

Effective estimates of ergodic quantities illustrated on the Bolyai-Rényi map

Mark Pollicott and Julia Slipantschuk*

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Abstract

We present a practical and effective method for rigorously estimating quantities associated to top eigenvalues of transfer operators to very high precision. The method combines explicit error bounds of the Lagrange-Chebyshev approximation with an established min-max method. We illustrate its applicability by significantly improving rigorous estimates on various ergodic quantities associated to the Bolyai-Rényi map.

1 Introduction

In a classic 1957 paper, A. Rényi described a general class of f -expansions. This included, as special cases, base d -expansions and β -expansions; continued fraction expansions; and (as Example 3 in [16]) the iterated radical expansions, which first appeared in Bolyai's 1832 book [3]. This last example gives an expansion for almost every $0 < x < 1$ in the form of an iterated square root:

$$x = -1 + \sqrt{a_1 + \sqrt{a_2 + \sqrt{a_3 + \cdots}}}, \quad (1.1)$$

where $a_1, a_2, a_3, \dots \in \{1, 2, 3\}$. Related infinite radical expansions appear in the notebooks of Ramanujan [1].

Statistical properties of the expansions in (1.1) are given by ergodic properties of the associated Bolyai-Rényi transformation $T: [0, 1) \rightarrow [0, 1)$ - a piecewise analytic expanding Markov map (see Section 2) with a unique T -invariant probability measure μ . Bosma, Dajani and Kraaikamp [5] conjectured that the metric entropy is $h(T, \mu) \approx 1.0545$, accurate to two decimal places. In [7] a rigorous estimate of 1.056313074 was given using a periodic

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point method. One of the aims of this note is to use estimation of this and other quantities to illustrate an alternative approach leading to significantly improved estimates.

In this and similar systems, estimating quantities of interest such as entropy, or digit frequencies, requires approximation of the pressure function arising from the maximal positive eigenvalue of an appropriately chosen (positive) transfer operator \mathcal{L} . This eigenvalue satisfies the so-called min-max principle, allowing to bound the pressure function in terms of bounds on $(\mathcal{L}f)/f$ for suitable positive functions f , see [12, Theorems 7.25-7.26] and [15].

Two crucial ingredients are needed to make these bounds rigorous and sufficiently tight: the first consists in choosing a suitable f , the second in obtaining rigorous and tight lower and upper bounds on $(\mathcal{L}f)/f$. For the first step, in order to obtain effective bounds on the pressure function, the function f should be chosen close to the leading eigenfunction of \mathcal{L} . This can be achieved by approximating \mathcal{L} by a suitable finite-rank operator \mathcal{L}_n and using its leading eigenfunction as a candidate for f . In [15, 14] this was accomplished using the classical Lagrange-Chebyshev interpolation scheme, for which results in [2] guarantee exponential convergence (in rank n) of eigenfunctions of \mathcal{L}_n to those of \mathcal{L} for analytic expanding maps T . For the second step, rigorous lower and upper bounds on $(\mathcal{L}f)/f$ were obtained using ball arithmetic, involving the use of higher-order derivatives for increased precision.

In this note we provide a refinement of this method, critically also using the Lagrange-Chebyshev interpolation method for enhancing the second step. In particular, the exponential convergence properties of this scheme allow us to rigorously bound $(\mathcal{L}f)/f$ to much greater precision, while being more efficient and using a single derivative of the quotient function. Overall, leveraging the strong convergence properties of Lagrange-Chebyshev interpolation for efficiently achieving accurate approximation, and ball arithmetic for rigorous bounding, this “Lagrange-Chebyshev min-max algorithm” provides a method for rigorous high-accuracy estimation of ergodic quantities for analytic expanding maps, at comparatively small computational cost.

This note is organized as follows. We introduce the Bolyai-Rényi map and state our approximation results for several of its ergodic quantities in §2, and summarize the well-known properties of the associated transfer operator and its leading eigenvalue in §3. In §4 we describe our min-max method: we derive the key estimate Theorem 4 in §4.1, describe the choice of approximate eigenfunctions f in §4.2 and the enhanced method for bounding $(\mathcal{L}f)/f$ in §4.3, and provide pseudo-code for the method’s implementation in §4.4. Proofs of the results from §2 based on this method are given in §5. Finally, in §6 we compare our algorithm with other available methods, and finish off with some generalizing remarks in §7.

2 Bolyai-Rényi transformation and results

Associated to (1.1) is the Bolyai-Rényi transformation $T: [0, 1) \rightarrow [0, 1)$ of the form

$$T(x) = \begin{cases} x^2 + 2x & \text{if } 0 \leq x < \sqrt{2} - 1 \\ x^2 + 2x - 1 & \text{if } \sqrt{2} - 1 \leq x < \sqrt{3} - 1 \\ x^2 + 2x - 2 & \text{if } \sqrt{3} - 1 \leq x \leq 1. \end{cases}$$

We can then write digits in the radical expansion (1.1) as

$$a_n = \begin{cases} 1 & \text{if } 0 \leq T^{n-1}(x) < \sqrt{2} - 1 \\ 2 & \text{if } \sqrt{2} - 1 \leq T^{n-1}(x) < \sqrt{3} - 1 \\ 3 & \text{if } \sqrt{3} - 1 \leq T^{n-1}(x) \leq 1. \end{cases}$$

Formally, it is often convenient to consider T as a piecewise expanding Markov map acting on three closed intervals. Since T is expanding (with $\inf_x T'(x) \geq 2$) there is a (unique) T -invariant probability measure μ which is absolutely continuous (even equivalent) to Lebesgue measure by, for example, the classical Lasota-Yorke theorem [10]. We denote the metric entropy of μ by $h(\mu) = h(T, \mu)$, and by the classical Rokhlin formula it is also conveniently given as the Lyapunov exponent, i.e.,

$$h(\mu) = \int \log T'(x) d\mu(x), \quad (2.1)$$

which, as usual, quantifies the sensitive dependence on initial conditions for typical orbits.

Theorem 1. *The metric entropy of T can be rigorously estimated to be*

$$h(\mu) = 1.0563130740\,7297055209\,9568877064\,0651679335\,4262184005\dots,$$

accurate to the number of decimal places presented.

The min-max method we use to obtain this new estimate is relatively undemanding on computer resources and so with more patience it would be relatively easy to get more decimal places.

Example 2.1. *As an application we have a variant on Lochs' theorem, which originally compared base 10 expansions with continued fraction expansions of typical digits. For a.e. (μ) point $x \in (0, 1)$ and $n \geq 1$ we can write*

$$\begin{aligned} x &= -1 + \sqrt{a_1 + \sqrt{a_2 + \sqrt{a_3 + \cdots}}} & \text{where } a_1, a_2, a_3, \dots \in \{1, 2, 3\} \\ &= \frac{b_1}{3} + \frac{b_2}{3^2} + \frac{b_3}{3^3} + \cdots & \text{where } b_1, b_2, b_3, \dots \in \{0, 1, 2\} \end{aligned}$$

and for each $n \geq 1$ we denote by $N_n(x)$ the minimal number of digits $(b_1, b_2, \dots, b_{N_n(x)})$ in base 3 expansion which are needed to specify the first n digits (a_1, a_2, \dots, a_n) in the radical expansion. The following is a simple consequence of Theorem 1.

