EXPERIMENT 9

RLC Step Response

9.1 Application

Parallel and series RLC circuits finds their use in many filter and oscillator designs. The resonant tank formed by an inductor and capacitor is also used widely in analog radios. A tunable capacitor in tandem with an inductor can be used to adjust the resonant frequency of a circuit to the desired radio station. RLC circuits are also used to model many non-idealities of wires and circuit board traces, so analysis of RLC circuit behavior can be used to identify potential non-ideal behavior in high frequency digital and analog systems.

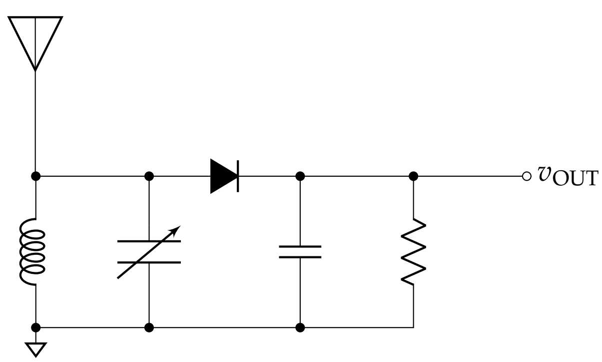

Figure 9.1: A simple AM radio detector using a resonant LC circuit

9.2 Second order systems

Resistor-Inductor-Capacitor (RLC) circuits are linear second order systems because they are fully described by linear second order differential equations. To successfully study these circuits, one must find the solution to the generalized second order system equation shown in equation (9.1). Note that $f(t)$ is the forcing function and is the input to the system. $x(t)$ is the output and is typically a current or voltage signal.

$$\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + b\frac{\mathrm{d}x(t)}{\mathrm{d}t} + cx(t) = f(t) \tag{9.1}$$

9.2.1 Unforced solution (homogeneous)

We begin our analysis with the homogeneous case, $f(t) = 0$. In this case, the system is un-forced and becomes

$$\frac{\mathrm{d}^2 x_0(t)}{\mathrm{d}t^2} + b\frac{\mathrm{d}x_0(t)}{\mathrm{d}t} + cx_0(t) = 0. \tag{9.2}$$

The solution, $x_0(t)$, takes the form Ke^{st} where K is a constant. It may not be immediately obvious that the solution is an exponential. It may not be immediately obvious, but we can see that it is possible to satisfy equation (9.2) with a solution that has a derivative equal to a constant multiple of itself.[(1)]

[(1)] A course or book on differential equations will have a much more thorough and rigorous treatment of this solution, but this approach works adequately for the circuit applications we are concerned with.

To find the specific solution, plug the general solution into equation (9.2) and solve.

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(Ke^{st}\right) + b\frac{\mathrm{d}}{\mathrm{d}t}\left(Ke^{st}\right) + ce^{st} &= 0\\
s^2Ke^{st} + bsKe^{st} + cKe^{st} &= 0\\
Ke^{st}\left(s^2 + bs + c\right) &= 0
\end{aligned}$$

Ignoring the trivial solution ($K = 0$), we can deduce that

$$s^2 + bs + c = 0 \tag{9.3}$$

must be true. Equation (9.3) is known as the characteristic equation and has two solutions described by the quadratic formula:

$$s = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

Therefore, a system described by equation (9.2) has two solutions

$$x_{0,1}(t) = K_1e^{s_1t} \qquad \text{and} \qquad x_{0,2}(t) = K_2e^{s_2t}$$

where $x_{0,i}(t)$ is the solution corresponding to the i-th root

$$s_1 = \frac{-b + \sqrt{b^2 - 4c}}{2} \qquad \text{and} \qquad s_2 = \frac{-b - \sqrt{b^2 - 4c}}{2}$$

This differential equation is linear, so the overall solution is the sum of the two individual ones. More generally, the overall answer space is spanned by the two.

In summary, the system's homogeneous solution, $x_0(t)$, is given by

$$x_0(t) = x_{0,1}(t) + x_{0,2}(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

where K_1 and K_2 are determined by the system's initial conditions.

9.2.2 Forced solution (nonhomogeneous)

For the purposes of this experiment, we limit our attention to nonhomogeneous case with *constant* force, $f(t) = F$. In this case, the system becomes

$$\frac{\mathrm{d}^2 x_0(t)}{\mathrm{d}t^2} + b\frac{\mathrm{d}x_0(t)}{\mathrm{d}t} + c x_0(t) = F. \tag{9.4}$$

The homogeneous solution, $x_0(t)$, no longer holds because the right hand side of the system has changed.[2] We must try a new form. Since the system is forced with a constant, let us try a solution that is the same but with a constant offset, G.

