Level-p-complexity of Boolean Functions

Using thinning, memoization, and polynomials

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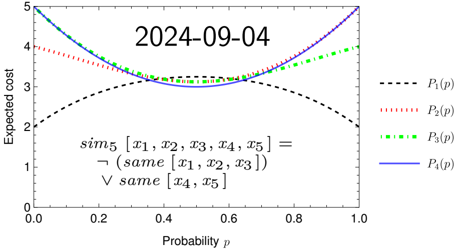
Functional Programming unit, Chalmers University of Technology



DSL → δσλ
DSLsofMath

[1] Joint with Julia Jansson, Math@Chalmers. Paper: doi:10.1017/S0956796823000102, code@GitHub.

Explain the setting: BoFun, DecTree, cost, expected cost

- Start with some boolean function $f : \mathbb{B}^n \to \mathbb{B}$

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\quad \mathit{Res}\ \ :: \mathbb{B} \to \mathit{DecTree}
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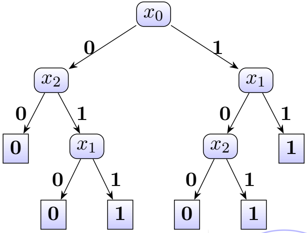

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- We compute the cost of a dectree (for our boolean function) $\mathit{cost} : \mathit{DecTree} \to \mathbb{B}^n \to \mathbb{N}$
  - for every n-bit word (there are 2^n such words)

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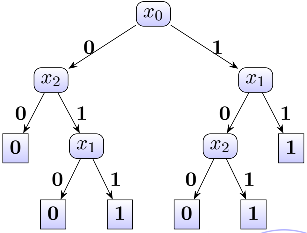

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- We compute the cost of a dectree (for our boolean function) $cost : DecTree \to \mathbb{B}^n \to \mathbb{N}$
  - for every n-bit word (there are 2^n such words)
- Then compute the *expected* cost over all 2^n words $\to$ a polynomial in the probability a bit is 1

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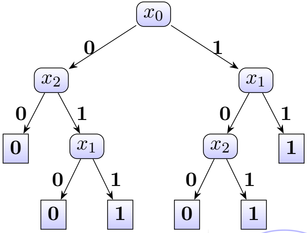

Examples of expected cost and level-p-complexity: $D_f(p)$

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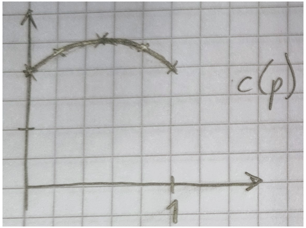

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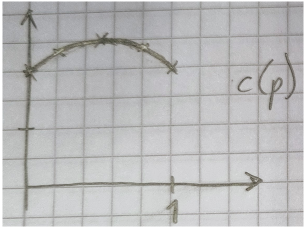

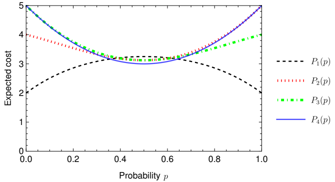

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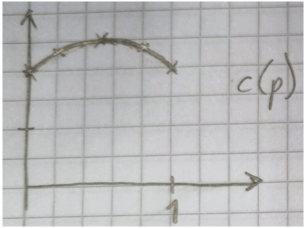

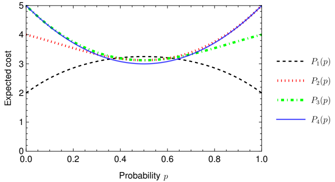

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- This minimum is the level-p-complexity $D_f(p)$ of f.
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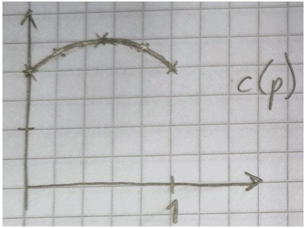
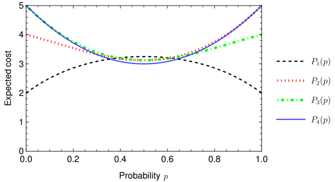
- Example 2: sim_5, $D_f(p) =$ three pieces, from P_1, P_4, P_1.





Core algorithm for computing the level-p-complexity of f

"generate all decision trees and pick the best one(s)"

- Informally: to generate all the decision trees:
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- How many $\mathit{DecTree}$s can there be? In the worst case we get $T_{n+1} = n \cdot T_n^2$
- An estimate of the number of different $\mathit{DecTree}$s for a function of n bits is $2^{2^{n-1}}$ or simply **FAR TOO MANY**.

