An Extension of the Construction of Power Conservative Equivalent Circuits for DC Networks containing Current Sources
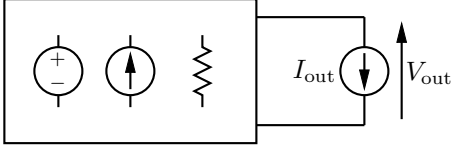
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Fig. 1. Electrical network with current load

I. Introduction

By the Thévenin Theorem, any electrical network between two terminals, containing only voltage sources, current sources, and resistors, can be replaced by an equivalent circuit consisting of a voltage source V_{th} and a series resistance R_{th}. However, the power dissipated by the series resistance is not necessarily the same as the combined power dissipated by the resistors of the original network. In a recent paper [1], it was shown that, in the case where the original network does not contain any current sources, a *power conservative* equivalent circuit can be constructed. The term 'power conservative' refers to the property of the equivalent circuit, that it exhibits the same amount of Joule heating as the original network under all conditions, i.e., the total power dissipated by the resistors of the equivalent circuit is the same as the total power dissipated by the resistors of the original network. In the following, an extension of this construction to the case of networks containing both voltage and current sources is developed. The construction is divided into two parts: in Section II an analysis of the power dissipated in the original network is performed and in Section III the equivalent circuit is introduced and the values of its elements determined.

II. Power dissipation in a general network

For the analysis of the power dissipated by the resistors in an electrical network consisting only of voltage sources, current sources, and resistors, a current load I_{out} is attached to the two terminals of the network as shown in Fig. 1. In the following, the voltage across this load is denoted by V_{out}. Furthermore, let R_k, $k = 0, \ldots, n-1$, denote the values of the resistors in the network and let V_{th} and R_{th} denote the values of the voltage source and resistor of the Thévenin equivalent of the network, respectively.

Now examine a modified version of the network where all the sources are switched off, i.e., all voltage sources are replaced by a short circuit and all current sources are replaced by an open circuit. The current through each of the resistors

R_k in this modified network is proportional to the current flowing through the terminals of the network, i.e., for each k, there exists a unique constant c_k such that the current through the resistor R_k is equal to $I_{k,\mathrm{off}} = I_{\mathrm{out}}\, c_k$. In particular, the power dissipated by R_k is equal to

$$P_{k,\mathrm{off}} = I_{k,\mathrm{off}}^2\, R_k = (I_{\mathrm{out}}\, c_k)^2\, R_k \ . \tag{1}$$

Furthermore, since the modified network consists purely of resistors and its equivalent resistance is, by the Thévenin theorem, equal to R_{th}, the total power dissipated by the entire network is given by

$$P_{\mathrm{total,off}} = I_{\mathrm{out}}^2\, R_{\mathrm{th}} \ . \tag{2}$$

Now since the total power dissipated by the entire network has to be equal to the sum of all the power dissipated in each individual resistor, combining (1) and (2), we obtain the equation

$$I_{\mathrm{out}}^2\, R_{\mathrm{th}} = \sum_k (I_{\mathrm{out}}\, c_k)^2\, R_k = I_{\mathrm{out}}^2\, \sum_k c_k^2\, R_k$$

and therefore

$$\sum_k c_k^2\, R_k = R_{\mathrm{th}} \ . \tag{3}$$

Returning to the original network, by the superposition principle, for each k, the current through the resistor R_k is given by $I_k = I_{k,\mathrm{off}} + I_{k,0} = I_{\mathrm{out}}\, c_k + I_{k,0}$, where $I_{k,0}$ denotes the current through the resistor at zero load current I_{out}. Correspondingly, for each k, the power dissipated by the resistor R_k is equal to

$$\begin{aligned}
P_k &= I_k^2\, R_k = (I_{\mathrm{out}}\, c_k + I_{k,0})^2\, R_k \\
&= I_{\mathrm{out}}^2\, c_k^2\, R_k + I_{\mathrm{out}}\, c_k\, I_{k,0} \cdot 2\, R_k + I_{k,0}^2\, R_k \ .
\end{aligned}$$

