

# ELEMENTS OF SUB-RIEMANNIAN GEOMETRY AND ITS APPLICATIONS

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ABSTRACT.

## INTRODUCTION

### 1. DISTRIBUTIONS

**1.1. Distributions and Horizontal Curves.** A manifold without boundary  $M$  of dimension  $m$  is said to be of class  $C^\infty$  if the differentiability class of the transition maps is  $C^\infty$ . In this case, for every  $p \in M$  one can define the tangent space to  $M$  at  $p$ ,  $T_p M$ , as the set of all derivations of the algebra of germs of smooth functions from  $M$  to  $\mathbb{R}$ , based at  $p$ . This definition of tangent space is equivalent (when the differentiability class of  $M$  is  $\infty$ ) to the equivalence class of germs of curves that passes through  $p$ . If a manifold  $M$  is  $C^r$ , with  $r < \infty$ , the definition for tangent space is the latter, so we use that definition for the tangent space throughout this text, independently of the differentiability class.

**Definition 1.** ([1], section 1.3) The triple  $(E, M, \pi : E \rightarrow M)$  is a *smooth vector bundle* if  $E, M$  are smooth manifolds,  $\pi$  is a smooth surjection; the fibers  $\pi^{-1}(p)$  for  $p \in M$  have the structure of finite dimensional vector spaces; and for every point  $p \in M$ , there is an open neighborhood  $U$  of  $p$ , a natural number  $k$  and a diffeomorphism  $\phi : U \times \mathbb{R}^k \rightarrow \pi^{-1}(U)$ , such that for all  $q \in U$  the following propositions hold:

- $(\pi \circ \phi)(q, v) = q$  for all  $v \in \mathbb{R}^k$ , and
- the map  $v \mapsto \phi(p, v)$  is a linear isomorphism between the vector spaces  $\pi^{-1}(p)$  and  $\mathbb{R}^k$ .

The first condition means that the following diagram is commutative,

$$\begin{array}{ccc} U \times \mathbb{R}^k & \xrightarrow{\phi} & \pi^{-1}(U) \\ p \downarrow & & \downarrow \pi \\ U & \xrightarrow{Id} & U \end{array}$$

where  $p : U \times \mathbb{R}^k \rightarrow U$ ,  $p(q, v) = q$  is the projection onto the first factor.

We take the disjoint union of the tangent spaces at every point of the manifold in order to define the tangent bundle,  $TM = \bigsqcup_{p \in M} T_p M$ , equipped with manifold structure given by the one of  $M$ . It is called a bundle because the triple  $(TM, M, \pi : TM \rightarrow M)$  satisfies the

definition of smooth vector bundle, where  $\pi : TM \rightarrow M$  is the projection function, that assigns each tangent vector to the point where it is based. Each fiber of this bundle is a tangent space based on a point, and the tangent bundle is in particular a vector bundle because each fiber has structure of finite dimensional real vector space.

There is another important class of smooth vector bundles, namely, the tensor bundles, that are defined following [2], examples 2.4 and 2.5.

**Definition 2.** If  $(E, M, \pi : E \rightarrow M)$  is a smooth vector bundle, the dual bundle of this bundle is another smooth vector bundle  $(E^*, M, \pi^* : E^* \rightarrow M)$ , known as *dual bundle*, whose fibers are the dual spaces to the fibers of  $E$ , that is,

$$(\pi^*)^{-1}(p) = (\pi^{-1}(p))^*, \text{ for all } p \in M,$$

and

$$E^* = \bigsqcup_{p \in M} (\pi^*)^{-1}(p).$$

**Definition 3.** If  $(E, M, \pi : E \rightarrow M)$  is a smooth vector bundle, the  $(k, l)$ -*tensor bundle* associated to this bundle, with  $k, l \geq 0$ , is defined as the smooth vector bundle  $(T_l^k E, M, \pi_{kl} : T_l^k E \rightarrow M)$ , whose fibers are the vector spaces of  $(k, l)$ -tensors of the fibers of  $E$ , that is,

$$\pi_{kl}^{-1}(p) = T_l^k \pi^{-1}(p) = \left( \bigotimes_{j=1}^l (\pi^*)^{-1}(p) \right) \otimes \left( \bigotimes_{j=1}^k \pi^{-1}(p) \right), \text{ for all } p \in M,$$

and

$$T_l^k E = \bigsqcup_{p \in M} \pi_{kl}^{-1}(p).$$

*Remark 1.* If  $k = 1, l = 0$ ,  $T_1^0 E = E$ , and if  $k = 0, l = 1$ ,  $T_0^1 = E^*$ . We associate  $T_0^0 = M \times \mathbb{R}$  to the *trivial line bundle* if both  $k = l = 0$ .

The formal proof that both definitions are consistent definitions of smooth vector bundles can be found in [1], section 1.3.

**Definition 4.** A *smooth section* of a smooth vector bundle  $(E, M, \pi : E \rightarrow M)$  is a (smooth) application  $s : M \rightarrow E$  such that  $\pi \circ s = \mathbb{1}_M$ .

For an arbitrary smooth vector bundle  $(E, B, \pi : E \rightarrow B)$ , a subbundle of this bundle  $(E', B', \pi' : E' \rightarrow B')$  is a smooth vector bundle bundle that satisfies  $E' \subset E$ ,  $B' \subset B$  and  $\pi|_{E'} = \pi'$ .

**Definition 5.** A *distribution*  $\mathcal{H}$  over a manifold  $M$  is a vector subbundle of the tangent bundle  $TM$ . If this vector subbundle is in addition smooth, the distribution is said to be smooth, and the rank  $k$  of this distribution is defined as the dimension of its fibers.

The fiber of  $\mathcal{H}$  at a point  $p \in M$  is denoted as  $H_p$ .

The concept of (smooth) distribution can be defined in an alternative but equivalent (see [3], Ch. 19, lemma 19.5) fashion using 1-forms, with the help of the following proposition.

**Proposition 1** (Smooth Distribution (using 1-forms)). *Suppose that  $M$  is a smooth  $m$ -dimensional manifold, and  $\mathcal{H} \subset TM$  is a distribution of rank  $k$ . Then  $\mathcal{H}$  is smooth if and only if for every point  $p \in M$  there is a neighborhood  $U$  of that point where exist  $n - k$  smooth 1-forms  $\omega^1, \dots, \omega^{n-k}$ , such that for all  $q \in U$ ,*

$$\mathcal{H}_q = \text{Ker } \omega^1|_q \cap \dots \cap \text{Ker } \omega^{n-k}|_q.$$

The following proposition gives an alternative way to represent a smooth distribution in terms of local frames ([3], Ch. 19, page 491).

**Proposition 2** (Smooth Distribution (using local frames)). *It is said that  $\mathcal{H} \subset TM$  is a smooth distribution over  $M$  if and only if for each point  $p \in M$ , there is an open neighborhood  $U$  of it on which there are smooth vector fields  $X_1, \dots, X_n : U \rightarrow TM$  that at each point of  $U$  form a base for the fiber of that point respect to the distribution.*

Some examples of distributions over manifolds are given below.

*Example 1* (Tangent Bundle). For any manifold  $M$ , the tangent bundle  $TM$  is by definition a distribution over it.

*Example 2* (Heisenberg Group). Take  $\mathbb{R}^3$  as the manifold, and define over it the distribution  $\mathcal{H} \subset T\mathbb{R}^3$ , whose fiber at an arbitrary  $(x, y, z) \in \mathbb{R}^3$  is

$$\mathcal{H}_{(x,y,z)} = \text{Ker } \omega|_{(x,y,z)},$$

where  $\omega = dz - (xdy - ydx)/2 \in \Omega^1(\mathbb{R}^3)$ . It is important to notice that  $\omega \neq 0$  for all  $(x, y, z) \in \mathbb{R}^3$ .

*Remark 2.* The last example is of particular interest because of its connection with the isoperimetrical problem. For more information see [4], Ch. 1.

*Example 3* (Vector field over a manifold). Any smooth vector field  $X : M \rightarrow TM$  (which does not vanish out at any point) on a manifold determines a distribution  $\mathcal{H}$  over the manifold  $M$  whose fiber at an arbitrary  $p \in M$  is the span of  $X_p$ .

Given a distribution  $\mathcal{H}$  over a manifold  $M$ , a curve or vector field is said to be *horizontal* if it is tangent to  $\mathcal{H}$ . This means that, in the case of a curve, its tangent vector at every point is contained in the fibers of  $\mathcal{H}$ ; and in the case of a vector field, its value at every point is contained in the fibers of  $\mathcal{H}$ .

For example, every curve over a manifold with distribution given by its tangent bundle is horizontal. In the example 2, the liftings of the solutions of the isoperimetrical problem are horizontal. In example 3, and in particular in the case of systems of linear differential equations, the solution curves of the system of differential equations are examples of horizontal curves for that distribution.

**1.2. Frobenius' Theorem.** Now the natural question that arises is if, given two points on a manifold with a given horizontal distribution, is there a smooth horizontal curve that joins them?

Given a smooth distribution  $\mathcal{H} \subset TM$ , we say that a nonempty immersed submanifold  $N \subseteq M$  is an *integral manifold* of  $\mathcal{H}$  if  $T_p N = \mathcal{H}_p$  for all  $p \in N$ , *i.e.*, if the tangent space of all points of the submanifold is the fiber of the horizontal distribution. As an example, in the distribution given in example 3, the image of any integral curve of the vector field that determines the distribution over the manifold is an integral manifold for that distribution. A particular example (taken from [3], example 19.1) of a distribution and its family of integral manifolds is given below.

*Example 4* (Orthogonal complement of a given vector field). Let  $\mathcal{H}$  be the distribution over  $\mathbb{R}^n$  determined by the radial vector field  $x^i \partial / \partial x^i$ , and let  $\mathcal{H}^\perp$  be its perpendicular bundle, *i.e.*, the distribution whose fibers are the orthogonal complement of the fibers of  $\mathcal{H}$ .  $\mathcal{H}^\perp$  is a distribution over  $\mathbb{R}^n$ , and the sphere centered at 0, of radius  $|x|$ , for all  $x \neq 0$  is an integral submanifold of  $\mathcal{H}^\perp$ .

There are distributions over a manifold that do not have integral manifolds, as is shown in the following example (taken from [3], Ch. 19, example 19.1).

*Example 5* (Distribution without integral manifolds). Consider the distribution  $\mathcal{H}$  over  $\mathbb{R}^3$  spanned by the following vector fields:

$$X = \frac{\partial}{\partial x} + y \frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y}.$$

This distribution does not have integral manifolds. To see this, suppose that  $N$  is an integral manifold that contains an arbitrary point  $\mathbf{x} \in \mathbb{R}^3$ . Because  $X$  and  $Y$  are tangent to  $N$ , any integral curve of  $X$  or  $Y$  that starts in  $N$  stays in  $N$ , at least for a short time (*i.e.*, the integral curve  $\gamma$  must be defined over a sufficiently small interval such that its image is fully contained in  $N$ ).

Therefore, as the integral curve of  $X$  is a straight line contained in the plane that is parallel to the  $xz$ -plane, there is a segment of this line that is contained in  $N$ , and for every point in this segment, the integral curves of  $Y$  that passes through them are straight lines parallel to the  $y$ -axis, so there must be an open neighborhood of  $\mathbf{x}$  that contains a plane generated by the segments of the integral curves of  $X$  and  $Y$  that passes through  $\mathbf{x}$  (see figure 1). However, the tangent plane at any point  $p$  of this plane off of the segment induced by the integral curve of  $X$  is not equal to  $\mathcal{H}_p$ , since  $X$  depends of  $y$ .

A smooth distribution  $\mathcal{H}$  is said to be *involutive* if given any pair of smooth vector fields  $X, Y$  defined on a open subset of  $M$  that satisfy  $X_p, Y_p \in \mathcal{H}_p$ , their Lie bracket also satisfies the same condition, *i.e.*, it is tangent to the distribution in the given open subset.

Let  $\Gamma(\mathcal{H})$  be the set of all smooth vector fields globally defined, tangent to  $\mathcal{H}$ . It is clear from the definition of involutivity that if  $\mathcal{H}$  is involutive,  $\Gamma(\mathcal{H}) \subset \mathfrak{X}(M)$  is a Lie subalgebra. The converse statement (if  $\Gamma(\mathcal{H})$  is a Lie algebra, then  $\mathcal{H}$  is involutive) is also true, and



FIGURE 1. Illustration of the planes generated at each point by the integral curves of the smooth vector fields that determine the distribution given in example 5.

can be proved extending any pair of smooth vector fields locally defined, tangent to  $\mathcal{H}$ , to a pair of vector fields in  $\Gamma(\mathcal{H})$ , using an adequate bump function.

A smooth distribution  $\mathcal{H}$  over a manifold  $M$  is called *integrable* if each point of  $M$  is contained in an integral manifold of  $\mathcal{H}$ . It is clear then that *every integrable distribution is involutive*, because every pair of smooth vectors tangent to  $\mathcal{H}$ , defined over an open set  $U \subset M$ , satisfy that their Lie bracket is also tangent to  $\mathcal{H}$  in that open set, since there exists a integral manifold  $N$  of  $\mathcal{H}$  for every point of  $U$  such that the pair of smooth vectors are tangent to  $N$ , and in consequence, their Lie bracket is also tangent to  $N$ .

In fact, the involutivity condition does not need to be checked for all smooth vector fields on  $\Gamma(\mathcal{H})$ , but it suffices to check if the Lie brackets of pairs of smooth vector fields of the local frame that determines the distribution are tangent to it, as the following lemma states ([3], lemma 19.4).

**Lemma 1** (Local frame criterion for involutivity). *Let  $\mathcal{H} \subset TM$  be a distribution. If for every point in  $M$  there is a neighborhood such that there exists a local smooth frame  $(X_1, \dots, X_k)$  for  $\mathcal{H}$  such that  $[X_i, X_j]$  is a section of  $\mathcal{H}$  for each  $i, j$ , then  $\mathcal{H}$  is involutive.*

**Definition 6.** Given a smooth distribution  $\mathcal{H} \subset TM$  of rank  $k$ , it is said that a smooth coordinate chart  $(U, \phi)$  of  $M$  is *flat for  $\mathcal{H}$*  if  $\phi(U)$  is a cube in  $\mathbb{R}^m$  (being  $m$  the dimension of  $M$ ), and at points of  $U$ ,  $\mathcal{H}$  is spanned by the first  $k$  coordinate vector fields  $\partial/\partial x^1, \dots, \partial/\partial x^k$  (see figure 2).

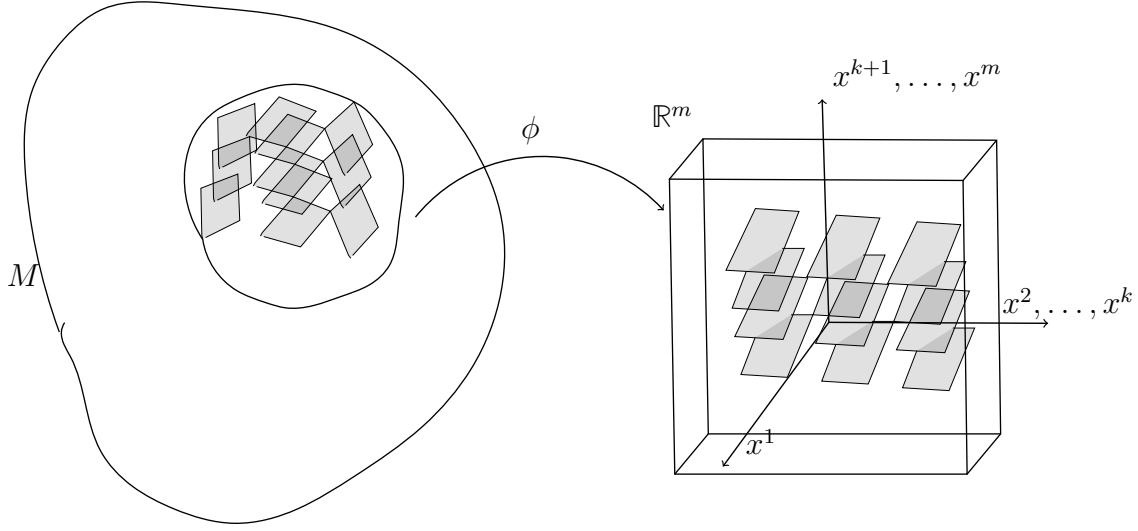


FIGURE 2. Flat chart for a distribution.

The geometrical meaning of this property for a distribution is that, locally (in that chart), the manifold is homeomorphic to a cube in  $\mathbb{R}^m$ , and the vector subspaces induced by the distribution at each point fit together as affine spaces of dimension given by the rank of the distribution in  $\mathbb{R}^m$ . Also, each slice of the form  $x^{k+1} = c^{k+1}, \dots, x^m = c^m$ , for constants  $c^{k+1}, \dots, c^m$  in the image of  $U$  of such a chart is an integral manifold of the distribution generated by  $\partial/\partial x^1, \dots, \partial/\partial x^k$ , and the preimage of this integral manifold is an integral manifold of  $\mathcal{H}$ , contained in  $U$ .

The case when every point in  $M$  has a coordinate chart centered in it, flat for  $\mathcal{H}$  is discussed in the following definition and geometrical interpretation.

**Definition 7.** A smooth distribution  $\mathcal{H} \subset TM$  is said to be *completely integrable* if there exists a flat chart for  $\mathcal{H}$  in a neighborhood of each point of  $M$ .

Therefore, if a distribution is completely integrable, is then integrable, since there exists an integral manifold for each point (that in particular is known as *maximal integral manifold*, as the dimension of this manifold coincides with the rank of the distribution), given by the integral manifold that contains this point in the image of the coordinate chart flat for  $\mathcal{H}$ . In this way, *if a distribution is completely integrable, it is integrable, and therefore involutive*. In fact, these implications are actually equivalences, thanks to Frobenius's theorem.