Corollary 2.2. *The quantity $\lim_{n \rightarrow +\infty} (N_n/n)$ evaluates to*

$$1.0400442024\ 5603917587\ 6064437681\ 7446547515\ 185552379\dots,$$

accurate to the number of decimal places presented.

Proof. The proof of Lochs' theorem is based on the Shannon-McMillan-Breman theorem [11], generalized to the present example to give that the limit exists and is equal to the ratio of the entropies of the two maps, that is, $\log(3)/h(\mu)$, which corresponds to the value stated. \square

In the following two subsections we will provide further applications of our method, presenting estimates for several other quantities related to the Bolyai-Rényi map.

2.1 Frequency of digits

We consider the frequency of digits that occur in typical expansions. In particular, we can define

$$f_i(x) := \lim_{n \rightarrow +\infty} \frac{1}{n} \#\{1 \leq k \leq n : a_k(x) = i\} \quad \text{for } i = 1, 2, 3$$

whenever the limit exists. By ergodicity of T the limit exists and is constant for a.e. (μ) $x \in [0, 1)$ and we simply denote it f_i .

Theorem 2. *The digit frequencies are*

$$\begin{aligned} f_1 &= 0.4640796294\ 4716719166\ 0214542662\ 4296264246\ 0872990983\dots, \\ f_2 &= 0.3044190449\ 4046044774\ 3959549801\ 4270582974\ 2520632111\dots, \\ f_3 &= 0.2315013256\ 1237236059\ 5825907536\ 1433152779\ 6606376905\dots, \end{aligned}$$

accurate to the number of decimal places presented.

These are an improvement on the accuracy of previous estimates in [7, Theorem 2], which were computed up to approximately $\pm 10^{-7}$.

2.2 Deleted digits

Another interesting problem is to consider the Cantor set whose radical expansions use only the digits 1 and 3, i.e., the “deleted digit” Cantor set

$$\mathcal{E} = \left\{ x = -1 + \sqrt{a_1 + \sqrt{a_2 + \sqrt{a_3 + \dots}}} : a_1, a_2, a_3, \dots \in \{1, 3\} \right\}.$$

Theorem 3. *The Hausdorff dimension of \mathcal{E} is*

$\dim_H(\mathcal{E}) = 0.6439131204\,7072945768\,7895134656\,7617073899\,0093573261\dots$,
accurate to the number of decimal places presented.

Similar problems have attracted interest in the context of continued fractions [6, 15] and this result might be considered as an analogy.

3 The transfer operators \mathcal{L}_t

We associate to T the three inverse branches $T_1, T_2, T_3: [0, 1] \rightarrow [0, 1]$ given by

$$T_i(x) = \sqrt{i+x} - 1 \quad \text{for } i = 1, 2, 3$$

(i.e., $T \circ T_i(x) = x$ for all $x \in [0, 1)$ and $i = 1, 2, 3$) and observe by direct computation that

$$T'_i(x) = \frac{1}{2(T_i(x) + 1)} \quad \text{for } i = 1, 2, 3.$$

Let $C([0, 1])$ denote the Banach space of continuous functions $f: [0, 1] \rightarrow \mathbb{R}$ with the supremum norm $\|f\|_\infty = \sup_{x \in [0, 1]} |f(x)|$. We recall the following well-known result.

Lemma 3.1 (Lasota-Yorke [10]). *The absolutely continuous T -invariant measure μ on $[0, 1]$ satisfies $h(x) := \frac{d\mu}{dx} \in C([0, 1])$, where h is the eigenvector associated to the maximal eigenvalue 1 for the linear operator $\mathcal{L}_0: C([0, 1]) \rightarrow C([0, 1])$ given by*

$$\mathcal{L}_0 f(x) = \sum_{i=1}^3 |T'_i(x)| f(T_i x), \quad \forall f \in C([0, 1]), x \in [0, 1].$$

More generally, we can consider the following parameterized family of bounded linear operators.

Definition 3.2. *For each $t \in \mathbb{R}$ we let $\mathcal{L}_t: C([0, 1]) \rightarrow C([0, 1])$ denote the bounded linear operator given by*

$$\mathcal{L}_t f(x) = \sum_{i=1}^3 |T'_i(x)|^{1+t} f(T_i x), \quad \forall f \in C([0, 1]), x \in [0, 1].$$

We can denote the spectral radius of \mathcal{L}_t by $e^{P(t)}$. The following formulation follows from the spectral radius theorem.

Definition 3.3. *We can associate the pressure function $P: \mathbb{R} \rightarrow \mathbb{R}$ defined by*

$$P(t) = \lim_{n \rightarrow +\infty} \frac{1}{n} \log \|\mathcal{L}_t^n 1\|_\infty,$$

where $1(x) = 1$ is the constant function.

As will be seen from the next lemma, the limit exists, and the pressure function defined this way has certain properties following from the transformation T being expanding and differentiable. We note that there are alternative definitions of pressure, e.g. via the variational principle applied to the function $-t \log |T'|$, see [20].

Lemma 3.4. *The function $P: \mathbb{R} \rightarrow \mathbb{R}$ has the following properties:*

1. $P(0) = 0$.
2. P is C^1 (even real analytic), monotone decreasing and convex.
3. $\frac{d}{dt}P(t)|_{t=0} = -h(\mu)$.

Proof. The first part follows from bounds on $\|\mathcal{L}_0^n 1\|_\infty$, for $n \geq 0$.

For the second part, the value $e^{P(t)}$ for the spectral radius of \mathcal{L}_t occurs as a maximal simple eigenvalue for the transfer operator $\mathcal{L}_t: C^1([0, 1]) \rightarrow C^1([0, 1])$ acting on the Banach space of C^1 functions $f: [0, 1] \rightarrow \mathbb{R}$ with the norm $\|f\| := \|f\|_\infty + \|f'\|_\infty$ (Ruelle Operator Theorem, see [13]). The analyticity of $P(t)$ then follows by perturbation theory applied to the maximal eigenvalue of \mathcal{L}_t , see [13, 17]. The monotone decreasing property follows from the negative derivative in the next part. Convexity follows from the observation that for each $n \geq 1$ the function $t \mapsto \|\mathcal{L}_t^n 1\|_\infty$ is convex.

For the third part, let $h_t \in C^1([0, 1])$ be an eigenfunction corresponding to the maximal eigenvalue of \mathcal{L}_t for $t \in \mathbb{R}$, and write $h'_t(x) = \frac{d}{dt}h_t(x)$. By analytic perturbation theory [9], we can differentiate the eigenvalue equation $\mathcal{L}_t h_t = e^{P(t)} h_t$ at $t = 0$ to write

$$\mathcal{L}_0 h'_0 - \mathcal{L}_0(\log |T'| h_0) = e^{P(0)} h'_0 + P'(0) h_0.$$

We can then integrate with respect to the measure $\nu_0 = \mathcal{L}_0^* \nu_0$ to write

$$P'(0) = -\nu_0(\log |T'| h_0) = -\mu(\log |T'|). \quad (3.1)$$

Finally, the Rokhlin identity gives $h(\mu) = \mu(\log |T'|)$. \square

We will make use of part 3 of Lemma 3.4, despite the unpromising formulation, to bound the entropy $h(\mu)$ by deriving inequalities on the derivative of $P(t)$ at $t = 0$. In particular, using elementary calculus, we can estimate $\frac{d}{dt}P(t)|_{t=0} = -h(\mu)$ from estimates of the two values of pressure $P(\epsilon)$ and $P(-\epsilon)$ for a fixed $\epsilon > 0$.

