Inner Mantles and Iterated HOD

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Set-theoretic geology

Usually, set theorists think of forcing from out an outward point of view. In geology, we reverse that perspective and look inward.

Definition

$W \subseteq \mathrm{V}$ is a ground if there is $\mathbb{P} \in W$ and $G \in \mathrm{V}$ a $\mathbb{P}$-generic over W so that $\mathrm{V} = W[G]$.

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Theorem (Laver, Woodin)

The grounds are uniformly first-order definable.

Grounds

Theorem (Usuba)

ZFC *proves that the grounds are strongly downward directed: If* $\{W_r : r \in I\}$ *is a set-sized collection of grounds, then there is a ground*

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The Bedrock Axiom asserts that the only ground is V itself.

Theorem (Reitz)

There is a class forcing notion which forces the Bedrock Axiom.

The Mantle

Definition

The mantle M is the intersection of the grounds.

The Bedrock Axiom can be equivalently phrased $\mathrm{V} = \mathrm{M}$.

Theorem (Fuchs–Hamkins–Reitz, Usuba)

The mantle satisfies ZFC*, is preserved by set forcing, and is the largest set forcing-invariant inner model.*

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The mantle satisfies ZFC*, is preserved by set forcing, and is the largest set forcing-invariant inner model.*

Theorem (Fuchs–Hamkins–Reitz)

There is a class forcing notion which forces V to be the mantle of the forcing extension.

Inner Mantles

Observation

Consistently, $\mathrm{M}^{\mathrm{M}} \neq \mathrm{M}$.

Proof.

Force $\mathrm{V} = \mathrm{M}$. Then use the Fuchs–Hamkins–Reitz forcing to get $\mathrm{V}[G]$ with $\mathrm{V} = \mathrm{M}^{\mathrm{V}[G]}$ and so $(\mathrm{M}^{\mathrm{M}})^{\mathrm{V}[G]} \neq \mathrm{M}^{\mathrm{V}[G]}$. □

Compare: consistently $\mathrm{HOD}^{\mathrm{HOD}} \neq \mathrm{HOD}$.

Inner Mantles

Definition

The sequence of inner mantles M^η is defined as follows.

- $\mathrm{M}^0 = \mathrm{V}$.
- $\mathrm{M}^{\eta+1} = \mathrm{M}^{\mathrm{M}^\eta}$.
- $\mathrm{M}^\lambda = \bigcap_{\eta<\lambda} \mathrm{M}^\eta$, for limit λ.

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Theorem (McAloon)

There is a model of ZFC so that HOD^ω is not a definable class.

Question

Can a similar result be proved for M^ω?

Inner Mantles

Fuchs, Hamkins, and Reitz asked: can we force V to be the η-th inner mantle of a forcing extension?
Compare:

Theorem (Zadrożny)

For each ordinal η or $\eta = \mathrm{Ord}$ there is a class forcing extension $\mathrm{V}[G]$ of V so that $\mathrm{V} = (\mathrm{HOD}^\eta)^{\mathrm{V}[G]}$.

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Reitz and I answered the question affirmatively.

Warmup: forcing $\mathrm{V} = \mathrm{M}^{\mathrm{V}[G]}$

For ease of presentation, I will assume GCH. The arguments can be made without this assumption, but choosing the coding points requires more care.

Let $\mathbb{P}$ the the product of $\mathrm{Add}(\alpha, \alpha^{++}) \oplus \mathbf{0}$ for regular cardinals α, with set support.

Theorem (Fuchs–Hamkins–Reitz)

The forcing $\mathbb{P}$ preserves ZFC *and if $G \subseteq \mathbb{P}$ is generic over V then $\mathrm{V} = \mathrm{M}^{\mathrm{V}[G]} = \mathrm{HOD}^{\mathrm{V}[G]}$.*

The forcing $\mathbb{M}(\eta)$

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- R is the class of regular cardinals $> \eta^+$. Partion R into η many cofinal classes:
- R_i consists of the elements of R whose index is equivalent to i modulo η.
- For $\alpha \in R$ the index $i(\alpha)$ of α is the unique $i < \eta$ so $\alpha \in R_i$.
- $R_{>i} = \bigcup_{j>i} R_j$.
- $R_{\geq i} = \bigcup_{j \geq i} R_j$.

The forcing $\mathbb{M}(\eta)$

Conditions in $\mathbb{M}(\eta)$ are set-sized functions p with $\operatorname{dom} p$ an initial segment of R. For each $\alpha \in \operatorname{dom} p$ we have $p(\alpha)$ is an $\mathbb{M}(\eta) \upharpoonright (R_{>i(\alpha)} \cap \alpha)$-name for a condition in $\mathrm{Add}(\alpha, \alpha^{++}) \oplus \mathbf{0}$. The support of p is arbitrary.

For $p, q \in \mathbb{M}(\eta)$, say $q \leq p$ if $\operatorname{dom} q \supseteq \operatorname{dom} p$ and for each $\alpha \in \operatorname{dom} p$ we have $p \upharpoonright (R_{>i(\alpha)} \cap \alpha)$ forces over $\mathbb{M}(\eta) \upharpoonright (R_{>i(\alpha)} \cap \alpha)$ that $q(\alpha) \leq p(\alpha)$.

