Matrix schema for forward/inverse, 1D/2D, non-normalized/normalized and cascade/packet Haar multi-resolution analysis

1.1 Haar matrix for a single resolution analysis

The N-point Haar matrix ($\mathbf{H}_N$) for decomposing a vector just once can be written as:

$$\mathbf{H}_N = f \cdot \mathbf{T}_N \tag{1}$$

where

$$\mathbf{T}_N = \begin{bmatrix} \mathbf{C}_{\tilde{N}} \\ \mathbf{D}_{\tilde{N}} \end{bmatrix}_{N \times N}, \tag{2}$$

$$\mathbf{C}_{\tilde{N}} = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix}_{\frac{N}{2} \times N} \tag{3}$$

and

$$\mathbf{D}_{\tilde{N}} = \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix}_{\frac{N}{2} \times N}. \tag{4}$$

The parameter f can be either $\frac{1}{2}$ or $\frac{1}{\sqrt{2}}$ for non-normalized or normalized Haar row vectors, respectively.

1.1.1 Forward and inverse transform applications

For decompose just once a column vector $\mathbf{v}$ of dimension N or a matrix $\mathbf{v}$ of dimension $N \times N$ through the Haar wavelet transform, we should apply, respectively for 1D and 2D cases:

$$\mathbf{w} = \mathbf{H}_N \cdot \mathbf{v} \tag{5}$$

$$\mathbf{w} = \mathbf{H}_N \cdot \mathbf{v} \cdot \mathbf{H}_N^{-1}. \tag{6}$$

Analogously, the inverse Haar transform for 1D and 2D cases, respectively, are:

$$\mathbf{v} = \mathbf{H}_N^{-1} \cdot \mathbf{w} \tag{7}$$

$$\mathbf{v} = \mathbf{H}_N^{-1} \cdot \mathbf{w} \cdot \mathbf{H}_N. \tag{8}$$

Note that, if we are considering the normalized Haar, i.e. $f = \frac{1}{\sqrt{2}}$, $\mathbf{H}_N^{-1} = \mathbf{H}_N^{\top}$.

1.2 Haar matrix for multiple resolution analysis

Before of all, when dealing with multi-resolution analysis (MRA), we need to decide between either the cascade or packet algorithm. Matrix formulation of both is separately explored in the following subsections.

1.2.1 Cascade algorithm for MRA

To pack l Haar decomposition levels into a single N-point matrix ($\mathbf{H}_N^l$) using the cascade algorithm, consider the following schema:

$$\mathbf{H}_N^l = (\prod_{k=\log_2(N)+1-l}^{\log_2(N)-1} \mathbf{H'}_{N,2^k}) \cdot (f \cdot \mathbf{T}_N) \tag{9}$$

where

$$\mathbf{H'}_{N,2^k} = \begin{bmatrix} f \cdot \mathbf{T}_{2^k} & \mathbf{Z}_{2^k,N-2^k} \\ \mathbf{Z}_{N-2^k,2^k} & \mathbf{I}_{N-2^k} \end{bmatrix}_{N\times N} \tag{10}$$

and $\mathbf{Z}_{m,n}$ and $\mathbf{I}_n$ are, respectively, a zeroed matrix with m rows and n columns and the Identity of order n.

An interesting option for Equations 9 and 10 is to consider, instead of the floating point arithmetics, the integer one. It can be accomplished once the individual factors f for each $\mathbf{H}'_{N,2^k}$ submatrix can be computed after multiplication of their $\{-1, 0, 1\}$-defined versions. It would save computational rounding errors mainly if $f = \frac{1}{\sqrt{2}}$ is considered. This solution is given by:

$$\mathbf{H}_N^l = \mathbf{S}_N \cdot \tilde{\mathbf{H}}_N^l, \tag{11}$$

where $\tilde{\mathbf{H}}_N^l$ is the same that Equation 9, but without f (or with $f = 1$). This removal or resetting of parameter f should be also applied to Equation 10, producing $\tilde{\mathbf{H}}'_{N,2^k}$ matrices. Then, by polar decomposition, $\mathbf{S}_N$ is a diagonal matrix defined by

$$\mathbf{S}_N = \sqrt{(\tilde{\mathbf{H}}_N^l \cdot \tilde{\mathbf{H}}_N^{l\top})^{-1}} \tag{12}$$

or

$$\mathbf{S}_N = (\tilde{\mathbf{H}}_N^l \cdot \tilde{\mathbf{H}}_N^{l\top})^{-1} \tag{13}$$

depending on the desired transformation: normalized (Equation 12) or non-normalized (Equation 13).