[2] This can be verified by plugging $x_0(t)$ into equation (9.4) and identifying the disagreement.

$$x_F(t) = x_0(t) + G = K_1 e^{s_1 t} + K_2 e^{s_2 t} + G \tag{9.5}$$

Substituting equation (9.5) into equation (9.4), we find that

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}t^2}(x_0(t) + G) + b\frac{\mathrm{d}}{\mathrm{d}t}(x_0(t) + G) + c(x_0(t) + G) &= F \\
\frac{\mathrm{d}^2 x_0(t)}{\mathrm{d}t^2} + \cancelto{0}{\frac{\mathrm{d}^2 G}{\mathrm{d}t^2}} + b\frac{\mathrm{d}x_0(t)}{\mathrm{d}t} + b\cancelto{0}{\frac{\mathrm{d}G}{\mathrm{d}t}} + c x_0(t) + cG &= F \\
\cancelto{0}{\frac{\mathrm{d}^2 x_0(t)}{\mathrm{d}t^2} + b\frac{\mathrm{d}x_0(t)}{\mathrm{d}t} + c x_0(t)} + cG &= F \\
cG &= F \\
G &= \frac{F}{c}
\end{aligned}$$

Therefore, the nonhomogeneous constant force solution, sometimes called the step response, is given by equation (9.6)

$$x_F(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} + \frac{F}{c} \tag{9.6}$$

where K_1 and K_2 are determined by the system's initial conditions and s_1 and s_2 are the same as the homogeneous case.

Constant forcing generalizes the homogeneous solution

For this experiment the constant forcing response of equation (9.6) is sufficient because we only consider *constant* input systems.(3) The homogeneous solution is just a special case of the nonhomogeneous one, where $F = 0$. That said, if a system were to have a forcing function that is not constant, then the solution changes.

(3) Even if the constant input is sometimes just zero.

9.2.3 Response types

The homogeneous portion of the response is defined by two roots, s_1 and s_2. Depending on the differential equation's coefficients, they can be either real or complex. This leads to three classes of solutions: overdamped (real and distinct roots), critically damped (real and repeated roots), and underdamped (complex roots).

Overdamped (real and distinct roots)

The overdamped case occurs when both of the roots of the homogeneous solution are real and distinct from each other. That is,

$$b^2 - 4c > 0.$$

The solution, $x_F(t)$, is identical to equation (9.6)

$$x_F(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} + \frac{F}{c} \tag{9.7}$$

where $s_1 = \frac{-b+\sqrt{b^2-4c}}{2}$ and $s_1 = \frac{-b-\sqrt{b^2-4c}}{2}$. K_1 and K_2 are found by equating the initial conditions $x(0)$ and $x'(0)$ with $x_F(t)$ and its derivative at $t = 0$, respectively. The resulting relationships are

$$x(0) = K_1 + K_2 + \frac{F}{c}$$
$$x'(0) = s_1 K_1 + s_2 K_2$$

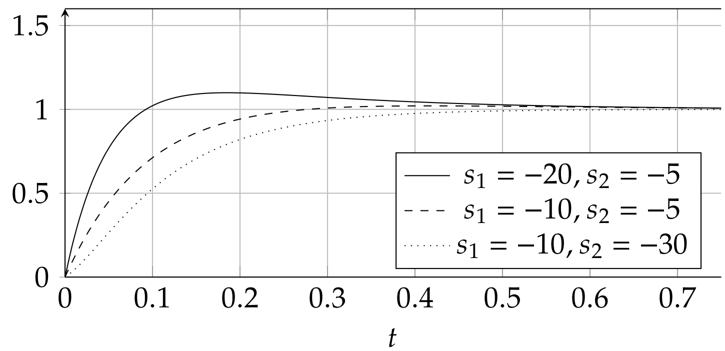

Figure 9.2: Normalized overdamped responses for constant input and initial conditions of $x(0) = 0$. In other words, the RLC overdamped step response.

Example overdamped responses for constant inputs can be found in figure 9.2. Notice how the responses smoothly rise to the final value with no oscillations with a speed based on the exponent. Interestingly, even though the system damps the response, it is still possible to overshoot the final value before settling down.

Critically damped (repeated real roots)

The critically damped case occurs when both roots are real and equal to each other. That is,

$$b^2 - 4c = 0.$$

The coincident roots causes a small hiccup in applying equation (9.6) directly. Repeated roots actually have a second solution of the form form Kte^{st}. Therefore, the critically damped response is

$$x_F(t) = (K_1 + tK_2)e^{st} + \frac{F}{c}, \tag{9.8}$$

where $s = -\frac{b}{2}$. K_1 and K_2 are found by equating the initial conditions $x(0)$ and $x'(0)$ with $x_F(t)$ and its derivative at $t = 0$, respectively. This resulting relationships are

$$\begin{aligned} x(0) &= K_1 + \frac{F}{c} \\ x'(0) &= sK_1 + K_2 \end{aligned}$$

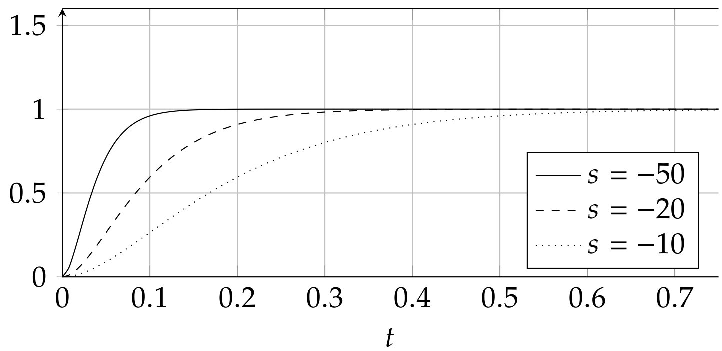

Figure 9.3: Normalized critically damped responses for constant input and initial conditions $x(0) = 0$ and $x'(0) = 0$. In other words, the RLC critically damped step response.