**Theorem 1** (Frobenius). *Every involutive distribution is completely integrable.*

*Proof.* ([3], Ch. 19, theorem 19.12) First, it will be shown that any involutive distribution is spanned by independent smooth commuting vector fields, and as a consequence of this fact, the distribution is completely integrable.

Let  $\mathcal{H}$  be an involutive distribution of rank  $k$  on an  $m$ -dimensional manifold  $M$ , and let  $p \in M$ . Let  $(U, \varphi)$  be a smooth coordinate chart centered in  $p$ . Then,  $\varphi(U) \subset \mathbb{R}^m$ , and let  $X_1, \dots, X_k$  be a smooth local frame for  $\mathcal{H}$  in  $U$ .

Now,  $\{d\varphi(X_i)\}_{i=1}^k$  is a set of linearly independent vector fields over  $\varphi(U)$ . One can reordinate the coordinates of  $\mathbb{R}^m$  to make  $\mathcal{H}'_{\varphi(p)}$  (understood as the fiber of the distribution whose local frame is given by  $\{d\varphi(X_i)\}_{i=1}^k$ ) complementary to the subspace generated by  $(\partial/\partial x^{k+1}|_{\varphi(p)}, \dots, \partial/\partial x^m|_{\varphi(p)})$ .

Let  $\pi : \mathbb{R}^m \rightarrow \mathbb{R}^k$  the projection onto the first  $k$  coordinates, that is,  $\pi(x^1, \dots, x^m) = (x^1, \dots, x^k)$ . This map induces a smooth (smooth because it is the composition  $\mathcal{H}' \hookrightarrow T\varphi(U) \xrightarrow{d\pi} T\pi(\varphi(U))$ ) bundle homomorphism  $d\pi : T\mathbb{R}^m \rightarrow \mathbb{R}^k$  that acts as follows:

$$d\pi \left( \sum_{i=1}^m v_i \frac{\partial}{\partial x^i} \Big|_{\varphi(q)} \right) = \sum_{i=1}^k v_i \frac{\partial}{\partial x^i} \Big|_{\pi(\varphi(q))}, \text{ for } q \in U.$$

By the choice of coordinates,  $\mathcal{H}'_{\varphi(p)} \subset T_{\varphi(p)}\mathbb{R}^m$  is complementary to the kernel of  $d\pi|_{\varphi(p)}$ , so the restriction  $d\pi|_{\mathcal{H}'_{\varphi(p)}}$  is bijective. By continuity, the same is true for  $d\pi|_{\mathcal{H}'_{\varphi(q)}}$ , with  $q \in U$ , and therefore, the matrix entries of  $\left(d\pi|_{\mathcal{H}'_{\varphi(q)}}\right)^{-1} : T_{\pi(\varphi(q))}\mathbb{R}^k \rightarrow \mathcal{H}'_{\varphi(q)}$  are smooth, which makes the map smooth. With this in mind, the set of vector fields  $V_1, \dots, V_k$ , defined by

$$V_i|_{\varphi(q)} = \left(d\pi|_{\mathcal{H}'_{\varphi(q)}}\right)^{-1} \frac{\partial}{\partial x^i} \Big|_{\pi(\varphi(q))}$$

are a local frame of  $\mathcal{H}'$ . It is in fact a smooth commuting local frame, as will be seen below.

First, notice that  $V_i$  and  $\partial/\partial x^i$  are  $\pi$ -related, for  $i = 1, \dots, k$ , because

$$\frac{\partial}{\partial x^i} \Big|_{\pi(\varphi(q))} = d\pi|_{\mathcal{H}'_{\varphi(q)}}(V_i|_{\varphi(q)}) = d\pi_{\varphi(q)}(V_i|_{\varphi(q)}),$$

so by the naturality of Lie brackets,

$$d\pi_{\varphi(q)}([V_i, V_j]_{\varphi(q)}) = \left[ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right]_{\pi(\varphi(q))} = 0,$$

but as  $d\pi|_{\mathcal{H}'_{\varphi(q)}}$  is injective and  $[V_i, V_j]_{\varphi(q)} \in \mathcal{H}'_{\varphi(q)}$ , since  $\mathcal{H}$  is involutive,  $[V_i, V_j]_{\varphi(q)} = 0$  for all  $q \in U$ , so  $\{V_i\}_{i=1}^k$  form a smooth commuting frame for  $\mathcal{H}'$ , and  $\{d\varphi^{-1}(V_i)\}_{i=1}^k$  are a smooth commuting local frame for  $\mathcal{H}$ .

Finally, to show that the existence of the commuting frame  $\{V_i\}_{i=1}^k$  for  $\mathcal{H}'$  is a sufficient condition for  $\mathcal{H}$  to be completely integrable, let  $\theta_i$  be the flow of  $V_i$ ,  $i = 1, \dots, k$ . There is a neighborhood  $W$  of  $\varphi(p)$ , contained in  $\varphi(U)$  such that the composition  $(\theta_1)_{t_1} \circ (\theta_2)_{t_2} \circ$

$\cdots \circ (\theta_k)_{t_k}$  is well defined, for sufficiently small  $t_1, \dots, t_k$  ( $|t_i| < \epsilon$ , for an adequate  $\epsilon > 0$ ). Define  $\Omega \subset \mathbb{R}^{m-k}$  as

$$\Omega = \{(s^{k+1}, \dots, s^m) \in \mathbb{R}^{m-k} : (0, \dots, 0, s^{k+1}, \dots, s^m) \in W\},$$

and  $\Phi : (-\epsilon, \epsilon) \times \Omega \rightarrow \varphi(U)$  as

$$\Phi(s^1, \dots, s^k, s^{k+1}, \dots, s^m) = (\theta_1)_{s^1} \circ \cdots \circ (\theta_k)_{s^k}(0, \dots, 0, s^{k+1}, \dots, s^m).$$

Notice that, by construction,  $\Phi(\{0\}^k \times \Omega) = (0^k \times \mathbb{R}^{m-k}) \cap W$ . Moreover,  $\partial/\partial s^i$  and  $V_i$  are  $\Phi$ -related for  $i = 1, \dots, k$ , because, as the flows commute, for a given  $s^0 = (s^1, \dots, s^m) \in (-\epsilon, \epsilon)^k \times \Omega$ ,

$$\begin{aligned} d\Phi_{s^0} \left( \frac{\partial}{\partial s^i} \Big|_{s^0} \right) f &= \frac{\partial}{\partial s^i} \Big|_{s^0} f(\Phi(s^1, \dots, s^m)) \\ &= \frac{\partial}{\partial s^i} \Big|_{s^0} f((\theta_1)_{s^1} \circ \cdots \circ (\theta_k)_{s^k}(0, \dots, 0, s^{k+1}, \dots, s^m)) \\ &= \frac{\partial}{\partial s^i} \Big|_{s^0} f((\theta_i)_{s^i} \circ \cdots \circ (\theta_{i-1})_{s^{i-1}} \circ (\theta_{i+1})_{s^{i+1}} \circ \cdots \\ &\quad \circ (\theta_k)_{s^k}(0, \dots, 0, s^{k+1}, \dots, s^m)), \end{aligned}$$

and for any  $q \in \varphi(U)$ ,  $t \mapsto (\theta_i)_t(q)$  is an integral curve of  $V_i$ , so the above expression is equal to  $V_i|_{\Phi(s^0)} f$ , which shows that  $\partial/\partial s^i|_{s^0}$  and  $V_i$  are  $\Phi$ -related.

By the previous computations,

$$d\Phi_0 \left( \frac{\partial}{\partial s^i} \Big|_0 \right) = V_i|_{\varphi(p)}, \quad i = 1, \dots, k,$$

and on the other hand, since  $\Phi(0, \dots, 0, s^{k+1}, \dots, s^m) = (0, \dots, 0, s^{k+1}, \dots, s^m)$ , it follows that

$$d\Phi_0 \left( \frac{\partial}{\partial s^i} \Big|_0 \right) = \frac{\partial}{\partial x^i} \Big|_{\varphi(p)}, \quad i = k+1, \dots, m.$$

Therefore,  $d\Phi_0$  takes the basis  $(\partial/\partial s^1|_0, \dots, \partial/\partial s^m|_0)$  of  $T_0\mathbb{R}^m$  to the basis  $(V_1|_{\varphi(p)}, \dots, V_k|_{\varphi(p)}, \partial/\partial x^{k+1}|_{\varphi(p)}, \dots, \partial/\partial x^m|_{\varphi(p)})$  of  $T_{\varphi(p)}\varphi(U)$ . By the inverse function theorem,  $\Phi$  is a local diffeomorphism (in a neighborhood of 0), and  $\phi = \Phi^{-1}$  is a smooth coordinate chart that takes  $V_i$  to  $\partial/\partial s^i$ , for  $i = 1, \dots, k$ , and takes  $\partial/\partial x^i$  to  $\partial/\partial s^i$ , for  $i = k+1, \dots, m$ . Thus, the smooth coordinate chart flat for  $\mathcal{H}$  in a neighborhood of  $p \in M$  is  $\phi \circ \varphi$ , and since this is independent of  $p$ , for every point there is a smooth coordinate chart flat for  $\mathcal{H}$ , what makes  $\mathcal{H}$  a completely integrable distribution.

□

The next proposition is one of the main consequences of Frobenius theorem, and it is fundamental to study foliations, topic that is going to be discussed below.



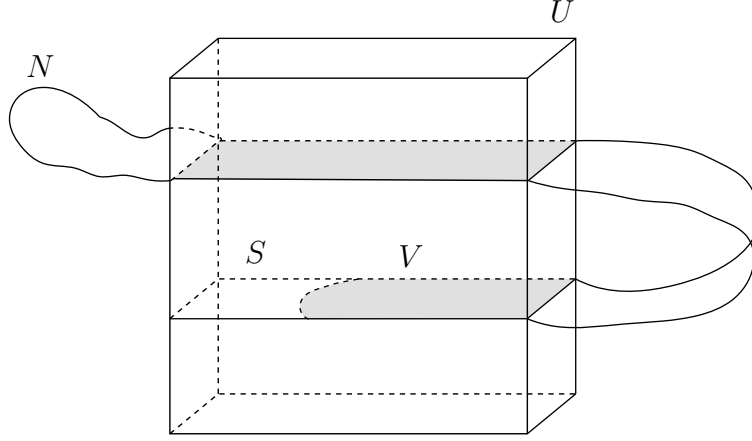


FIGURE 3. Local structure of an integral manifold.

**Proposition 3** (Local structure of integral manifolds). *Let  $\mathcal{H}$  be an involutive distribution of rank  $k$  on a smooth manifold  $M$ , and let  $(U, \varphi)$  be a flat chart for  $\mathcal{H}$ . If  $N$  is any connected integral manifold of  $\mathcal{H}$ , then  $\varphi(U \cap N)$  is the union of countably many disjoint open subsets of parallel  $k$ -dimensional slices of  $\varphi(U)$ , whose preimages are open in  $N$  and embedded in  $M$ .*

*Proof.* ([3], Ch. 19, proposition 19.16) Let  $N$  be an integral manifold of  $\mathcal{H}$ . Since the inclusion map  $\iota : N \hookrightarrow M$  is continuous,  $\varphi(\iota^{-1}(U)) = \varphi(N \cap U)$  is open in  $\varphi(N)$ . Then,  $\varphi(U \cap N)$  is the union of open slices (*i.e.*, open subsets of a slice of  $\varphi(U)$ ) with  $x_i = \text{constant}$ , for  $i = k+1, \dots, m$ . Moreover, this union is at most countable, because  $N$  is second countable, and since  $\varphi(N \cap U)$  is a union of open slices,  $\pi(\varphi(U \cap N))$  consists of a countable number of points (recall that  $\pi : \varphi(U) \rightarrow \mathbb{R}^{m-k}$  is the projection onto the last  $m-k$  coordinates) in  $\mathbb{R}^{m-k}$ , which implies that the union of open slices is at most countable.

Finally, if  $V$  is an open slice in the slice  $S$ , the inclusion map  $\varphi^{-1}(V) \hookrightarrow M$  is a smooth embedding, because it is the composition of smooth embeddings  $\varphi^{-1}(V) \hookrightarrow \varphi^{-1}(S) \hookrightarrow M$ .  $\square$

The last proposition implies that one can put all the maximal integral manifolds of an involutive distribution of rank  $k$  together, to obtain a partition on  $M$  into  $k$ -dimensional submanifolds, that satisfy the following definition.

**Definition 8.** A smooth chart  $(U, \varphi)$  for  $M$  is called *flat for a collection  $\mathcal{F}$  of  $k$ -dimensional submanifolds of  $M$*  if  $\varphi(U)$  is a cube in  $\mathbb{R}^m$ , and the image of each submanifold via  $\varphi$  intersects  $\varphi(U)$  in either the empty set or in a countable union of  $k$ -dimensional slices of the form  $x^{k+1} = c^{k+1}, \dots, x^m = c^m$ .

**Definition 9.** A *foliation of dimension  $k$  on a smooth manifold  $M$*  is a collection  $\mathcal{F}$  of disjoint, connected, nonempty, immersed  $k$ -dimensional submanifolds of  $M$  (called the *leaves* of the foliation), whose union is  $M$ , and such that in a neighborhood of each point  $p \in M$  there is a flat chart for  $\mathcal{F}$ .

The following are examples of foliations over a manifold, mainly taken from [3], Ch. 19, example 19.18.

*Example 6* (Collection of affine subspaces). The collection of all  $k$ -dimensional affine subspaces of  $\mathbb{R}^m$  parallel to  $\mathbb{R}^k \times \{0\}$  is a  $k$ -dimensional foliation for  $\mathbb{R}^m$ .

*Example 7* (Spheres centered at the origin). The collection of all spheres centered at 0 is an  $(m - 1)$ -dimensional foliation of  $\mathbb{R}^m \setminus \{0\}$ .

*Example 8* (Cartesian product of manifolds). If  $M$  and  $N$  are connected smooth manifolds, the collection of subsets of the form  $M \times \{q\}$ , with  $q \in N$ , is a foliation of  $M \times N$ , each of whose leaves is diffeomorphic to  $M$ .

*Example 9* (Foliations on a torus). The torus  $T = \mathbb{S}^1 \times \mathbb{S}^1$  can be endowed with the distribution induced by the cartesian product of manifolds (see example 8). In this case, the foliation is conformed by copies of  $\mathbb{S}^1$ . The horizontal curves are segments of this copies, and if two points lay in different copies, there is not a horizontal curve that connects them.

From the definition of foliations, it is clear that if  $\mathcal{F}$  is a foliation on a smooth manifold  $M$ , the collection of tangent spaces to the leaves of  $\mathcal{F}$  forms an involutive distribution on  $M$ . In a reciprocal way, the collection of maximal integral manifolds of an involutive distribution forms a foliation over the manifold, as it is stated in the global Frobenius theorem.

**Theorem 2** (Global Frobenius theorem). *Let  $\mathcal{H}$  be an involutive distribution on a smooth manifold  $M$ . The collection of all maximal connected integral manifolds of  $\mathcal{H}$  forms a foliation of  $M$ .*

The next lemma ([3], Ch. 19, lemma 19.22) is going to be used to prove this theorem.

**Lemma 2.** *Suppose that  $\mathcal{H} \subset TM$  is an involutive distribution, and let  $\{N_\alpha\}_{\alpha \in A}$  be any collection of connected integral manifolds of  $\mathcal{H}$  with a point in common. Then,  $N = \bigcup_\alpha N_\alpha$  has a unique smooth manifold structure making it into a connected integral manifold of  $\mathcal{H}$ .*

*Proof of the global Frobenius theorem.* ([3], Ch. 19, theorem 19.21) For each  $p \in M$ , let  $L_p$  be the union of all connected integral manifolds of  $\mathcal{H}$  containing  $p$ . By lemma 2,  $L_p$  is a connected integral manifold of  $\mathcal{H}$  containing  $p$ , and it is clearly maximal. By maximality, if  $L_p$  and  $L_{p'}$  intersect,  $L_p = L_{p'}$ , because  $L_p \cup L_{p'}$  is an integral manifold containing both  $p$  and  $p'$ . Thus, the maximal connected integral manifolds are either disjoint or identical.

If  $(U, \varphi)$  is any flat chart for  $\mathcal{H}$ , then  $\varphi(L_p \cap U)$  is a countable union of open subsets of slices (by proposition 3). For any such slice  $S$ , if  $\varphi(U \cap L_p) \cap S$  is neither empty nor all of  $S$ , then  $L_p \cup \varphi^{-1}(S)$  is a connected integral manifold properly containing  $L_p$ , which contradicts the maximality of  $L_p$ . Therefore,  $\varphi(L_p \cap U)$  is precisely a countable union of slices, so the collection  $\{L_p : p \in M\}$  is the desired foliation.  $\square$

With the global Frobenius theorem, one can answer the question at the start of this subsection: in general, for an given distribution there is no smooth horizontal curve that connects an arbitrary pair of points, because the points can be in different leaves of the foliation given by the distribution.

## 2. SUBRIEMANNIAN GEOMETRY

### 2.1. SubRiemannian Structure and Geodesics.

**Definition 10.** A *subriemannian structure* over a manifold  $M$  is a pair  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ , where  $\mathcal{H} \subset TM$  is a distribution and  $\langle \cdot, \cdot \rangle$  is a section of the bundle  $T_2^0 \mathcal{H} \xrightarrow{\pi} M$ , whose values are positive definite symmetric bilinear forms (see [5], Ch. 1, page 12).

The distribution  $\mathcal{H}$  is called *horizontal*, and the tensor field  $\langle \cdot, \cdot \rangle$  is called *metric*.

Some examples of subriemannian structures are given below.

*Example 10* (Riemannian Structure). Every Riemannian structure is in particular a subriemannian structure, where the distribution is the entire tangent bundle.

