(a). Equation (1)

$$\ell = \sum_{w\in V}\sum_{c\in V} \#(w,c)\left(\log\left(\sigma(v_w^T v_c)\right) + k\, E_{c_n}\left[\log\left(\sigma(-v_w^T v_{c_n})\right)\right]\right)$$

for an arbitrary $\hat{c}$.

$$\ell = \sum_{w\in V}\sum_{c\in V} \#(w,c)\log\left(\sigma(v_w^T v_c)\right) + \sum_{w\in V}\sum_{c\in V} \#(w,c)\, k\, E_{c_n}\left[\log\left(\sigma(-v_w^T v_{c_n})\right)\right])$$

let $b = \sum_{w\in V}\sum_{c\in V} \#(w,c)\, k\, E_{c_n}\left[\log\left(\sigma(-v_w^T v_{c_n})\right)\right]$

Since $P(c) = \frac{\#(c)}{|D|}$

$\therefore E_{c_n}\left[\log\left(\sigma(-v_w^T v_{c_n})\right)\right] = \frac{\#(\hat{c})}{|D|}\log\left(\sigma(-v_w^T v_{\hat{c}})\right) + \sum_{c_n \in V-\{\hat{c}\}} \frac{\#(c_n)}{|D|}\log\left(\sigma(-v_w^T v_{c_n})\right)$

$\because \#(w) = \sum_{c\in V} \#(w,c)$

$$b = \sum_{w\in V} k\cdot\#(w)\left(\frac{\#(\hat{c})}{|D|}\log\left(\sigma(-v_w^T v_{\hat{c}})\right) + \sum_{c_n\in V-\{\hat{c}\}} \frac{\#(c_n)}{|D|}\log\left(\sigma(-v_w^T v_{c_n})\right)\right)$$

$$\therefore \ell = \sum_{w\in V}\sum_{c\in V} \#(w,c)\log\left(\sigma(v_w^T v_c)\right)$$
$$+ \sum_{w\in V} k\,\#(w)\left(\frac{\#(\hat{c})}{|D|}\log\left(\sigma(-v_w^T v_{\hat{c}})\right) + \sum_{c_n\in V-\{\hat{c}\}} \frac{\#(c_n)}{|D|}\log\left(\sigma(-v_w^T v_{c_n})\right)\right)$$

Since ℓ is the sum of all (w,c) pairs,
the local objective for a specific (w,c) pair is only the terms that are dependent on w and c.

Thus. $\ell(w,c) = \#(w,c)\left(\log\left(\sigma(v_w^T v_c)\right) + k\,\#(w)\frac{\#(c)}{|D|}\log\left(\sigma(-v_w^T v_c)\right)\right)$

(b) Equation (2) : $\ell(w,c) = \#(w,c) \log(\sigma(v_w^T v_c)) + k \cdot \#(w) \frac{\#(c)}{|D|} \log(\sigma(-v_w^T v_c))$

$X = v_w^T v_c$

$$\sigma(x) = \frac{1}{1+e^{-x}} \qquad \sigma(-x) = \frac{1}{1+e^{x}}$$

$$\frac{d\sigma(x)}{dx} = -(1+e^{-x})^{-2} \cdot -e^{-x} \qquad \frac{d\sigma(-x)}{dx} = -(1+e^{x})^{-2} \cdot e^{x}$$

$$= (1+e^{-x})^{-2} \cdot e^{-x}$$

$$\ell(w,c) = \#(w,c) \log\left(\frac{1}{1+e^{-x}}\right) + \frac{k\#(w)\#(c)}{|D|} \log\left(\frac{1}{1+e^{x}}\right)$$

$$= -\#(w,c) \cdot \log(1+e^{-x}) - \frac{k\#(w)\#(c)}{|D|} \log(1+e^{x})$$

$$\frac{d\ell}{dx} = \frac{+\#(w,c)}{1+e^{-x}} \cdot e^{-x} - \frac{k\#(w)\#(c)}{|D|} \cdot \frac{e^{x}}{1+e^{x}} = 0$$

$$\#(w,c) \cdot e^{-x}(1+e^{x}) - \frac{k\#(w)\#(c)}{|D|} \cdot e^{x}(1+e^{-x}) = 0$$

$$\#(w,c) \cdot e^{\cancel{-x}} + \#(w,c) - \frac{k\#(w)\#(c)}{|D|} \cdot e^{x} - \frac{k\#(w)\#(c)}{|D|} = 0$$

$$-\frac{k\#(w)\#(c)}{|D|} \cdot e^{2x} + e^{x}\left(\#(w,c) - \frac{k\#(w)\#(c)}{|D|}\right) + \#(w,c) = 0$$

$$e^{2x} - e^{x}\left(\frac{\#(w,c)}{k\#(w)\frac{\#(c)}{|D|}} - 1\right) - \frac{\#(w,c)}{k \cdot \#(c)\frac{\#(w)}{|D|}} = 0$$

(c). Equation (4)

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$e^{2x} - \left(\frac{\#(w,c)}{k \cdot \#(w) \cdot \frac{\#(c)}{|D|}} - 1 \right) e^x - \frac{\#(w,c)}{k \cdot \#(w) \cdot \frac{\#(c)}{|D|}} = 0.$$

$$y = e^x$$

$$y^2 - \left(\frac{\#(w,c)}{k \cdot \frac{\#(w) \cdot \#(c)}{|D|}} - 1 \right) y - \frac{\#(w,c)}{k \cdot \frac{\#(w) \cdot \#(c)}{|D|}} = 0$$

$$y = \frac{\left(\frac{\#(w,c)|D|}{k\#(w)\#(c)} - 1 \right)^2 \pm \sqrt{\left(\frac{\#(w,c)|D|}{k\#(w)\#(c)} - 1 \right)^2 + 4 \cdot \frac{\#(w,c)|D|}{k\#(w)\#(c)}}}{2}$$

$$\Delta = \frac{\#(w,c)^2 |D|^2}{k^2 \#(w)^2 \#(c)^2} + \frac{2\#(w,c)|D|}{k\#(w)\#(c)} + 1$$

$$y_1 = \frac{\#(w,c)|D|}{k\#(w)\#(c)} \qquad y_2 = -1$$

$$e^{v_w^T v_c} = \frac{\#(w,c)|D|}{k\#(w)\#(c)}$$

$$v_w^T v_c = \log\left(\frac{\#(w,c) \cdot |D|}{\#(w) \cdot \#(c)} \right) - \log k.$$