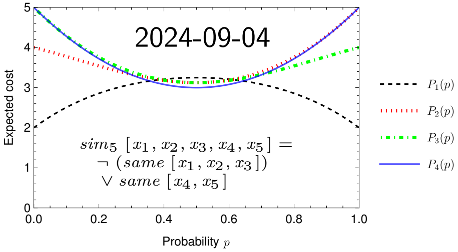
Level-p-complexity of Boolean Functions

Using thinning, memoization, and polynomials

Patrik Jansson[1]

Functional Programming unit, Chalmers University of Technology



[1] Joint with Julia Jansson, Math@Chalmers. Paper: doi:10.1017/S0956796823000102, code@GitHub.

Explain the setting: BoFun, DecTree, cost, expected cost

- Start with some boolean function $f : \mathbb{B}^n \to \mathbb{B}$

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  - query one bit (index) at a time
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&\textbf{data}\ \mathit{DecTree}\ \textbf{where}\\
&\quad \mathit{Pick} :: \mathit{Index} \to \mathit{DecTree} \to \mathit{DecTree} \to \mathit{DecTree}\\
&\quad \mathit{Res}\ \ :: \mathbb{B} \to \mathit{DecTree}
\end{aligned}
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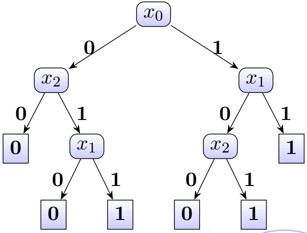
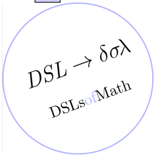
Example: a $\mathit{DecTree}$ for maj_3:





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  $cost : DecTree \to \mathbb{B}^n \to \mathbb{N}$
  - for every n-bit word (there are 2^n such words)

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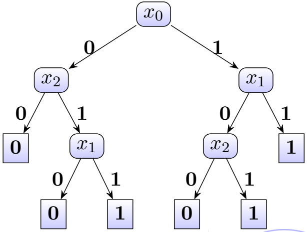

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- We compute the cost of a dectree (for our boolean function) $cost : DecTree \to \mathbb{B}^n \to \mathbb{N}$
  - for every n-bit word (there are 2^n such words)
- Then compute the *expected* cost over all 2^n words $\to$ a polynomial in the probability a bit is 1

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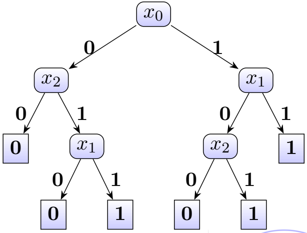

Examples of expected cost and level-p-complexity: $D_f(p)$

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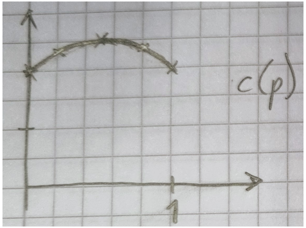

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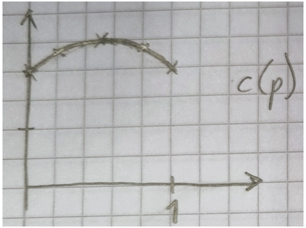

- Example 1: maj_3, $expCost\ t = 2 + 2p(1-p)$ (for all t)
- Example 2: sim_5,
  - $P_1\ p = expCost\ t_1 = 2 + 6p - 10p^2 + 8p^3 - 4p^4$
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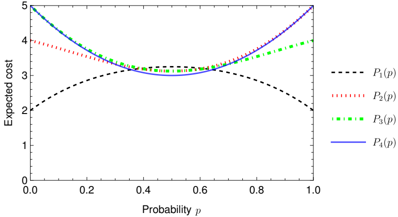

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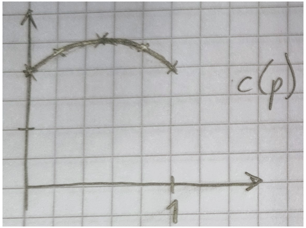