*Example 11* (Heisenberg Group). The distribution of the Heisenberg group is commented in example 2. The inner product over a fiber  $\mathcal{H}_{(x,y,z)}$ , with  $(x, y, z) \in \mathbb{R}^3$  is given by  $\langle \cdot, \cdot \rangle : \mathcal{H}_{(x,y,z)} \times \mathcal{H}_{(x,y,z)} \rightarrow \mathbb{R} : (v, w) \mapsto v_1 w_1 + v_2 w_2$ , where  $v = (v_1, v_2, v_3)$  and  $w = (w_1, w_2, w_3)$ .

*Example 12* (Vector Field over a Manifold). As seen in example 3, any smooth vector field  $X : M \rightarrow TM$  that does not cancel out in any point determines a distribution. The fiber inner-product  $\langle \cdot, \cdot \rangle : \mathcal{H}_p \times \mathcal{H}_p \rightarrow \mathbb{R}$  for  $p \in M$  is given by  $\langle \lambda_1 X_p, \lambda_2 X_p \rangle = \lambda_1 \lambda_2$ .

In the particular case of the horizontal curves, one can define the *length of a horizontal smooth curve*  $\gamma$  (denoted by  $\ell(\gamma)$ ) as in the case of Riemannian geometry:

$$\ell(\gamma) = \int \|\dot{\gamma}\| dt,$$

since the tangent vectors of the curve at all points are in the fibers of the horizontal distribution, where there is a inner product defined, and  $\|\dot{\gamma}\| = \sqrt{\langle \dot{\gamma}, \dot{\gamma} \rangle}$ .

**Definition 11.** The *distance between two points*  $p, q \in M$ , denoted by  $d(p, q)$ , is defined as the infimum of the lengths of all absolutely continuous horizontal curves that begin in  $p$  and end in  $q$ , that is,

$$d(p, q) = \inf \{ \ell(\gamma) \mid \gamma : [0, 1] \rightarrow M \text{ is absolutely continuous and } \gamma(0) = p, \gamma(1) = q \}.$$

The distance between two points is said to be infinite if there is no horizontal curve joining them.

The curves that are considered in the previous definition of distance are required to be absolutely continuous, that is a weaker condition of smoothness, and refers to a curve that is differentiable in almost all points of its domain. Nevertheless, the distance remains invariant if we consider instead the set of all smooth curves (see [4], page 23), but the motivation to consider the bigger set of absolutely continuous curves is the fact that the curves that realize the distance between two points are not always smooth.

**Definition 12.** Given a subriemannian structure  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$  over a manifold  $M$  (or Riemannian structure, if  $\mathcal{H} = TM$ ), it is said that an absolutely continuous horizontal curve  $\gamma : [a, b] \rightarrow M$ , with  $\gamma(a) = p, \gamma(b) = q$  is a *geodesic* if it realizes the distance between  $p$  and  $q$ , that is,

$$\ell(\gamma) = d(p, q).$$

**2.2. Chow's Theorem.** The section 1.2 provides a class of (involutive) distributions where it is not true that every two points can be connected through a horizontal curve. Now, it is worthwhile to ask for a sufficient condition for a distribution to have the connectedness property through horizontal curves. To discuss such a condition, we need to cover some preliminaries.

**Definition 13.** A distribution  $\mathcal{H} \subset TM$  is called *bracket generating* if for every  $p \in M$ , there is a local frame  $X_1, \dots, X_k : U \rightarrow TM$  of  $\mathcal{H}$  such that

$$TU = \text{span} \{ [X_{i_1}, \dots, [X_{i_{j-1}}, X_{i_j}]] : i_1, \dots, i_j = 1, \dots, k; j \in \mathbb{N} \}.$$

**Lemma 3.** Let  $X_1, X_2$  be two smooth vector fields over the manifold  $M$ , with respective local flows  $\Phi_1, \Phi_2 : (-\epsilon, \epsilon) \times U \rightarrow U$ , and let  $p \in U$ . Then, in any coordinate system the following relation holds:

$$[\Phi_1(t), \Phi_2(t)](p) := (\Phi_1(t) \circ \Phi_2(t) \circ \Phi_1(t)^{-1} \circ \Phi_2(t)^{-1})(p) = p + t^2 [X_1, X_2](p) + O(t^3).$$

*Proof.* By Taylor's theorem, we have

$$\Phi_i(t)(p) = p + tX_i(p) + \frac{t^2}{2}X_i(X_i(p)) + O(t^3), \text{ for } i = 1, 2.$$

In the other hand, as  $\{\Phi_i(t)\}_{t \in (-\epsilon, \epsilon)}$  is the family of diffeomorphisms associated with the flow of  $X_i$ ,  $\Phi_i(t)^{-1} = \Phi_i(-t)$ . Then,

$$\begin{aligned} (\Phi_1(t)^{-1} \circ \Phi_2(t)^{-1})(p) &= p - t(X_1(p) + X_2(p)) + t^2 X_1(X_2(p)) \\ &\quad + t^2(X_1(X_1(p)) + X_2(X_2(p))) + O(t^3), \end{aligned}$$

and evaluating  $\Phi_1(t) \circ \Phi_2(t)$  in this point gives

$$\begin{aligned} (\Phi_1(t) \circ \Phi_2(t) \circ \Phi_1(t)^{-1} \circ \Phi_2(t)^{-1})(p) &= p + t^2 [X_1, X_2](p) + t^2 (X_1(X_1(p)) + X_2(X_2(p))) \\ &\quad - t(X_1(p) + X_2(p)) - t^2 (X_1(X_1(p)) + X_2(X_2(p))) \\ &\quad + t(X_1(p) + X_2(p)) + O(t^3) \\ &= p + t^2 [X_1, X_2](p) + O(t^3), \end{aligned}$$

which is the desired result.  $\square$

Chow's theorem guarantees that any bracket generating over a connected manifold has the connectedness property through horizontal curves, and we going to derive this result in the rest of this section, following closely the procedure of [4], Section 2.4. First of all, choose a base point  $p \in M$  and a local orthonormal frame  $X_i$ ,  $i = 1, \dots, k$  for the distribution  $\mathcal{H}$ . Let  $\Phi_i$  be their respective flows. These flows can be used to move in the horizontal directions (points connected to  $p$  through integral curves of the orthonormal frame), using the formula  $\Phi_i(t)(q) = q + tX_i(q) + O(t^2)$  for sufficiently small  $t$ .

The assignments  $t \mapsto \Phi_i(t)(q)$  are horizontal curves for the distribution  $\mathcal{H}$ , and are called *simple horizontal curves*. As the frame is orthonormal, the length of a simple horizontal curve with  $0 \leq t \leq \epsilon$  is  $\epsilon$ .

Applying  $\Phi_k(t_k) \circ \dots \circ \Phi_2(t_2) \circ \Phi_1(t_1)$  to  $p$  and letting  $t_1, \dots, t_k$  vary over the cube  $|t_i| \leq \epsilon$ , we move from  $p$  to another point inside the  $k$ -dimensional cube of volume  $(2\epsilon)^k$  in the coordinate chart, whose sides are determined by the local frame in the image of  $p$ . Each point of this cube can be reached from  $p$  through the concatenation of  $k$  or fewer flows, and this cube is inside of the subriemannian ball of radius  $k\epsilon$ , that is,

$$\begin{aligned} \text{Box}_p(\epsilon) &:= \{\Phi_k(t_k) \circ \dots \circ \Phi_2(t_2) \circ \Phi_1(t_1)(p) \mid |t_i| < \epsilon, \ i = 1, \dots, k\} \\ &\subset B(k\epsilon, p) := \{q \in M \mid d(p, q) < k\epsilon\}. \end{aligned}$$

We can move in the remaining directions along horizontal paths by successive applications of the commutators of flows, based on the bracket generating property of the distribution in the following way. Let  $\mathcal{H}$  denote also the sheaf of smooth vector fields tangent to  $\mathcal{H}$ , that is, the association between open sets  $U \subset M$  and horizontal smooth vector fields  $\mathcal{H}(U)$  defined locally on  $U$ , and define the new vector sheaves

$$\mathcal{H}^2 := \mathcal{H} \oplus [\mathcal{H}, \mathcal{H}], \quad \mathcal{H}^{r+1} := \mathcal{H}^r \oplus [\mathcal{H}, \mathcal{H}^r],$$

where

$$[\mathcal{H}, \mathcal{H}^j] := \text{span} \{[X, Y] \mid X \in \mathcal{H}, \ Y \in \mathcal{H}^j\}, \text{ for } j = 1, 2, \dots$$

The movement in the  $\mathcal{H}^2/\mathcal{H}$  directions can be made along horizontal paths by applying the commutators  $\Phi_{ij}(t) := [\Phi_i(t), \Phi_j(t)]$  to  $p$ . As  $\Phi_{ij}(t)(p) = p + t^2[X_i, X_j](p)$  (by lemma 3) if  $|t| \leq \epsilon$ , we will move by an amount  $\epsilon^2$  in the  $\mathcal{H}^2/\mathcal{H}$  directions. This process is realized inductively until exhausting the tangent space. Let  $I = (i_1, i_2, \dots, i_n)$ ,  $1 \leq i_j \leq k$ , and  $X_I := [X_{i_1}, X_J]$ , where  $J = (i_2, \dots, i_n)$ , so that  $X_I$  is the iteration of  $n$  Lie brackets.

Similarly, define flows  $\Phi_I(t) := [\Phi_{i_1}(t), \Phi_{i_2}(t)]$ . By successive application of lemma 3, we have that

$$\Phi_I(t) = \mathbb{1} + t^n X_I + O(t^{n+1}).$$

Due to the bracket generating condition over any local frame of the distribution, we can select a local frame for the entire tangent bundle amongst the  $X_I$ , letting  $n$  and  $i_1, \dots, i_n$  vary accordingly. We choose such a frame and relabel it as  $Y_i$ ,  $i = 1, \dots, m$ , in a way that it satisfies:  $Y_1 = X_1, \dots, Y_k = X_k$  span  $\mathcal{H}$  near  $p$ ,  $\{Y_1, \dots, Y_{n_2}\}$  span  $\mathcal{H}^2$  near  $p$ ,  $\{Y_1, \dots, Y_{n_2}, \dots, Y_{n_3}\}$  span  $\mathcal{H}^3$  near  $p$ , and so on. The tuple  $(k, n_2, n_3, \dots, n_r)$  is called the *growth vector of the distribution at  $p$* , and the smallest integer  $r$  is called the *step or degree of nonholonomy of the distribution at  $p$* .

**Definition 14** ([4], page 49.). Let  $\{Y_1, \dots, Y_m\}$  be the local frame of the tangent bundle in a neighborhood of  $p \in M$  constructed as stated before using the local frame of a bracket generating distribution. Let  $Y_i = X_I$ ,  $I = (i_1, \dots, i_{n_I})$ . We denote the length of  $I$  as  $w_i := |I|$ , and define the assignment  $i \mapsto w_i$ , called the *weighting associated to the growth vector*.

We relabel the flows  $\Phi_I$  in a similar fashion, so that  $\Phi_i$  is the flow associated to  $Y_i$ . In this case, each point  $\Phi_i(t)(p)$  is the endpoint of the concatenation of  $w_i$  simple horizontal curves, each one of length  $t$ . If we impose that  $|t| \leq \epsilon$ , then  $\Phi_i(t)(p)$  lies in the ball of radius  $w_i \epsilon$  centered at  $p$ . Moreover, in coordinates we have that

$$\Phi_i(t)(p) = p + t^{w_i} X_i(p) + O(t^{w_i+1}),$$

so  $\Phi_i(t)(p)$  lies in the Euclidean box of volume  $\epsilon^{w_i}$ , in the  $\mathcal{H}^{w_i}$  directions. From this we can deduct that the subriemannian ball  $B(\epsilon, p)$  contains an Euclidean coordinate box whose sides are of order  $\epsilon^{w_i}$  in the  $i$ -th direction. This result is known as the ball-box theorem, and it will be stated and proved after the following definitions.

**Definition 15** ([4], definition 2.4.1). Coordinates  $y_1, \dots, y_m$  are said to be *linearly adapted to the distribution  $\mathcal{H}$  at  $p$*  if  $\mathcal{H}^i(p) := \{X(p) \mid X \text{ is a section of } \mathcal{H} \text{ defined on } p\}$  is annihilated by the differentials  $dy_{n_i+1}, \dots, dy_m$  at  $p$ , where  $n_i = n_i(p)$  are the coordinates of the growth vector at  $p$ .

**Definition 16.** The  $w$ - weighted box of size  $\epsilon$  is the point set

$$\text{Box}^w(\epsilon) := \{y \in \mathbb{R}^m \mid |y_i| \leq \epsilon^{w_i}; i = 1, \dots, m\}.$$

Set  $y_i$  as the coordinates centered at  $p$  such that  $dy_i(p)$  are the dual basis to the  $Y_i(p)$ , so that these coordinates are linearly adapted to the distribution by definition.

**Theorem 3** (Ball-Box Theorem). *There exist linearly adapted coordinates  $y_1, \dots, y_m$  and positive constants  $c < C$ ,  $\epsilon_0 > 0$ , such that for all  $\epsilon < \epsilon_0$ ,*

$$\text{Box}^w(c\epsilon) \subset B(\epsilon, p) \subset \text{Box}^w(C\epsilon).$$

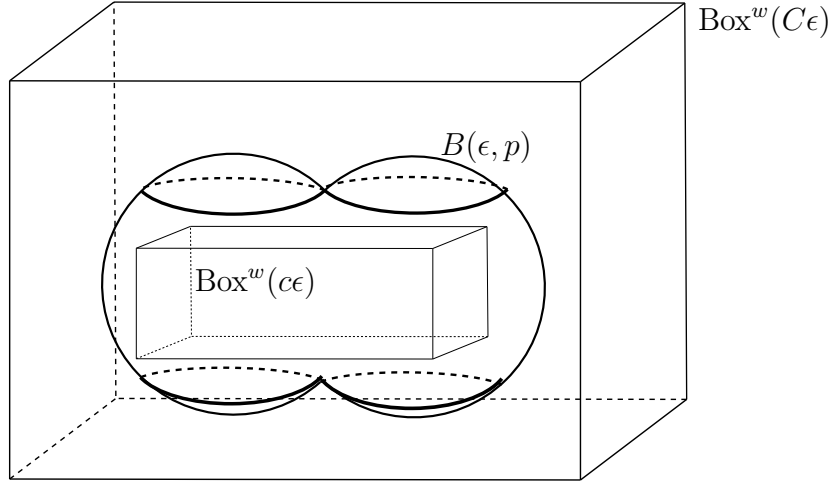


FIGURE 4. Ball box theorem.

*Proof of  $\text{Box}^w(c\epsilon) \subset B(\epsilon, p)$ .* ([4], theorem 2.4.2). As the approximation  $\Phi_{ij}(t) = \mathbb{1} + t^2 [X_i, X_j] + O(t^3)$  has the coefficient  $t^2$  that is always positive, we can not move in the negative  $[X_i, X_j]$  direction. To circumvent this, we define

$$\Psi_{ij}(t) = \begin{cases} [\Phi_i(t), \Phi_j(t)] & t \geq 0 \\ [\Phi_j(t), \Phi_i(t)] & t < 0, \end{cases}$$

so that

$$\Psi_{ij}(t) = \begin{cases} I + t^2 X_{ij} + O(t^3) & t \geq 0 \\ I - t^2 X_{ij} + O(t^3) & t < 0. \end{cases}$$

This problem occurs whenever the number  $w_i$  associated to the flow  $\Phi_I$  of  $Y_i$  is even. To solve this in general, we set

$$\Psi_I(t) = \begin{cases} \Phi_I(t) & t \geq 0 \\ [\Phi_J(t), \Phi_{i_1}(t)] & t < 0. \end{cases}$$

If  $w_i$  is odd, we keep  $\Psi_I = \Phi_I$ , and we relabel  $\Psi_I$  as  $\Psi_i$  accordingly. Now, we introduce the functions

$$\sigma_i(t) = \begin{cases} t^{w_i} & w_i \text{ even}, t \geq 0 \\ -t^{w_i} & w_i \text{ even}, t < 0 \\ t^{w_i} & w_i \text{ odd} \end{cases}$$

to simplify the notation of the approximation

$$\Psi_I(t) = \mathbb{1} + \sigma_i(t)Y_i + O(t^{w_i+1}).$$

Define the map  $F : \mathbb{R}^m \rightarrow M$  as  $F(t_1, \dots, t_m) = (\Psi_m(t_m) \circ \dots \circ \Psi_1(t_1))(p)$ , and let  $y_i$  be linearly adapted coordinates for which the  $dy_i$  are dual to the  $Y_i$  of the basis of the tangent space at the base point  $p$ . Then, in coordinates,

$$F_i(t_1, \dots, t_m) = \sigma_i(t_i) + o(t_i^{w_i}) = \pm t_i^{w_i} + o(t_i^{w_i}),$$

where the error  $o(t_i^{w_i})$  corresponds to the composition of the previous  $\Psi_1(t), \dots, \Psi_{i-1}(t)$  flows evaluated in the big  $O$ -errors of the previous sequential arguments. It remains to understand and bound these error terms, and for that we introduce the new variables  $s_i = \sigma_i(t)$ , so that  $F_i(t_1, \dots, t_m) = s_i + o(s_i)$ .

The map  $\sigma = (\sigma_1, \dots, \sigma_m)$  is a homeomorphism between neighborhoods of 0 in  $\mathbb{R}^m$ , so let  $S$  be the inverse of  $\sigma$  that is given by  $S = (s_1, \dots, s_m)$ , where  $s_i(t_i) = \pm |t_i|^{1/w_i}$ . Then, it follows that

$$y \circ F \circ S(s_1, \dots, s_m) = (s_1, \dots, s_m) + o(|s|),$$

from where we deduce that  $F \circ S$  is differentiable at the origin because of the differentiability of both  $y$  and  $y \circ G \circ S$ , and its derivative is the identity in the  $s - y$  coordinates, so it is also  $C^1$  near 0. We will proceed applying the inverse function theorem to invert  $F \circ S$  in a neighborhood of  $p$ .