By convexity we can (rigorously) bound the entropy (cf. Figure 1):

Corollary 3.5. *For any $\epsilon > 0$ we have the bounds*

$$\frac{P(-\epsilon)}{\epsilon} \geq h(\mu) \geq \frac{|P(\epsilon)|}{\epsilon}. \quad (3.2)$$

Remark 3.6. *To make the bounds in (3.2) effective, we need to choose $\epsilon > 0$ appropriately small. On the other hand, for large values of ϵ we observe that*

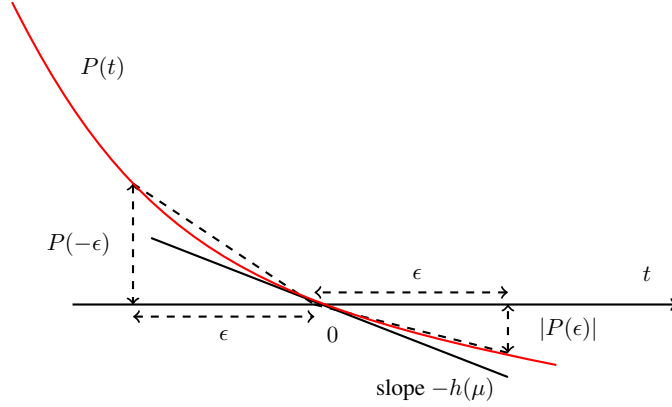


Figure 1: The pressure function $P(t)$ and the inequalities in Corollary 3.5.

the bounds become the asymptotic derivatives $P'(t)$ of the pressure $P(t)$ as $t \rightarrow \pm\infty$. Interestingly these become

$$\inf_{\nu} \nu(\log |T'|) \leq h(\mu) \leq \sup_{\nu} \nu(\log |T'|),$$

where the supremum and infimum range over all T -invariant probability measures, although these are clearly not useful for practical estimates.

4 Algorithm for estimating properties of $P(\pm\epsilon)$

The bounds in (3.2) may at first sight seem unpromising since we need to choose $\epsilon > 0$ appropriately small. Therefore it is important to find an efficient and accurate method to estimate $P(\pm\epsilon)$, or equivalently the leading eigenvalue of $\mathcal{L}_{\pm\epsilon}$. We shall now describe a simple min-max method to give an upper bound on this eigenvalue.

4.1 Min-max method to bound the top eigenvalue of $\mathcal{L}_{\pm\epsilon}$

We will use a “min-max” estimate which is based on the following lemma.

Lemma 4.1. *Let $t, \alpha \in \mathbb{R}$, and assume there exists a positive continuous function $g: [0, 1] \rightarrow \mathbb{R}^+$ such that $\sup_x \frac{\mathcal{L}_t g(x)}{g(x)} \leq e^\alpha$. Then $P(t) \leq \alpha$.*

Proof. Since the transfer operator \mathcal{L}_t is positive, applying it multiple times to the inequality in the hypothesis yields

$$0 < \mathcal{L}_t^n g(x) \leq e^\alpha \mathcal{L}_t^{n-1} g(x) \leq e^{2\alpha} \mathcal{L}_t^{n-2} g(x) \leq \dots \leq e^{n\alpha} g(x)$$

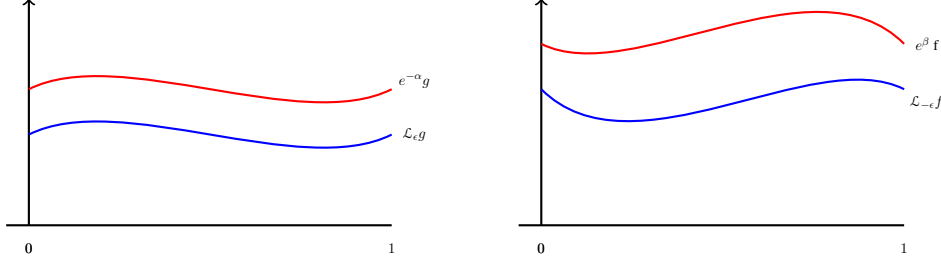


Figure 2: Illustrations of the hypotheses of (4.1) (left) and (4.2) (right).

for all $x \in [0, 1]$. In particular, since g is positive we have

$$e^{P(t)} = \lim_{n \rightarrow +\infty} \|\mathcal{L}_t^n g\|_\infty^{1/n} \leq e^\alpha,$$

using our definition of $P(t)$. The claim follows. \square

Using the fact that $P(t)$ is positive for $t < 0$ and negative for $t > 0$, the lemma yields the following implications for any $\epsilon, \alpha, \beta > 0$ and any positive continuous functions f, g (cf. Figure 2):

$$\sup_x \frac{\mathcal{L}_\epsilon g(x)}{g(x)} \leq e^{-\alpha} \implies |P(\epsilon)| \geq \alpha, \quad (4.1)$$

$$\sup_x \frac{\mathcal{L}_{-\epsilon} f(x)}{f(x)} \leq e^\beta \implies P(-\epsilon) \leq \beta. \quad (4.2)$$

Combining with the inequalities in (3.2), we have the following useful bound.

Theorem 4 (Bounds on the entropy $h(\mu)$). *Let $\epsilon > 0$, and assume there are positive continuous functions $g, f: [0, 1] \rightarrow \mathbb{R}^+$ and $\alpha, \beta > 0$ satisfying the hypotheses of (4.1) and (4.2), respectively, then*

$$\frac{\beta}{\epsilon} \geq h(\mu) \geq \frac{\alpha}{\epsilon}. \quad (4.3)$$

In practice, dividing by a small value ϵ needs to be more than compensated for by the efficiency of the approach in estimating $P(-\epsilon)$ and $P(\epsilon)$, to allow α and β to be chosen very close (compared to $\epsilon > 0$). There are two crucial ingredients to make the bounds (4.3) effective, rigorous and computationally efficient. The first consists in choosing g, f in (4.1), (4.2) as close as possible to the leading eigenfunctions of $\mathcal{L}_{\pm\epsilon}$, the second in efficiently computing tight rigorous bounds on $(\mathcal{L}g)/g$ and $(\mathcal{L}f)/f$. We now describe effective schemes for both, based on Lagrange-Chebyshev interpolation.

