An Extension of the Construction of Power Conservative Equivalent Circuits for DC Networks containing Current Sources

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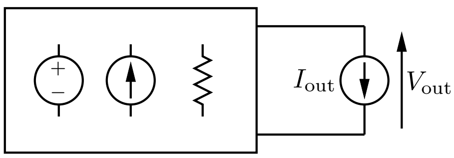

Fig. 1. Electrical network with current load

I. Introduction

By the Thévenin Theorem, any electrical network between two terminals, containing only voltage sources, current sources, and resistors, can be replaced by an equivalent circuit consisting of a voltage source V_{th} and a series resistance R_{th}. However, the power dissipated by the series resistance is not necessarily the same as the combined power dissipated by the resistors of the original network. In a recent paper [1], it was shown that, in the case where the original network does not contain any current sources, a *power conservative* equivalent circuit can be constructed. The term 'power conservative' refers to the property of the equivalent circuit, that it exhibits the same amount of Joule heating as the original network under all conditions, i.e., the total power dissipated by the resistors of the equivalent circuit is the same as the total power dissipated by the resistors of the original network. In the following, an extension of this construction to the case of networks containing both voltage and current sources is developed. The construction is divided into two parts: in Section II an analysis of the power dissipated in the original network is performed and in Section III the equivalent circuit is introduced and the values of its elements determined.

II. Power dissipation in a general network

For the analysis of the power dissipated by the resistors in an electrical network consisting only of voltage sources, current sources, and resistors, a current load I_{out} is attached to the two terminals of the network as shown in Fig. 1. In the following, the voltage across this load is denoted by V_{out}. Furthermore, let R_k, $k = 0, \ldots, n-1$, denote the values of the resistors in the network and let V_{th} and R_{th} denote the values of the voltage source and resistor of the Thévenin equivalent of the network, respectively. The output voltage of the network is thus given by

$$V_{\mathrm{out}} = V_{\mathrm{th}} - I_{\mathrm{out}}\, R_{\mathrm{th}} \ . \tag{1}$$

In order to simplify the analysis, only the case where R_{th} is non-zero will be treated. The case where R_{th} is zero is fairly simple as in that case the power dissipated in the resistors of the network is independent of the load current I_{out}.

Now examine a modified version of the network where all the sources are switched off, i.e., all voltage sources are replaced by a short circuit and all current sources are replaced by an open circuit. The current through each of the resistors R_k in this modified network is proportional to the current flowing through the terminals of the network, i.e., for each k, there exists a unique constant c_k such that the current through the resistor R_k is equal to $I_{k,\mathrm{off}} = I_{\mathrm{out}}\, c_k$. In particular, the power dissipated by R_k is equal to

$$P_{k,\mathrm{off}} = I_{k,\mathrm{off}}^2\, R_k = (I_{\mathrm{out}}\, c_k)^2\, R_k \ . \tag{2}$$

Furthermore, since the modified network consists purely of resistors and its equivalent resistance is, by the Thévenin theorem, equal to R_{th}, the total power dissipated by the entire network is given by

$$P_{\mathrm{total,off}} = I_{\mathrm{out}}^2\, R_{\mathrm{th}} \ . \tag{3}$$

Now since the total power dissipated by the entire network has to be equal to the sum of all the power dissipated in each individual resistor, combining (2) and (3), we obtain the equation

$$\begin{aligned} I_{\mathrm{out}}^2\, R_{\mathrm{th}} &= P_{\mathrm{total,off}} \\ &= \sum_k P_{k,\mathrm{off}} = \sum_k (I_{\mathrm{out}}\, c_k)^2\, R_k = I_{\mathrm{out}}^2 \sum_k c_k^2\, R_k \end{aligned}$$

and therefore

$$\sum_k c_k^2\, R_k = R_{\mathrm{th}} \ . \tag{4}$$

Returning to the original network, by the superposition principle, for each k, the current through the resistor R_k is given by $I_k = I_{k,\mathrm{off}} + I_{k,0} = I_{\mathrm{out}}\, c_k + I_{k,0}$, where $I_{k,0}$ denotes the current through the resistor at zero load current I_{out}. Correspondingly, for each k, the power dissipated by the resistor R_k is equal to

$$\begin{aligned} P_k = I_k^2\, R_k &= (I_{\mathrm{out}}\, c_k + I_{k,0})^2\, R_k \\ &= I_{\mathrm{out}}^2\, c_k^2\, R_k + I_{\mathrm{out}}\, c_k\, I_{k,0} \cdot 2\, R_k + I_{k,0}^2\, R_k \ . \end{aligned}$$

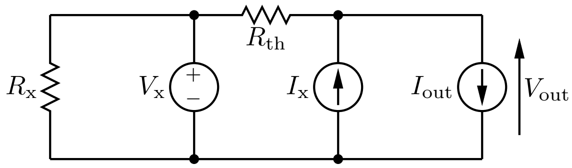

Fig. 2. Power conservative equivalent circuit with current load

Summing over all the resistors and applying (4) yields the total power dissipated by the network:

$$\begin{aligned} P_{\mathrm{total}} &= \sum_k P_k \\ &= \sum_k \left(I_{\mathrm{out}}^2 \, c_k^2 \, R_k + I_{\mathrm{out}} \, c_k \, I_{k,0} \cdot 2 \, R_k + I_{k,0}^2 \, R_k \right) \\ &= I_{\mathrm{out}}^2 \sum_k c_k^2 \, R_k + I_{\mathrm{out}} \cdot 2 \sum_k c_k \, I_{k,0} \, R_k \\ &\quad + \sum_k I_{k,0}^2 \, R_k \\ &= I_{\mathrm{out}}^2 \, R_{\mathrm{th}} + I_{\mathrm{out}} \cdot 2 \sum_k c_k \, I_{k,0} \, R_k + \sum_k I_{k,0}^2 \, R_k \quad . \end{aligned} \tag{5}$$

The key observation to be made here is that the power dissipated in the network is a quadratic polynomial in I_{out} with leading coefficient R_{th}.