Write  $F \circ S(s_1, \dots, s_m)$  in coordinates as  $(y_1, \dots, y_m)$ . Then, there exists a constant  $\epsilon_0$  and constants  $c = c(\epsilon_0)$ ,  $C = C(\epsilon_0)$  such that  $c|s_i| \leq |y_i(s_1, \dots, s_m)| \leq C|s_i|$  whenever  $|s_i| \leq \epsilon_0$ . Apply the constraint  $|s_i| \leq \epsilon^{w_i}$  to the  $s_i$ . It is clear that the inverse image of the point set  $|s_i| \leq \epsilon^{w_i}$  under the map  $S$  is the  $\epsilon$ -cube box  $\text{Box}(\epsilon) = \{(t_1, \dots, t_m) \mid |t_i| \leq \epsilon\}$ , so that applying  $F$  to the  $\epsilon$ -cube gives

$$\text{Box}^w(c\epsilon) \subset F(\text{Box}(\epsilon)) \subset \text{Box}^w(C\epsilon).$$

Each curve in  $F(\text{Box}(\epsilon))$  is the endpoint of a horizontal curve starting at  $p$  whose length is less than  $M\epsilon$ , where  $M = M(w)$  counts the number of concatenations involved in  $F$ . Consequently,

$$F(\text{Box}(\epsilon)) \subset B(M\epsilon, p),$$

from where we get that  $\text{Box}^w(c\epsilon) \subset B(\epsilon, p)$ . □

The remaining containment in the Ball-Box theorem will not be proved in this text, as the proof of Chow's theorem only requires the first containment. The proof of the remaining containment can be found in [4], section 2.7.

With the Ball-Box theorem, Chow's theorem can be derived easily.

**Theorem 4** (Chow's Theorem). *If  $\mathcal{H}$  is a bracket generating distribution on a connected manifold  $M$ , then any two points of  $M$  can be connected by a horizontal path.*

*Proof.* ([4], page 52). The containment  $\text{Box}^w(c\epsilon) \subset B(\epsilon, p)$  shows that  $B(\epsilon, p)$  is a neighborhood of  $p$ , which means that the accesible points from  $p$  through horizontal curves form a neighborhood of  $p$ . Let  $q \in M$  be another arbitrary point on the connected manifold, and take a smooth curve  $\gamma$  connecting  $p$  and  $q$ .

The image of  $\gamma$  is compact, so we can cover it with finitely many successive box neighborhoods, where the Ball-Box theorem holds. Denote these neighborhoods as  $U_1, \dots, U_n$ , with each one of these boxes centered on successive points  $p_i$  along the image of  $\gamma$ , and



$p_1 = p, p_n = q$ . Moreover, these open sets can be chosen in order to satisfy  $U_i \cap U_{i+1} \neq \emptyset$ , so that there exists  $q_i \in U_i \cap U_{i+1}$ . By the Ball-Box theorem, we have paths connecting  $p_i$  with  $q_i$ , and  $q_i$  with  $p_{i+1}$ . Concatenating these paths yields a piecewise horizontal path connecting  $p$  and  $q$ .

□

*Example 13* (Cartan's Distribution). Consider the distribution over  $\mathbb{R}^3$  determined by the 1-form  $\omega = dy - zdx$ , or in dual manner, by the smooth vector fields  $\partial/\partial x + z\partial/\partial y, \partial/\partial z$ . From its definition it is clear that Cartan's distribution is of rank 2, so we can not know beforehand if any two pair of points can be connected through a horizontal curve. Nevertheless, as

$$\left[ \frac{\partial}{\partial z}, \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} \right] = \frac{\partial}{\partial y}$$

and  $T\mathbb{R}^3 = \text{span}\{\partial/\partial x + z\partial/\partial y, \partial/\partial z, \partial/\partial y\}$ , Cartan's distribution is bracket generating, so by Chow's theorem any two points in  $\mathbb{R}^3$  can be connected by a horizontal path.

### 3. SUBRIEMANNIAN METRICS ON BUNDLES

**3.1. Ehresmann connection.** Let  $\pi : Q \rightarrow M$  be a submersion, which means that for all  $q \in Q$ ,  $d\pi_q : T_q Q \rightarrow T_{\pi(q)} M$  is a surjective map. The fiber  $Q_m = \pi^{-1}(m)$ , with  $m = \pi(q)$  is a submanifold because  $m$  is a regular value of  $\pi$ .

**Definition 17.** Given a submersion  $\pi : Q \rightarrow M$ , the *vertical space* at  $q \in Q$ , denoted as  $V_q$ , is the tangent space to the fiber  $Q_m$ , with  $m = \pi(q)$ , that is,

$$V_q = \text{Ker}(d\pi_q) = T_q Q_m.$$

The collection of vertical spaces is a distribution  $V \subset TQ$  that assigns to each  $q \in Q$  the space  $V_q$ . The distribution is by construction integrable, and its integral manifolds are the fibers  $Q_m, m \in \pi(Q)$ .

**Definition 18** ([4], definition 11.1.1). A *connection* for a submersion  $\pi : Q \rightarrow M$  is a distribution that is everywhere transverse to the vertical one, that is,

$$V_q \oplus \mathcal{H}_q = T_q Q, \text{ for all } q \in Q.$$

The restriction of  $d\pi_q$  to  $\mathcal{H}_q$  is a linear isomorphism between  $\mathcal{H}_q$  and  $T_{\pi(q)} M$  because  $d\pi_q$  is surjective, and by the isomorphism theorem we have that  $\mathcal{H}_q \cong T_{\pi(q)} M$ .

**Definition 19.** If  $M$  is endowed with a metric  $g$ , we can use the previous linear isomorphism to pull the inner product back to the horizontal distribution  $\mathcal{H}$ : the subriemannian structure  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$  induced by the Riemannian structure  $(M, g)$  is given by

$$\langle v, w \rangle_q = g_{\pi(q)}(d\pi_q(v), d\pi_q(w)), \text{ for all } q \in Q \text{ and } v, w \in \mathcal{H}_q.$$

This subriemannian metric varies smoothly with  $q \in Q$  ([4], page 194), and is referred as the *induced subriemannian structure* over  $Q$  by the submersion  $\pi : Q \rightarrow M$  and the Riemannian metric over  $M$ .

**Definition 20.** The *horizontal lift* of a curve  $c : I \rightarrow M$  starting at  $m \in M$  is defined as the unique curve  $\gamma : I \rightarrow Q$  that is tangent to  $\mathcal{H}$ , starts at  $q \in Q_m$ , and projects to  $c$ , that is,  $\pi \circ \gamma = c$ .

The uniqueness of the horizontal lift follows from the fact that  $\gamma : I \rightarrow Q$  is solution of the initial value problem

$$\frac{d\gamma(t)}{dt} = d\pi_{c(t)}^{-1} \left( \frac{dc(t)}{dt} \right), \quad \gamma(0) = q.$$

*Example 14* (Canonical projection from  $\mathbb{R}^{m+n}$  to  $\mathbb{R}^m$ ). Consider the canonical projection  $\pi : \mathbb{R}^{m+n} \rightarrow \mathbb{R}^m$  that maps the first  $m$ -coordinates of a point in  $\mathbb{R}^{m+n}$  to  $\mathbb{R}^m$ . This map is a submersion, and its vertical space at a point  $q \in \mathbb{R}^{m+n}$  is  $V_q = \{(0, \dots, 0, v_1, \dots, v_n) : v_1, \dots, v_n \in \mathbb{R}\} \cong \mathbb{R}^n$ . A connection for this submersion is  $\mathcal{H}_q = \{(v_1, \dots, v_m, 0, \dots, 0) : v_1, \dots, v_m \in \mathbb{R}\} \cong \mathbb{R}^m$ . Given an arbitrary curve  $c : I \rightarrow \mathbb{R}^m$ , starting at  $\mathbf{p} = (p_1, \dots, p_m)$ , its horizontal lift passing through the point  $(p_1, \dots, p_m, q_1, \dots, q_n) \in \pi^{-1}(\mathbf{p})$  is the curve  $\gamma : I \rightarrow \mathbb{R}^{m+n}$ ,  $t \mapsto (c(t), q_1, \dots, q_n)$ .

*Remark 3.* The connection  $\mathcal{H}$  is called *complete* or is referred as an *Ehresmann connection* if every smooth curve  $c : I \rightarrow M$  has a horizontal lift.

**Proposition 4.** *The induced subriemannian structure satisfies the following properties:*

- (1) *The subriemannian length of a horizontal path on  $Q$  equals the Riemannian length of its projection to  $M$ .*
- (2) *The horizontal lift of a Riemannian geodesic in  $M$  is a subriemannian geodesic in  $Q$ .*
- (3) *The projection  $\pi$  is distance decreasing, that is,  $d_M(\pi(q_1), \pi(q_2)) \leq d_Q(q_1, q_2)$ , for all  $q_1, q_2 \in Q$ .*

*Proof.* (1) Let  $\gamma : I \rightarrow Q$  be a horizontal path to  $\mathcal{H}$  starting at  $q \in Q$ . Then, its longitude is

$$\ell_Q(\gamma) = \int_0^1 \|\dot{\gamma}(t)\|_Q dt = \int_0^1 \|d\pi_{\gamma(t)}(\dot{\gamma}(t))\|_M dt,$$

having in mind that  $d\pi_{\gamma(t)}$  induces an isomorphism between  $\mathcal{H}_{\gamma(t)}$  and  $T_{\pi(\gamma(t))}M$ , and this isomorphism determines the induced subriemannian structure over  $Q$ . As  $c = \pi \circ \gamma$  and  $d/dt(\pi \circ \gamma) = d\pi_{\gamma(t)}(\dot{\gamma}(t))$ ,  $\ell_Q(\gamma) = \ell_M(c)$ .

- (2) Suppose that  $c : I \rightarrow M$  is a Riemannian geodesic, and let  $\gamma : I \rightarrow Q$  be its respective horizontal lift. Take any other horizontal curve  $\eta : I \rightarrow Q$  with the same endpoints than  $\gamma$ . Then, by the first affirmation of this proposition and the fact that  $c$  is a geodesic, we have

$$\ell_Q(\eta) = \ell_M(\pi \circ \eta) \geq \ell_M(c) = \ell_Q(\gamma),$$

from where we get that  $\gamma : I \rightarrow Q$  is a subriemannian geodesic.

- (3) Let  $q_1, q_2 \in Q$ . If  $d_Q(q_1, q_2) = \infty$  the result is immediate. Else, let  $\{\gamma_i\}_{i \in \mathbb{N}}$  be a minimizing sequence of  $d_Q(q_1, q_2)$ . Now consider the sequence of projected curves  $\{\pi \circ \gamma_i\}_{i \in \mathbb{N}}$ , from where there are two possibilities: if this last sequence is minimizing,  $d_Q(q_1, q_2) = d_M(\pi(q_1), \pi(q_2))$ , as  $\ell_Q(\gamma_i) = \ell_M(\pi \circ \gamma_i)$ ; if it is not minimizing,  $d_Q(q_1, q_2) = \lim_{i \rightarrow \infty} \ell_Q(\gamma_i) = \lim_{i \rightarrow \infty} \ell_M(\pi \circ \gamma_i) > d_M(\pi(q_1), \pi(q_2))$  by definition. Therefore,  $d_Q(q_1, q_2) \geq d_M(\pi(q_1), \pi(q_2))$ .

□

**Definition 21.** A Riemannian metric on  $Q$  is said to be *compatible* with the induced subriemannian metric if the algebraic splitting  $TQ = V \oplus \mathcal{H}$  is an orthogonal decomposition with respect to the Riemannian metric on  $Q$ .

*Remark 4.* Thanks to the orthogonal algebraic splitting, a compatible Riemannian metric is determined by its restriction to  $V$ , and the induced subriemannian structure restricted to  $\mathcal{H}$ . This fact can be turned around in the following sense. Take a Riemannian metric  $g_Q$  on  $Q$ . Let  $V_q^\perp$  be the orthogonal complement of the vertical space  $V_q$  in  $q \in Q$  with respect to the Riemannian metric, and set  $\mathcal{H}_q = V_q^\perp$ . The Riemannian metric defines inner products in  $V_q^\perp$  and  $T_m M$ , being defined in the first as the restriction of the Riemannian metric, and in the second is given by the metric and the isomorphism  $\mathcal{H}_q \cong T_m M$ . Nevertheless, this inner product on  $T_m M$  can depend on the point  $q \in Q_m$ , as the metric could vary with the point of the fiber. If that is the case, we cannot construct a Riemannian metric on  $M$ , as there is no consistent way to choose which  $q \in Q_m$  is going to be used to define the inner product ([4], page 195).

**Definition 22.** Given a Riemannian manifold  $(Q, g_Q)$ ,  $\pi : Q \rightarrow M$  a submersion, we say that the pair  $(\pi, g_Q)$  is a *Riemannian submersion* if for every  $m \in M$  and  $q \in Q_m$ , the isomorphism  $d\pi_q|_{\mathcal{H}_q}$  is an isometry between  $\mathcal{H}_q$  and  $T_m M$ .

*Example 15.* Note that the canonical projection  $\pi : \mathbb{R}^{m+n} \rightarrow \mathbb{R}^m$  given in example 14 is a Riemannian submersion if  $\mathbb{R}^{m+n}$  is endowed with the cartesian metric.

### 3.2. Metrics on Principal Bundles.

**Definition 23** ([3], page 268). Let  $M, G, Q$  be manifolds. A *(locally trivial) fiber bundle* over  $M$  with fiber  $G$  is a manifold  $Q$ , together with a smooth surjective map  $\pi : Q \rightarrow M$  with the property that for each  $x \in M$  there is a neighborhood  $U$  of  $x$  and a diffeomorphism  $\phi : \pi^{-1}(U) \rightarrow U \times G$ , called a local trivialization of  $Q$  over  $U$ , such that the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\phi} & U \times G \\ & \searrow \pi \quad \swarrow \text{proj}_U & \\ & U & \end{array}$$

$Q$  is called the *total space of the bundle*,  $M$  is the *base*,  $G$  is the *standard fiber* and  $\pi$  the *projection*.

The following definition will require some definitions related to group actions over manifolds, that we will cover next.

**Definition 24.** For a Lie group  $G$  and a manifold  $Q$ , a *group action* is a smooth application  $\alpha : Q \times G \rightarrow Q$ , denoted by  $\alpha(q, g) = q \cdot g$ , such that:

- (1) For all  $q \in Q$ ,  $q \cdot e = q$ , where  $e$  is the identity element of  $G$ .
- (2) For all  $g, h \in G$  and  $q \in Q$ ,  $(q \cdot g) \cdot h = q \cdot (gh)$ .

If this action exists, we say that  $G$  acts on  $Q$  through  $\alpha$ . As a consequence of that definition, we can see that the map  $\alpha_g : Q \rightarrow Q : q \mapsto q \cdot g$  is a diffeomorphism for all  $g \in G$ , as it is smooth and its inverse is  $\alpha_{g^{-1}}$ .

We say that the action  $\alpha$  is *free* (or that  $G$  acts *freely* on  $Q$ ) if for all  $q \in Q$ ,  $qg = q$  implies that  $g = e$ . The *orbit*  $G_q$  of the action for a point  $q \in Q$  is defined as the image of the restriction  $\alpha|_{\{q\} \times G}$ , that is,  $G_q := \{q \cdot g \mid g \in G\}$ . We can construct an equivalence relation over  $Q$  in the following way:  $q \sim p$  if and only if  $G_p = G_q$ , that is, if both points belong to the same orbit, or written explicitly, if there is an element  $g \in G$  such that  $q = p \cdot g$ . The set of all equivalence classes is denoted as  $Q/G$ , and is known as the *orbit space of the action*.

*Remark 5.* The orbit space of an action is not, in general, a manifold, but because of the *quotient manifold theorem* (see [3], theorem 21.10), it is sufficient to have a free, proper and smooth action in order to guarantee that  $Q/G$  is a topological manifold.

**Definition 25.** The submersion  $\pi : Q \rightarrow M$  is called a *principal  $G$ -bundle* if its is a fiber bundle whose fiber  $G$  is a Lie group, and this group acts on  $Q$  in such a way that the following properties hold:

- (1)  $G$  acts freely on  $Q$ .
- (2) The action orbits are the fibers of  $\pi : Q \rightarrow M$ , that is, for all  $m \in M$ ,  $\pi^{-1}(m) = G_q$ , with  $q \in Q_m$ .
- (3) The local trivializations of the fiber bundle are equivariant with respect to the action, that is, for every local trivialization  $\phi : U \times G \rightarrow \pi^{-1}(U)$ , the following property holds:

$$\phi(q, g \cdot h) = \phi(q, g) \cdot h,$$

for every  $q \in U$ , and  $g, h \in G$ .

From this we can conclude that if  $\pi : Q \rightarrow M$  is a principal  $G$ -bundle, then  $Q/G$  is a topological manifold, and  $M$  is homeomorphic to  $Q/G$ . This homeomorphism  $\phi : M \rightarrow Q/G$  is given by  $\phi(m) = [q]$ , with  $\pi(q) = m$ .

**Definition 26** ([4], page 196). Given a principal  $G$ -bundle  $\pi : Q \rightarrow M$ , we say that a horizontal distribution  $\mathcal{H}$  is a *connection for this principal  $G$ -bundle* if it is a connection for  $\pi$  and the  $G$ -action on  $Q$  preserves the horizontal distribution  $\mathcal{H}$  in the sense that  $\mathcal{H}_{q \cdot g} = d(\alpha_g)_q(\mathcal{H}_q)$ , for all  $q \in Q$ ,  $g \in G$ . In this case, we say that the pair  $(\pi, \mathcal{H})$  is a *principal  $G$ -bundle with connection*.