4.2 Choosing the functions f and g via interpolation

Given $\epsilon > 0$, our aim is to choose suitable functions $f, g: [0, 1] \rightarrow \mathbb{R}^+$ for which we can associate values $\alpha, \beta > 0$ satisfying (4.1) and (4.2) and then apply the bounds in (4.3). Although there is plenty of scope in how they are actually arrived at, the most promising way is to choose f, g as close as possible to the leading eigenfunctions of $\mathcal{L}_{-\epsilon}$ and \mathcal{L}_ϵ , respectively. As these eigenfunctions are not known explicitly, we approximate them by approximating the respective transfer operators by finite-rank operators.

More precisely, denoting by $L: B \rightarrow B$ any bounded operator on a Banach space B , we can approximate it by a finite-rank operator $P_m L P_m$ where $P_m: B \rightarrow B$ is a bounded projection operator of finite rank given by

$$P_m f = \sum_{l=0}^{m-1} e_l^*(f) e_l,$$

with $e_l \in B$ and $e_l^* \in B^*$ for $l = 0, \dots, m-1$. The operator $P_m L P_m$ is represented by an $m \times m$ matrix M , whose (k, l) -entry is given by

$$M_{kl} = e_k^*(L e_l).$$

Definition 4.2. For $B = C([-1, 1])$, the Lagrange-Chebyshev interpolation scheme consists in using P_m as defined above with

1. e_l ($0 \leq l \leq m-1$) the Chebyshev polynomial of the first kind and degree l ; and
2. e_k^* ($0 \leq k \leq m-1$) the linear functional given by

$$e_k^*(f) = \frac{2 - \delta_{0,k}}{m} \sum_{j=0}^{m-1} f(x_j) e_k(x_j),$$

with x_0, \dots, x_{m-1} the Chebyshev nodes of order m , $x_j = \cos((2j+1)\pi/(2m))$.

For an arbitrary interval $I = [c-r, c+r] = \gamma([-1, 1])$, with $\gamma(x) = \gamma_{r,c}(x) = rx + c$ for some $c \in \mathbb{R}$, $r > 0$, we denote $P_{m,\gamma}$ the corresponding Lagrange-Chebyshev interpolation operator on $B = C(I)$, defined as above with e_l and x_j replaced with $\hat{e}_l = e_l \circ \gamma^{-1}$ and $\hat{x}_j = \gamma(x_j)$, respectively.

We apply the above scheme to our transfer operator $L = \mathcal{L}_\epsilon: C([0, 1]) \rightarrow C([0, 1])$, using the change of coordinates $\gamma = \gamma_{r,c}: [-1, 1] \rightarrow [0, 1]$ with $r = c = 1/2$. We obtain an $m \times m$ matrix M , whose right eigenvector $v = (v_0, \dots, v_{m-1})^T$ corresponding to the largest eigenvalue of M yields a candidate function $g := g^{(m)} = \sum_{l=0}^{m-1} v_l \hat{e}_l$ to use in (4.1). A priori it may not be obvious why g is positive and a good candidate function, however we observe that \mathcal{L}_ϵ is not an arbitrary bounded operator. The contractions T_i involved in the definition of \mathcal{L}_ϵ are analytic on $[0, 1]$, hence extend holomorphically to a complex neighbourhood, in particular to a suitable ellipse

$E_{\gamma,R} = \gamma(E_R) \supset [0, 1]$, where $E_R \supset [-1, 1]$ is a standard Bernstein ellipse¹ for some $R > 0$. Now, Theorem 3.3 and Corollary 3 of [2] guarantee that the (generalized) eigenfunctions of $P_{m,\gamma}\mathcal{L}_\epsilon P_{m,\gamma}$ converge (in supremum norm) exponentially fast in m to those of \mathcal{L}_ϵ . In particular, for large enough m , the function $g = g^{(m)}$ is positive on $[0, 1]$ and approximates the eigenfunction for the maximal positive eigenvalue $e^{P(\epsilon)}$ of \mathcal{L}_ϵ .

4.3 Rigorous bounds on the supremum/infimum

The main step in establishing the left-hand sides of (4.1) and (4.2) comes down to rigorously estimating $\sup_{x \in I} h(x)$ for $h(x) = (\mathcal{L}_\epsilon f)(x)/f(x)$ and $I = [0, 1] = [c - r, c + r]$ with $c = r = 1/2$. By the mean value theorem we have

$$\sup_{x \in I} h(x) \leq h(c) + \sup_{x \in I} |h'(x)| \cdot r, \quad (4.4)$$

and using the quotient rule we can write $h' = \psi/f^2$ with

$$\psi = (\mathcal{L}_\epsilon f)' \cdot f - f' \cdot (\mathcal{L}_\epsilon f),$$

which we use to rewrite (4.4) as

$$\sup_{x \in I} h(x) \leq h(c) + \frac{r}{\inf_{x \in I} f(x)^2} \cdot \sup_{x \in I} |\psi(x)|. \quad (4.5)$$

To obtain a tight bound on $\sup_{x \in I} |\psi(x)|$, we can approximate ψ using the Lagrange-Chebyshev interpolation projector $P_{n,\gamma}$, $n \in \mathbb{N}$, introduced in the previous section, and use the fact that

$$\sup_{x \in I} |\psi(x)| \leq \sup_{x \in I} |(\psi - P_{n,\gamma}\psi)(x)| + \sup_{x \in I} |(P_{n,\gamma}\psi)(x)|. \quad (4.6)$$

We note that the function ψ is holomorphic on $E_{\gamma,R} = \gamma(E_R)$ for a suitable $R > 1$, whenever f is a polynomial (or more generally, any entire function). Based on results in [2], the following lemma shows that the error incurred by this approximation decays exponentially in n , and provides all relevant constants explicitly, allowing for a rigorous implementation.

Lemma 4.3. *Let $R > 1$, and let ψ be a holomorphic function on a scaled Bernstein ellipse $E_{\gamma,R}$. Then for any $\rho \in (1, R)$ and $n \in \mathbb{N}$, the approximation error under Lagrange-Chebyshev interpolation obeys*

$$\sup_{x \in I} |(\psi - P_{n,\gamma}\psi)(x)| \leq c_{\rho,R} \cdot \frac{\cosh(n \log \rho)}{\sinh(n \log R)} \cdot \sup_{x \in I} |(\psi \circ \gamma \circ \sigma)(Re^{2\pi i x})|,$$

where $c_{\rho,R} = \frac{\sinh(\log R)}{\cosh(\log R) - \cosh(\log \rho)}$ and $\sigma(z) = (z + z^{-1})/2$.

¹A Bernstein ellipse is an ellipse $E_R \subset \mathbb{C}$ with foci -1 and 1 and lengths of major and minor semi-axes given by $a = \cosh(\log R)$ and $b = \sinh(\log R)$, respectively.