Examples of critically damped responses given constant input can be found in figure 9.3. The critically damped point is directly between the over and underdamped cases. At this point, the response is as fast as possible without oscillation. Practically, a system can not be built to consistently operate in the critically damped region. Component tolerances and random noise will push the response into the over or underdamped regions.

Underdamped (complex roots)

The underdamped case occurs when the roots are complex(4) conjugate roots. That is,

(4) To avoid confusion with current, we use $j = \sqrt{-1}$ rather than i.

$$b^2 - 4c < 0.$$

In this case the roots are

$$s_1 = \frac{-b + j\sqrt{4c - b^2}}{2} \qquad s_2 = \frac{-b - j\sqrt{4c - b^2}}{2}$$

For future convenience, we separate the roots into their real and imaginary parts such that

$$s = -\frac{b}{2} \pm j\frac{\sqrt{4c - b^2}}{2} = -\sigma \pm j\omega_d$$

Where σ and ω_d can be defined:

$$\sigma = \frac{b}{2} \qquad \omega_d = \frac{\sqrt{4c - b^2}}{2}$$

The solution, $x_F(t)$, is found by plugging the roots into equation (9.6)

$$\begin{aligned} x_0(t) &= K_1 e^{(-\sigma+j\omega_d)t} + K_2 e^{(-\sigma-j\omega_d)t} + \frac{F}{c} \\ &= e^{-\sigma t}\left(K_1 e^{j\omega_d t} + K_2 e^{-j\omega_d t}\right) + \frac{F}{c} \\ &= e^{-\sigma t}(A\cos\omega_d t + B\sin\omega_d t) + \frac{F}{c} \end{aligned}$$

where the final equality is by Euler's formula.[5] The coefficients are $A = K_1 + K_2$ and $B = j(K_1 - K_2)$.

[5] $e^{j\theta} = \cos(\theta) + j\sin(\theta)$

In our study of second order systems, we are modeling the response of voltages and currents, so it may surprise you that B involves an imaginary number. All is well, it just means that real RLC circuits have a K_1 and K_2 such that A and B end up real. Following this line of thinking, we will no longer attempt to find K_1 and K_2. Instead, we will focus on the real valued A and B. With that, the solution has two convenient forms:

$$\begin{aligned} x_F(t) &= e^{-\sigma t}(A\cos\omega_d t + B\sin\omega_d t) + \frac{F}{c} \\ &= Re^{-\sigma t}\cos\big(\omega_d t - \phi\big) + \frac{F}{c} \end{aligned} \tag{9.9}$$

where $R = \sqrt{A^2 + B^2}$ and $\phi = \arctan(B/A)$. A and B are found by equating the initial conditions $x(0)$ and $x'(0)$ with $x_F(t)$ and its derivative at $t = 0$, respectively. The resulting relationships are

$$\begin{aligned} x(0) &= A + \frac{F}{c} \\ x'(0) &= B\omega_d - A\sigma \end{aligned}$$

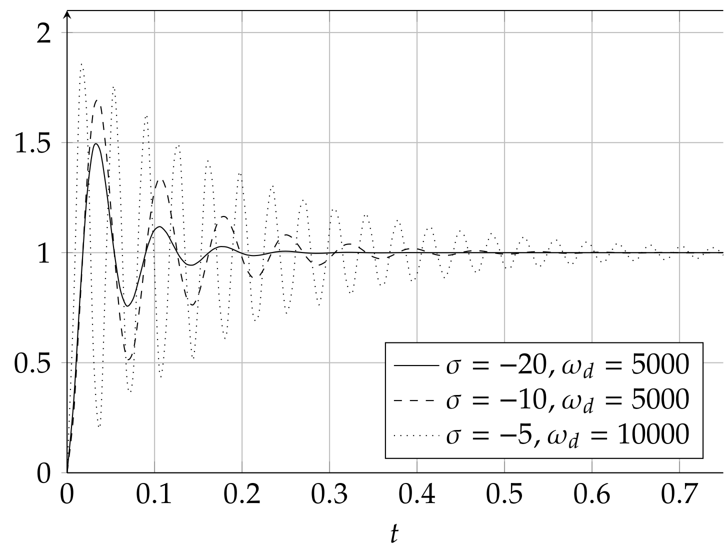

Figure 9.4: Normalized underdamped responses for constant input and initial conditions $x(0) = 0$ and $x'(0) = 0$. In other words, the RLC underdamped step response.

We can see from the simplified version of equation (9.9) that we expect the response to be sinusoidal with decaying amplitude. Figure 9.4 presents some example constant input and underdamped responses. The underdamped response oscillates around the final value and can significantly overshoot the final value. However, it has the fastest rise time of all three cases.