*Remark 6.* An extremely useful property that any principal  $G$ -bundle with connection is that its connection is an Ehresmann connection, *i.e.*, it is complete, as it is stated in the following result.

**Proposition 5** ([6], proposition 3.1). *Let  $\pi : Q \rightarrow M$  be a principal  $G$ -bundle with connection and  $c : I \rightarrow M$  a smooth curve. For an arbitrary  $q \in Q_{c(0)} = G_q$ , there is a unique horizontal lift  $\gamma : I \rightarrow Q$  of  $c$  ( $\pi \circ \gamma = c$ ,  $\dot{\gamma}(t) \in \mathcal{H}_{\gamma(t)}$  for all  $t \in I$ ) that starts from  $q$ .*

**Definition 27** ([4], definition 11.2.1). A subriemannian structure  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$  on the total space of the principal  $G$ -bundle  $\pi : Q \rightarrow M$  is called a *metric of bundle type* if it is an induced subriemannian structure with respect to the submersion  $\pi$  and the metric on  $M$  (see definition 19).

### 3.3. Examples of principal $G$ -bundles.

**3.3.1. Trivial bundle.** Let  $M$  be a manifold and  $G$  a Lie group. Define the projection onto the first factor  $\pi : M \times G \rightarrow M$ . It is clear that this map is surjective and a submersion. Moreover, it is a fiber bundle, whose local (global) trivialization is the identity map  $\iota : M \times G \rightarrow M \times G$ .

If we define the action  $\alpha : (M \times G) \times G \rightarrow M \times G$  as  $((p, g), q) \mapsto (p, gq)$ , it is clear that  $G$  acts freely on  $M \times G$ ,  $\iota(p, gh) = \iota(p, g) \cdot h$  (the local trivialization is equivariant with respect to the action) and  $\pi^{-1}(p) = \{p\} \times G = G_{(p,g)}$ , so that  $\pi$  is a principal  $G$ -bundle. A connection for this bundle is the distribution  $\mathcal{H}_{(p,g)} = \text{Ker } d\tilde{\pi}(p, g) \subset T(M \times G)$ , with  $\tilde{\pi} : M \times G \rightarrow G$  the projection onto the second factor.

**3.3.2. Hopf Fibration.** All rotations in  $\mathbb{R}^3$  can be expressed using quaternions. The set of quaternions is denoted as  $\mathbb{H} := \{a + bi + cj + dj \mid a, b, c, d \in \mathbb{R}\}$ , and in it we can define sum and product operations. The sum of two quaternions  $a_1 + b_1i + c_1j + d_1k$ ,  $a_2 + b_2i + c_2j + d_2k$  is  $(a_1 + a_2) + (b_1 + b_2)i + (c_1 + c_2)j + (d_1 + d_2)k$ . The product is defined in order to satisfy the distributive property with respect to the sum, and it also satisfies the following rules:

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

In analogy with the complex plane and its isomorphism with  $\mathbb{R}^4$ , we define the norm of  $z = a + bi + cj + dk$  as  $|z| = \sqrt{a^2 + b^2 + c^2 + d^2}$ , and the conjugate of  $z$  as  $z^* = a - bi - cj - dk$ . Moreover, every quaternion  $z \neq 0$  has a multiplicative inverse, given by  $z^{-1} = z^*/|z|^2$ . One remarkable property of quaternions is the relation  $|z_1 z_2| = |z_1| |z_2|$ . Every rotation around an axis given by a vector  $\vec{v} = (b, c, d)$  and an angle  $\theta$  can be identified with an

unitary quaternion  $r = a + bi + cj + dk$  ( $|z|=1$ ), where  $\theta = 2 \arccos(a)$  (see [7], page 6). This rotation  $R_r : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is given by  $x \mapsto r x r^{-1} = r x r^*$ , and  $x$  identified as a pure quaternion, that is,  $x = x_1 i + x_2 j + x_3 k$ . To show explicitly that this map is the desired rotation, one can first show that this linear map preserves norm and the vector  $bi + cj + dk$ , that is, it is an eigenvector with eigenvalue 1, so it must be a rotation with respect the axis given by  $(b, c, d)$ . The idea of the proof that the rotation angle is  $2 \arccos(a)$  can be found in [7], page 6, investigation step 3.

The set of all unitary quaternions is  $\mathbb{S}^3$ , so that  $\mathbb{S}^3$  acts on  $\mathbb{R}^3$  via rotations. The Hopf projection is defined as the map  $\pi : \mathbb{S}^3 \rightarrow \mathbb{S}^2$  that sends the point  $(1, 0, 0)$  (identified as the quaternion  $i$ ) to its respective image after rotating  $\mathbb{R}^3$  with the associated quaternion, as an element of  $\mathbb{S}^3$ . That is,  $\pi(a, b, c, d) = R_r(1, 0, 0)$ , where  $r = a + bi + cj + dk$ . The function  $\pi$  previously defined is surjective, as every point  $x \in \mathbb{S}^2$  can be reached from  $(1, 0, 0)$  through an adequate rotation. Explicitly, it suffices to take a normal vector to both  $(1, 0, 0), x$  as the axis of rotation, and the angle of rotation is the angle between both vectors.

If we identify  $\mathbb{R}^4 = \mathbb{C} \times \mathbb{C}$  and  $\mathbb{R}^3 = \mathbb{R} \times \mathbb{C}$ , Hopf projection from  $\mathbb{S}^3 = \{(z_1, z_2) \mid z_1, z_2 \in \mathbb{C}, |z_1|^2 + |z_2|^2 = 1\}$  to  $\mathbb{S}^2 = \{(x, z) \mid z \in \mathbb{C}, x \in \mathbb{R}, |z|^2 + x^2 = 1\}$  can be written as  $\pi : \mathbb{S}^3 \rightarrow \mathbb{S}^2$ ,  $\pi(z_1, z_2) = (|z_1|^2 - |z_2|^2, 2z_1 z_2^*)$ , because writing explicitly  $r i r^{-1}$  with  $r = a + bi + cj + dk$  gives  $(a^2 + b^2 - c^2 - d^2)i + 2(ad + bc)j + 2(bd - ac)k$ . In this form one can show that this map is a submersion analyzing the minors of the jacobian matrix in cartesian coordinates (see [8], proof of theorem 3.11).

To recognize the fibers of this submersion, notice that  $\pi^{-1}(1, 0) = \{(w_1, 0) \mid |w_1| = 1\} = \{(e^{i\theta}, 0) \mid \theta \in (0, 2\pi]\}$ . Now, in quaternion notation, if  $s, r \in \pi^{-1}(x)$ ,  $r i r^{-1} = s i s^{-1} = x$ , that is,  $(s^{-1}r)i(r^{-1}s) = i$ . Then,  $s^{-1}r \in \pi^{-1}(1, 0)$ , and  $r = s e^{i\theta}$ , so  $\pi^{-1}(x) = \{r e^{i\theta} \mid \theta \in (0, 2\pi]\}$ , and each fiber is diffeomorphic to  $\mathbb{S}^1$ .

With this,  $\mathbb{S}^3$  can be seen as the disjoint union of these circles, and endowing this disjoint union with the standard manifold structure of  $\mathbb{S}^3$ , we identify  $\mathbb{S}^2 = \mathbb{C}P^1$ , and take the open sets  $U_i = \{[z_0 : z_1] \mid z_i \neq 0\}$ , with  $i = 0, 1$ . The local trivializations are (see [1], page 68)  $\varphi_i : U_i \times \mathbb{S}^1 \rightarrow \pi^{-1}(U_i)$ ,

$$\varphi_i([z_0 : z_1], \lambda) = \left( \frac{\lambda}{\sqrt{1 + |\omega_i|^2}}, \frac{\lambda \omega_i}{\sqrt{1 + |\omega_i|^2}} \right),$$

with  $\omega_0 = z_1/z_0$  and  $\omega_1 = z_0/z_1$ . Now, we define an action  $\alpha : \mathbb{S}^3 \times \mathbb{S}^1 \rightarrow \mathbb{S}^3$ , given by  $\alpha(z_1, z_2, \lambda) = (z_1 \lambda, z_2 \lambda)$ , where  $(z_1, z_2) \in \mathbb{S}^3 \subset \mathbb{C}^2$  and  $\lambda \in \mathbb{S}^1 \subset \mathbb{C}$ . This action acts freely on  $\mathbb{S}^3$ , and its orbits coincide with the fibers of  $\pi$ , so that  $\mathbb{S}^3/\mathbb{S}^1 \cong \mathbb{S}^2$ . From this point it is clear that the Hopf bundle is a fiber bundle, and its local trivializations are equivariant with respect to the action, as  $\varphi_i([z_0 : z_1], \lambda_1 \lambda_2) = \varphi_i([z_0 : z_1], \lambda_1) \lambda_2$ . Therefore,  $\pi : \mathbb{S}^3 \rightarrow \mathbb{S}^2$  is a principal  $\mathbb{S}^1$ -bundle.

To find a connection for this submersion, we recall that, by the preimage of regular value theorem,  $\pi^{-1}(m) = \mathbb{S}_q^1$  is a submanifold of  $\mathbb{S}^3$ , of dimension  $\dim(\mathbb{S}^3) - \dim(\mathbb{S}^2) = 1$ , and its tangent space (vertical space of the submersion)  $T_q \mathbb{S}_q^1$  is  $\text{Ker}(d\pi_q)$ . Now, if

$\mathbb{S}^3$  is endowed with the standard Riemannian structure, there is only one connection compatible with the metric, and is given by  $\mathcal{H}_q = \text{Ker}(d\pi_q)^\perp$ . The vector that determines  $\text{Ker}(d\pi_q)$  can be found using the action if we consider the tangent vector of the curve  $\gamma : (-\epsilon, \epsilon) \rightarrow \mathbb{S}^3$ ,  $t \mapsto \gamma(z_1, z_2, e^{it}) = (z_1 e^{it}, z_2 e^{it})$  at  $t = 0$ , where  $q = (z_1, z_2)$ . With this,  $\text{Ker}(d\pi_q) = \langle (iz_1, iz_2) \rangle$ , and the connection at  $q \in \mathbb{S}^3$ ,  $\mathcal{H}_q$ , will be the orthogonal complement (with respect to  $\mathbb{R}^4$  provided with the usual norm) of the vector space  $\langle (iz_1, iz_2), (z_1, z_2) \rangle = \langle (x_1, y_1, x_2, y_2), (-y_1, x_1, -y_2, x_2) \rangle$ , that is,  $\mathcal{H}_q = \langle (x_2, -y_2, -x_1, y_1), (y_2, x_2, -y_1, -x_1) \rangle$ , where  $q = (x_1 + iy_1, x_2 + iy_2)$ .

This principal  $\mathbb{S}^1$ -bundle has connection because the kernel of  $d(\pi \circ \alpha_g)_q$  is the same of  $d\pi_q$ , as  $d\alpha_g$  is an isomorphism. Therefore,  $d(\alpha_g)_q(\mathcal{H}_q) = \mathcal{H}_{q \cdot g}$ , and the  $\mathbb{S}^1$ -action preserves  $\mathcal{H}$ . By proposition 5, this connection  $\mathcal{H}$  is complete.

**3.4. Theorem on normal geodesics of bundle type subriemannian metrics.** Before proceeding with one of the principal results on subriemannian geometry and metrics on bundles, we need to recall some definitions and important results of symplectic geometry. Therefore, this subsection is dedicated to the study of these prerequisites and to the statement and proof of the theorem on normal geodesics of bundle type subriemannian metrics. To start with, we define the concept of symplectic structure, and continue to explain the ideas of cotangent bundle and Hamilton's equations.

**Definition 28** ([9], section 8A). Let  $M$  be an even-dimensional manifold. A *symplectic structure on  $M$*  is a closed non-degenerate differential 2-form  $\omega$  on  $M$ , that is,

- (1)  $d\omega = 0$ ,
- (2)  $\forall \xi \neq 0 : \omega_m(\xi, \cdot) \neq 0$  ( $\xi \in T_m M$ ),  $\forall m \in M$ .

The non-degeneracy condition is equivalent to  $\det \omega_m \neq 0, \forall m \in M$ . The pair  $(M, \omega)$  is called a *symplectic manifold*.

*Example 16* (Canonical symplectic structure on  $\mathbb{R}^{2n}$ ). Consider  $\mathbb{R}^{2n}$  with canonical coordinates  $(q_1, q_2, \dots, q_n, p_1, p_2, \dots, p_n)$ . The 2-form  $\omega = dp_i \wedge dq_i$  is closed and non-degenerate, so that the pair  $(\mathbb{R}^{2n}, dp_i \wedge dq_i)$  is a symplectic manifold.

**Definition 29** ([4], definition A.4.1). Given a symplectic manifold  $(M, \omega)$ , an *observable* is a smooth function  $H : M \rightarrow \mathbb{R}$ . Depending of the context, a given observable can induce some structure on the symplectic manifold, and in these cases the observable will be referred as *Hamiltonian*.

Historically, the definition of a function over a symplectic manifold receives the name of observable because the physical space has been modeled through history as a manifold, and the dual tangent space of this manifold represents the position and momentum space, which is a symplectic manifold (as will be discussed below). In principle, all observables in a physical phenomena can be calculated with the variables of the dual tangent space of the manifold that represents physical space, and they are functions of the position and momentum. From this fact, a real function over the symplectic space receives the

name of observable. Over all the physical observables, one of the most important is the Hamiltonian, as it describes the evolution of the system through Hamilton's equations as a function of time. Thus, a real function over the symplectic manifold that is important in the sense of giving additional structure to the manifold is called Hamiltonian.

**Definition 30** ([9], page 203). For each vector  $\xi \in T_m M$  tangent to a symplectic manifold  $(M, \omega)$ , we associate a 1-form  $\omega_\xi \in T_m^* M$  by the formula

$$\omega_\xi(\eta) = \omega(\xi, \eta).$$

This last definition induces an isomorphism between  $T_m M$  and  $T_m^* M$ , given by the map  $\xi \mapsto \omega_\xi$ . If  $H : M \rightarrow \mathbb{R}$  is a Hamiltonian, then  $dH = \partial_i H dq^i$  is a differential 1-form on  $M$ , and by the previous isomorphism, at every point  $m \in M$  there is a tangent vector  $(X_H)_m \in T_m M$  such that  $(dH)_m = \omega_{(X_H)_m}$ .

**Definition 31.** For a given Hamiltonian  $H : M \rightarrow \mathbb{R}$ , the vector field  $X_H \in \mathfrak{X}(M)$  that satisfies

$$dH = \omega(X_H, \cdot)$$

is called *Hamiltonian vector field* for  $H$ , or *symplectic gradient* of  $H$ .

From any manifold  $M$  we can construct a symplectic structure: the cotangent bundle. Suppose that  $M$  is an  $n$ -dimensional manifold.  $T^*M = \bigsqcup_{p \in M} T_p^* M$  is then a  $2n$ -dimensional manifold. There is a canonical choice of coordinates for this space, namely, given  $m \in M$ , there is a coordinate chart  $(U_m, \phi)$  on  $M$  such that  $q = \phi(m) = (q_1, \dots, q_n) \in \mathbb{R}^n$ , and for a given 1-form  $\alpha \in T_m^* M$ , in these coordinates we have that  $\alpha = p_i dq^i$ ; the cross product of these two coordinates in a suitable neighborhood of an element  $\alpha_m \in T^* M$  gives the *canonical coordinates* of  $T^* M$ , where the element  $\alpha_m \in T^* M$  can be represented locally as the pair of coordinates  $(q, p)$ .

**Theorem 5** ([9], page 202). *The cotangent bundle  $T^* M$  has a natural symplectic structure given in coordinates by  $\omega = dq^i \wedge dp_i$ .*

*Proof.* Let  $v \in T_\alpha(T^* M)$  be a vector tangent to the cotangent bundle at  $\alpha \in T_m^* M$ . If  $\pi : T^* M \rightarrow M$  is the natural projection, the differential  $d\pi : T(T^* M) \rightarrow TM$  takes  $v$  to  $d\pi(v) = d\pi_\alpha(v)$ , which is a vector tangent to  $M$  at  $m$ . We define a 1-form  $\theta$  on  $T^* M$  by the relation  $\theta_\alpha(v) = \alpha(d\pi_\alpha(v))$ . In local coordinates this form is  $\theta = p_i dq^i$ , and the form  $\omega = d\theta$  is non-degenerate.  $\square$

Given the symplectic structure  $(T^* M, \omega = dq^i \wedge dp_i)$ , for a Hamiltonian  $H : T^* M \rightarrow \mathbb{R}$ , there is a vector field  $X_H \in \mathfrak{X}(T^* M)$  such that  $dH = \omega(X_H, \cdot)$ . Writing  $X_H$  in coordinates,  $X_H = X_i \partial / \partial q^i + X^i \partial / \partial p_i$ , and evaluating  $\partial / \partial q^j$  and  $\partial / \partial p_j$  in  $dH$  gives

$$dH(\partial / \partial q^j) = \frac{\partial H}{\partial q^j} = \omega(X_H, \partial / \partial q^j) = -\omega(\partial / \partial q^j, X_H) = -X^j,$$



$$\text{and } dH(\partial/\partial p_j) = \frac{\partial H}{\partial p_j} = \omega(X_H, \partial/\partial p_j) = X_j,$$

so that  $X_H = (\partial H/\partial p_j)\partial/\partial q^j - (\partial H/\partial q^j)\partial/\partial p_j$ , and its integral curves are defined by the following system of differential equations:

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}. \quad (1)$$

This system of differential equations is called the *Hamilton equations for  $H$* . Despite this formulation on the cotangent bundle came after Hamilton, this method turns out to be a generalization of Hamiltonian mechanics to symplectic manifolds, as one obtains the classical Hamiltonian equations of the classical mechanics using  $M = \mathbb{R}^n$ , and taking the Hamiltonian as the energy of the system. That is the reason why they are referred as Hamilton equations.

