SEMINAR
HW 1

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Theorem. *(3.6.10)*
If $ne^{-r/n} \to \lambda \in [0, \infty)$ *the number of empty boxes approaches a Poisson distribution with mean* λ.

Proof. To see where the answer comes from, notice that in the Poisson approximation the probability that a given box is empty is $e^{-r/n} \approx \lambda/n$, so if the occupancy of the various boxes were independent, the result would follow from Theorem 3.6.1. To prove the result, we begin by observing

$$P(\text{ boxes } i_1, i_2, \dots, i_k \text{ are empty }) = \left(1 - \frac{k}{n}\right)^r$$

If we let $p_m(r, n) =$ the probability exactly m boxes are empty when r balls are put in n boxes, then $P(\text{ no empty box }) = 1 - P(\text{ at least one empty box })$. So by inclusion-exclusion

$$p_0(r, n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(1 - \frac{k}{n}\right)^r \tag{a}$$

By considering the locations of the empty boxes

$$p_m(r, n) = \binom{n}{m} \left(1 - \frac{m}{n}\right)^r p_0(r, n - m) \tag{b}$$

To evaluate the limit of $p_m(r, n)$ we begin by showing that if $ne^{-r/n} \to \lambda$ then

$$\binom{n}{m} \left(1 - \frac{m}{n}\right)^r \to \lambda^m/m! \tag{c}$$

One half of this is easy. Since $(1 - x) \leq e^{-x}$ and $ne^{-r/n} \to \lambda$

$$\binom{n}{m} \left(1 - \frac{m}{n}\right)^r \leq \frac{n^m}{m!} e^{-mr/n} \to \lambda^m/m! \tag{d}$$

For the other direction, observe $\binom{n}{m} \geq (n - m)^m/m!$ so

$$\binom{n}{m} \left(1 - \frac{m}{n}\right)^r \geq \left(1 - \frac{m}{n}\right)^{m+r} n^m/m!$$

Now $(1-m/n)^m \to 1$ as $n \to \infty$ and $1/m!$ is a constant. To deal with the rest, we note that if $0 \leq t \leq 1/2$ then

$$\begin{aligned} log(1 - t) &= -t - t^2/2 - t^3/3 \dots \\ &=\geq -t - \frac{t^2}{2}(1 + 2^{-1} + 2^{-2} + \cdots) = -t - t^2 \end{aligned}$$

So we have

$$\log\left(n^m\left(1-\frac{m}{n}\right)^r\right) \geq m\log n - rm/n - r(m/n)^2$$

Our assumption $ne^{-r/n} \to \lambda$ means

$$r = n\log n - n\log\lambda + o(n)$$

so $r(m/n)^2 \to 0$. Multiplying the last display by m/n and rearranging gives $m\log n - rm/n \to m\log\lambda$. Combining the last two results shows

$$\liminf_{n\to\infty} n^m\left(1-\frac{m}{n}\right)^r \geq \lambda^m$$

and (c) follows. From (a), (c), and the dominated convergence theorem(using (d) to get the domination) we get

(e) if $ne^{-r/n} \to \lambda$ then $p_0(r,n) \to \sum_{k=0}^{\infty}(-1)^k\frac{\lambda^k}{k!} = e^{-\lambda}$

For fixed m, $(n-m)e^{-r/(n-m)} \to \lambda$, so it follows from (e) that $p_0(r,n-m) \to e^{-\lambda}$. Combining this with (b) and (c) completes the proof. $\square$