III. Power conservative equivalent circuit

In order to obtain a power conservative equivalent circuit, a circuit exhibiting both the same electrical properties as the original network and the same power dissipation in its resistors has to be found. The Thévenin theorem reduces the problem of electrical equivalence to the problem of finding a circuit satisfying (1). As shown in Section II, the power dissipated by the resistors in the original network is given by (5) and hence the problem of finding a power conservative equivalent circuit reduces to the problem of finding an electrically equivalent circuit such that the total power dissipated by its resistors is equal to the right-hand side of (5).

One possibility of obtaining a power conservative equivalent circuit is shown in Fig. 2. The resistor R_{x} contained in this circuit does not contribute anything to the electrical properties of the circuit, its sole purpose being the dissipation of a fixed amount of power which in the following will be referred to as P_{x}. The current flowing through the resistor R_{th} is given by $I_{\mathrm{out}} - I_{\mathrm{x}}$, resulting in an output voltage of

$$V_{\mathrm{out}} = V_{\mathrm{x}} - (I_{\mathrm{out}} - I_{\mathrm{x}}) \, R_{\mathrm{th}} = V_{\mathrm{x}} - I_{\mathrm{out}} \, R_{\mathrm{th}} + I_{\mathrm{x}} \, R_{\mathrm{th}} \quad . \tag{6}$$

Furthermore, the power dissipated by R_{th} is given by $(I_{\mathrm{out}} - I_{\mathrm{x}})^2 \, R_{\mathrm{th}}$ and therefore the total power dissipated by the resistors R_{th} and R_{x} amounts to

$$\begin{aligned} P_{\mathrm{total}} &= (I_{\mathrm{out}} - I_{\mathrm{x}})^2 \, R_{\mathrm{th}} + P_{\mathrm{x}} \\ &= I_{\mathrm{out}}^2 \, R_{\mathrm{th}} - I_{\mathrm{out}} \, I_{\mathrm{x}} \cdot 2 \, R_{\mathrm{th}} + I_{\mathrm{x}}^2 \, R_{\mathrm{th}} + P_{\mathrm{x}} \quad . \end{aligned} \tag{7}$$

In order to achieve power conservative equivalence to the original network analyzed in Section II, the values of the elements V_{x}, I_{x}, and R_{x} have to be chosen such that the right-hand sides of equations (1) and (6) and the right-hand sides of equations (5) and (7) become equal. The right-hand sides of (5) and (7) are both quadratic polynomials in I_{out} with leading coefficient R_{th} and achieving equality reduces to equating the remaining coefficients, yielding

$$2 \sum_k c_k \, I_{k,0} \, R_k = -I_{\mathrm{x}} \cdot 2 \, R_{\mathrm{th}}$$

and hence

$$I_{\mathrm{x}} = -\sum_k c_k \, I_{k,0} \, \frac{R_k}{R_{\mathrm{th}}} \tag{8}$$

and

$$\sum_k I_{k,0}^2 \, R_k = I_{\mathrm{x}}^2 \, R_{\mathrm{th}} + P_{\mathrm{x}}$$

and hence

$$P_{\mathrm{x}} = \sum_k I_{k,0}^2 \, R_k - I_{\mathrm{x}}^2 \, R_{\mathrm{th}} \quad . \tag{9}$$

Finally, in view of (1) and (6), V_{x} has to be chosen such that

$$V_{\mathrm{th}} - I_{\mathrm{out}} \, R_{\mathrm{th}} = V_{\mathrm{x}} - I_{\mathrm{out}} \, R_{\mathrm{th}} + I_{\mathrm{x}} \, R_{\mathrm{th}} \quad ,$$

i.e.,

$$V_{\mathrm{x}} = V_{\mathrm{th}} - I_{\mathrm{x}} \, R_{\mathrm{th}} \quad . \tag{10}$$

All that remains is to choose the resistor R_{x} such that the power dissipated by it is equal to P_{x}. However, this is only possible in the case where P_{x} and V_{x} are both non-zero, as then R_{x} can be chosen to be $V_{\mathrm{x}}^2/P_{\mathrm{x}}$, leading to a power dissipation of $V_{\mathrm{x}}^2/R_{\mathrm{x}} = P_{\mathrm{x}}$. In the remaining cases, the circuit has to be modified slightly. In the case where P_{x} is zero, R_{x} can simply be replaced by an open circuit. In the case where P_{x} is non-zero and V_{x} is zero, R_{x} and V_{x} can be replaced by a short circuit and a separate source with a suitable resistor connected across its terminals added to the circuit so that the power dissipated in that resistor is equal to P_{x}. Note that none of these modifications increase the number of elements in the circuit, i.e., a power conservative equivalent circuit can always be constructed using at most four elements.

References

[1] I. Barbi, "Power conservative equivalent circuit for DC networks," *IEEE Access*, vol. 8, pp. 113 667–113 674, Jun. 2020.