**Theorem 6.** *The Hamiltonian  $H : T^*M \rightarrow \mathbb{R}$  is constant along the integral curves of the vector field  $X_H$ .*

*Proof.* Suppose that  $c : I \subseteq \mathbb{R} \rightarrow T^*M$  is an integral curve of  $X_H$ . Then, the composition  $(H \circ c)(t)$  can be differentiated: let  $t \in I$ , and  $(U_{c(t)}, \varphi)$  be a coordinate chart, then,

$$\begin{aligned} \frac{d}{dt}(H \circ c)(t) &= \frac{d}{dt}((H \circ \varphi^{-1}) \circ (\varphi \circ c))(t) = \frac{d(\varphi \circ c)^i}{dt} \cdot \frac{\partial(H \circ \varphi^{-1})}{\partial q^i} = X_H^i \partial_i H \\ &= dH(X_H) = \omega(X_H, X_H) = 0, \end{aligned}$$

and  $H$  is constant throughout the curve  $c$ , given the skew-symmetry of  $\omega$ .  $\square$

**Definition 32** ([4], example 2, section A.4). For any vector field  $\xi \in \mathfrak{X}(M)$ , we define its *momentum function*  $P_\xi : T^*M \rightarrow \mathbb{R}$  by

$$P_\xi(p_q) = p(\xi_q).$$

*Example 17* ([4], example 1, section A.4). If  $\xi = \partial/\partial q^i \in \mathfrak{X}(\mathbb{R}^n)$ , its momentum function is

$$P_{\partial/\partial q^i}(p_q) = p(\partial/\partial q^i) = p_i.$$

On the other hand, if we take the Hamiltonian  $H(p, q) = p_i$ , we get that  $X_H = \partial/\partial q^i$ .

In local coordinates,  $P_\xi = p_i \xi_q^i$ , hence, computing its Hamilton equations gives

$$\dot{q}^i = \xi_q^i, \quad \dot{p}_i = -p_j \frac{\partial \xi_q^j}{\partial q^i}.$$

The first Hamilton equation says that the flow of the Hamiltonian vector field  $X_{P_\xi}$  on  $T^*M$  projects onto the flow of the original vector field  $\xi$  on  $M$  ([4], page 278). To understand the second Hamilton equation, recall that the *cotangent lift*  $\Phi_{T^*} : T^*M \rightarrow T^*M$  of any diffeomorphism  $\Phi : M \rightarrow M$  is given by

$$\Phi_{T^*}(p) = (\Phi(q), (\Phi^{-1})_{\Phi(q)}^*(p)), \quad p \in T_q^*M,$$

where  $(\Phi^{-1})^*$  is

$$(\Phi^{-1})_{\Phi(q)}^*(p)(\eta) = p\left(d\Phi_{\Phi(q)}^{-1}(\eta)\right)$$

for every  $\eta \in T_{\Phi(q)}M$ . If  $\Phi_t : M \rightarrow M$  is the flow of the vector field  $\xi$ , then the flow of  $X_{P_\xi}$  is  $(\Phi_t)_{T^*}$ , as

$$(\Phi_t)_{T^*}(p) = (\Phi_t(q), p(d(\Phi_{-t})_{\Phi(q)}(\cdot))) ,$$

and the first component represents the flow of  $\xi$ , while the second component is in coordinates  $p_i(\partial(\Phi_{-t})^i/\partial q^j)dq^j$ , so differentiating this expression with respect to  $t$  at  $t = 0$  gives  $-p_i(\partial\xi_q^i/\partial q^j)dq^j$ , which is the 1-form whose coordinates are equal to the coordinates on  $p$  of  $X_{P_\xi}$ . Therefore, the second Hamilton equation for  $P_\xi$  asserts that the Hamiltonian flow is the cotangent lift of this flow on the configuration space  $M$  ([4], page 278). Summarizing, we have the formula

$$X_{P_\xi} = \left. \frac{d(\Phi_t)_{T^*}}{dt} \right|_{t=0}.$$

**Definition 33.** Let  $f, g : T^*M \rightarrow \mathbb{R}$  be two observables. We define the *Poisson bracket* of  $f$  and  $g$  as

$$\{f, g\} = X_g(f),$$

where  $X_g$  is the Hamiltonian vector field associated with  $g$ .

In coordinates, the Poisson bracket is

$$\{f, g\} = \frac{\partial g}{\partial p_i} \frac{\partial f}{\partial q^i} - \frac{\partial g}{\partial q^i} \frac{\partial f}{\partial p_i}.$$

Another way to write the Poisson bracket of  $f$  and  $g$  using the symplectic form  $\omega$  of  $T^*M$  is the following:

$$\{f, g\} = df(X_g) = \omega(X_f, X_g).$$

**Proposition 6.** *The Poisson bracket satisfies the following properties:*

- (1)  $\{f, g\} = -\{g, f\}$ ,
- (2)  $\{af + bg, h\} = a\{f, h\} + b\{g, h\}$ ,
- (3)  $\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0$ ,
- (4)  $\{f, gh\} = g\{f, h\} + \{f, g\}h$ ,

where  $f, g, h : T^*M \rightarrow \mathbb{R}$  are observables, and  $a, b \in \mathbb{R}$ .

*Proof.* ([10], proof of proposition 1.3) Properties 1, 2 and 4 are trivial. The third property, known as the *Jacobi identity*, is equivalent to the fact that  $\omega$  is a closed form, as will be shown next. Recall that

$$d\chi(X, Y, Z) = X\chi(Y, Z) - Y\chi(X, Z) + Z\chi(X, Y) - \chi([X, Y], Z) + \chi([X, Z], Y) - \chi([Y, Z], X)$$

for any 2-form  $\chi$  and  $X, Y, Z \in \mathfrak{X}(T^*M)$ . Therefore,

$$\begin{aligned}
d\omega(X_f, X_g, X_h) &= X_f\{g, h\} - X_g\{f, h\} + X_h\{f, g\} + [X_f, X_g]h - [X_f, X_h]g + [X_g, X_h]f \\
&= \{f, \{g, h\}\} - \{g, \{f, h\}\} + \{h, \{f, g\}\} + \{f, \{g, h\}\} - \{g, \{f, h\}\} \\
&\quad - \{f, \{h, g\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} - \{h, \{g, f\}\} \\
&= 3(\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\}).
\end{aligned}$$

From the last expression it is clear that  $d\omega = 0$  is equivalent to the Jacobi identity.  $\square$

The first three properties of last proposition guarantee that  $(C^\infty(T^*M), \{\cdot, \cdot\})$  is a Lie algebra. The last property is known as Leibniz identity, because due to it the operator  $f \mapsto \{f, H\}$ , with  $H$  a Hamiltonian, is a derivation of  $C^\infty(T^*M)$ :

$$\{fg, H\} = f\{g, H\} + g\{f, H\}.$$

Therefore,  $f \mapsto \{f, H\}$  determines a vector field over  $T^*M$ : the Hamiltonian vector field of  $H$ ,  $X_H$  ([4], page 279).

**Proposition 7** (Hamilton equations in bracket form). *If  $c(t)$  is an integral curve for  $X_H$  and  $f \in C^\infty(T^*M)$  is an observable, then*

$$\frac{d}{dt}(f \circ c)(t) = \{f, H\}(c(t)). \quad (2)$$

*Proof.* Following the steps of the proof of theorem 6, we obtain that

$$\frac{d}{dt}(f \circ c)(t) = \omega(X_f, X_H) = \{f, H\}(c(t)).$$

$\square$

We write  $\dot{f} = \{f, H\}$  to refer to the expression (2), valid for any observable  $f$ . Equation (2) also receives the name of Hamilton equations, and when  $f$  runs over the canonical coordinates  $q^i, p_i$ , we get the Hamilton equations previously stated in (1).

**Proposition 8** (Poisson bracket of momentum functions). *For any two vector fields  $\xi, \eta \in \mathfrak{X}(M)$  with momentum functions  $P_\xi, P_\eta : T^*M \rightarrow \mathbb{R}$ , we have that*

$$\{P_\xi, P_\eta\} = -P_{[\xi, \eta]},$$

where  $[\xi, \eta]$  is the Lie bracket between  $\xi$  and  $\eta$ , and  $P_{[\xi, \eta]}$  is the momentum function of  $[\xi, \eta] \in \mathfrak{X}(M)$ .

*Proof.* Follows directly from calculus in coordinates.  $\square$

If  $(M, g)$  is a Riemannian manifold, we define the *Riemannian Hamiltonian*  $H_R : T^*M \rightarrow \mathbb{R}$  as

$$H_R(q, p) = \frac{1}{2}g_q(p, p).$$

In local coordinates,  $H_R(q, p) = g^{ij}(q)p_i p_j / 2$ . This Hamiltonian has the following Hamilton equations:

$$\dot{q}^i = \frac{\partial H_R}{\partial p_i} = g^{ij} p_j, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i} = -\frac{1}{2} \frac{\partial g^{jk}(q)}{\partial q^i} p_j p_k.$$

From the first Hamilton equation we deduce that  $p_i = g_{ij} \dot{q}^j$ , and substituting this expression in the second Hamilton equation gives

$$\dot{p}_i = \frac{1}{2} \frac{\partial g_{ml}}{\partial q^i} \dot{q}^m \dot{q}^l,$$

so that

$$\ddot{q}^i = \frac{\partial g^{ij}(q)}{\partial q^k} \dot{q}^k p_j + g^{ij} \dot{p}_j = \frac{1}{2} g^{ij} \left( \frac{\partial g_{kl}}{\partial q^j} - 2 \frac{\partial g_{jl}}{\partial q^k} \right) \dot{q}^k \dot{q}^l = -\Gamma_{kl}^i \dot{q}^k \dot{q}^l,$$

and  $q$  satisfies the geodesic equation, from where we deduce that the integral curves of  $X_{H_R}$  are geodesics. Moreover, any geodesic  $q(t)$  satisfies Hamilton equations, given that  $p_i = g_{ij} \dot{q}^j$ . From this discussion we derive the following proposition.

**Proposition 9.** *If  $(M, g)$  is a Riemannian manifold, and  $H_R : T^*M \rightarrow \mathbb{R}$  is its Riemannian Hamiltonian,  $q(t)$  is a geodesic if and only if  $(q^i(t), g_{ij}(q(t))\dot{q}^j(t))$  is an integral curve of  $X_{H_R}$ .*

Recall that given a Riemannian manifold  $(M, g)$ , we can construct a metric  $g_q^*$  over every cotangent fiber  $T_q^*M$ , using the identification

$$g_q^*(v^*, w^*) := g_q(v, w),$$

where  $v^*, w^* \in T_q^*M$ , and  $v, w \in T_qM$  are the preimages of  $v^*, w^*$  via the isomorphism  $T_q^*M \cong T_qM$  induced by the metric  $g$ , that is,  $v^* = g_q(v, \cdot)$  and  $w^* = g_q(w, \cdot)$ . In coordinates, if  $g_q = g_{ij}(q)$ , then  $g_q^* = g^{ij}(q)$ , that is, the representation in coordinates of the metric in the cotangent fiber is equivalent to the inverse matrix of the representation in coordinates of the metric in the tangent fiber.

Now, if  $M$  is a manifold with a subriemannian structure  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$  over it, we can define a *cometric*, that is just a metric over the fibers of the cotangent bundle as follows. Given that  $\langle \cdot, \cdot \rangle_q$  is a metric for  $\mathcal{H}_q$ , for any  $v^* \in T_q^*M$  there exists a single  $V \in \mathcal{H}_q$  such that  $v^*|_{\mathcal{H}_q} = \langle V, \cdot \rangle_q$ . With this property, we can define the cometric  $(\cdot, \cdot) \in \mathfrak{X}_2^0(M)$  as a bilinear form (in general it is not a metric) over the fibers of the cotangent space as follows:

$$(v^*, w^*)_q = \langle V, W \rangle_q,$$

where  $v^*, w^* \in T_q^*M$ , and  $V, W \in \mathcal{H}_q$  are such that  $v^*|_{\mathcal{H}_q} = \langle V, \cdot \rangle_q$ ,  $w^*|_{\mathcal{H}_q} = \langle W, \cdot \rangle_q$ .

**Definition 34** ([4], definition 1.5.2). If  $M$  is a manifold, and  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$  is a subriemannian structure over it, we define the subriemannian Hamiltonian  $H_{sR} : T^*M \rightarrow \mathbb{R}$  as

$$H_{sR}(p, q) = \frac{1}{2} (p, p)_q,$$

where  $(\cdot, \cdot)$  is the cometric induced by  $\langle \cdot, \cdot \rangle$ .

If  $\pi : Q \rightarrow M$  is a submersion endowed with a connection, and  $(M, g)$  is a Riemannian manifold, we know that the submersion induces a subriemannian structure on  $Q$ ,  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ , where  $\mathcal{H}$  is the Ehresmann connection associated with  $\pi$ , and  $\langle \cdot, \cdot \rangle$  is defined by the isomorphism  $d\pi_q|_{\mathcal{H}_q} : \mathcal{H}_q \rightarrow T_{\pi(q)}M$  and the metric  $g$  on  $M$ .

The subriemannian Hamiltonian  $H_{sR} : T^*Q \rightarrow \mathbb{R}$  can be written in terms of the Riemannian Hamiltonian  $H_R : T^*M \rightarrow \mathbb{R}$  in the following way: given the splitting  $TQ = \mathcal{H} \oplus V$ , we have that  $T^*Q = \mathcal{H}^* \oplus V^*$ , and so we have the canonical projection  $\text{pr}_1 : T^*Q \rightarrow \mathcal{H}^*$ . Now, we define a second projection  $\text{pr}_2 : \mathcal{H}^* \rightarrow T^*M$  by the formula  $\text{pr}_2(\alpha)(v) = \alpha(h_q v)$ , where  $\alpha \in \mathcal{H}_q^*$ ,  $v \in T_{\pi(q)}M$ , and  $h_q v$  is the horizontal lift of  $v$  into  $\mathcal{H}_q$  (well-defined and unique, because  $\mathcal{H}$  is an Ehresmann connection).

**Proposition 10** ([4], proposition 11.1.5). *The induced subriemannian Hamiltonian is given by  $H_{sR} = H_R \circ \text{pr}_2 \circ \text{pr}_1$ .*

*Proof.* We can horizontally lift any  $Y \in \mathfrak{X}(M)$  in a unique way to a vector field  $hY \in \mathfrak{X}(Q)$  that satisfies the following properties:  $(hY)_q \in \mathcal{H}_q$  and  $d\pi_q((hY)_q) = Y_{\pi(q)}$  for all  $q \in Q$ . If  $Y_a$  is an orthonormal frame for the Riemannian structure on  $M$ , then  $hY_a$  is a horizontal orthonormal frame for the induced subriemannian structure on  $Q$ , and therefore

$$H_R = \frac{1}{2} \sum_a P_{Y_a}^2 \text{ and } H_{sR} = \frac{1}{2} \sum_a P_{(hY)_a}^2,$$

where  $P_{Y_a}$  and  $P_{(hY)_a}$  are the momentum functions associated to  $Y_a$  and  $(hY)_a$ , respectively. Now, for any  $Y \in \mathfrak{X}(M)$ ,  $\text{pr}^* P_Y = P_{hY}$  because if  $\alpha : T^*Q \rightarrow \mathbb{R}$ , then

$$\begin{aligned} (\text{pr}^* P_Y)(\alpha) &= P_Y(\text{pr}(\alpha)) = \text{pr}(\alpha)(Y_{\pi(q)}) = \text{pr}(\alpha)(d\pi_q((hY)_q)) \\ &= \text{pr}_1(\alpha)((hY)_q) = \alpha((hY)_q) = P_{hY}(\alpha). \end{aligned}$$

With this,

$$H_R \circ \text{pr} = \frac{1}{2} \sum_a (P_{Y_a} \circ \text{pr})^2 = \frac{1}{2} \sum_a (\text{pr}^* P_{Y_a})^2 = \frac{1}{2} \sum_a P_{(hY)_a}^2 = H_{sR}.$$

□

**Definition 35.** Let  $(M, g)$  be a Riemannian manifold, and  $G$  a Lie group that acts on  $M$  via the application  $\alpha : M \times G \rightarrow M$ ,  $\alpha(m, h) = m \cdot h$ . We say that *the metric  $g$  is  $G$ -invariant* if

$$g_{m \cdot h}((d\alpha_h)_m(v), (d\alpha_h)_m(w)) = g_m(v, w),$$

for all  $m \in M$ ,  $v, w \in T_m M$  and  $h \in G$ .

Let  $\pi : Q \rightarrow M$  be a principal  $G$ -bundle with connection  $\mathcal{H}$ . If  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$  is a metric of bundle type, we say that a metric  $h$  on  $Q$  is a *compatible Riemannian metric* if  $h$  is  $G$ -invariant and compatible (see definition 21). From this definition, it is clear that  $h|_{G_q} : TG_q \oplus TG_q \rightarrow \mathbb{R}$  is a right-invariant metric, and we can use it to define a metric on  $G$  as follows:

Let  $\mathfrak{g} = \text{Lie}(G) = T_e G$ , and  $\exp : \mathfrak{g} \rightarrow G$  be the exponential application. *The infinitesimal generator map for the group action  $\sigma_v : Q \rightarrow T_q Q$ , for  $v \in \mathfrak{g}$ , is defined as*

$$\sigma_v(q) = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} (q \cdot \exp(\epsilon v)).$$

**Definition 36** ([4], definition 11.2.3). For a given  $q \in Q$ , the bilinear form on  $\mathfrak{g}$ ,  $\mathbb{I}_q : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ , given by

$$\mathbb{I}_q(v, w) = h_q(\sigma_v q, \sigma_w q)$$

is called the *moment of inertia tensor at  $q$* .

