Math 321: Relations and functions

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These functions and relations are themselves objects of mathematical study.

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The standard in mathematics is to take the extension of a relation as *the* definition of relations.

- A binary relation on a set A is a subset of A², the set of ordered pairs from A.
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(You can talk about relations between more than two objects, but binary relations are used the most.)

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- mod 3 on Z (that is, a and b have the same remainder when divided by 3)
 This is called equivalence modulo 3.

Some properties a relation can have

Let \star be a binary relation on A.

- \star is reflexive if $a \star a$ for all $a \in A$.
- \star is symmetric if $a \star b$ implies $b \star a$ for all $a, b \in A$.
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- (RST) We checked earlier that = on \mathbb{R} has all three properties.
- (RT) We checked earlier that \leq on \mathbb{R} is reflexive and transitive but not symmetric.

- ullet (T) We checked earlier that < on $\mathbb R$ is transitive but neither reflexive nor symmetric.
- (S) We checked earlier that \neq on \mathbb{R} is symmetric but neither reflexive nor transitive.

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We have examples for all eight cases, completing the proof of the theorem.

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Let \star be a relation on a set A. We can add new instances to \star to make it satisfy these properties.

- The reflexive closure of * is the smallest reflexive relation on A which contains *, i.e. as a subset.
- The symmetric closure of ★ is the smallest symmetric relation on A which contains ★.
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- What is the reflexive closure of the empty relation on R?

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Instead, we have to close off † in a recursive process with infinitely many steps.

- Start with $\dagger_0 = \dagger$.
- Given \dagger_n define \dagger_{n+1} as:

$$\dagger_{n+1} = \dagger_n \cup \{(a,c) \in A^2 : a \dagger_n b \dagger_n c$$
 for some $b \in A\}$.

Another way to define closures

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• Then the transitive closure of † is

$$\bar{\dagger} = \bigcup_{n=0}^{\infty} \dagger_n \\
= \dagger_0 \cup \dagger_1 \cup \dagger_2 \cup \cdots$$

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- The rearrangement relation on finite lists of integers: two lists are equivalent if they are rearrangements of each other.
 This is the equivalence relation used in the statement of the fundamental theorem of arithmetic: when we proved that any two prime factorizations of *n* must be the same, what we meant is that the two lists were related in this way.

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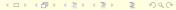
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Notation: Mathematicians write A/\sim for the family of \sim -equivalence classes.

