals

Fix a cardinal $\mathfrak{p}$. Then X is p-mediate if

- $\mathfrak{q} \le |X|$ for all $\mathfrak{q} < \mathfrak{p}$;
- $\mathfrak{p} \not\le |X|$; and
- $|X| \not\le \mathfrak{p}$.

A p-mediate cardinal is a cardinal number of a $\mathfrak{p}$-mediate set.

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- Dedekind-finite infinite $\Leftrightarrow \aleph_0$-mediate.
- ZF proves there are no finite degrees of mediacy.

Wrinch's theorem

Theorem (Wrinch 1923)

Over ZF, *the following are equivalent.*

1. AC;
2. *There are no mediate cardinals; and*
3. *There are no κ-mediate cardinals for well-ordered κ.*

(Wrinch originally formulated this result in the framework of *Principia Mathematica*.)

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Prove $(1 \Rightarrow 2)$ by contrapositive.

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- (Hartogs 1915) AC iff Cardinal Trichotomy.

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- (Hartogs) For any $\mathfrak{p}$ there is a smallest well-orderable cardinal $\aleph(\mathfrak{p})$ so that $\aleph(\mathfrak{p}) \nleq \mathfrak{p}$.
- If $\mathfrak{p}$ is not well-orderable then $\mathfrak{p}$ is $\aleph(\mathfrak{p})$-mediate.

Dependent choice

Dependent choice (DC) informally says you can make ω many choices where each choice depends on the previous ones.

- Suppose R is a relation on a set X so that for each $x \in X$ there is $y \in X$ with $x \, R \, y$. Then there is a branch $\langle x_i : i \in \omega \rangle$ through R: for each i have $x_i \, R \, x_{i+1}$.

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DC_κ says:

- Suppose R is a relation on $X^{<\kappa} \times X$ so that for each $s \in X^{<\kappa}$ there is $y \in X$ with $s \, R \, y$. Then there is a branch $b = \langle x_i : i < \kappa \rangle$ through R: for each i have $(b \restriction i) \, R \, b_i$.

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Facts:

- AC is equivalent to $\forall \kappa \, DC_\kappa$.
- $\lambda < \kappa$ implies $DC_\kappa \Rightarrow DC_\lambda$.
- $ZF + DC_{<\kappa} + \neg DC_\kappa$ is consistent.
- DC implies AC_ω over ZF, but not vice versa.
- DC is equivalent to "a relation is well-founded iff it has no infinite descending sequence".
- (Solovay) $ZF + DC +$ "every set of reals is Lebesgue-measurable" is consistent.

DC and mediate cardinals

Can get a level-by-level version of Wrinch's theorem.

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Lemma: DC_κ implies there are no κ-mediates.

Corollary: AC iff for all κ there are no κ-mediates.

- Suppose $\lambda \leq \mathfrak{p}$ for all $\lambda < \kappa$ but $\mathfrak{p} \not\leq \kappa$.
- Consider the collection of all injections $\alpha \to \mathfrak{p}$ for $\alpha < \mathfrak{p}$.
- None of the injections are onto, so you can always extend them to an injection $\alpha + 1 \to \mathfrak{p}$.
- By DC_κ there's a branch, which gives an injection $\kappa \to \mathfrak{p}$.

Refining mediacy

Observation:

- If $\mathfrak{p}$ is κ-mediate and $\lambda > \kappa$ then
  $\mathfrak{p} + \lambda$ is λ^+-mediate.
- So if there is a degree of mediacy
  then every larger successor cardinal is
  also a degree of mediacy.

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Lemma: If $\mathfrak{p}$ is κ-mediate where κ is smallest such that κ-mediates exist, then $\mathfrak{p}$ is exact κ-mediate.

Consistency questions

> **Question**
> - *Consistently, what can be the smallest degree of mediacy?*
> - *Consistently, what can be the class of degrees of exact mediacy?*

Symmetric extensions

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- $\mathbb{P} = \mathrm{Add}(\omega, \omega)$ is the poset. Conditions are finite partial functions $\omega \times \omega \to 2$.

- Changing the order is permuting the columns in the $\omega \times \omega$ grid.

- Any permutation ϖ of ω generates an automorphism of $\mathbb{P}$:
$\varpi p(n, i) = p(\varpi n, i)$.

- Also generates an automorphism on the $\mathbb{P}$-names:
$\varpi \sigma = \{(\varpi \tau, \varpi p) : (\tau, p) \in \sigma\}$

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- "Forgetting the order" is restricting to names fixed by a 'large' group of automorphisms:
  A group H of automorphisms is large if there is finite $e \subseteq \omega$ so that each $\varpi \in H$ fixes e pointwise: $H \supseteq \mathrm{fix}(e)$.