Recall that for a Lie Group  $G$ , the map  $\text{ad}_g : G \rightarrow G$  given by  $h \mapsto ghg^{-1}$  is a diffeomorphism, and its differential at identity,  $d(\text{ad}_g)_e$ , is denoted as  $\text{Ad}_g : \mathfrak{g} \rightarrow \mathfrak{g}$ .

**Proposition 11.** *The family of vector fields  $\{\sigma_v\}_{v \in \mathfrak{g}}$  satisfies the following property:*

$$\sigma_v(q \cdot g) = d(\alpha_g)_q (\sigma_{\text{Ad}_g(v)}(q)),$$

where  $\alpha : Q \times G \rightarrow Q$  is the action of  $G$  on  $Q$ .

*Proof.* Let  $\gamma : (-\epsilon, \epsilon) \rightarrow G$  be the one parameter subgroup such that  $\gamma(0) = e$ ,  $\dot{\gamma}(0) = \xi \in \mathfrak{g}$ . Then,  $\text{Ad}_g(\xi) = d/dt|_{t=0}(g\gamma(t)g^{-1})$ , and evaluating this on  $\sigma_q$  gives

$$\sigma_\xi(q) = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} q \cdot \exp \left( \epsilon \left. \frac{d}{dt} \right|_{t=0} g\gamma(t)g^{-1} \right) = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} q \cdot g\gamma(\epsilon)g^{-1}.$$

Finally, if we evaluate this on  $d(\alpha_g)_q$ , we get that

$$\begin{aligned} d(\alpha_g)_q \left( \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} q \cdot g\gamma(\epsilon)g^{-1} \right) &= \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} q \cdot g\gamma(\epsilon) \\ &= \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} q \cdot g \exp(\epsilon \xi) \\ &= \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} (q \cdot g) \cdot \exp(\epsilon \xi) \\ &= \sigma_\xi(q \cdot g). \end{aligned}$$

□

Thanks to the last proposition and the fact that  $h$  is  $G$ -invariant, we have that

$$\begin{aligned} \mathbb{I}_{q \cdot g}(v, w) &= h_{q \cdot g} (d(\alpha_g)_q (\sigma_{\text{Ad}_g(v)}(q)), d(\alpha_g)_q (\sigma_{\text{Ad}_g(w)}(q))) \\ &= h_q (\sigma_{\text{Ad}_g(v)}(q), \sigma_{\text{Ad}_g(w)}(q)) \\ &= \mathbb{I}_q(\text{Ad}_g(v), \text{Ad}_g(w)). \end{aligned}$$

**Definition 37.** Let  $G$  be a Lie group. A bilinear form  $\beta \in \Lambda^2(\mathfrak{g})$  is called *bi-invariant* or *adjoint invariant* if

$$\beta(\text{Ad}_g v, \text{Ad}_g w) = \beta(v, w) \text{ for all } g \in G, v, w \in \mathfrak{g}.$$

Any bilinear form  $\beta \in \Lambda^2(\mathfrak{g})$  can be extended to all  $TG$  using left or right translations  $L_g : G \rightarrow G, h \mapsto gh, R_g : G \rightarrow G, h \mapsto hg$  in the following way:

$$\beta_g(v, w) = \beta(d(L_{g^{-1}})_g(v), d(L_{g^{-1}})_g(w)), \text{ or}$$

$$\beta_g(v, w) = \beta(d(R_{g^{-1}})_g(v), d(R_{g^{-1}})_g(w)),$$

for all  $g \in G, v, w \in T_g G$ . If  $\beta$  is bi-invariant, both extensions coincide, because

$$\begin{aligned} \beta(d(R_{g^{-1}})_g(v), d(R_{g^{-1}})_g(w)) &= \beta(d(R_{g^{-1}})_g \circ d(L_g)_e \circ d(L_{g^{-1}})_g(v), d(R_{g^{-1}})_g \circ d(L_g)_e \\ &\quad \circ d(L_{g^{-1}})_g(w)) \\ &= \beta(\text{Ad}_g \circ d(L_g^{-1})_g(v), \text{Ad}_g \circ d(L_g^{-1})_g(w)) \\ &= \beta(d(L_{g^{-1}})_g(v), d(L_{g^{-1}})_g(w)). \end{aligned}$$

Therefore, an adjoint invariant positive definite bilinear form on the Lie algebra defines a *bi-invariant Riemannian metric on  $G$* .

**Definition 38** ([4], definition 11.2.4). The  $G$ -invariant Riemannian metric (or pseudo-metric)  $h$  on  $Q$  is said to be of *constant bi-invariant type* if its inertia tensor  $\mathbb{I}_q$  equals the same bi-invariant bilinear form on  $\mathfrak{g}$ , for all  $q \in Q$ .

Note that if  $G$  is a Lie group acting on itself via the multiplication map  $\alpha : G \times G \rightarrow G, (k, h) \mapsto kh, \alpha_h = R_h$ , and if  $g$  is a  $G$ -invariant metric on  $G$  we have that

$$g_{kh}(d(R_h)_k(v), d(R_h)_k(w)) = g_k(v, w)$$

for all  $k, h \in G, v, w \in T_k G$ . Therefore, a metric on  $G$  is  $G$ -invariant under the canonical action of  $G$  on itself if and only if this metric is right invariant. Moreover,  $g$  is of constant bi-invariant type if and only if it is bi-invariant, because for any  $v, w \in T_k G$ , there exist  $\xi, \eta \in \mathfrak{g}$  such that  $d(L_k)_e \xi = v, d(L_k)_e \eta = w$ , and

$$\mathbb{I}_k(\xi, \eta) = g_k(v, w) = \mathbb{I}_{hk}(\xi, \eta) = g_{hk}(d(L_h)_k v, d(L_h)_k w)$$

as  $\sigma_\chi(k) = d(L_k)_e \chi$  for all  $k \in G, \chi \in \mathfrak{g}$ .

**Proposition 12.** *If  $g$  is a bi-invariant metric on a Lie group  $G$ , its geodesics through the identity coincide with its one-parameter subgroups.*

*Proof.* If  $m : G \times G \rightarrow G$  is the multiplication map, let  $h, k \in G$ , and  $\gamma, \eta : (-\epsilon, \epsilon) \rightarrow G$  be two curves such that  $\gamma(0) = h, \dot{\gamma}(0) = v \in T_h G, \eta(0) = k, \dot{\eta}(0) = w \in T_k G$ . The differential  $dm_{(h,k)} : T_{(h,k)} G \times G \cong T_h G \oplus T_k G \rightarrow T_{hk} G$  is bilinear, so that if  $l, s : (-\epsilon, \epsilon) \rightarrow H$  are constant maps with  $l(t) = h, s(t) = k$ , then

$$dm_{(h,k)}(v, 0) = \left. \frac{d}{dt} \right|_{t=0} (\gamma(t)k) = d(R_k)_h(v),$$

and

$$dm_{(h,k)}(0, w) = \left. \frac{d}{dt} \right|_{t=0} (h\eta(t)) = d(L_h)_k(w).$$

Therefore,  $dm_{(h,k)}(v, w) = d(R_k)_h(v) + d(L_h)_k(w)$  by linearity. Using this result, if  $\gamma : (-\epsilon, \epsilon)$  is a curve such that  $\gamma(0) = 0, \dot{\gamma}(0) = v \in T_h G$ , differentiating the expression  $\gamma(t) [\gamma(t)]^{-1} = e$  gives

$$0 = d(R_{h^{-1}})_h(v) + d(L_h)_{h^{-1}}(d\iota_h(v)),$$

where  $\iota : G \rightarrow G$  is the inversion map  $h \mapsto h^{-1}$ . From last expression we get that  $d\iota_h(v) = -d(L_{h^{-1}})_e \circ d(R_{h^{-1}})_h(v)$ . As  $g$  is a bi-invariant metric,  $g_{h^{-1}}(d\iota_h(v), d\iota_h(w)) = g_h(v, w)$ , and as  $\iota : G \rightarrow G$  is a diffeomorphism, it is an isometry.

Now, if  $\gamma : (-\epsilon, \epsilon) \rightarrow G$  is a geodesic with  $\gamma(0) = e, \dot{\gamma}(0) = v \in T_e G$ , as  $\iota$  is an isometry,  $\iota \circ \gamma(t) = [\gamma(t)]^{-1}$  is also a geodesic. Moreover,  $d\iota_e(v) = -v$ , so that  $\gamma(t)$  and  $[\gamma(t)]^{-1}$  are the same geodesic, with  $[\gamma(t)]^{-1} = \gamma(-t)$ . For small  $t_0 \in \mathbb{R}$  (sufficiently small in order to have that  $\text{Im } \gamma|_{[0, t_0]} \subset U$ , where  $U$  is a totally normal neighborhood of  $e$ ), consider the curve  $\eta : (-t_0, 0) \rightarrow G$ ,  $t \mapsto \gamma(t_0)\gamma(t) = L_{\gamma(t_0)}(\gamma(t))$ . This curve is a geodesic because  $g$  is bi-invariant, with  $\gamma(0) = \gamma(t_0), \eta(-t_0) = e$ . Due to the uniqueness of minimizing geodesics in a totally normal neighborhood,  $\eta(t) = \gamma(t_0 + t)$ , that is,  $\gamma(t_0)\gamma(t) = \gamma(t_0 + t)$ , for  $t \in (-t_0, 0)$ . Using the relation  $\gamma(s+t) = \gamma(s)\gamma(t)$  and the local uniqueness of minimizing geodesics we can extend this property over all the domain of the geodesic  $\gamma$ . With this,  $\gamma$  is a one-parameter subgroup, and

$$\exp(v) = \gamma(1) = (\exp_R)_e(v),$$

where  $(\exp_R)_e : T_e G \rightarrow G$  is the Riemannian exponential.  $\square$

**Proposition 13.** *If  $G$  is a Lie group endowed with a bi-invariant metric  $g$ , the Riemannian Hamiltonian  $H_R : T^*G \rightarrow \mathbb{R}, (q, p) \mapsto g_q(p, p)/2$  commutes with every right or left invariant observable  $f \in C^\infty(T^*G)$ .*

*Proof.* From proposition 7, if  $f : T^*G \rightarrow \mathbb{R}$  is an observable, then  $d/dt(f \circ c)(t) = \{f, H_R\}(c(t))$ , where  $c(t)$  is an integral curve of the Hamiltonian vector field associated to  $H_R$ . By proposition 9,  $c(t) = (q(t), g_{q(t)}(\cdot, \dot{q}(t)))$ , where  $q(t)$  is a geodesic. As  $G$  is complete (because  $g$  is bi-invariant), there is a geodesic that connects  $e$  with every  $h \in G$ , and by last proposition, this geodesic is a one-parameter subgroup, which implies that  $d/dt(f \circ c)(t) = 0$ , because the quantity  $f \circ c(t)$  is constant since  $f$  is left or right invariant.  $\square$

**Theorem 7** (Theorem on normal geodesics of bundle type subriemannian metrics, [4], theorem 11.2.5). *Let  $\pi : Q \rightarrow M$  be a principal  $G$ -bundle with connection  $\mathcal{H}$ . Suppose that  $Q$  is a Riemannian manifold such that its metric is a Riemannian submersion (see definition 22), compatible with  $\mathcal{H}$ ,  $G$ -invariant, and its associated vertical metric has constant bi-invariant type. Therefore, given any geodesic  $\gamma : I \rightarrow Q$ , the horizontal lift of  $\pi \circ \gamma$  is a subriemannian geodesic. All subriemannian geodesics can be obtained in this manner. Moreover, the projected curve  $\pi \circ \gamma$  is a geodesic in the base space  $M$  if and only if  $\gamma$  is a horizontal geodesic in  $(Q, \mathcal{H})$ .*



*Remark 7.* The assumptions of this theorem are equivalent to suppose that  $\pi : Q \rightarrow M$  is a principal  $G$ -bundle endowed with a subriemannian metric of bundle type, and that this metric can be extended to a compatible Riemannian metric whose associated vertical metric has constant bi-invariant type.

*Proof.* Let  $H_R$  be the Riemannian Hamiltonian of  $Q$ ,  $H_{sR}$  its subriemannian Hamiltonian, and let  $H_G$  be the vertical part of the Riemannian Hamiltonian, that is,  $H_R = H_{sR} + H_G$ . We will need the following two claims for this proof:

- (1)  $H_{sR}$  and  $H_G$  Poisson-commute, that is,  $\{H_{sR}, H_G\} = 0$ .
- (2) The projection to  $Q$  of any integral curve in  $T^*Q$  for the Hamiltonian vector field defined by  $H_G$  is of the form  $q \exp(tv)$ , for some fixed  $v \in \mathfrak{g}$ , which depends of the initial conditions of the integral curve.

*Proof of the claims.* It suffices to show that the claims hold over any locally trivialized neighborhood  $\pi^{-1}(U)$  in  $Q$ . Let  $\phi_U : \pi^{-1}(U) \rightarrow U \times G$  be the corresponding local trivialization, which induces a diffeomorphism  $T^*Q \cong T^*U \times T^*G$ .

Let  $(x, p) \in T^*U$  and  $(g, \mu) \in T^*G$ , being  $\mu \in \mathfrak{g}^*$ . Then, we have that

$$H_G(x, p, g, \mu) = \beta^*(\mu) \quad (3)$$

and

$$H_{sR}(x, p, g, \mu) = f(x, p, \mu). \quad (4)$$

Equations (3) and (4) come from the fact that the metric is of constant bi-invariant type and  $G$ -invariant. Using these equations, the Poisson bracket of  $H_{sR}$  and  $H_G$  is

$$\{f, \beta^*\} = \underbrace{\{f, \beta^*\}_U}_0 + \{f, \beta^*\}_G = \sum_a \frac{\partial f}{\partial g^a} \frac{\partial \beta^*}{\partial \mu_a} = \sum_a \frac{\partial f}{\partial \mu_a} \frac{\partial \mu_a}{\partial g^a} \frac{\partial \beta^*}{\partial \mu_a} = \sum_a \frac{\partial f}{\partial \mu_a} \{\mu_a, \beta^*\}_G,$$

using both the Leibniz rule and the product Poisson structure,  $\mu_a$  are coordinates on  $\mathfrak{g}^*$ , and  $g^a$  are coordinates on  $G$ . As the  $\mu_a$  are right-invariant functions on  $T^*G$ , by proposition 13,  $\{\mu_a, \beta^*\}_G = 0$ , and this proves the first claim.

To prove the second claim, note that the flow of  $H_G$  is of the form  $\Phi_G(t)(x, p, g, \mu) = (x, p; \Phi_\beta(t)(g, \mu))$ , where  $\Phi_\beta(t) : T^*G \rightarrow T^*G$  is the flow associated with  $\beta^*$ , that is,  $\Phi_\beta(t)(g, \mu) = (\exp_g(tv), \mathbf{v}_{\exp_g(tv)}(\cdot, d/dt \exp_g(tv)))$ , where  $\mathbf{v}$  is the vertical metric, and  $v \in \mathfrak{g}$  satisfies  $\mu = \mathbf{v}_g(\cdot, v)$ . With this, the projection to  $Q$  of  $\Phi_G(t)(x, p, g, \mu)$  is  $x \exp_g(tv)$ .  $\square$

With the claims proved, we proceed with the proof of the theorem. The Riemannian exponential map is the composition of four maps:

$$T_q Q \xrightarrow{\iota} TQ \longrightarrow T^*Q \xrightarrow{\Phi_R(1)} T^*Q \xrightarrow{\text{proj}} Q,$$

where the first map  $\iota$  is the inclusion, the second map is the metric induced identification,  $\Phi_R(1)$  is the time 1-flow of  $H_R$ , and  $\text{proj}$  is the canonical projection. The Riemannian geodesics are obtained by the same sequence of compositions, changing  $\Phi_R(1)$  to  $\Phi_R(t)$ . Since  $H_{sR} = H_R - H_G$ , and the flows for  $H_R$ ,  $H_{sR}$  and  $H_R$  all commute according to claim (1), we have that  $\Phi_{sR}(t) = \Phi_R(t) \circ \Phi_G(-t)$ . Applying the projection  $\text{proj} : T^*Q \rightarrow Q$  to this formula, and considering the claim (2), we find that the normal subriemannian geodesics all have the form

$$\exp_R(tv) \exp_G(-tw). \quad (5)$$

These subriemannian geodesics must be horizontal, which leads to  $w = A(v) \in \mathfrak{g}$ , where  $A \in \Omega^1(Q, \mathfrak{g})$  is the connection one-form annihilating the horizontal space.

As the flow  $\Phi_G$  is purely vertical ( $\pi(qg) = \pi(q)$  for any  $g \in G$ ), (5) projects to  $\pi(\exp_R(tv))$  in the quotient  $M = Q/G$ , and its horizontal lift will be (5). This proves that the lifting of any projected geodesic for the extended metric is a normal subriemannian geodesic.

Finally, observe that if  $\gamma : I \rightarrow Q$  is a horizontal geodesic, its projection  $\pi \circ \gamma$  is a geodesic in  $M$ . Reciprocally, if  $\pi \circ \gamma$  is a geodesic, as the distance between points in  $M$  is the distance between fibers in  $Q$ ,  $\pi(\gamma(t))$  is a geodesic if and only if  $\gamma(t)$  minimizes distances between fibers, but this implies that this curve is normal to the fibers (if it were not normal, we could construct a shorter curve, leading to a contradiction), and therefore it is a subriemannian geodesic.  $\square$

#### 4. CLASSICAL PARTICLES IN YANG-MILLS FIELDS

What we aim to show in this section is that, given a principal  $G$ -bundle  $\pi : Q \rightarrow M$ , there is a family of differential equations that describe the projected normal geodesics  $\pi \circ \gamma$ . When the fiber is a circle, these projected equations are the *Lorentz equations of electromagnetism*. When the fiber is a general Lie group  $G$ , these equations are the *nonabelian Lorentz equations*, also known as *Wong's equations* ([4], p. 209). Moreover, the results of this section are useful to find the subriemannian geodesics in cases where Theorem 7 can not be applied.