9.3 Second order systems for circuits

The system equations for an RLC circuit can be found using Kirchhoff's laws and the fundamental component equations. The system responses are found by determining which case it is and applying equations (9.7) to (9.9) as needed.

Examples 9.3.1 and 9.3.2 will demonstrate computing a second order circuit's system equation as well as predicting the responses.

Example 9.3.1: Solving a parallel RLC circuit in a underdamped response

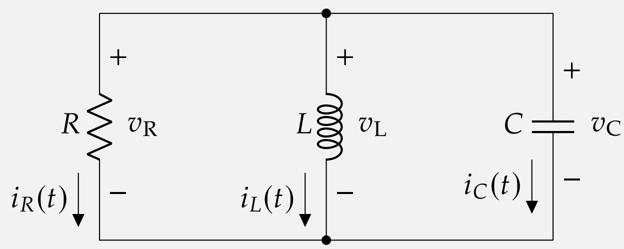

Figure 9.5: A parallel RLC circuit.

Determine the voltage, $v_c(t)$. Let $L = 10\,\mu\text{H}$, $C = 40\,\mu\text{F}$, and $R = 20\,\Omega$. The initial conditions are $v_c(0) = 10\,\text{V}$ and $i_C(0) = 0\,\text{A}$.

Find the system equation We begin by finding the system equation. Notice that all of the components in figure 9.5 are in parallel, so

$$v(t) = v_R(t) = v_L(t) = v_C(t)$$

Using Kirchhoff's current law, we have

$$\begin{aligned} i_R(t) + i_L(t) + i_C(t) &= 0 \\ \frac{v(t)}{R} + \frac{1}{L}\int v(t)\,\mathrm{d}t + C\frac{\mathrm{d}v(t)}{\mathrm{d}t} &= 0 \end{aligned}$$

Now we take the derivative of both sides to eliminate the integral.

$$C\frac{\mathrm{d}^2 v(t)}{\mathrm{d}t^2} + \frac{1}{R}\frac{\mathrm{d}v(t)}{\mathrm{d}t} + \frac{1}{L}v(t) = 0$$

Finally, we divide through by C so that it takes the same form as equation (9.1)

$$\frac{\mathrm{d}^2 v(t)}{\mathrm{d}t^2} + \frac{1}{RC}\frac{\mathrm{d}v(t)}{\mathrm{d}t} + \frac{1}{LC}v(t) = 0$$

Determine the response type To determine the type of response (under, over, critically damped), check the value of $b^2 - 4c$

$$\left(\frac{1}{RC}\right)^2 - \frac{4}{LC} = \left(\frac{1}{20\,\Omega \times 40\,\mu\text{F}}\right)^2 - \frac{4}{10\,\mu\text{H} \times 40\,\mu\text{F}} = -9998437500 < 0$$

The value is negative, so this is the underdamped case, and the response follows equation (9.9).

Underdamped parameters From our treatment of the underdamped system, we know that

$$\sigma = \frac{b}{2} = \frac{1}{2RC} = 625$$

$$\omega_d = \frac{\sqrt{4c - b^2}}{2} = \frac{\sqrt{\frac{4}{LC} - \left(\frac{1}{RC}\right)^2}}{2} = 49996.1$$

$$v(0) = 10\,\mathrm{V} = A + \frac{F}{c} = A + \frac{0}{1/LC} \implies A = 10$$

$$v'(0) = \frac{i_c(0)}{C} = 0 = B\omega_d - A\sigma = 49996.1B - 10 \times 625 \implies B = 0.125$$

The response Putting it all together, the expected response is

$$\begin{aligned} v(t) &= e^{-625t}(10\cos 49996.1t + 0.125\sin 49996.1t)\ \mathrm{V} \\ &= 10e^{-625t}\cos(49996.1t - 0.0125)\ \mathrm{V} \end{aligned}$$

In other words, the voltage response is a decaying sinusoid with frequency 7957.1 Hz and 0.72° phase shift, as seen in figure 9.6.

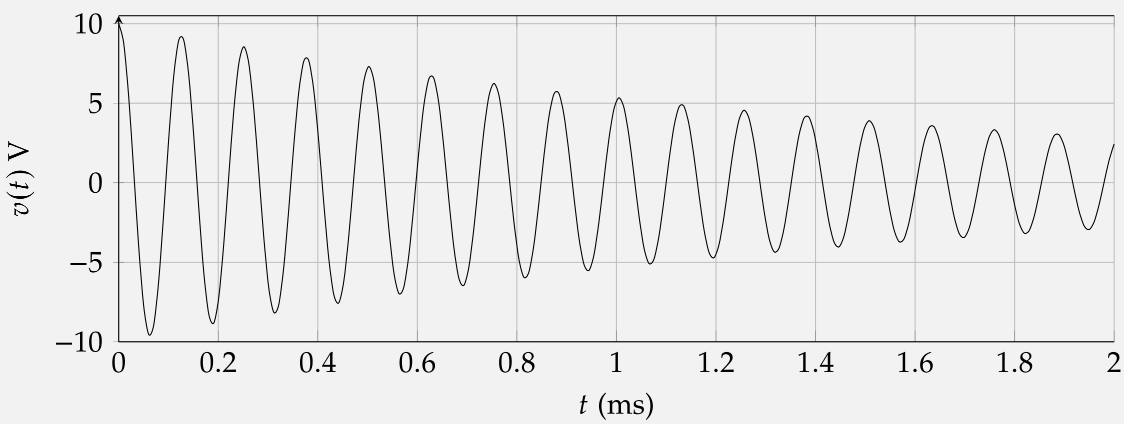

Figure 9.6: Predicted voltage response of the circuit shown in figure 9.5 with $v_c(0) = 10\,\mathrm{V}$ and $i_c(0) = 0\,\mathrm{A}$.