Properties of $\mathbb{M}(\eta)$

Lemma

- $\mathbb{M}(\eta)$ *is a progressively distributive iteration and thus preserves* ZFC. *That is, for arbitrarily large* κ *we can factor* $\mathbb{M}(\eta)$ *as* $\mathbb{Q}_\kappa * \dot{\mathbb{Q}}^{\mathrm{tail}}$ *where* $\mathbb{Q}_\kappa$ *is a set and* $\mathbb{Q}_\kappa \Vdash \dot{\mathbb{Q}}^{\mathrm{tail}}$ *is* $<\kappa$*-distributive.*
- $\mathbb{M}(\eta)$ *is* $\leq\eta^+$*-closed.*
- *Forcing with* $\mathbb{M}(\eta)$ *preserves R.*

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- *$\mathbb{M}(\eta)$ is $\leq\eta^+$-closed.*
- *Forcing with $\mathbb{M}(\eta)$ preserves R.*

Analogous facts hold for $\mathbb{M}(\eta) \restriction R_{\geq i}$ for each $i \leq \eta$.

Forcing V to be an inner mantle

Theorem (Reitz–W.)

Let $G \subseteq \mathbb{M}(\eta)$ be generic over V. Then $\mathrm{V} = (\mathrm{M}^\eta)^{V[G]} = (\mathrm{HOD}^\eta)^{\mathrm{V}[G]}$.

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Set $\mathbb{P} = \mathbb{M}(\eta)$ and $\mathbb{P}_i = \mathbb{M}(\eta) \upharpoonright R_{\geq i}$. Then $\mathbb{P}_i$ canonically embeds into $\mathbb{P}$ giving

$$\mathbb{P} = \mathbb{P}_0 \supseteq \mathbb{P}_1 \supseteq \cdots \supseteq \mathbb{P}_i \supseteq \cdots \qquad i < \eta$$

a descending chain of complete subposets.

Set $G_i = G \cap \mathbb{P}_i$.

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Set $G_i = G \cap \mathbb{P}_i$.

Claim

For each $i \leq \eta$ we have $(\mathrm{M}^i)^{\mathrm{V}[G]} = (\mathrm{HOD}^i)^{\mathrm{V}[G]} = V[G_i]$.

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The successor case is essentially the Fuchs–Hamkins–Reitz argument.

The limit case goes through a technical lemma, a variant of a result due to Jech about continuous descending sequences of complete boolean subalgebras.

The technical lemma

Lemma

i is a limit ordinal, $\mathbb{P}$ is a $<i^+$-closed pretame class forcing notion, and

$$\mathbb{P} = \mathbb{P}_0 \supseteq \mathbb{P}_1 \supseteq \cdots \supseteq \mathbb{P}_j \supseteq \cdots \supseteq \mathbb{P}_i$$

is a continuous descending sequence of complete suborders, coded as a single class. Further suppose that $\mathbb{P}$ is a progressively distributive iteration, factoring as $\mathbb{Q}_\kappa * \dot{\mathbb{Q}}^{\mathrm{tail}}$ for arbitrary large κ. Even further suppose $\mathbb{P}_j \cap \mathbb{Q}_\kappa$ is a complete suborder of $\mathbb{P}_j$ for each j, and the intersections form a continuous descending sequence of complete suborders:

$$(\mathbb{P} \cap \mathbb{Q}_\kappa) = (\mathbb{P}_0 \cap \mathbb{Q}_\kappa) \supseteq (\mathbb{P}_1 \cap \mathbb{Q}_\kappa) \supseteq \cdots \supseteq (\mathbb{P}_j \cap \mathbb{Q}_\kappa) \supseteq \cdots \supseteq (\mathbb{P}_i \cap \mathbb{Q}_\kappa).$$

Then, if $G \subseteq \mathbb{P}$ is generic over V and $G_j = G \cap \mathbb{P}_j$, then for any set of ordinals $X \in \mathrm{V}[G]$ we have $X \in \bigcap_{j<i} \mathrm{V}[G_j]$ if and only if $X \in \mathrm{V}[G_i]$.

Separating the inner mantles from the iterated HODs

Theorem (Reitz–W.)

Let ζ and η be ordinals. There are class forcings $\mathbb{A}$ and $\mathbb{B}$ uniformly definable in ζ and η so that:

- *Forcing with $\mathbb{A}$ gives a model where the sequence of iterated HODs has length exactly ζ and the sequence of inner mantles has length exactly $\zeta + \eta$.*
- *Forcing with $\mathbb{B}$ gives a model where the sequences of inner mantels has length exactly ζ and the sequence of iterated HODs has length exactly $\zeta + \eta$.*

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- *Forcing with $\mathbb{A}$ gives a model where the sequence of iterated HODs has length exactly ζ and the sequence of inner mantles has length exactly $\zeta + \eta$.*
- *Forcing with $\mathbb{B}$ gives a model where the sequences of inner mantels has length exactly ζ and the sequence of iterated HODs has length exactly $\zeta + \eta$.*

Fuchs–Hamkins–Reitz had forcings for separating the mantle and HOD. We modify their constructions similar to the definition of $\mathbb{M}(\eta)$.

Some open questions

The forcings $\mathbb{A}$ and $\mathbb{B}$ separate the sequences of inner mantles and iterated HODs. But they each make one sequence an initial segment of the other.

How independent are the two sequences?

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How independent are the two sequences?

Question

η an ordinal. Can we force V to be the η-th inner mantle and η-th iterated HOD of the extension, but $\mathrm{M}^i \neq \mathrm{HOD}^i$ for each $0 < i < \eta$?

Question

η an ordinal. Can we force that $\mathrm{M}^i = \mathrm{HOD}^{2i}$ for each $i < \eta$? What about getting $\mathrm{M}^{2i} = \mathrm{HOD}^i$ for each $i < \eta$?

Thank you!