- This gives a normal filter $\mathcal{F}$ on the lattice of subgroups.

- A name σ is $\mathcal{F}$-symmetric if
  $\mathrm{sym}(\sigma) = \{\varpi : \varpi\sigma = \sigma\} \in \mathcal{F}$.

- The symmetric extension consists of the interpretations of all hereditarily symmetric names.

Symmetric extensions, in general

A symmetric system is $(\mathbb{P}, G, \mathcal{F})$ so that

- $\mathbb{P}$ is a forcing poset;
- $G \leq \text{Aut}(\mathbb{P})$; and
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A $\mathbb{P}$-name σ is symmetric if $\operatorname{sym}(\sigma) \in \mathcal{F}$.

- (Symmetry lemma) $p \Vdash \varphi(\sigma)$ iff $\varpi p \Vdash \varphi(\varpi\sigma)$.

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The symmetric extension by $(\mathbb{P}, G, \mathcal{F})$ via a generic $g \subseteq \mathbb{P}$:

- Consists of the interpretations of hereditarily symmetric names.
- $\mathrm{V}[g/\mathcal{F}] = \{\sigma^g : \sigma$ is hereditarily symmetric$\}$.

$\mathrm{V}[g/\mathcal{F}] \models \mathsf{ZF}$, but the point is to make AC fail in a controlled way.

The Cohen symmetric extension

Fix regular κ and assume $\kappa^{<\kappa} = \kappa$.

- $\mathbb{P}_\kappa = \operatorname{Add}(\kappa, \kappa)$;
- $G_\kappa \leq \operatorname{Aut}(\mathbb{P}_\kappa)$ is generated by permutations of κ;
- $H \in \mathcal{F}_\kappa$ if $\exists e \in [\kappa]^{<\kappa}$ so that $\operatorname{fix}(e) \subseteq H$.

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In $V[g_\kappa / \mathcal{F}_\kappa]$ the set $A = \{c_i : i < \kappa\}$ of Cohen subsets of κ is not well-orderable.

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Facts:

- $\mathbb{P}_\kappa$ is κ-closed and has the κ^+-cc.
- $\mathcal{F}_\kappa$ is κ-complete.

Thus, $(\mathbb{P}_\kappa, G_\kappa, \mathcal{F}_\kappa)$ will preserve $\mathrm{DC}_{<\kappa}$.

In particular, there will be no λ-mediates for $\lambda < \kappa$.

Symmetric extensions and DC

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- Consider appropriate $R \subseteq X^{<\lambda} \times X$ in $\mathrm{V}[g/\mathcal{F}]$. We need a branch through R in $\mathrm{V}[g/\mathcal{F}]$.
- By κ-closure λ remains a cardinal in $\mathrm{V}[g]$.
- In $\mathrm{V}[g]$, by DC_λ there is a branch $b = \langle x_i : i < \lambda \rangle$.
- Each x_i comes from a symmetric name $\dot{x}_i$.
- By κ-completeness $H = \bigwedge_{i<\lambda} \mathrm{sym}(\dot{x}_i)$ is in $\mathcal{F}$.
- Can get a name $\dot{b}$ for b with $\mathrm{sym}(\dot{b}) \supseteq H$.
- So the branch b is in $\mathrm{V}[g/\mathcal{F}]$.

The smallest mediate can be anything

Theorem (W.)

Suppose $\kappa = \kappa^{<\kappa}$ is regular. In the symmetric extension by $(\mathbb{P}_\kappa, G_\kappa, \mathcal{F}_\kappa)$:

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We've already seen $DC_{<\kappa}$ and so there are no λ-mediates for $\lambda < \kappa$.

Claim: Let A be the set of the Cohen subsets of κ added by $\mathbb{P}_\kappa$. Then $V[g/\mathcal{F}_\kappa] \models A$ is κ-mediate.

Like getting DFI set in $(\mathbb{P}_\omega, G_\omega, \mathcal{F}_\omega)$.

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- $\lambda < \kappa$ injects by κ-closure of $\mathbb{P}_\kappa$ and κ-completeness of $\mathcal{F}_\kappa$
- $|A| \nleq \kappa$ because A can't be well-ordered.
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- $\lambda < \kappa$ injects by κ-closure of $\mathbb{P}_\kappa$ and κ-completeness of $\mathcal{F}_\kappa$
- $|A| \not\leq \kappa$ because A can't be well-ordered.
- $\kappa \not\leq |A|$:
  - Suppose $\dot{f}$ is hereditarily symmetric, $\text{sym}(f) \supseteq \text{fix}(e)$, and $p \Vdash \dot{f} : \kappa \to A$ is one-to-one.
  - Extend p to q deciding $\dot{f}(\alpha) = c_i$ for some $\alpha \neq i$ both $\notin e$.
  - Find ϖ fixing $e \cup \{i\}$, moving α, and $q \parallel \varpi q$.
  - So $q \cup \varpi q \Vdash \dot{f}$ is not one-to-one. Contradiction.