Proof. Writing $\tilde{\psi} = \psi \circ \gamma$, we have

$$\sup_{x \in I} |(\psi - P_{n,\gamma}\psi)(x)| = \sup_{x \in [-1,1]} |(\tilde{\psi} - P_n\tilde{\psi})(x)| \leq \sup_{z \in E_\rho} |(\tilde{\psi} - P_{n,\gamma}\tilde{\psi})(z)|,$$

for any $\rho \in (1, R)$. Using [2, Lemma 2.6], we obtain the bound

$$\sup_{z \in E_\rho} |(\tilde{\psi} - P_{n,\gamma}\tilde{\psi})(z)| \leq c_{\rho,R} \cdot \frac{\cosh(n \log \rho)}{\sinh(n \log R)} \cdot \sup_{z \in E_R} |\tilde{\psi}(z)|,$$

with the constant $c_{\rho,R} = \sinh(\log R) / (\cosh(\log R) - \cosh(\log \rho))$. The assertion follows by application of the maximum modulus principle, and observing that σ maps a circle of radius R to the boundary of E_R :

$$\sup_{z \in E_R} |\tilde{\psi}(z)| \leq \sup_{z \in \partial E_R} |\tilde{\psi}(z)| = \sup_{x \in I} |(\tilde{\psi} \circ \sigma)(Re^{2\pi i x})|. \quad \square$$

Remark 4.4. Expressing a quantity as the extremum of a function over an interval (as done in the previous lemma) allows for rigorous computational bounding by using ball arithmetic, as, for example, implemented in the Arb library [8]. In practice, in order to obtain rigorous bounds on a function f over the interval $I = [0, 1]$, say, we can evaluate f on $k \in \mathbb{N}$ equal-sized subintervals $I_l = [c_l - r, c_l + r]$, $r = |I|/k$, $c_l = (l - 1/2) \cdot r$, $l = 1, \dots, k$. The image intervals $J_l = [c'_l - r'_l, c'_l + r'_l]$ output by the ball arithmetic calculation are then guaranteed to satisfy $f(I_l) \subseteq J_l$, yielding effective lower and upper bounds on f .

Combining (4.5), (4.6) and Lemma 4.3 yields the inequality

$$\sup_{x \in I} \frac{\mathcal{L}_\epsilon f(x)}{f(x)} \leq \frac{\mathcal{L}_\epsilon f(c)}{f(c)} + \frac{r}{\inf_{x \in I} f(x)^2} \left(\delta_{n,\rho,R}(\psi) + \sup_{x \in I} |(P_{n,\gamma}\psi)(x)| \right),$$

where $\delta_{n,\rho,R}(\psi)$ denotes the upper bound from Lemma 4.3 which decays exponentially with n . We note that, as in Remark 4.4, all suprema and infima on the right-hand side of this inequality can be rigorously bounded via ball arithmetic.

Remark 4.5. We note that with the same method as above, a rigorous lower bound on the infimum of $(\mathcal{L}_\epsilon f)/f$ can be obtained, yielding

$$\inf_{x \in I} \frac{\mathcal{L}_\epsilon f(x)}{f(x)} \geq \frac{\mathcal{L}_\epsilon f(c)}{f(c)} - \frac{r}{\inf_{x \in I} f(x)^2} \left(\delta_{n,\rho,R}(\psi) + \sup_{x \in I} |(P_{n,\gamma}\psi)(x)| \right).$$

4.4 Algorithmic implementation of rigorous bounds

With the results of the previous sections, we are now ready to describe our algorithm for computing rigorous and tight estimates on $(\mathcal{L}f)/f$ for a given transfer operator $\mathcal{L} = \mathcal{L}_\epsilon$, which will form the key ingredient in the proofs of Theorems 1-3; see Algorithm 1.

Algorithm 1 Rigorous lower/upper bounds on $(\mathcal{L}f)/f$

```

function COMPUTE_BOUNDS( $\mathcal{L} : g \mapsto \mathcal{L}g, d\mathcal{L} : (g, g') \mapsto (\mathcal{L}g)', m, n, c, r$ )
     $\{e_l\}, \{e'_l\} \leftarrow \text{CHEB\_BASIS}(m, c, r)$   $\triangleright$  basis functions and their derivatives
     $M \leftarrow \text{COMPUTE\_MATRIX}(\mathcal{L}, \{e_l\})$ 
     $v \leftarrow \text{POWER\_METHOD}(M)$   $\triangleright$  e.vec. corresponding to leading e.val.
     $f \leftarrow \sum_l v_l \cdot e_l$ ;  $f' \leftarrow \sum_l v_l \cdot e'_l$ 
     $f_{\min} \leftarrow \text{INF\_BALL}(f)$ 
     $\text{ASSERT}(f_{\min} > 0)$   $\triangleright$  abort if eigenfunction non-positive
     $\mathcal{L}f \leftarrow \mathcal{L}(f)$ ;  $(\mathcal{L}f)' \leftarrow d\mathcal{L}(f, f')$ 
     $\psi \leftarrow (\mathcal{L}f)' \cdot f - \mathcal{L}f \cdot f'$ 
     $P_{n,\gamma}(\psi), \text{error}(P_{n,\gamma}, \psi) \leftarrow \text{PN\_APPROXIMATE}(\psi, n)$ 
     $S \leftarrow \text{SUP\_BALL}(|P_{n,\gamma}(\psi)|) + \text{error}(P_{n,\gamma}, \psi)$ 
    return  $\mathcal{L}f(c)/f(c) \mp S \cdot r / f_{\min}^2$ 

```

The algorithm requires access to mappings which, for a given differentiable function g and its derivative g' , produce $\mathcal{L}g$ and $(\mathcal{L}g)'$. Following Section 4.2, it begins by computing a matrix representation M in the Chebyshev basis for a finite-rank approximation of \mathcal{L} (`COMPUTE_MATRIX`) and obtaining the eigenvector v corresponding to its leading eigenvalue via the power method (`POWER_METHOD`).

The corresponding eigenfunction f and its derivative are then used to define $\psi = (\mathcal{L}f)' \cdot f - \mathcal{L}f \cdot f'$. The algorithm proceeds by computing $P_{n,\gamma}(\psi)$ and the upper bound on the approximation error $\text{error}(P_{n,\gamma}, \psi) = \sup |\psi - P_{n,\gamma}(\psi)|$ described in Lemma 4.3 (`PN_APPROXIMATE`). Finally, it returns the (rigorous) lower and upper bounds, following the respective formulas derived in Section 4.3.

We note that all calculations are performed using ball arithmetic to account for potential accumulation of errors. In particular, as in Remark 4.5, all `sup` (`SUP_BALL`) and `inf` (`INF_BALL`) computations are performed by function evaluation on k equal-sized subintervals of $[c - r, c + r]$.

5 Proofs of results

5.1 Proof of Theorem 1

In order to complete the proof of Theorem 1 we combine the rigorous computational estimation presented in Sections 4.2-4.4 with Theorem 4.