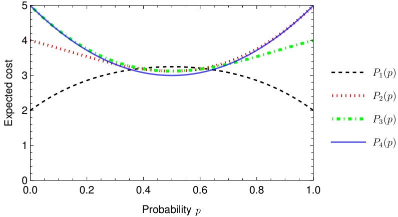

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- This minimum is the level-p-complexity $D_f(p)$ of f.
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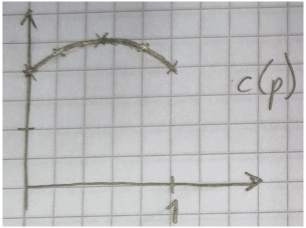

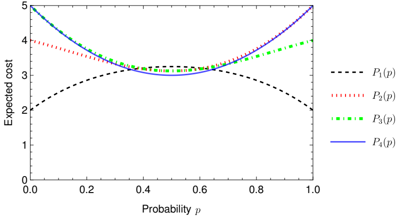

Core algorithm for computing the level-p-complexity of f

Spec: "generate all decision trees and pick the best one(s)"

- Informally: to generate all the decision trees:
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- How many $\mathit{DecTree}$s can there be? In the worst case we get $T_0 = 1$; $T_n = n \cdot T^2_{n-1}$.
- An estimate of the number of different $\mathit{DecTree}$s for a function of n bits is $2^{2^{n-1}}$ or simply **FAR TOO MANY**.

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- Success? Not yet.

Reuse of intermediate results (memoization)

- We can now easily compute up to n=5 but n=9 is still beyond reach.
- In a motivating example (iterated majority on $3 \cdot 3 = 9$ bits) we now get just 1 polynomial at the end, but we still have $2 \cdot n$ immediate subfunctions (recursive calls) and around **100 million** recursive calls in total.

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- This turns out to be a great idea - for our running example the number of subfunctions (unique recursive calls) is only 215.
- Finally we can compute the Level-p-complexity for maj_3^2 in around a few seconds. (Still 100 million recursive calls, but almost all are just a lookup in a memo-table.)

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- To implement comparison of polynomials we need to know (exactly) where they intersect. In effect this requires computing with Algebraic numbers (symbolic computations with polynomials: root counting, factoring, gcd, etc). Details in the paper and the code.
- To go further, beyond $n = 9$, we have used more properties of special classes of boolean functions (iterated threshold functions) to reduce the computational cost even further. (In the repo we have reached $n = 27$ - thanks to help from Christian Sattler.)

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Conclusions / Questions?

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Paper: doi:10.1017/S0956796823000102, code: github:juliajansson/BoFunComplexity.

Joint work with Julia Jansson (PhD student @ Math.chalmers.se)

Motivating example: two-level iterated majority

Our running example is a simple case of two-level majority ($maj_3^2 : \mathbb{B}^9 \to \mathbb{B}$).

$$\underbrace{\underbrace{x_{(1,1)}, x_{(1,2)}, x_{(1,3)}}_{m_1 = maj_3\ (\ldots)}, \underbrace{x_{(2,1)}, x_{(2,2)}, x_{(2,3)}}_{m_2 = maj_3\ (\ldots)}, \underbrace{x_{(3,1)}, x_{(3,2)}, x_{(3,3)}}_{m_3 = maj_3\ (\ldots)}}_{maj_3(m_1, m_2, m_3)}$$

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Example 1: Five $\mathbf{0}$ votes, four $\mathbf{1}$ votes, even distribution:

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Example 2: Five $\mathbf{0}$ votes, four $\mathbf{1}$ votes, regrouped (perhaps through gerrymandering):

$$\underbrace{\underbrace{\mathbf{1}, \mathbf{1}, \mathbf{0}}_{m_1 = \mathbf{1}}, \underbrace{\mathbf{1}, \mathbf{0}, \mathbf{1}}_{m_2 = \mathbf{1}}, \underbrace{\mathbf{0}, \mathbf{0}, \mathbf{0}}_{m_3 = \mathbf{0}}}_{maj_3 = \mathbf{1}}$$