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Work in $V[g]$:

- Consider the tree of hereditarily symmetric names for injections $\alpha \to X$ for $\alpha < \lambda$.
- Lemma implies the tree has a branch.
- But why should the branch be in $V[g/\mathcal{F}_\kappa]$?
- Branch has size $\lambda > \kappa$ and $|\mathcal{F}_\kappa| = \kappa$, so λ many names $\dot{f}_\alpha$ on the branch have the same $\text{sym}(\dot{f}_\alpha)$.
- Can build a branch b so every injection on branch has same $\text{sym}(\dot{f}_\alpha)$.
- Then b has a hereditarily symmetric name.

Thus $V[g/\mathcal{F}_\kappa] \models \lambda \leq |X|$. Contradiction.

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- Karagila has a framework for iterations of symmetric extensions.
- It's complicated, with scary group theoretic objects like wreath products.
- We are lucky and can get away with products, where the details are significantly less technical.

Products of symmetric extensions

Suppose $(\mathbb{P}, G, \mathcal{F})$ and $(\mathbb{Q}, H, \mathcal{E})$ are symmetric systems. Can define their product $(\mathbb{P}, G, \mathcal{F}) \times (\mathbb{Q}, H, \mathcal{E})$:

- $\mathbb{P} \times \mathbb{Q}$ is usual product of posets;
- $G \times H$ is generated by (ϖ, ϱ) with $\varpi \in G$, $\varrho \in H$; and
- $\mathcal{F} \times \mathcal{E}$ is generated by $G_0 \times H_0$ for $G_0 \in \mathcal{F}$ and $H_0 \in \mathcal{E}$.

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Like with forcing, we have a product lemma stating that the extension by the product is the same as the two-step extensions, in either order.

Can also do this for infinite products, with a notion of support.

- Suppose $(\mathbb{P}_\kappa, G_\kappa, \mathcal{F}_\kappa)$ are symmetric systems for $\kappa \in M$.
- Then there is a product $\prod_{\kappa \in M}(\mathbb{P}_\kappa, G_\kappa, \mathcal{F}_\kappa)$ with that support.
- Again we get a product lemma stating we can split the full extension into two-step extensions.

Refining earlier ideas

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Suppose $\lambda < \kappa$ are regular.

- $(\mathsf{ZF} + \mathsf{DC}_\kappa)$ If $\mathbb{P}$ is κ-closed and $\mathcal{F}$ is κ-complete then $(\mathbb{P}, G, \mathcal{F})$ preserves DC_λ.
- $(\mathsf{ZF} + \mathsf{DC}_\kappa)$ Suppose $\mathbb{P}$ has the λ^+-cc and $\mathcal{F}$ is generated by a basis of size $\leq \lambda$. Then $\mathrm{V}[g/\mathcal{F}] \models$ there are no exact κ-mediates.

The pattern of the exact mediates

Theorem (W.)

Assume GCH *and fix a class M of regular cardinals. Do the Easton support product of the* $(\mathbb{P}_\kappa, G_\kappa, \mathcal{F}_\kappa)$ *for* $\kappa \in M$. *In the symmetric extension, M is exactly the class of regular degrees of exact mediacy.*

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Sketch:

- $\mathbb{P}_{>\alpha}$ is α-closed and $\mathcal{F}_{>\alpha}$ is α-complete.
- $\mathbb{P}_{<\alpha}$ has the α^+-cc and $\mathcal{F}_{<\alpha}$ is generated by a basis of cardinality $\leq \alpha$.
- In $V[g_{>\alpha}/\mathcal{F}_{>\alpha}]$: DC_α is true. So there are no α-mediates.
- In $V[g_{>\alpha}/\mathcal{F}_{>\alpha}][g_{<\alpha}/\mathcal{F}_{<\alpha}]$: there are no exact α-mediates.
- So the only way there could be an exact α-mediate is if it was added by $(\mathbb{P}_\alpha, G_\alpha, \mathcal{F}_\alpha)$ for $\alpha \in M$.
- But we already know that adds an exact mediate.

Open questions

- What's up with singular cardinals?
- What if we don't make such strong cardinal arithmetic assumptions?
- What happens if AC fails badly in the ground model?
- Is it consistent to have DC_κ fail but there be no κ-mediates?

Thank you!

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