Summing over all the resistors and applying (3) yields the total power dissipated by the network:

$$\begin{aligned}
P_{\mathrm{total}} &= \sum_k P_k \\
&= \sum_k \left(I_{\mathrm{out}}^2\, c_k^2\, R_k + I_{\mathrm{out}}\, c_k\, I_{k,0} \cdot 2\, R_k + I_{k,0}^2\, R_k \right) \\
&= I_{\mathrm{out}}^2\, \sum_k c_k^2\, R_k + I_{\mathrm{out}} \cdot 2 \sum_k c_k\, I_{k,0}\, R_k \\
&\quad + \sum_k I_{k,0}^2\, R_k \\
&= I_{\mathrm{out}}^2\, R_{\mathrm{th}} + I_{\mathrm{out}} \cdot 2 \sum_k c_k\, I_{k,0}\, R_k + \sum_k I_{k,0}^2\, R_k \ .
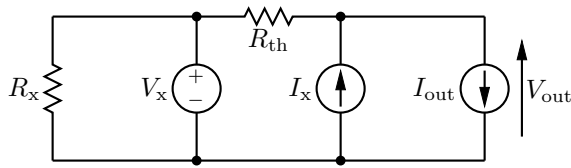\end{aligned}$$

$$\tag{4}$$

Fig. 2. Power conservative equivalent circuit with current load

In summary, the power dissipated in the network is a non-negative quadratic polynomial in I_{out} with leading coefficient R_{th}.

III. POWER CONSERVATIVE EQUIVALENT CIRCUIT

In order to obtain a power conservative equivalent circuit, a circuit exhibiting both the same electrical properties as the original network and the same power dissipation in its resistors has to be found. The Thévenin theorem reduces the problem of electrical equivalence to the problem of finding a network satisfying $V_{\text{out}} = V_{\text{th}} - I_{\text{out}} R_{\text{th}}$.

As shown in Section II, the power dissipated by the resistors in the network is a non-negative quadratic polynomial in I_{out} with leading coefficient R_{th}. Elementary algebra shows that such a polynomial can always be parametrized as

$$P_{\text{total}} = (I_{\text{out}} - I_{\text{x}})^2 R_{\text{th}} + P_{\text{x}} \qquad (5)$$

The parameters $P_{\text{x}} \geq 0$ and I_{x} are equal to the minimum power dissipated by the resistors in the network and the current at which this minimum is attained, respectively.

One possibility of obtaining a power conservative equivalent circuit is shown in Fig. 2. In order to achieve equivalence, the values of the elements R_{th} and I_{x} have to be chosen as the values appearing in (5) and the values of the elements R_{x} and V_{x} have to be chosen according to the following equations:

$$V_{\text{x}} = V_{\text{th}} - I_{\text{x}} R_{\text{th}} \qquad R_{\text{x}} = \frac{V_{\text{x}}^2}{P_{\text{x}}}$$

In the case $P_{\text{x}} = 0$, the resistor R_{x} can be omitted.

It is a straightforward exercise to verify that, as long as $V_{\text{x}} \neq 0$ or $P_{\text{x}} = 0$, the obtained circuit is indeed both power conservative and electrically equivalent. The construction fails in the case where $V_{\text{x}} = 0$ and $P_{\text{x}} \neq 0$ since then the resistor R_{x} is unable to dissipate power. However, as long as $I_{\text{x}} \neq 0$, a resistor of value $P_{\text{x}}/I_{\text{x}}^2$ may be connected in series with I_{x} instead. If both $V_{\text{x}} = 0$ and $I_{\text{x}} = 0$ and yet still $P_{\text{x}} \neq 0$, a separate source has to be added to the circuit and a suitable resistor connected across its terminals. Since in each of these special cases some of the elements disappear from the circuit, a power conservative equivalent circuit can nevertheless always be constructed from at most four elements. These four elements clearly correspond to the four degrees of freedom V_{th}, R_{th}, P_{x}, and I_{x} in the above characterization of the network.

REFERENCES

[1] I. Barbi, "Power conservative equivalent circuit for dc networks," *IEEE Access*, vol. 8, pp. 113 667–113 674, Jun. 2020.