Circuit currents We can compute the circuit currents from $v(t)$.

The resistor's current is simply

$$i_\mathrm{R}(t) = \frac{v(t)}{R} = \frac{1}{2}e^{-625t}\cos(49996.1t - 0.0125)\ \mathrm{A}$$

The capacitor's current can be found with the capacitor equation

$$\begin{aligned} i_C(t) = C\frac{\mathrm{d}v(t)}{\mathrm{d}t} &= C\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{-625t}(10\cos 49996.1t + 0.125\sin 49996.1t)\right] \\ &= e^{-625t}(0\cos 4996.1t - 20.0016\sin 49996.1t) \\ &= 20.0016e^{-625t}\cos(49996.1t + 1.5708)\ \mathrm{A} \end{aligned}$$

Finally, the inductor's current can be found using Kirchhoff's current law

$$\begin{aligned} i_\mathrm{L}(t) &= -i_\mathrm{R}(t) - i_\mathrm{C}(t) = \\ &= -\frac{1}{2}e^{-625t}\cos(49996.1t) + 19.9953e^{-625t}\sin(49996.1t) \\ &= 20.0016e^{-625t}\cos(49996.1t - 1.59580) \end{aligned}$$

Example 9.3.2: Solving a series RLC circuit in an overdamped response

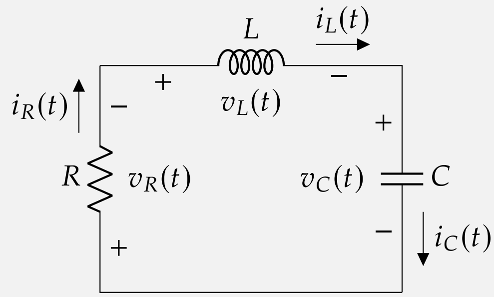

Figure 9.7: A series RLC circuit.

Determine the voltage, $v_\mathrm{C}(t)$. Let $L = 20\,\mu\mathrm{H}$, $C = 10\,\mu\mathrm{F}$, and $R = 100\,\Omega$. The initial conditions are $i_\mathrm{C}(0) = 0.25\,\mathrm{A}$ and $v_\mathrm{L}(0) = 0\,\mathrm{V}$.

Find the system equation We begin by finding the system equation. Notice that all of the components in figure 9.7 are in series, therefore

$$i(t) = i_\mathrm{R}(t) = i_\mathrm{L}(t) = i_\mathrm{C}(t)$$

Using Kirchhoff's voltage law, we have

$$\begin{aligned} v_\mathrm{R}(t) + v_\mathrm{L}(t) + v_\mathrm{C}(t) &= 0 \\ Ri(t) + L\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \frac{1}{C}\int i(t)\,\mathrm{d}t &= 0 \end{aligned}$$

Now we take the derivative of both sides to eliminate the integral.

$$L\frac{\mathrm{d}^2 i(t)}{\mathrm{d}t^2} + R\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \frac{1}{C}i(t) = 0$$

Finally, we divide through by L so that it takes the same form as equation (9.1)

$$\frac{\mathrm{d}^2 i(t)}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \frac{1}{LC}i(t) = 0$$

Determine the response type To determine the type of response (under, over, critically damped), check the value of $b^2 - 4c$

$$\left(\frac{R}{L}\right)^2 - \frac{4}{LC} = \left(\frac{100\,\Omega}{20\,\mu\mathrm{H}}\right)^2 - \frac{4}{20\,\mu\mathrm{H} \times 10\,\mu\mathrm{F}} = 2.498 \times 10^{13} > 0$$

The value is positive, so this is the overdamped case and the response follows equation (9.7).