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- We aim to filter (minimize) at the end - can we take advantage of this?
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- Success? Not yet.

Reuse of intermediate results (memoization)

- We can now easily compute up to n=5 but n=9 is still beyond reach.
- In a motivating example (iterated majority on $3 \cdot 3 = 9$ bits) we now get just 1 polynomial at the end, but we still have $2 \cdot n$ immediate subfunctions (recursive calls) and around **100 million** recursive calls in total.

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- This turns out to be a great idea - for our running example the number of subfunctions (unique recursive calls) is only 215.
- Finally we can compute the Level-p-complexity for maj_3^2 in around a few seconds. (Still 100 million recursive calls, but almost all are just a lookup in a memo-table.)

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- To implement comparison of polynomials we need to know (exactly) where they intersect. In effect this requires computing with Algebraic numbers (symbolic computations with polynomials, root counting, interval halving, etc). Details in the paper and the code.
- To go further, beyond $n = 9$, we have used more properties of special classes of boolean functions (iterated threshold functions) to reduce the computational cost even further. (In the repo we have reached $n = 27$ - thanks to help from Christian Sattler.)

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Conclusions / Questions?

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Paper: doi:10.1017/S0956796823000102, code: github:juliajansson/BoFunComplexity.

Joint work with Julia Jansson (PhD student @ Math.chalmers.se)

Motivating example: two-level iterated majority

Our running example is a simple case of two-level majority ($maj_3^2 : \mathbb{B}^9 \to \mathbb{B}$).

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Example 1: Five $\mathbf{0}$ votes, four $\mathbf{1}$ votes, even distribution:

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Example 2: Five $\mathbf{0}$ votes, four $\mathbf{1}$ votes, regrouped (perhaps through gerrymandering):

$$\underbrace{\underbrace{\mathbf{1}, \mathbf{1}, \mathbf{0}}_{m_1 = \mathbf{1}}, \underbrace{\mathbf{1}, \mathbf{0}, \mathbf{1}}_{m_2 = \mathbf{1}}, \underbrace{\mathbf{0}, \mathbf{0}, \mathbf{0}}_{m_3 = \mathbf{0}}}_{maj_3 = \mathbf{1}}$$

Result for maj_3^2 and sim_5

$$
\begin{aligned}
&ps_9 = genAlgThinMemo\ 9\ \ maj_3^2 :: Set\ (Poly\ \mathbb{Q})\\
&check_9 = \ ps_9 == fromList\ [\ P\ [4, 4, 6, 9, -61, 23, 67, -64, 16]]
\end{aligned}
$$

$$
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&ps5\ = genAlgThinMemo\ 5\ sim_5\ :: Set\ (Poly\ \mathbb{Q})\\
&check5\ = ps5\ == fromList\ [P\ [2, 6, -10, 8, -4], P\ [4, -2, -3, 8, -2],\\
&\qquad\qquad\qquad\qquad\qquad P\ [5, -8, 9, 0, -2],\ \ P\ [5, -8, 8]]
\end{aligned}
$$

Subfunctions illustrated

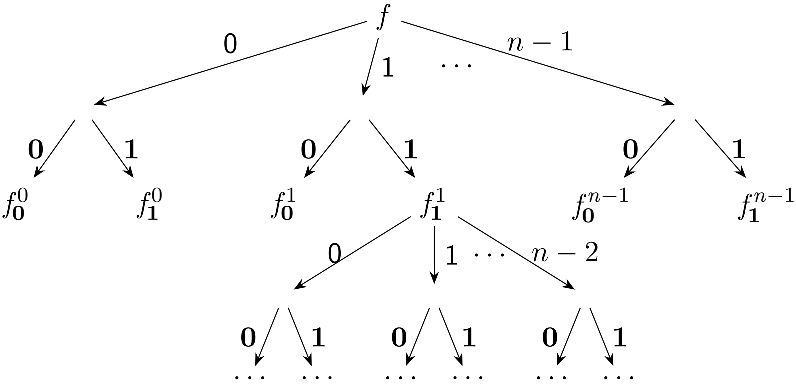

Figure: The tree of subfunctions of a Boolean function f. For brevity $setBit\ i\ b\ f$ is denoted f^i_b. This tree structure is also the call-graph for our generation of decision trees. Note that this tree structure is related to, but not the same as, the decision trees.

A class of Boolean functions

We abstract from the actual type for Boolean functions to a type class:

$$
\begin{array}{l}
\mathbf{class}\ \mathit{BoFun}\ \mathit{bf}\ \mathbf{where} \\
\quad \mathit{isConst} :: \mathit{bf} \to \mathit{Maybe}\ \mathbb{B} \\
\quad \mathit{setBit} \ \ :: \mathit{Index} \to \mathbb{B} \to \mathit{bf} \to \mathit{bf}
\end{array}
$$

We only use one instance: Binary Decision Diagrams (BDDs) which allows good sharing and fast operations.

