Simulations on N-Body problems with Relativistic Corrections

Stability on the restricted three bodies problem

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Potential Energy in Schwarszchild de Sitter metric

The potential associated to the Schwarszchild de Sitter metric, is given by

$$U(r) = \frac{k}{r} + \frac{A}{r^2} + \frac{B}{r^3} \tag{1}$$

where $k = GM$, $A = \frac{\Lambda c^2}{6}$ and $B = \frac{GML^2}{c^2}$.

Augmented and Effective Potential

Augmented Potential

An infinitesimal mass in the presence of two masses M_1, M_2 such that $M_1 + M_2 = 1$, with relativistic coefficients A_1, A_2, and oblateness parameters B_1 and B_2, respectively; has an augmented potential

$$U_a(r) = -\frac{1}{r}\left(1 + \frac{A_1 + A_2}{r} + \frac{B_1 + B_2}{r^2}\right) + \frac{r^2\omega^2}{2} \tag{2}$$

Effective Potential

and an effective potential

$$U_{eff}(r) = -\frac{1}{r}\left(1 + \frac{A_1 + A_2}{r} + \frac{B_1 + B_2}{r^2}\right) + \frac{l^2}{2r^2} \tag{3}$$

Constraint for ω

ω

Under an escalation of the space, we look for the critical points located at $r = 1$. For that, the equation

$$U'_{eff}(r)|_{r=1} = 0$$

must holds. This implies that

$$\omega^2 = 1 + 2(A_1 + A_2) + 3(B_1 + B_2) \tag{4}$$

Constraint for oblateness coefficients

Equilibrium points

In order to guarantee the stability of the orbit, we look for the minimal points of the potential, *i.e.*

$$U''_{eff}(r)|_{r=1}>0,$$

where we get

$$B_1 + B_2<1/3, \tag{5}$$

as expected[1].

Restricted Three Bodies Problem

Problem Statement

On the rotating system, the primaries are localized in $-\mu$ and $1-\mu$, and have parameters $(1\mu, B_1, A_1)$ and (μ, B_2, A_2), respectively[3]. The potential energy is given by

$$U(x_1,x_2) = -(1-\mu)\Big(\frac{1}{\rho_1} + \frac{A_1}{\rho_1^2} + \frac{B_1}{\rho_1^3}\Big) - \mu\Big(\frac{1}{\rho_2} + \frac{A_2}{\rho_2^2} + \frac{B_2}{\rho_2^3}\Big) \quad (6)$$

where

$$\rho_1 = \sqrt{(x_1+\mu)^2 + x_2^2} \quad , \quad \rho_2 = \sqrt{(x_1+\mu-1)^2 + x_2^2}$$

Movement Equations

Hamiltonian

Denoting $x = (x_1, x_2)$ and $y = (y_1, y_2)$, the dynamics of the rotating system are determined by the Hamiltonian

$$H(x, y) = \frac{1}{2} y^2 - \omega(x_1 y_2 - x_2 y_1) + U(x).$$

Movement Equations

The movement equations would be:

$$\frac{d^2 x}{dt^2} = -\omega K y + \frac{\partial U(x)}{\partial x} \quad , \quad \frac{d^2 x}{dt^2} = -\omega K y + \frac{\partial U(x)}{\partial x}$$

Rotating Equilibria

Rotating Equilibria

Additionally, the rotating equilibria are found in the points that satisfy[2]

$$-\omega^2 x + \frac{\partial U}{\partial x} = 0. \tag{7}$$

From the last equation we get a system of two equations that, due to the linear independence of the powers of ρ_i, can be resumed into the following equality

$$\frac{1}{\rho_1^3} + \frac{2A_1}{\rho_1^4} + \frac{3B_1}{\rho_1^5} = \frac{1}{\rho_2^3} + \frac{2A_1}{\rho_2^4} + \frac{3B_1}{\rho_2^5} = \omega^2 \tag{8}$$

Linearization

The linearization at equilibrium is

$$\begin{bmatrix} 0 & \omega & 1 & 0 \\ -\omega & 0 & 0 & 1 \\ U_{x_1,x_1} & U_{x_1,x_2} & 0 & \omega \\ U_{x_1,x_2} & U_{x_2,x_2} & -\omega & 0 \end{bmatrix}$$

whose characteristic equation is

$$\begin{aligned} \lambda^4 + \lambda^2(2\omega^2 - U_{x_1,x_1} - U_{x_2,x_2}) + U_{x_1,x_1}U_{x_2,x_2} \\ + \omega^2(U_{x_1,x_1} + U_{x_2,x_2}) - U_{x_1,x_2}^2 + \omega^4 = 0 \end{aligned} \tag{9}$$

Spectral Stability

Due to the characteristic polynom, in order to get spectral stability, we require

$$2\omega^2 - U_{x_1,x_1} - U_{x_2,x_2} > 0 \tag{10}$$

$$\begin{split}(2\omega^2 - U_{x_1,x_1} - U_{x_2,x_2})^2 \\ - 4(U_{x_1,x_1}U_{x_2,x_2} + \omega^2(U_{x_1,x_1} + U_{x_2,x_2}) - U^2_{x_1,x_2} + \omega^4) > 0\end{split} \tag{11}$$

References

References

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References

References

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