The computational step is summarized in Algorithm 2. Making use of `COMPUTE_BOUNDS` (see Algorithm 1) to compute rigorous upper bounds

$$\frac{\mathcal{L}_\epsilon g(x)}{g(x)} \leq e^{-\alpha} \quad \text{and} \quad \frac{\mathcal{L}_{-\epsilon} f(x)}{f(x)} \leq e^\beta,$$

Algorithm 2 Rigorous lower/upper bounds on $P'(0) = -h(\mu)$

```

function COMPUTE_PRESSURE_DERIVATIVE( $\epsilon, m, n, c, r$ )
   $\mathcal{L}_\epsilon, d\mathcal{L}_\epsilon \leftarrow \text{OPERATORS}(\epsilon); \mathcal{L}_{-\epsilon}, d\mathcal{L}_{-\epsilon} \leftarrow \text{OPERATORS}(-\epsilon)$ 
   $b_{\min}, b_{\max} \leftarrow \text{COMPUTE\_BOUNDS}(\mathcal{L}_{-\epsilon}, d\mathcal{L}_{-\epsilon}, m, n, c, r)$ 
   $a_{\min}, a_{\max} \leftarrow \text{COMPUTE\_BOUNDS}(\mathcal{L}_\epsilon, d\mathcal{L}_\epsilon, m, n, c, r)$ 
   $\beta \leftarrow \log(b_{\max})$ ;  $\alpha \leftarrow -\log(a_{\max})$ 
  return  $\alpha/\epsilon, \beta/\epsilon$ 

```

bounds on $h(\mu)$ are obtained via Theorem 4 as $\alpha/\epsilon \leq h(\mu) \leq \beta/\epsilon$. In order to obtain the given precision, we make the following set of parameter choices in executing the algorithm:

- (a) We take the value $\epsilon = 10^{-50}$.
- (b) We choose $m = 160$ for the rank of the approximation of \mathcal{L}_ϵ .
- (c) We choose $n = 200$ for the rank of the Lagrange-Chebyshev approximation of the function ψ in Section 4.3.
- (d) For estimating suprema and infima with ball arithmetic as described in Section 4.3, we use $k = 250$ intervals.
- (e) We use the value² $R = 5.5$ and $\rho = 1.001$ for computing the bound on the approximation error in Lemma 4.3.

The resulting values are

$$\begin{aligned} \alpha/\epsilon &= 1.0563130740\,7297055209\,9568877064\,0651679335\,4262184005\,66\dots \\ \beta/\epsilon &= 1.0563130740\,7297055209\,9568877064\,0651679335\,4262184005\,75\dots \end{aligned}$$

yielding the asserted value for $h(\mu)$.

Remark 5.1. We note that in (4.6), the approximation error incurred by $P_{n,\gamma}$ is negligible compared to the actual size of the approximated supremum. With our choices of parameters (in particular, approximation rank $n = 200$), the error is $\sup_{x \in I} |(\psi - P_{n,\gamma}\psi)(x)| < 2 \cdot 10^{-142}$, whereas $\sup_{x \in I} |\psi(x)| \approx \sup_{x \in I} |(P_{n,\gamma}\psi)(x)| \approx 10^{-112}$.

5.2 Proof of Theorem 2

By a simple application of the Birkhoff ergodic theorem, the frequency f_i ($i = 1, 2, 3$) of each of the three digits can be expressed in terms of the T -invariant measure μ as $f_i = \mu(I_i)$, with

$$I_1 = [0, \sqrt{2} - 1), \quad I_2 = [\sqrt{2} - 1, \sqrt{3} - 1), \quad I_3 = [\sqrt{3} - 1, 1).$$

²We note that ψ is holomorphic on $E_{\gamma,R}$ for any $1 < R < \exp(\text{arccosh}(3)) \approx 5.8$.

Thus we need to estimate the respective measure for each of these digits. To this end we want to replace the family of operators \mathcal{L}_t by the following.

Definition 5.2. Given $t \in \mathbb{R}$, we define operators $\mathcal{N}_{i,t}: C([0, 1]) \rightarrow C([0, 1])$ for $i = 1, 2, 3$ by

$$\mathcal{N}_{i,t}f(x) = \mathcal{L}_t f(x) + (e^t - 1)|T'_i(x)|f(T_i).$$

We can denote by $e^{R_i(t)}$ the spectral radius of the operator $\mathcal{N}_{i,t}$.

Definition 5.3. For $i = 1, 2, 3$ we define $R_i: \mathbb{R} \rightarrow \mathbb{R}$ as

$$R_i(t) = \limsup_{n \rightarrow +\infty} \frac{1}{n} \log \|\mathcal{N}_{i,t}^n 1\|_\infty,$$

where $1(x) = 1$ is the constant function and $\|f\|_\infty = \sup_{x \in [0,1]} |f(x)|$.

The following results are analogues of Lemma 3.4 and Corollary 3.5.

Lemma 5.4. For $i = 1, 2, 3$ the function $R_i: \mathbb{R} \rightarrow \mathbb{R}$ has the following properties:

1. $R_i(0) = 0$.
2. R_i is C^∞ (even real analytic), monotone decreasing and convex.
3. $\frac{d}{dt} R_i(t)|_{t=0} = -f_i$.

Corollary 5.5. For $\epsilon > 0$ and $i = 1, 2, 3$ we can bound

$$\frac{R_i(-\epsilon)}{\epsilon} \geq \mu(I_i) \geq \frac{|R_i(\epsilon)|}{\epsilon}.$$

In order to convert these bounds into practical estimates we need the analogue of Lemma 4.1, now for the operators $\mathcal{N}_{i,t}$ ($i = 1, 2, 3$).

Lemma 5.6. Let $t, \alpha \in \mathbb{R}$, $i = 1, 2, 3$. If there exists a positive continuous function $g: [0, 1] \rightarrow \mathbb{R}^+$ such that $\sup_x \frac{\mathcal{N}_{i,t} g(x)}{g(x)} \leq e^\alpha$, then $R_i(t) \leq \alpha$.

The remainder of the proof of the theorem is completely analogous to that of Theorem 1, where for the computational step we also use the exact same parameter values.

5.3 Proof of Theorem 3

We can adapt the method of the proof of Theorem 1 to prove Theorem 3. To begin we need to replace the operators \mathcal{L}_t by the following operators.

Definition 5.7. For any $t \in \mathbb{R}$, let $\mathcal{M}_t: C([0, 1]) \rightarrow C([0, 1])$ be the bounded linear operator defined by

$$\mathcal{M}_t f(x) = |T'_0(x)|^t f(T_0 x) + |T'_2(x)|^t f(T_2 x) \quad \forall f \in C([0, 1]), x \in [0, 1].$$

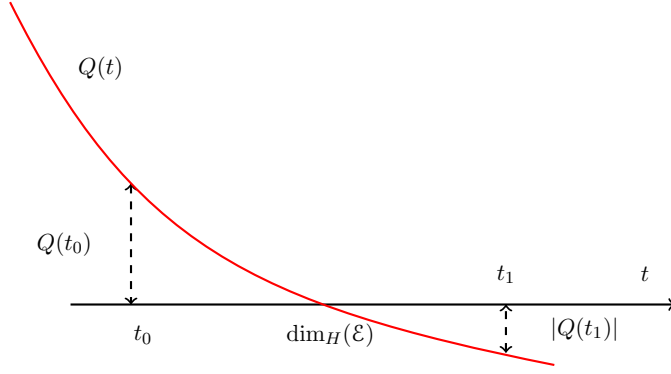


Figure 3: $t_0 \leq \dim_H(\mathcal{E}) \leq t_1$

Denoting the spectral radius of \mathcal{M}_t by $e^{Q(t)}$, the following re-formulation follows from the spectral radius theorem.