Overdamped parameters From our treatment of the overdamped system, we know that

$$s_1 = \frac{-b + \sqrt{b^2 - 4c}}{2} = \frac{-\frac{R}{L} + \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}}{2} = \frac{-\frac{100}{20\,\mu\mathrm{H}} + \sqrt{\left(\frac{100\,\Omega}{20\,\mu\mathrm{H}}\right)^2 - \frac{4}{20\,\mu\mathrm{H}\times 10\,\mu\mathrm{F}}}}{2} = -1000.2$$

$$s_2 = \frac{-b - \sqrt{b^2 - 4c}}{2} = \frac{-\frac{R}{L} - \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}}{2} = \frac{-\frac{100}{20\,\mu\mathrm{H}} - \sqrt{\left(\frac{100\,\Omega}{20\,\mu\mathrm{H}}\right)^2 - \frac{4}{20\,\mu\mathrm{H}\times 10\,\mu\mathrm{F}}}}{2} = -4.99 \times 10^6$$

$$i(0) = 0.25\,\mathrm{A} = K_1 + K_2 + \frac{F}{1/LC} = K_1 + K_2 + \frac{0}{1/LC} \implies K_1 + K_2 = 0.25\,\mathrm{A}$$

$$i'(0) = \frac{v_\mathrm{L}(0)}{L} = 0\,\mathrm{V} = s_1 K_1 + s_2 K_2 \implies -1000.2K_1 - 4.99 \times 10^6 K_2 = 0\,\mathrm{V}$$

Solving the last two conditions simultaneously, we have

$$K_1 \approx 0.25005 \qquad\qquad K_2 \approx -0.00005$$

The current response Putting it all together, the expected response for the current is

$$i(t) = 0.25005e^{-1000.2t} - 0.00005e^{-4.999\times 10^6 t}\ \mathrm{A}$$

The circuit voltages We can compute the circuit voltages from $i(t)$.

The resistor's voltage is simply

$$v_\mathrm{R}(t) = i_\mathrm{R}(t)R = 25.005e^{-1000.2t} - 0.005e^{-4.99\times 10^6 t}\ \mathrm{V}$$

The inductor's voltage can be found with the inductor equation

$$v_\mathrm{L}(t) = L\frac{\mathrm{d}i(t)}{\mathrm{d}t} = -0.5002e^{-1000.2t} + 0.5002e^{-4.99\times 10^6 t}\ \mathrm{V}$$

Finally, the capacitor voltage can be found using Kirchhoff's voltage law

$$\begin{aligned} v_C(t) &= -v_R(t) - v_L(t) \\ &\approx 25e^{-1000.2t}\ \mathrm{V} \end{aligned}$$

Overall the response is the sum of two decaying exponentials. The current response is shown in figure 9.8.

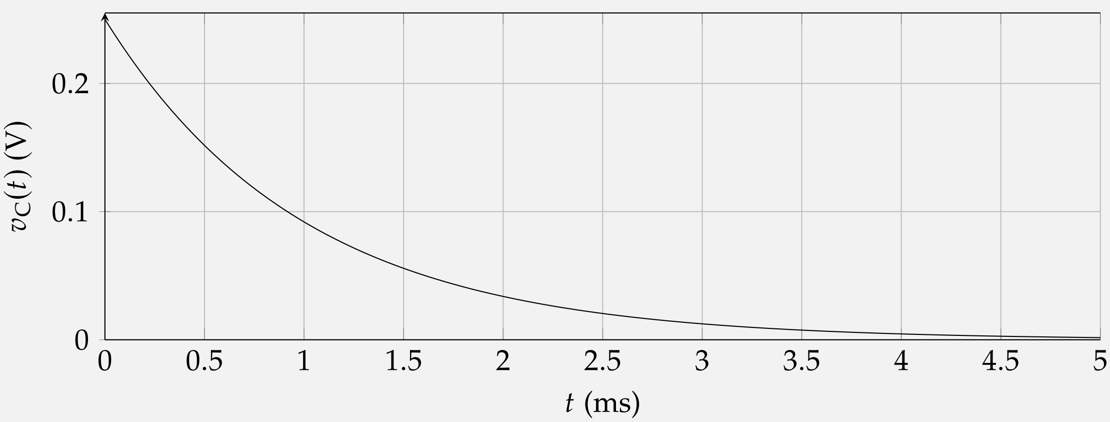

Figure 9.8: Predicted voltage response of the circuit shown in figure 9.7 with $i_C(0) = 0.25$ A and $v_L(0) = 0$ V.

9.3.1 Series and parallel RLC response summary

All second order RLC circuits will either follow the parallel or series response even if the circuit does not at first appear to be parallel or series. In order to identify if a circuit is parallel or series, observe the connection between the inductor and capacitor. Whether or not the capacitor and inductor are in series or parallel defines the operation of the circuit. When in doubt, you can always use Kirchhoff's voltage law (KVL) or Kirchhoff's current law (KCL) to write the differential equation for the circuit and compare it to the series and parallel equations. Additionally, source transforms and Thévenin equivalent circuits can be used to simplify the sources and resistors so that it looks like one of the ideal cases in figure 9.9.