Result for maj_3^2 and sim_5

$$
\begin{aligned}
&ps_9 = genAlgThinMemo\ 9\ \ maj_3^2 :: Set\ (Poly\ \mathbb{Q})\\
&check_9 = ps_9 == fromList\ [\ P\ [4,4,6,9,-61,23,67,-64,16]]\\
&ps5 = genAlgThinMemo\ 5\ sim_5\ :: Set\ (Poly\ \mathbb{Q})\\
&check5 = ps5 == fromList\ [P\ [2,6,-10,8,-4], P\ [4,-2,-3,8,-2],\\
&\qquad\qquad\qquad\qquad\qquad P\ [5,-8,9,0,-2],\ \ P\ [5,-8,8]]
\end{aligned}
$$

Subfunctions illustrated

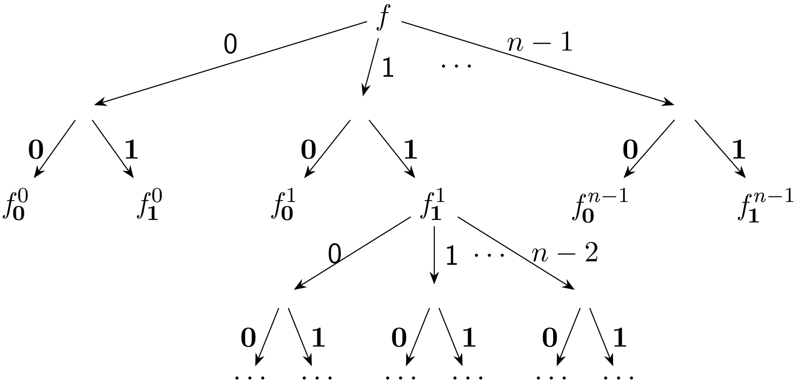

Figure: The tree of subfunctions of a Boolean function f. For brevity $setBit\ i\ b\ f$ is denoted f^i_b. This tree structure is also the call-graph for our generation of decision trees. Note that this tree structure is related to, but not the same as, the decision trees.

A class of Boolean functions

We abstract from the actual type for Boolean functions to a type class:

$$
\begin{array}{l}
\textbf{class}\ \mathit{BoFun}\ \mathit{bf}\ \textbf{where} \\
\quad \mathit{isConst} :: \mathit{bf} \ \rightarrow\ \mathit{Maybe}\ \mathbb{B} \\
\quad \mathit{setBit} \quad :: \mathit{Index} \rightarrow \mathbb{B} \rightarrow \mathit{bf} \rightarrow \mathit{bf}
\end{array}
$$

We only use one instance: Binary Decision Diagrams (BDDs) which allows good sharing and fast operations.

A class of decision tree algebras

Similarly, for decision trees, we use a class of tree algebras:

$$\begin{array}{l} \textbf{class}\ \mathit{TreeAlg}\ a\ \textbf{where} \\ \quad \mathit{res} :: \mathbb{B} \to a \\ \quad \mathit{pic} :: \mathit{Index} \to a \to a \to a \end{array}$$

$$\begin{array}{l} \mathit{fold}_{DT} :: \mathit{TreeAlg}\ a \Rightarrow \mathit{DecTree} \to a \\ \mathit{fold}_{DT}\ (\mathit{Res}\ b) \qquad\quad = \mathit{res}\ b \\ \mathit{fold}_{DT}\ (\mathit{Pick}\ i\ t_0\ t_1) = \mathit{pic}\ i\ (\mathit{fold}_{DT}\ t_0)\ (\mathit{fold}_{DT}\ t_1) \end{array}$$

$$\begin{array}{l} \mathit{ex1} :: \mathit{TreeAlg}\ a \Rightarrow a \\ \mathit{ex1} = \mathit{pic}\ 0\ (\mathit{pic}\ 2\ (\mathit{res}\ \mathbf{0})\ (\mathit{pic}\ 1\ (\mathit{res}\ \mathbf{0})\ (\mathit{res}\ \mathbf{1}))) \\ \qquad\qquad\qquad (\mathit{pic}\ 1\ (\mathit{pic}\ 2\ (\mathit{res}\ \mathbf{0})\ (\mathit{res}\ \mathbf{1}))\ (\mathit{res}\ \mathbf{1})) \end{array}$$