1.2.2 Packet algorithm for MRA

Here, not so different from Equation 9, the single matrix that performs the packet algorithm in l levels of Haar decomposition is given by:

$$\mathbf{H}_N^l = (\prod_{k=\log_2(N)+1-l}^{\log_2(N)-1} \mathbf{H'}_{N,2^k}) \cdot (f \cdot \mathbf{T}_N) \tag{14}$$

where

$$\mathbf{H'}_{N,2^k} = \begin{bmatrix} f \cdot \mathbf{T}_{2^k} & \mathbf{Z}_{2^k} & \cdots & \mathbf{Z}_{2^k} \\ \mathbf{Z}_{2^k} & f \cdot \mathbf{T}_{2^k} & \cdots & \mathbf{Z}_{2^k} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{Z}_{2^k} & \mathbf{Z}_{2^k} & \cdots & f \cdot \mathbf{T}_{2^k} \end{bmatrix}_{N \times N} \tag{15}$$

is the concatenation of all possibilities of submatrices $\mathbf{T}_{2^k}$ inside a N-point matrix with their diagonal aligned to the diagonal of the larger matrix.

Not surprisingly, matrix $\mathbf{H}_N^l$ can also be computed using integer arithmetic, generating $\tilde{\mathbf{H}}_N^l$. Once again, it is done by removing (or setting to one) the parameter f from Equations 14 and 15. Interestingly, Equations 11, 12 and 13, as their interpretations, remain the same.

1.2.3 Forward and inverse transform applications

Analogously to Section 1.1.1, the same vector $\mathbf{v}$ can be decomposed now l times through:

$$\mathbf{w} = \mathbf{H}_N^l \cdot \mathbf{v} \tag{16}$$

$$\mathbf{w} = \mathbf{H}_N^l \cdot \mathbf{v} \cdot \mathbf{H}_N^{l\,-1}. \tag{17}$$

and reconstructed through:

$$\mathbf{v} = \mathbf{H}_N^{l\,-1} \cdot \mathbf{w} \tag{18}$$

$$\mathbf{v} = \mathbf{H}_N^{l\,-1} \cdot \mathbf{w} \cdot \mathbf{H}_N^l. \tag{19}$$

Note again that $\mathbf{H}_N^l$ can be computed either by Equation 9 and 11 for cascade algorithm and Equation 14 and 11 for packet algorithm. Figure 1 illustrates the construction of a $(N = 8)$-point Haar integer transform in three levels, i.e. $\tilde{\mathbf{H}}_8^3$. Note that if $\mathbf{A}_4$ is equal to the top block of order 4 and $\mathbf{B}_2$ are equal to the top block of order 2, then we are applying the packet algorithm. By other hand, if $\mathbf{A}_4$ and $\mathbf{B}_2$ blocks are the Identity of orders 4 and 2, respectively, so we have the cascade algorithm.

1.3 Final considerations

This matrix-based framework allows to simplify the computations for the Haar transform in following aspects: forward and inverse, one- and bi-dimensional, non-normalized and normalized and cascade and packet algorithms. It also allows to construct an analysis that combines, by example, the two algorithms for decomposing differently depending on the level l. The possibility of creating this all-in-one MRA matrix with only integer numbers also can be applied in image and video coding for reducing the computational complexity of the transformation step, once $\mathbf{S}_N$ matrix can be merged to the quantization step.

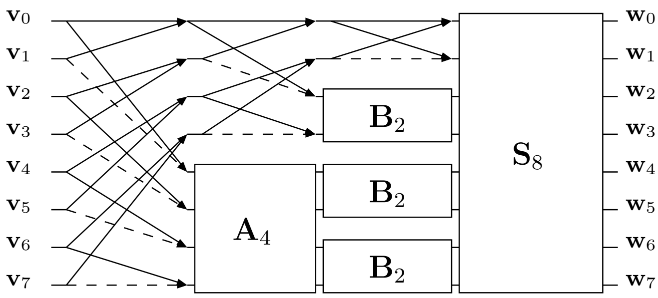

Figure 1: Signal-flow graph for $\mathbf{H}_N^l$ with $l = 3$ and $N = 8$. The input data $\mathbf{v}_i$, $i = 0, 1, \ldots, 7$ relates to the output data $\mathbf{w}_j$, $j = 0, 1, \ldots, 7$ according to Equation 16. Dashed arrows represent multiplications by -1.