Definition 5.8. We can write $Q: \mathbb{R} \rightarrow \mathbb{R}$ as

$$Q(t) = \lim_{n \rightarrow +\infty} \frac{1}{n} \log \|\mathcal{M}_t^n \mathbf{1}\|_\infty,$$

where $\mathbf{1}(x) = 1$ is the constant function.

The following result is the analogue of Lemma 3.4.

Lemma 5.9. The function $Q: \mathbb{R} \rightarrow \mathbb{R}$ is C^∞ (even real analytic), monotone decreasing and convex.

The following lemma can be proved analogously to Lemma 4.1.

Lemma 5.10. Let $t > 0, \alpha \in \mathbb{R}$, and let $g: [0, 1] \rightarrow \mathbb{R}^+$ be a positive continuous function. Then the following implications hold:

$$\inf_{x \in [0, 1]} \frac{\mathcal{M}_t g(x)}{g(x)} \geq e^\alpha \implies Q(t) \geq \alpha, \quad (5.1)$$

$$\sup_{x \in [0, 1]} \frac{\mathcal{M}_t g(x)}{g(x)} \leq e^\alpha \implies Q(t) \leq \alpha. \quad (5.2)$$

The connection between $\dim(\mathcal{E})$ and the function Q is the following classical result [4, 18].

Lemma 5.11 (Bowen's Theorem). The Hausdorff dimension of \mathcal{E} corresponds to the solution $t = \dim_H(\mathcal{E})$ to $Q(t) = 0$.

Lemmas 5.10 and 5.11 and the monotonicity of $Q(t)$ immediately yield the following corollary (see Figure 3).

Corollary 5.12. *Let $t_1 > t_0 > 0$ and assume there are positive continuous functions $f, g: [0, 1] \rightarrow \mathbb{R}^+$ and $\alpha, \beta > 0$ such that*

$$\inf_x \frac{\mathcal{M}_{t_0} g(x)}{g(x)} \geq e^\beta \quad \text{and} \quad \sup_x \frac{\mathcal{M}_{t_1} f(x)}{f(x)} \leq e^{-\alpha}.$$

Then $t_0 < \dim_H(\mathcal{E}) < t_1$.

Remark 5.13. *It is worth noting that in the proofs of Theorems 1 and 2 our goal was to estimate the derivative of the pressure function, which required obtaining two upper bounds on expressions of the form $(\mathcal{L}f)/f$. Here, on the other hand, we are aiming to accurately estimate the zero of such function, which is achieved by computing a pair of lower and upper bounds.*

Corollary 5.12 provides us with a criterion that we can easily rigorously verify with the computational method described in Section 4, in particular using Algorithm 1.

Algorithm 3 Rigorous lower/upper bounds on zero of Q

```

function COMPUTE_PRESSURE_ZERO( $t_0, t_1, m, n, c, r, \delta$ )
  while  $t_1 - t_0 > \delta$  ▷ iterate to desired precision  $\delta$ 
     $t \leftarrow (t_0 + t_1)/2$ 
     $\mathcal{M}_t, d\mathcal{M}_t \leftarrow \text{OPERATORS}(t)$ 
     $\alpha_{\min}, \alpha_{\max} \leftarrow \text{COMPUTE\_BOUNDS}(\mathcal{M}_t, d\mathcal{M}_t, m, n, c, r)$ 
    if  $\alpha_{\min} > 0$  then
       $t_0 \leftarrow t$ 
    else if  $\alpha_{\max} < 0$  then
       $t_1 \leftarrow t$ 
    else ▷ also invoke this case if COMPUTE_BOUNDS fails ( $f_{\min} \leq 0$ )
       $m \leftarrow m + 1$ 
  return  $t_0, t_1$ 

```

The entire procedure is described in Algorithm 3. It is based on a binary search principle: beginning with a small initial approximation rank m and an interval $[t_0, t_1]$ guaranteed to contain $\dim_H(\mathcal{E})$, we compute rigorous bounds on \inf and \sup of $\mathcal{M}_t f/f$ for t the mid-point of $[t_0, t_1]$ (COMPUTE_BOUNDS). If $\inf(\mathcal{M}_t f)/f > 1$, then $\dim_H(\mathcal{E}) > t$ and the search continues on the sub-interval $[t, t_1]$; if $\sup(\mathcal{M}_t f)/f < 1$, then $\dim_H(\mathcal{E}) \leq t$ and the search continues on $[t_0, t]$. If neither of the conditions hold, or if the approximate eigenfunction f computed in COMPUTE_BOUNDS fails to be positive (both of which can occur due to insufficiently accurate finite-rank approximation of the operator), then we increment the approximation rank m and repeat the procedure on $[t_0, t_1]$. This iteration proceeds until the required accuracy $\delta > 0$ is reached, that is, until $t_1 - t_0 < \delta$.

We perform this computation with the initial values $t_0 = 0.5$, $t_1 = 0.8$ and $m = 10$, and the parameters $n = 100$, $k = 250$, $\rho = 1.001$ and $R = 5.5$, which yields the asserted value and accuracy for $\dim_H(\mathcal{E})$, proving Theorem 3. We note that in practice the algorithm reaches this accuracy after incrementing the approximation rank for \mathcal{M}_t to $m = 75$.

6 Comparison with other approaches

In this section we compare the estimate in Theorem 1 with the results obtained using other methods.

6.1 The finite section method

There is a more classical approach to estimating the entropy of the Bolyai-Rényi map directly. We have seen that the metric entropy $h(\mu)$ (or equivalently the Lyapunov exponent) can be described using the maximal eigenvalue and the associated eigenfunction (and eigenmeasure) of a transfer operator, see (3.1). We write

$$h(\mu) = \mu(\log |T'|) = \ell^*(\eta \cdot \rho),$$

where ρ, ℓ^* are the eigenfunction and eigenfunctional of \mathcal{L}_0 corresponding to the leading eigenvalue 1, normalized so that $\ell^*(f) = 1$, and $\eta(x) = \log(|T'(x)|)$ for $x \in I = [0, 1]$.

Using the Lagrange-Chebyshev interpolation method (Definition 4.2), we approximate \mathcal{L}_0 by a rank m operator (for some $m \in \mathbb{N}$), represented by an $m \times m$ matrix M . The right and left eigenvectors $v, w \in \mathbb{R}^m$ corresponding to the leading eigenvalue of M give rise to the corresponding eigenfunction ρ_m and the eigenfunctionals ℓ_m of the approximating rank m operator via

$$\rho_m(x) = \sum_{l=0}^{m-1} v_l e_l(x) \quad \text{and} \quad \ell_m(f) = \sum_{l=0}^{m-1} w_l e_l^*(f),$$

where e_m and e_m^* are as in Definition 4.2, up to an (affine) change of coordinates, taking into account the (non-standard) interval $I = [0, 1]$. Writing $b_l = e_l^*(\eta \cdot \rho_m)$, the approximation to the entropy can now be computed as

$$h_m = \frac{\ell_m(\eta \cdot \rho_m)}{\ell_m(\rho_m)} = \frac{\sum_{l=0}^{m-1} b_l w_l}{\sum_{l=0}^{m-1} v_l w_l}.$$

The denominator in this expression is needed for normalization.