$$\begin{array}{llll} \textbf{instance} & \mathit{TreeAlg}\ \mathit{DecTree} & \textbf{where}\ \mathit{res} = \mathit{Res}; & \mathit{pic} = \mathit{Pick}; \\ \textbf{instance} & \mathit{TreeAlg}\ \mathit{CostFun} & \textbf{where}\ \mathit{res} = \mathit{resC}; & \mathit{pic} = \mathit{pickC} \\ \textbf{instance}\ \mathit{Ring}\ a \Rightarrow & \mathit{TreeAlg}\ (\mathit{ExpCost}\ a) & \textbf{where}\ \mathit{res} = \mathit{resPoly}; & \mathit{pic} = \mathit{pickPoly} \end{array}$$

A class of decision tree algebras

Similarly, for decision trees, we use a class of tree algebras:

$$\begin{array}{l} \textbf{class}\ \mathit{TreeAlg}\ a\ \textbf{where} \\ \quad \mathit{res} :: \mathbb{B} \to a \\ \quad \mathit{pic} :: \mathit{Index} \to a \to a \to a \end{array}$$

$$\begin{array}{l} \mathit{fold}_{DT} :: \mathit{TreeAlg}\ a \Rightarrow \mathit{DecTree} \to a \\ \mathit{fold}_{DT}\ (\mathit{Res}\ b) \qquad\quad = \mathit{res}\ b \\ \mathit{fold}_{DT}\ (\mathit{Pick}\ i\ t_0\ t_1) = \mathit{pic}\ i\ (\mathit{fold}_{DT}\ t_0)\ (\mathit{fold}_{DT}\ t_1) \end{array}$$

$$\begin{array}{l} \mathit{ex1} :: \mathit{TreeAlg}\ a \Rightarrow a \\ \mathit{ex1} = \mathit{pic}\ 0\ (\mathit{pic}\ 2\ (\mathit{res}\ \mathbf{0})\ (\mathit{pic}\ 1\ (\mathit{res}\ \mathbf{0})\ (\mathit{res}\ \mathbf{1}))) \\ \qquad\qquad\quad (\mathit{pic}\ 1\ (\mathit{pic}\ 2\ (\mathit{res}\ \mathbf{0})\ (\mathit{res}\ \mathbf{1}))\ (\mathit{res}\ \mathbf{1})) \end{array}$$

$$\begin{array}{l} \textbf{instance}\ \mathit{TreeAlg}\ \mathit{DecTree}\ \textbf{where}\ \mathit{res} = \mathit{Res}; \mathit{pic} = \mathit{Pick}; \\ \textbf{data}\ \mathit{DecTree}\ \textbf{where} \\ \quad \mathit{Res}\ \ :: \mathbb{B} \to \mathit{DecTree} \\ \quad \mathit{Pick} :: \mathit{Index} \to \mathit{DecTree} \to \mathit{DecTree} \to \mathit{DecTree} \end{array}$$

A class of decision tree algebras

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$$\begin{aligned}&\textbf{class}\ \mathit{TreeAlg}\ a\ \textbf{where}\\ &\quad \mathit{res} :: \mathbb{B} \to a\\ &\quad \mathit{pic} :: \mathit{Index} \to a \to a \to a\end{aligned}$$

$$\begin{aligned}&\mathit{fold}_{DT} :: \mathit{TreeAlg}\ a \Rightarrow \mathit{DecTree} \to a\\ &\mathit{fold}_{DT}\ (\mathit{Res}\ b) = \mathit{res}\ b\\ &\mathit{fold}_{DT}\ (\mathit{Pick}\ i\ t_0\ t_1) = \mathit{pic}\ i\ (\mathit{fold}_{DT}\ t_0)\ (\mathit{fold}_{DT}\ t_1)\end{aligned}$$

$$\begin{aligned}&\mathit{ex1} :: \mathit{TreeAlg}\ a \Rightarrow a\\ &\mathit{ex1} = \mathit{pic}\ 0\ (\mathit{pic}\ 2\ (\mathit{res}\ \mathbf{0})\ (\mathit{pic}\ 1\ (\mathit{res}\ \mathbf{0})\ (\mathit{res}\ \mathbf{1})))\\ &\qquad\qquad (\mathit{pic}\ 1\ (\mathit{pic}\ 2\ (\mathit{res}\ \mathbf{0})\ (\mathit{res}\ \mathbf{1}))\ (\mathit{res}\ \mathbf{1}))\end{aligned}$$

