Mathematical Foundations

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1 Numerical Estimates

1.1 Estimates on the Gaussian function and derived functions

Definition 1.1.1. (Radial Gaussian function) Given the *Gaussian function*

$$g_{\sigma,\mu}: \begin{array}{c} \mathbb{R} \to \mathbb{R}_{>} \\ x \mapsto \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \end{array}$$

with parameters $\sigma \in \mathbb{R}_{>}$ and $\mu \in \mathbb{R}$, we define, with σ as before and now $\mu \in \mathbb{R}^n$, the (n-dimensional) radial Gaussian function for $n \in \mathbb{N}$

$$G_{\sigma,\mu}: \begin{array}{c} \mathbb{R}^n \to \mathbb{R}_{>} \\ x \mapsto g_{\sigma,0}(|x-\mu|) \end{array},$$

where $|\cdot|$ is Euclidean distance.

Remark. Since the Gaussian function is symmetric in μ, the 1-dimensional radial Gaussian function is identical to the Gaussian function.

Proposition 1.1.2. *Let $G_{\sigma,\mu}$ be a n-dimensional radial Gaussian function (cf. Definition 1.1.1) with parameters σ and μ. For $\varepsilon < \frac{\sigma}{\sqrt{2\pi}}$, we have for all $x \in \mathbb{R}^n$:*

$$|x-\mu| > \sigma\sqrt{-2\ln\left(\frac{\sqrt{2\pi}\varepsilon}{\sigma}\right)} \iff G_{\sigma,\mu}(x) < \varepsilon.$$

Proof. Let $\tilde{x} := |x-\mu|$ for $x \in \mathbb{R}^n$. The proof is due to a simple calculation:

$$G_{\sigma,\mu}(x) = g_{\sigma,\mu}(\tilde{x}) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{\tilde{x}^2}{2\sigma^2}\right),$$

so

$$\begin{aligned} & G_{\sigma,\mu}(x) < \varepsilon \\ \iff & g_{\sigma,\mu}(\tilde{x}) < \varepsilon \\ \iff & -\frac{\tilde{x}^2}{2\sigma^2} < \ln\left(\frac{\sqrt{2\pi}\varepsilon}{\sigma}\right) \end{aligned}$$

We use our assumption on ε to obtain $-\ln\left(\frac{\sqrt{2\pi}\varepsilon}{\sigma}\right) > 0$. Equivalence uses $\tilde{x} \geq 0$.

$$\iff \tilde{x} > \sigma\sqrt{-2\ln\left(\frac{\sqrt{2\pi}\varepsilon}{\sigma}\right)}$$

□

1.2 Estimates on Gaussian processes