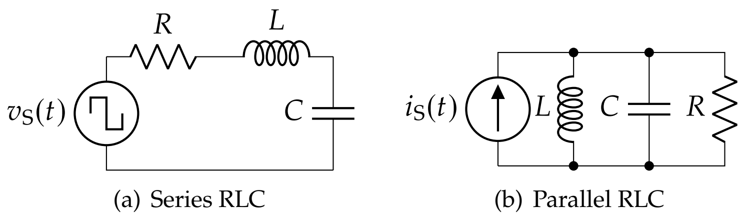

Figure 9.9: Simplest series and parallel circuits

The characteristic equation for an RLC circuit takes one of two forms:

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \quad \text{(Series)}$$
$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0 \quad \text{(Parallel)}$$

The roots of the characteristic equation are:

$$s_1, s_2 = -\frac{R}{2L} \pm \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}} \qquad \text{(Series)} \qquad (9.10)$$
$$s_1, s_2 = -\frac{1}{2RC} \pm \frac{1}{2}\sqrt{\left(\frac{1}{RC}\right)^2 - \frac{4}{LC}} \qquad \text{(Parallel)} \qquad (9.11)$$

All voltages and currents in a series or parallel circuit will have one of the following three forms depending on the solution to the characteristic equation.

$$x(t) = \begin{cases} K_1 e^{s_1 t} + K_2 e^{s_2 t} + x(\infty) & s_1, s_2 \in \mathbb{R} \\ (K_1 + tK_2)e^{st} + x(\infty) & s = s_1 = s_2 \in \mathbb{R} \\ e^{-\sigma t}(A\cos(\omega_d t) + B\sin(\omega_d t)) + x(\infty) & s_1, s_2 \in \mathbb{C} \end{cases}$$

As a reminder, these responses are called overdamped, critically damped, and underdamped, respectively.

Each solution contains an $x(\infty)$ term. This term represents the "final" value that the circuit will reach given unlimited time. The final value can be found by replacing inductors with short circuits and capacitors with open circuits then solving for the desired voltage or current.

Recall that inductors have continuous current and capacitors have continuous voltage. That is, $i_L(t^-)i = i_L(t^+)$ and $v_C(t^-) = v_C(t^+)$.

The overdamped coefficients K_1 and K_2 are calculated by using the initial conditions.

$$x(0) = K_1 + K_2 + x(\infty)$$
$$x'(0) = s_1 K_1 + s_2 K_2$$

The initial conditions for $x'(0)$ can be found by using the characteristic equations for the inductor and capacitor. This yields $v_C'(0) = \frac{1}{C} i_C(0^+)$ for a capacitor and $i_L'(0) = \frac{1}{L} v_L(0^+)$. Note that both of these are the current or voltage evaluated at time $t = 0^+$.

In the critically damped case, the coefficients are:

$$x(0) = K_1 + x(\infty)$$
$$x'(0) = sK_1 + K_2$$

For the underdamped case, the roots have the form $s_1, s_2 = -\sigma + j\omega_d$, so the simplified relations are

$$\sigma = \frac{R}{2L} \qquad \omega_d = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} \qquad \text{(Series)}$$
$$\sigma = \frac{1}{2RC} \qquad \omega_d = \sqrt{\frac{1}{LC} - \left(\frac{1}{2RC}\right)^2} \qquad \text{(Parallel)}$$

ω_d is called the "damped" oscillation frequency while $\omega_0 = \frac{1}{\sqrt{LC}}$ is the "natural" oscillation frequency. ω_d and ω_0 are related by $\omega_d^2 = \omega_0^2 - \sigma^2$.

The coefficients A and B are found using the initial conditions:

$$x(0) = A + x(\infty)$$
$$x'(0) = -A\sigma + B\omega_d$$

9.4 Prelab

Task 9.4.1: Prelab Questions

1. *Calculate* the step response $v_C(t)$ for $t > 0$ for the circuit in figure 9.10.

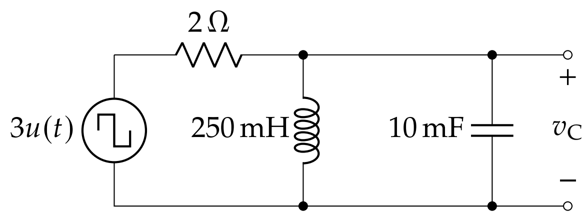

Figure 9.10: Parallel RLC Circuit

2. *Calculate* the Thévenin equivalent resistance shown in figure 9.11 by solving for the ratio $v_{\text{test}}/i_{\text{test}}$.

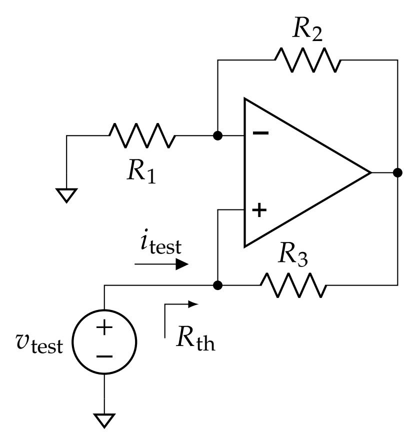

Figure 9.11: A negative impedance converter circuit

3. *Plot* the first 200 µs of $v_C(t)$ in figure 9.12 if $v_C(0) = 1\,\text{mV}$ and $i_L(0) = 0$ using Python's NumPy, MATLAB®, or a similar plotting tool.