$$\begin{aligned}&\textbf{instance}\ \mathit{TreeAlg}\ \mathit{CostFun}\ \textbf{where}\ \mathit{res} = \mathit{resC}; \mathit{pic} = \mathit{pickC}\\ &\textbf{type}\ \mathit{CostFun} = \mathbb{B}^n \to \mathit{Int}\\ &\mathit{resC}\ b = \mathit{const}\ 0\\ &\mathit{pickC}\ i\ c_0\ c_1 = \lambda t \to 1 + \textbf{if}\ \mathit{index}\ t\ i\ \textbf{then}\ c_1\ t\ \textbf{else}\ c_0\ t\end{aligned}$$

A class of decision tree algebras

Similarly, for decision trees, we use a class of tree algebras:

$$\begin{array}{l} \textbf{class } \mathit{TreeAlg}\ a\ \textbf{where} \\ \quad \mathit{res} :: \mathbb{B} \to a \\ \quad \mathit{pic} :: \mathit{Index} \to a \to a \to a \end{array}$$

$$\begin{array}{l} \mathit{fold}_{DT} :: \mathit{TreeAlg}\ a \Rightarrow \mathit{DecTree} \to a \\ \mathit{fold}_{DT}\ (\mathit{Res}\ b) \qquad\quad = \mathit{res}\ b \\ \mathit{fold}_{DT}\ (\mathit{Pick}\ i\ t_0\ t_1) = \mathit{pic}\ i\ (\mathit{fold}_{DT}\ t_0)\ (\mathit{fold}_{DT}\ t_1) \end{array}$$

$$\begin{array}{l} \mathit{ex1} :: \mathit{TreeAlg}\ a \Rightarrow a \\ \mathit{ex1} = \mathit{pic}\ 0\ (\mathit{pic}\ 2\ (\mathit{res}\ \mathbf{0})\ (\mathit{pic}\ 1\ (\mathit{res}\ \mathbf{0})\ (\mathit{res}\ \mathbf{1}))) \\ \qquad\qquad\quad (\mathit{pic}\ 1\ (\mathit{pic}\ 2\ (\mathit{res}\ \mathbf{0})\ (\mathit{res}\ \mathbf{1}))\ (\mathit{res}\ \mathbf{1})) \end{array}$$

$$\begin{array}{l} \textbf{instance } \mathit{Ring}\ a \Rightarrow \mathit{TreeAlg}\ (\mathit{ExpCost}\ a)\ \textbf{where } \mathit{res} = \mathit{resPoly}; \mathit{pic} = \mathit{pickPoly} \\ \textbf{type } \mathit{ExpCost}\ a = \mathit{Poly}\ a \\ \mathit{resPoly}\ \_b = \mathit{zero} \\ \mathit{pickPoly}\ \_\ p_0\ p_1 = \mathit{one} + (\mathit{one} - \mathit{xP}) \times p_0 + \mathit{xP} \times p_1 \end{array}$$

Memoization reminder

- Naively computing $\mathit{fib}\ (n+2) = \mathit{fib}\ (n+1) + \mathit{fib}\ n$ leads to exponential growth in the number of function calls.
- But if we fill in a table indexed by n with already computed results we get a the result in linear time.

Conjecture from the MSc thesis

These are the polynomials from the MSc thesis (P_t) and the best one (P_*):

$$P_t = P\,[4, 4, 7, 6, -57, 20, 68, -64, 16]$$
$$P_* = P\,[4, 4, 6, 9, -61, 23, 67, -64, 16]$$

Comparing the two polynomials shows that the new one has strictly lower expected cost than the one from the thesis. The difference is $p^2(1-p)^2(1-p+p^2)$ is illustrated here:
It is non-negative in the whole interval.
The value of both polynomials at the endpoints is 4 and the maximum of P_* is ≈ 6.14 compared to the maximum of P_t which is ≈ 6.19.