A class of decision tree algebras

Similarly, for decision trees, we use a class of tree algebras:

$$
\begin{array}{l}
\textbf{class}\ \mathit{TreeAlg}\ a\ \textbf{where} \\
\quad \mathit{res} :: \mathbb{B} \to a \\
\quad \mathit{pic} :: \mathit{Index} \to a \to a \to a
\end{array}
$$

$$
\begin{array}{l}
\mathit{fold}_{DT} :: \mathit{TreeAlg}\ a \Rightarrow \mathit{DecTree} \to a \\
\mathit{fold}_{DT}\ (\mathit{Res}\ b) \qquad\quad = \mathit{res}\ b \\
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$$
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\mathit{ex1} :: \mathit{TreeAlg}\ a \Rightarrow a \\
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\begin{array}{llll}
\textbf{instance} & \mathit{TreeAlg}\ \mathit{DecTree} & \textbf{where}\ \mathit{res} = \mathit{Res}; & \mathit{pic} = \mathit{Pick}; \\
\textbf{instance} & \mathit{TreeAlg}\ \mathit{CostFun} & \textbf{where}\ \mathit{res} = \mathit{resC}; & \mathit{pic} = \mathit{pickC} \\
\textbf{instance}\ \mathit{Ring}\ a \Rightarrow & \mathit{TreeAlg}\ (\mathit{ExpCost}\ a) & \textbf{where}\ \mathit{res} = \mathit{resPoly}; & \mathit{pic} = \mathit{pickPoly}
\end{array}
$$

A class of decision tree algebras

Similarly, for decision trees, we use a class of tree algebras:

$$\begin{aligned}
&\textbf{class}\ \mathit{TreeAlg}\ a\ \textbf{where}\\
&\quad \mathit{res} :: \mathbb{B} \to a\\
&\quad \mathit{pic} :: \mathit{Index} \to a \to a \to a
\end{aligned}$$

$$\begin{aligned}
&\mathit{fold}_{DT} :: \mathit{TreeAlg}\ a \Rightarrow \mathit{DecTree} \to a\\
&\mathit{fold}_{DT}\ (\mathit{Res}\ b) \qquad\quad = \mathit{res}\ b\\
&\mathit{fold}_{DT}\ (\mathit{Pick}\ i\ t_0\ t_1) = \mathit{pic}\ i\ (\mathit{fold}_{DT}\ t_0)\ (\mathit{fold}_{DT}\ t_1)
\end{aligned}$$

$$\begin{aligned}
&\mathit{ex1} :: \mathit{TreeAlg}\ a \Rightarrow a\\
&\mathit{ex1} = \mathit{pic}\ 0\ (\mathit{pic}\ 2\ (\mathit{res}\ \mathbf{0})\ (\mathit{pic}\ 1\ (\mathit{res}\ \mathbf{0})\ (\mathit{res}\ \mathbf{1})))\\
&\qquad\qquad\quad (\mathit{pic}\ 1\ (\mathit{pic}\ 2\ (\mathit{res}\ \mathbf{0})\ (\mathit{res}\ \mathbf{1}))\ (\mathit{res}\ \mathbf{1}))
\end{aligned}$$

$$\begin{aligned}
&\textbf{instance}\ \mathit{TreeAlg}\ \mathit{DecTree}\ \textbf{where}\ \mathit{res} = \mathit{Res}; \mathit{pic} = \mathit{Pick};\\
&\textbf{data}\ \mathit{DecTree}\ \textbf{where}\\
&\quad \mathit{Res}\ \ :: \mathbb{B} \to \mathit{DecTree}\\
&\quad \mathit{Pick} :: \mathit{Index} \to \mathit{DecTree} \to \mathit{DecTree} \to \mathit{DecTree}
\end{aligned}$$