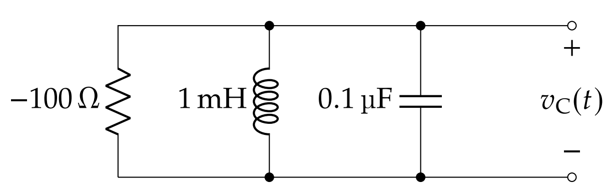

Figure 9.12: A parallel RLC circuit

9.5 Tasks

Task 9.5.1: Series RLC response measurements

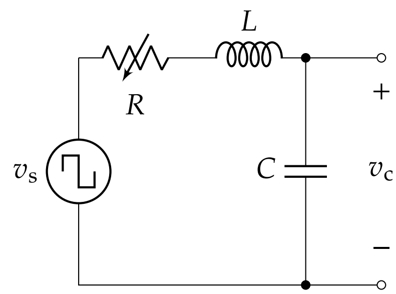

Figure 9.13: A series RLC circuit.

For this task, use $L = 1\,\text{mH}$ and $C = 0.001\,\mu\text{F}$. Assume zero initial conditions. Even though the input square wave *is* a time varying signal, if the period is much longer than the response, then it is effectively constant. One half of the square wave sets the initial conditions, and the other half provides a constant input.

1. Let v_s be a square wave that varies between 0 V and 3 V at 1 kHz. *Adjust* the potentiometer R to obtain and *capture* an underdamped and overdamped v_c response.

2. *Calculate* the peak to peak value of v_s and a resistance value[a] to obtain the response

$$v_C(t) = 3.5e^{-1.558\times10^6 t} - 8.5e^{-642\times10^3 t} + 4$$

3. *Replace* the potentiometer with the fixed resistor computed in the last step. *Capture* the waveform v_c by *applying* the correct input and *downloading* the data from the oscilloscope.

4. *Use* Python's NumPy or MATLAB®, to *plot* and *compare* data collected from the oscilloscope with the expected response.

[a] Hint: compute $s_1 + s_2$ using equation (9.10).

Task 9.5.2: Parallel RLC response measurements

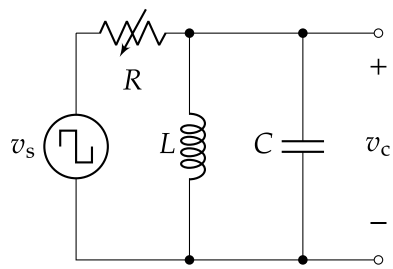

Figure 9.14: A parallel RLC circuit.

For this task, use $L = 1\,\mathrm{mH}$ and $C = 2\,\mathrm{nF}$. Assume zero initial conditions. Even though the input square wave *is* a time varying signal, if the period is much longer than the response, then it is approximately constant. One half of the square wave sets the initial conditions and the other half provides a constant input.

1. Let v_s be a square wave that varies between $0\,\mathrm{V}$ and $3\,\mathrm{V}$ at $1\,\mathrm{kHz}$. *Adjust* the potentiometer R to obtain and *capture* an underdamped and overdamped v_c response.

2. *Determine* the peak value of v_s and *compute* a resistance value to obtain the response

$$v_C(t) = 1.3e^{-9.2\times 1\times 10^4 t}(\sin 7 \times 1 \times 10^5 t)$$

3. *Replace* the potentiometer with the fixed resistor computed in the last step. *Capture* the waveform v_c by applying the correct input and *downloading* the data from the scope.

4. *Use* Python's NumPy or MATLAB® to *plot* and *compare* data collected from the oscilloscope with the expected response.

Task 9.5.3: Oscillator

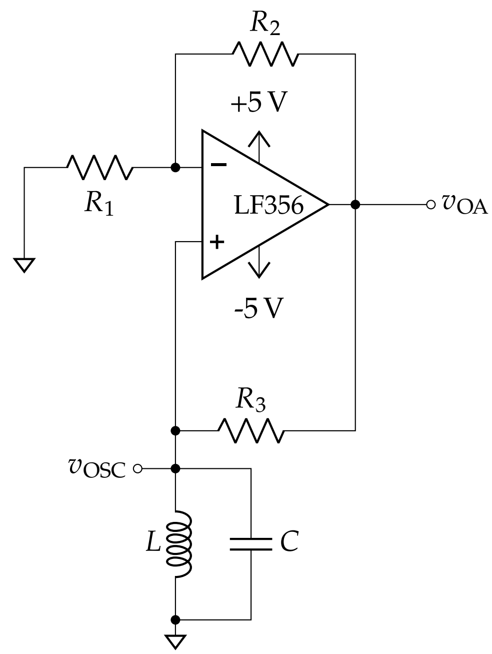

Figure 9.15: An oscillator circuit.

For this task, use $R_1 = R_2 = 1\,\mathrm{k\Omega}$, $R_3 = 100\,\Omega$, $L = 1\,\mathrm{mH}$, and $C = 0.1\,\mu\mathrm{F}$.

1. *Measure* the ESR of the inductor.
2. *Simulate* the circuit in figure 9.15. Use the measured ESR values for the inductor. *Obtain* plots of v_{OA} and v_{OSC}.
3. *Construct* the oscillator circuit.
4. *Capture* an oscilloscope screenshot showing v_{OUT} and v_{OSC} with frequency and peak to peak measurements.
5. *Calculate* the error between the simulated frequency and measured frequency. How do these relate to ω_d for the circuit?