Example 6.1. *If we let $m = 100$ and carry out the above calculation we have the following heuristic estimate for the entropy*

$$h_{100} = 1.0563130740\,7297055209\,9568877064\,0651679335\,4262184005\,7092244740\,0283696700\,9505655203\,1501166170\,438688675\dots$$

This agrees with the value in Theorem 1. Theorem 3.3 and Corollary 3 of [2] guarantee exponential convergence (in m) of h_m to $h(\mu)$, where the convergence rate can be bounded using the complex contraction ratios of the inverse branches of T on suitable Bernstein ellipses. However the implied constant depends on the resolvent of the operator, making rigorous error estimates more difficult to obtain in this case.

6.2 The periodic point method

The approach in [7] was based on using the data $(T^n)'(x)$ for fixed points $T^n x = x$ for $T^n: [0, 1] \rightarrow [0, 1]$ ($n \geq 1$). More precisely, one considers a determinant function of two variables formally defined by

$$d(z, s) = \exp \left(- \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{T^n x = x} \frac{|(T^n)'(x)|^{-s}}{1 - 1/(T^n)'(x)} \right),$$

which converges for any $z \in \mathbb{C}$ and $\operatorname{Re}(s)$ sufficiently large, and has an analytic extension to \mathbb{C}^2 . The entropy can then be written in the form

$$h(\mu) = \frac{\partial d(z, s)}{\partial s} / \frac{\partial d(z, s)}{\partial z} \Big|_{s=0, z=1}.$$

In order to convert this into a useful algorithm we truncate the Taylor series for $d(z, s)$ in z . In particular, for any $\frac{1}{3-\sqrt{3}} < \theta < 1$ and $M \geq 1$, we have

$$d(z, s) = 1 + \sum_{n=1}^M a_n(s) z^n + O(\theta^{M^2}),$$

where $a_n(s)$ depends only on the weights associated to periodic points of period at most n .

Example 6.2. With the choice $M = 12$ the entropy was rigorously estimated to be³ $h(\mu) = 1.05631307402 \pm 4.1 \times 10^{-6}$ (see [7, §5]) and heuristically estimated to 9 decimal places as 1.056313074. This required computing the $3^M = 531441$ periodic points $T^n x = x$ and their weights $(T^n)'(x)$ for $1 \leq n \leq 12$.

Although the error terms can be bounded with comparative ease, an issue for this method is the exponential growth in data that needs to be managed as M increases.

7 Final remarks

In this last section, we briefly comment on some directions in which our method and results generalize, as well as on some of their limitations.

³Here the notation $x = A \pm 10^{-\kappa}$ signifies that $A - 10^{-\kappa} \leq x \leq A + 10^{-\kappa}$ holds rigorously.

Remark 7.1. In the interest of clarity, we have chosen to define the operators \mathcal{L}_t , \mathcal{M}_t and $\mathcal{N}_{j,t}$ separately. Instead, we could have introduced a single more general class of operators of the form

$$\widehat{\mathcal{L}}_t f(x) = \sum_{i=0}^2 |T'_i(x)| e^{g(T_i x)} f(T_i x),$$

with different choices of $g \in C^1([0, 1])$ giving rise to the three operators of interest.

Remark 7.2. Our method readily applies to generalizations of the Bolyai-Rényi transformation [19]. As an example, for $m \in \mathbb{N} \setminus \{1\}$ we consider $\tilde{T}_m: [0, 1) \rightarrow [0, 1)$ given by

$$\tilde{T}_m(x) = (x + 1)^m - 1,$$

noting that for $m = 2$ this reduces to the classical Bolyai-Rényi map $T = \tilde{T}_2$. For each m , the map \tilde{T}_m has exactly $2^m - 1$ contractive inverse branches given by $x \mapsto (x + i)^{1/m} - 1$, $i = 1, 2, \dots, 2^m - 1$. Completely analogously to Theorem 1, we can rigorously estimate the entropy $h(\tilde{T}_m, \mu_m)$ (where μ_m denotes the respective \tilde{T}_m -invariant probability measure absolutely continuous to Lebesgue measure). For $m = 3, 4, \dots, 10$ we obtain the following values, each accurate to the number of decimal places presented:

m	$h(\tilde{T}_m, \mu_m)$
3	1.83495 44938 47482...
4	2.50156 90070 03226...
5	3.10685 89449 66953...
6	3.67309 55489 06997...
7	4.21213 20147 18818...
8	4.73108 63064 39220...
9	5.23459 49760 98698...
10	5.72585 67503 35337...

Remark 7.3. We have presented our results in the specific setting of expansions by iterated radicals. This is in part to illustrate the method, and as a paradigm of a more general method, which originated in [15], while also giving better results on the map T , interesting in its own right. The important properties that we need (and which are easy to check for T) are the following:

1. T is conformal (which is automatic since it is one dimensional);
2. T is C^1
3. T has a finite number of inverse branches;
4. T is expanding, i.e., there exists $\kappa > 1$ such that $\inf_x |T'(x)| \geq \kappa$;

5. T is Markov (even Bernoulli, in this case).

This fits naturally in the setting of f -expansions [16]. These properties allow us to make use of Theorem 4, Corollaries 5.5 and 5.12. In order to bound the relevant quantities rigorously to a very high precision using the algorithm presented, we crucially used the fact that T is analytic, ensuring exponential convergence of the Lagrange-Chebyshev interpolation scheme [2].

Remark 7.4. While as above our method generalizes beyond the exact setting of this paper, we should also acknowledge its limitations. In particular, the application of our method presented here is specifically tailored for estimating quantities relating to the top eigenvalue of a transfer operator. It is not directly applicable to estimating quantities such as the rate of mixing of a transformation, which is determined by the transfer operator’s spectral gap and has been previously estimated with the periodic point method [7].

Remark 7.5. A real number is called computable if there is an algorithm (corresponding to a Turing machine) computing a value arbitrarily close to that number. The set of computable numbers is countable. It is a consequence of the algorithm presented that the entropy $h(\mu)$ is a computable number.

Remark 7.6. Our algorithm can be used to obtain any specified accuracy on the estimated quantities (given sufficient computational resources). It is natural to ask how its time complexity grows with an increase in desired accuracy. This boils down to the time complexity of COMPUTE_BOUNDS (Algorithm 1), which is driven by computing the top eigenvalue and corresponding eigenvector of an $m \times m$ matrix, and evaluating the approximation $P_n \psi$ on k intervals. The first of these has a complexity of $O(p \cdot m^2)$, with p the number of steps of the power method (both p and m can be expected to be $O(\log(1/\delta))$ for a desired eigenvector accuracy of δ), the second has a complexity of $O(k \cdot n)$.

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M. Pollicott, Department of Mathematics, University of Warwick, Coventry, CV4 7AL, UK.

E-mail: masdbl@warwick.ac.uk

J. Slipantschuk, Department of Mathematics, University of Warwick, Coventry, CV4 7AL, UK.

E-mail: julia.slipantschuk@warwick.ac.uk