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\end{aligned}$$

$$\begin{aligned}
&\mathit{fold}_{DT} :: \mathit{TreeAlg}\ a \Rightarrow \mathit{DecTree} \to a\\
&\mathit{fold}_{DT}\ (\mathit{Res}\ b) \qquad\ \ = \mathit{res}\ b\\
&\mathit{fold}_{DT}\ (\mathit{Pick}\ i\ t_0\ t_1) = \mathit{pic}\ i\ (\mathit{fold}_{DT}\ t_0)\ (\mathit{fold}_{DT}\ t_1)
\end{aligned}$$

$$\begin{aligned}
&\mathit{ex1} :: \mathit{TreeAlg}\ a \Rightarrow a\\
&\mathit{ex1} = \mathit{pic}\ 0\ (\mathit{pic}\ 2\ (\mathit{res}\ \mathbf{0})\ (\mathit{pic}\ 1\ (\mathit{res}\ \mathbf{0})\ (\mathit{res}\ \mathbf{1})))\\
&\qquad\qquad\quad (\mathit{pic}\ 1\ (\mathit{pic}\ 2\ (\mathit{res}\ \mathbf{0})\ (\mathit{res}\ \mathbf{1}))\ (\mathit{res}\ \mathbf{1}))
\end{aligned}$$

$$\begin{aligned}
&\mathbf{instance}\ \mathit{TreeAlg}\ \mathit{CostFun}\ \mathbf{where}\ \mathit{res} = \mathit{resC}; \mathit{pic} = \mathit{pickC}\\
&\mathbf{type}\ \mathit{CostFun} = \mathbb{B}^n \to \mathit{Int}\\
&\mathit{resC}\ b = \mathit{const}\ 0\\
&\mathit{pickC}\ i\ c_0\ c_1 = \lambda t \to 1 + \mathbf{if}\ \mathit{index}\ t\ i\ \mathbf{then}\ c_1\ t\ \mathbf{else}\ c_0\ t
\end{aligned}$$

A class of decision tree algebras

Similarly, for decision trees, we use a class of tree algebras:

$$\begin{aligned}&\textbf{class}\ \mathit{TreeAlg}\ a\ \textbf{where}\\&\quad \mathit{res} :: \mathbb{B} \to a\\&\quad \mathit{pic} :: \mathit{Index} \to a \to a \to a\end{aligned}$$

$$\begin{aligned}&\mathit{fold}_{DT} :: \mathit{TreeAlg}\ a \Rightarrow \mathit{DecTree} \to a\\&\mathit{fold}_{DT}\ (\mathit{Res}\ b) = \mathit{res}\ b\\&\mathit{fold}_{DT}\ (\mathit{Pick}\ i\ t_0\ t_1) = \mathit{pic}\ i\ (\mathit{fold}_{DT}\ t_0)\ (\mathit{fold}_{DT}\ t_1)\end{aligned}$$

$$\begin{aligned}&\mathit{ex1} :: \mathit{TreeAlg}\ a \Rightarrow a\\&\mathit{ex1} = \mathit{pic}\ 0\ (\mathit{pic}\ 2\ (\mathit{res}\ \mathbf{0})\ (\mathit{pic}\ 1\ (\mathit{res}\ \mathbf{0})\ (\mathit{res}\ \mathbf{1})))\\&\qquad\qquad (\mathit{pic}\ 1\ (\mathit{pic}\ 2\ (\mathit{res}\ \mathbf{0})\ (\mathit{res}\ \mathbf{1}))\ (\mathit{res}\ \mathbf{1}))\end{aligned}$$

$$\begin{aligned}&\textbf{instance}\ \mathit{Ring}\ a \Rightarrow \mathit{TreeAlg}\ (\mathit{ExpCost}\ a)\ \textbf{where}\ \mathit{res} = \mathit{resPoly}; \mathit{pic} = \mathit{pickPoly}\\&\textbf{type}\ \mathit{ExpCost}\ a = \mathit{Poly}\ a\\&\mathit{resPoly}\ \_b = \mathit{zero}\\&\mathit{pickPoly}\ \_\ p_0\ p_1 = \mathit{one} + (\mathit{one} - \mathit{xP}) \times p_0 + \mathit{xP} \times p_1\end{aligned}$$

Memoization reminder

- Naively computing $\mathit{fib}\ (n+2) = \mathit{fib}\ (n+1) + \mathit{fib}\ n$ leads to exponential growth in the number of function calls.
- But if we fill in a table indexed by n with already computed results we get a the result in linear time.

Conjecture from the MSc thesis

These are the polynomials from the MSc thesis (P_t) and the best one (P_*):

$$P_t = P\,[4, 4, 7, 6, -57, 20, 68, -64, 16]$$
$$P_* = P\,[4, 4, 6, 9, -61, 23, 67, -64, 16]$$

Comparing the two polynomials shows that the new one has strictly lower expected cost than the one from the thesis. The difference is $p^2(1-p)^2(1-p+p^2)$ is illustrated here:
It is non-negative in the whole interval.
The value of both polynomials at the endpoints is 4 and the maximum of P_* is ≈ 6.14 compared to the maximum of P_t which is ≈ 6.19.