Q_1 1(a). $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$$\therefore \hat{m}_n(x) = \begin{pmatrix} \hat{m}_n(x_1) \\ \hat{m}_n(x_2) \\ \vdots \\ \hat{m}_n(x_n) \end{pmatrix} = \begin{pmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \vdots \\ \hat{y}_n \end{pmatrix} = \begin{pmatrix} \quad \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

[1] If $\hat{m}_h()$ is a Smoothing kernel estimator.

$$\hat{m}_h(x) = \frac{\sum_{i=1}^n Y_i k_h(X_i, x)}{\sum_{i=1}^n k_h(X_i, x)} \qquad k_h(x, z) = \exp\left(-\frac{\|x-z\|^2}{2\underbrace{h^2}_{\text{bandwidth}}}\right)$$

$$= \frac{\sum_{i=1}^n \exp\left(-\frac{\|X_i - x\|^2}{2h^2}\right) y_i}{\sum_{i=1}^n \exp\left(-\frac{\|X_i - x\|^2}{2h^2}\right)} = LY$$

[2] If $\hat{m}_h$ is a mercer kernel.

$$\hat{m}(x) = \sum \hat{\alpha}_i K(X_i, x) \qquad \hat{\alpha} = (\mathbb{K} + \lambda I)^{-1} Y$$

$$\mathbb{K}_{ij} = k(X_j, X_k)$$

$$= \underbrace{\sum_{i=1}^n k(X_i, x) \cdot (\mathbb{K} + \lambda I)^{-1}}_{} Y$$

$\because$ Both kernel estimator are linear. L

$\therefore$ both kernel estimator can be written as

$$\hat{m}_h(x) = \sum_{i=1}^n l_i(x) Y_i$$

$$\begin{pmatrix} \hat{m}_n(x_1) \\ \hat{m}_n(x_2) \\ \vdots \\ \hat{m}_n(x_n) \end{pmatrix} = L \cdot \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}$$

$$\Rightarrow \hat{m}_n(X) = LY$$

(b). $\hat{R}(h) = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \hat{Y}_{(i)})^2$

$$= \frac{1}{n}\sum_{i=1}^{n}(Y_i - \hat{m}_{(-i)}(x_i))^2$$

$\hat{m}_{(-i)}$ is the estimator obtained by omitting ith pair.

$$\hat{m}_{(-i)}(x) = \sum_{j=1}^{n} Y_i \, l_{j(-i)}(x).$$

$$l_{j,(-i)}(x) = \begin{cases} 0 & \text{if } j = i. \\ \dfrac{l_j(x)}{\sum_{k\neq i} l_k(x)} & \text{if } j \neq i \end{cases}$$

$$\therefore \hat{R}(h) = \frac{1}{n}\sum_{i=1}^{n}\Big| Y_i - \overbrace{\sum_{j=1}^{n} Y_i \, l_{j,(-i)}(x)}^{\text{sum of off diagonal elements of } L.}\Big|^2.$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(\frac{Y_i - \hat{m}_n(x_i)}{1 - L_{ii}}\right)^2.$$