1(a).

$$P(z_n \mid z_{-n}, \beta, w_{1:N}, \alpha)$$

$$= \frac{P(z_n, w_n \mid z_{-n}, \beta, w_{-n}, \alpha)}{P(w_n \mid z_{-n}, \beta, w_{-n}, \alpha)}$$

$$= \frac{P(z_n, w_n \mid z_{-n}, \beta, w_{-n}, \alpha)}{\sum_{z_i} P(w_n, z_i \mid z_{-i}, \beta, w_{-n}, \alpha)} \quad \begin{matrix} ① \\ ② \end{matrix}$$

$$① = \int P(z_n, w_n, \theta \mid z_{-n}, \beta, w_{-n}, \alpha)\, d\theta$$

$$= \int P(z_n \mid \theta)\, P(w_n \mid z_n, \beta)\, P(\theta \mid z_{-n}, \alpha)\, d\theta$$

$$= \int P(w_n \mid z_n, \beta) \cdot \theta_{z_n} P(\theta \mid z_{-n}, \alpha)\, d\theta$$

$$= (\beta_{z_n})_{w_n} \underbrace{\int \theta_{z_n} P(\theta \mid z_{-n}, \alpha)\, d\theta}_{E_{posterior}(\theta) = E(\theta \mid z_{-n}, \alpha)} \quad ✲$$

$\because$ the prior distribution of θ is dirichlet.

$\therefore P(\theta \mid z_{-n}, \alpha) \sim \text{Dirichlet}(\alpha + n_1(z_{-n}), \ldots, \alpha + n_k(z_{-n}))$.

$$\therefore E_{posterior}(\theta) = \frac{\alpha + n_{z_n}(z_{-n})}{k\alpha + N-1}$$

$$\therefore ① = (\beta_{z_n})_{w_n} \cdot \frac{\alpha + n_{z_n}(z_{-n})}{k\alpha + N-1}$$

$$\therefore P(z_n \mid z_{-n}, \beta, w_{1:N}, \alpha) = \frac{(\beta_{z_n})_{w_n} \frac{\alpha + n_{z_n}(z_{-n})}{k\alpha + N-1}}{\sum_{z_i} P(w_n, z_i \mid z_i, \beta, w_{-n}, \alpha)}$$

$$= \frac{(\beta_{z_n})_{w_n} (\alpha + n_{z_n}(z_n))}{\sum_i (\beta_i)_{w_n} (\alpha + N_i(z_{-n}))}.$$

(b). $E(\theta_d \mid \beta_{1:k}, w_{1:N}, \alpha)$.

Since we've shown.

$$P(z_n, w_n \mid z_n, \beta, w_n, \alpha) = \int P(z_n, w_n, \theta \mid z_{-n}, \beta, w_{-n}, \alpha)\, d\theta.$$

$$= (\beta_{z_n})_{w_n} \cdot E(\theta \mid z_{-n}, \alpha).$$

$$E(\theta \mid \beta_{1..k}, w_{1..n}, \alpha) = \frac{1}{(\beta_{z_n})_{w_n}} P(z_n, w_n \mid z_{-n}, \beta, w_{-n}, \alpha)$$

$\therefore E(\theta_d \mid \beta_{1..k}, w_{1..n}, \alpha)$ Could be estimated by sampling from ↓.

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1(c). $P(w_{1:N}, z_{1:N} \mid \beta, \alpha)$

$$= \int P(w_{1:N}, z_{1:N}, \theta \mid \beta, \alpha)\, d\theta$$

$$= \int P(w_{1:N} \mid z_{1:N}, \beta)\, P(z_{1:N} \mid \theta)\, p(\theta \mid \alpha)\, d\theta$$

$$= \int \prod_{n=1}^{n} P(w_n \mid z_n, \beta)\, P(z_n \mid \theta)\, P(\theta \mid \alpha)\, d\theta$$

$$= \int \prod_{n=1}^{N} \beta_{z_n, w_n} \cdot \theta_{z_n}\, Dir(\theta \mid \alpha)\, d\theta$$

$$= \prod_{n=1}^{N} \beta_{z_n, w_n} \cdot \frac{\Gamma(k\alpha)}{\Gamma(\alpha)^k} \cdot \frac{\prod_{k=1}^{k} \Gamma(n_k(z_{1:n}) + \alpha)}{\Gamma(k\alpha + N)}$$

(d). $P(z_n \mid z_{-n}, w, \alpha, \eta)$

$$= \frac{P(z_n, w_n \mid w_{-n}, z_{-n}, \alpha, \eta) \quad ①}{\sum_{z_n} P(z_n, w_n \mid z_{-n}, w_{-n}, \alpha, \eta) \quad ②}$$

$$① = \iint P(z_n, w_n, \beta, \theta \mid z_{-n}, w_{-n}, \alpha, \eta)\, d\beta\, d\theta$$

$$= \iint P(w_n \mid \beta, z_n)\, P(z_n \mid \theta)\, p(\beta \mid z_{-n}, w_{-n}, \eta)\, P(\theta \mid z_{-n}, \alpha)\, d\beta\, d\theta$$

$$(\beta_{z_n})_{w_n} \cdot P(\beta \mid z_{-n} \ldots) \quad \theta_{z_n} P(\theta \mid z_{-n}, \alpha)$$

$$= E(\beta_{z_n, w_n} \mid \underline{z_{-n}, w_{-n}, \eta})\ \underline{E(\theta \mid z_{-n}, \alpha)}$$

$$\hookrightarrow Dir(\alpha + n_1(z_{-n}), \ldots \alpha + n_k(z_{-n}))$$

$$P(\beta_{z_n} \mid z_{-n}, w_{-n}, \eta)$$

$$\propto P(\beta_{z_n}, w_n \mid z_{-n}, \eta)$$

$$\propto P(w_{-n} \mid \beta_{z_n}, z_{-n}, \eta) \cdot P(\beta_{z_n} \mid z_{-n}, \eta)$$

$$\propto P(w_{-n} \mid \beta_{z_n}, z_{-n})\, P(\beta_{z_n} \mid \eta)$$

$$\propto \prod_{n=1}^{N} \beta_{z_{-n}, w_{-n}}\, Dir(\beta_{z_n} \mid \eta)$$

$$= Dir(\beta_{z_n} \mid \eta + m_{z_n,1}(z_{-n}, w_{-n}), \ldots \eta + w_{z_n,V}(z_{-n}, w_{-n}))$$

$$P(z_n \mid z_{-n}, w, \alpha, \eta) = \frac{\dfrac{\alpha + n_{z_n}(z_{-n})}{k\alpha + N - 1} \cdot \dfrac{\eta + m_{z_n, w_n}(z_{-n}, w_{-n})}{V\eta + \sum_v m_{z_n,v}(z_{-n}, w_{-n})}}{\sum_{z_n} P(z_n, w_n \mid z_{-n}, w_{-n}, \alpha, \eta)}$$

$$= \frac{(\alpha + n_{z_n}(z_{-n}))\, \dfrac{\eta + m_{z_n, w_n}(z_{-n}, w_{-n})}{V\eta + \sum_v m_{z_n,v}(z_{-n}, w_{-n})}}{\sum_{k=1}^{k} (\alpha + n_k(z_{-n}))\, \dfrac{\eta + m_{k, w_n}(z_{-n}, w_{-n})}{V\eta + \sum_v m_{k,v}(z_{-n}, w_{-n})}}$$