The subriemannian geodesic equations are ordinary differential equations on  $T^*Q$ . For metrics of bundle type, the Hamiltonian  $H : T^*Q \rightarrow \mathbb{R}$  is  $G$ -invariant, and hence the action of  $G$  on  $T^*Q$  takes subriemannian geodesics to subriemannian geodesics. Therefore, we should be able to write the subriemannian geodesic equations on  $T^*Q/G$ , and this is going to be the objective of this dissertation.

**4.1. Construction of a vector bundle through its transition maps.** The first step in our dissertation to write the subriemannian geodesic equations in the quotient space is to build the associated bundle of a given bundle. To achieve this we will introduce how to construct a vector bundle using its transition maps. The main reference of this section is [1], chapter 1.1.

For a vector bundle  $\pi : Q \rightarrow M$  with fiber  $E$ , we know that for all  $m \in M$  there is a neighborhood  $U \subset M$  such that  $U \times E$  is diffeomorphic to  $\pi^{-1}(U)$  through the map  $\varphi : U \times E \rightarrow \pi^{-1}(U)$  that satisfies  $\pi \circ \varphi(x, v) = x$ , for all  $x \in U$ . According to this,

we can cover the base space  $M$  with a family of open sets  $\{U_\alpha\}_\alpha$ , such that  $\pi^{-1}(U_\alpha)$  is diffeomorphic to  $U_\alpha \times E$  through the diffeomorphism  $\varphi_\alpha : U_\alpha \times E \rightarrow \pi^{-1}(U_\alpha)$ . This gives a coordinate chart  $\{(U_\alpha, \varphi_\alpha : U_\alpha \times E \rightarrow \pi^{-1}(U_\alpha))\}_\alpha$  for the vector bundle. Given two coordinates  $\varphi_\alpha : U_\alpha \times E \rightarrow \pi^{-1}(U_\alpha)$ ,  $\varphi_\beta : U_\beta \times E \rightarrow \pi^{-1}(U_\beta)$ , the restricted maps  $\varphi_\alpha : (U_\alpha \cap U_\beta) \times E \rightarrow \pi^{-1}(U_\alpha \cap U_\beta)$ ,  $\varphi_\beta : (U_\alpha \cap U_\beta) \times E \rightarrow \pi^{-1}(U_\alpha \cap U_\beta)$  are fiberwise  $(\pi \circ \varphi_i(x, v) = x)$  diffeomorphisms. We denote the as  $\varphi_{\beta\alpha} := \varphi_\beta \circ \varphi_\alpha^{-1} : (U_\alpha \cap U_\beta) \times E \rightarrow (U_\alpha \cap U_\beta) \times E$  the diffeomorphism that maps  $(U_\alpha \cap U_\beta) \times E$  onto itself, and we call them *transition functions* of the coordinate chart  $\{(U_\alpha, \varphi_\alpha)\}_\alpha$  of the vector bundle.

For any three indices  $\alpha, \beta, \gamma$ , the family of transition maps satisfies the following properties:

$$\varphi_{\alpha\alpha} = 1_{U_\alpha \times E} \quad (6)$$

$$\varphi_{\alpha\gamma} \circ \varphi_{\gamma\beta} \circ \varphi_{\beta\alpha} = 1_{(U_\alpha \cap U_\beta \cap U_\gamma) \times E}. \quad (7)$$

These conditions are sufficient for a vector bundle to be reconstructed from the base  $M$ , fiber  $E$ , open cover  $\{U_\alpha\}_\alpha$  of  $M$  and transition functions  $\varphi_{\beta\alpha}$ . To see this, let  $S' = \sqcup_\alpha (U_\alpha \times E)$ , and introduce the following equivalence relation over it:

$$U_\alpha \times E \ni (x, v) \sim (y, w) \in U_\beta \times E \iff x = y \in U_\alpha \cap U_\beta \text{ and } (y, w) = \varphi_{\beta\alpha}(x, v).$$

Conditions (6) and (7) guarantee that this is an equivalence relation. Let  $S = S'/\sim$ , and define the map  $\pi : S \rightarrow M$  as  $\pi([x, v]) = x$ . Now, we define the maps  $\varphi_\alpha = \phi|_{U_\alpha \times E} : U_\alpha \times E \rightarrow S$ , where  $\phi : S' \rightarrow S$  is the canonical projection, that maps each element of  $S'$  to its equivalence class. As each class in  $\pi^{-1}(U_\alpha)$  has a unique representative on  $U_\alpha \times E$ ,  $\varphi_\alpha$  is injective, and  $\varphi_\alpha : U_\alpha \times E \rightarrow \pi^{-1}(U_\alpha)$  is a fiberwise diffeomorphism. Therefore,  $\pi : S \rightarrow M$  is a vector bundle with coordinate chart  $\{(U_\alpha, \phi|_{U_\alpha \times E})\}_\alpha$ .

#### 4.2. Associated bundles.

**Definition 39.** Let  $G$  be a Lie group. A *representation of  $G$  on a vector space  $E$*  is a Lie group homomorphism  $\rho : G \rightarrow \text{GL}(E)$ .

**Definition 40** ([4], p. 210). Let  $\pi : Q \rightarrow M$  be a principal  $G$ -bundle, and  $\rho : G \rightarrow \text{GL}(E)$  be a representation of  $G$  on a vector space  $E$ . Consider the right action on  $Q \times E$  given by  $(q, v) \cdot g = (q \cdot g, \rho(g^{-1})v)$ . The orbit space associated to this action is denoted as  $(Q \times E)/G$ ,  $Q \times_\rho E$  or  $\rho(Q)$ . A point of  $\rho(Q)$  corresponding to the orbit of  $(q, v)$  is written as  $[q, v]$  or  $qv$ .

The associated bundle of  $\pi : Q \rightarrow M$  is the vector bundle  $\phi : \rho(Q) \rightarrow M$  with fiber  $E$ , where  $\phi(qv) = \pi(q)$ . The sum between two elements  $qv, qw$  in the same fiber is  $q(v + w)$ .

*Example 18* ([4], p. 210). Let  $\xi : R \rightarrow M$  be a  $n$ -real vector bundle, and  $\pi : Q \rightarrow M$  is the principal  $\text{GL}(n, \mathbb{R})$ -bundle whose fibers are  $\pi^{-1}(m) = \{q : \mathbb{R}^n \rightarrow \xi^{-1}(m) \mid q \in \text{Isom}(\mathbb{R}^n, \xi^{-1}(m))\}$ , that is, each fiber consists of all frames for  $\xi^{-1}(m)$ , and each element  $q \in Q_m$  is a linear isomorphism between  $\mathbb{R}^n$  and  $R_m$ , with the

given basis being the image of the standard basis of  $\mathbb{R}^n$ . The principal  $\mathrm{GL}(n, \mathbb{R})$ -bundle  $\pi$  receives the name of *frame bundle*.

The action of  $\mathrm{GL}(n, \mathbb{R})$  on  $Q_m$  is given by  $q \mapsto qg := q \circ g$ , and  $\mathrm{GL}(n, \mathbb{R})$  acts on  $\mathbb{R}^n$  in the standard way. Now, let us show that the associated bundle of  $\pi : Q \rightarrow M$  is  $\xi : R \rightarrow M$ . We consider the trivial representation of  $\mathrm{GL}(n, \mathbb{R})$ , that is, the identity map  $\rho : \mathrm{GL}(n, \mathbb{R}) \rightarrow \mathrm{GL}(n, \mathbb{R})$ , so that the action on  $Q \times \mathbb{R}^n$  is given by  $(q, v) \cdot g = (q \circ g, g^1(v))$ , where  $q \in Q$ ,  $v \in \mathbb{R}^n$  and  $g \in \mathrm{GL}(n, \mathbb{R})$ . Now, the orbit space associated to this action,  $\rho(Q)$ , is diffeomorphic to  $R$  through the maps

$$\psi : \rho(Q) \rightarrow R, \quad qv \mapsto q(v), \quad \psi^{-1} : R \rightarrow \rho(Q), \quad r \mapsto [q, q^{-1}(r)], \quad \text{where } q \in \pi^{-1}(\xi(r)),$$

and the following diagram commutes because  $\xi \circ \psi(qv) = \xi(q(v)) = \pi(q) = \phi(qv)$ . Explicitly,  $\xi(q(v)) = \pi(q)$  because  $q \in \pi^{-1}(m)$  and  $q(v) \in \xi^{-1}(m)$  for a given  $m \in M$ , so that  $\xi(q(v)) = m = \pi(q)$ . Therefore, the associated bundle of the frame bundle  $\pi : Q \rightarrow M$  is  $\xi : R \rightarrow M$ .

$$\begin{array}{ccc} \rho(Q) & \xrightarrow{\psi} & R \\ \phi \downarrow & & \downarrow \xi \\ M & \xrightarrow{1_M} & M \end{array}$$

This example can be extended to vector bundles endowed with additional structure. For example, for complex vector bundles we take  $Q_m$  as the space of complex linear maps  $\mathbb{C}^n \rightarrow R_m$ ; for a vector bundle  $\xi$  with a fiber metric we take  $Q_m$  to be the space of linear isometries  $\mathbb{R}^n \rightarrow R_m$ ; and if  $\xi$  is complex with a fiber-Hermitian structure we take  $Q_m$  to consist of the unitary maps  $\mathbb{C}^n \rightarrow R_m$ .

The adjoint representation of a Lie group  $G$  on its Lie algebra  $\mathfrak{g}$  is given by the adjoint map  $\mathrm{Ad} : G \rightarrow \mathrm{Aut}(G)$ , where  $\mathrm{Ad}(g) = \mathrm{Ad}_g = d(\mathrm{ad}_g)_e : \mathfrak{g} \rightarrow \mathfrak{g}$ . Its dual representation is  $\mathrm{Ad}^* : G \rightarrow \mathrm{Aut}(\mathfrak{g}^*)$ ,  $\mathrm{Ad}^*(g) = \mathrm{Ad}(g^{-1})^T$ , defined on the dual Lie algebra  $\mathfrak{g}^*$ . From these representations, given a principal  $G$ -bundle  $\pi : Q \rightarrow M$ , we can construct the associated bundles  $\mathrm{Ad}(Q)$  and  $\mathrm{Ad}^*(Q)$ , called *adjoint and co-adjoint bundles over  $M$* , respectively. These two bundles are *dual vector bundles*, which means that fibers of  $\mathrm{Ad}^*(Q)$  are the dual spaces to the fibers of  $\mathrm{Ad}(Q)$ .

*Example 19* ([4], p. 211). If  $\pi : Q \rightarrow M$  is the frame bundle of some vector bundle  $\xi : R \rightarrow M$ , then  $\mathrm{Ad}(Q)$  is identified with the bundle whose fiber over  $m \in M$  is the space of linear maps on  $\xi_m$ , that is,  $\mathrm{Ad}(Q)_m \cong \mathrm{Hom}(\xi_m, \xi_m)$ , because  $\mathrm{Ad}(Q)_m \cong \mathfrak{gl}(n, \mathbb{R})$ .

**Definition 41.** Let  $Q$  be a manifold,  $G$  a Lie group that acts on  $Q$  and  $E$  a vector space. We say that  $F : Q \rightarrow E$  is  *$G$ -equivariant* if  $F(q \cdot g) = \rho(g^{-1})F(Q)$ . The set of all  $G$ -equivariant maps will be denoted as  $C_G(Q, E)$ .

If  $\pi : Q \rightarrow M$  is a principal  $G$ -bundle and  $\rho : G \rightarrow \mathrm{GL}(E)$  is a representation of  $G$ , the module of sections of  $\rho(Q) \rightarrow M$  is in one-to-one correspondence with  $C_G(Q, E)$ . To see this, assign to each  $F \in C_G(Q, E)$  the section  $s$  defined by  $s(m) = [q, F(q)] \in \rho(Q)_m$

whenever  $\pi(q) = m \in M$ . This is well-defined because if  $h = q \cdot g$ ,  $[h, F(h)] = [q \cdot g, \rho(g^{-1})F(q)] = [q, F(q)]$ .

*Example 20* ([4], p. 211). A function  $\xi : Q \rightarrow \mathfrak{g}$  which is Ad-invariant, meaning that  $\xi(q \cdot g) = \text{Ad}_{g^{-1}} \xi(q)$  defines a section of the adjoint bundle.

This relation between bundles and sections extends to forms, as will be seen next.

**Definition 42.** A  $k$ -form  $\omega$  with values in the associated bundle  $\rho(Q)$  is a multilinear totally skew assignment, taking  $v_1, \dots, v_k \in T_m M$  to a vector  $\omega_m(v_1, \dots, v_k) \in \rho(Q)_m$ . We denote as  $\Omega^k(M, \rho(Q))$  the space of these forms.

Any  $k$ -form  $\omega \in \Omega^k(M, \rho(Q))$  is *horizontal* if it annihilates any vertical vector, that is,  $\omega_m(V, \cdot, \dots) = 0$  whenever  $V$  is a vertical vector in  $Q$ . For such a form,  $\omega_m(X_1 + V_1, \dots, X_k + V_k) = \omega_m(X_1, \dots, X_k)$  for  $V_i$  vertical.

**Proposition 14.** The space of  $k$ -forms on  $Q$  with values in  $E$  which are  $G$ -equivariant and horizontal (denoted as  $\Omega_G^k(Q, E)$ ) is isomorphic to  $\Omega^k(M, \rho(Q))$ .

*Proof.* For a given  $\tilde{\omega} \in \Omega_G^k(Q, E)$ , we assign to it the form  $\omega \in \Omega^k(M, \rho(Q))$  given by  $\omega_m(v_1, \dots, v_k) = [q, \tilde{\omega}(z_1, \dots, z_k)]$ ,  $v_i \in T_m M$ , where the  $z_i$  are any  $k$  vectors in  $T_q Q$  projecting onto the  $v_i$ , that is,  $d\pi_q(z_i) = v_i$ .  $\square$

Given a projection  $\pi : Q \rightarrow M$  and a Lie group  $G$  acting on the total space  $Q$ , the infinitesimal generators of the  $G$ -action  $\sigma_v : Q \rightarrow T_q Q$ ,  $v \in \mathfrak{g}$ ,

$$\sigma_v(q) = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} q \cdot \exp(\epsilon v),$$

satisfy  $V_q = \text{Ker } d\pi_q = \{\sigma_v(q) : v \in \mathfrak{g}\}$ . From this,  $\sigma(q)$  is a linear isomorphism from  $\mathfrak{g}$  to  $V_q$ . Moreover, the map  $\eta_q(v) = \sigma_v(q)$  is equivariant (see definition 41)

$$\eta_{q \cdot g} \circ \text{Ad}_{g^{-1}} = d\alpha_g \circ \eta_q$$

thanks to Proposition 11. Thinking  $\sigma$  as a map from  $Q \times \mathfrak{g}$  to  $V$ ,  $\sigma : Q \times \mathfrak{g} \rightarrow V$ ,  $\sigma(q, v) = \sigma_v(q)$ , we have that it is an *equivariant bundle map* over  $Q$ , that is, it is a fiber bundle such that  $G$  acts on the total and base spaces, and the following equivariant property holds:

$$\sigma((q, v) \cdot g) = \sigma(q, v) \cdot g.$$

**Proposition 15.** The map  $\sigma : \text{Ad}(Q) = (Q \times \mathfrak{g})/G \rightarrow V/G$  is an isomorphism between bundles over  $M$ .

*Proof.* The group  $G$  acts on  $V = \bigsqcup_{q \in Q} V_q$  through the assignment

$$w_q \cdot g = (d\alpha_g)_q(w_q),$$

where  $w_q \in V_q$ . From this, we can consider the equivalence classes of this action, denoted by  $V/G := \{[w_q] : w_q \in V\}$ . The map  $\sigma : \text{Ad}(Q) \rightarrow V/G$  sends an element  $[q, v] \in \text{Ad}(Q)$

to  $[\sigma_v(q)] \in V/G$ . From the equivariant property of  $\sigma$ , this map is well defines. The inverse map  $\sigma^{-1} : V/G \rightarrow \text{Ad}(Q)$  is  $\sigma^{-1}[w_q] = [q, \sigma_{w_q}^{-1}]$ , given that  $\sigma.(q) : \mathfrak{g} \rightarrow V_q$  is a linear isomorphism. To conclude this proof, we need to show that the following diagram is commutative,

$$\begin{array}{ccccc} \text{Ad}(Q) & \xrightarrow{\sigma} & V/G & \xrightarrow{\sigma^{-1}} & \text{Ad}(Q) \\ & \searrow \pi_{\text{Ad}(Q)} & \downarrow \pi_{V/G} & \swarrow \pi_{\text{Ad}(Q)} & \\ & & M & & \end{array}$$

but this is immediate, since  $\pi_{\text{Ad}(Q)}[q, v] = \pi_{V/G}[w_q] = \pi(q)$ .  $\square$

Under the isomorphism of this Proposition, given the one-to-one correspondence between sections of  $\text{Ad}(Q) \rightarrow M$  and  $G$ -equivariant maps  $C_G(Q, \mathfrak{g})$ , sections of the adjoint bundle correspond to  $G$ -invariant vertical vector fields. This is because such a vector defines and is uniquely defined by a section of  $V(G \rightarrow M)$ . To see this, let  $X : Q \rightarrow V$  be a  $G$ -invariant vertical vector field, that is,  $(d\alpha_g)_q(X_q) = X_{q \cdot g}$ , and  $X_q \in V_q$  for all  $q \in Q$ . The map  $\phi : M \rightarrow V/G$ ,  $\phi(m) = [X_q]$ , where  $q \in \pi^{-1}(m)$  is well defined because of the  $G$ -invariant property of  $X$ . Now, if  $\phi : M \rightarrow V/G$  is a section of  $V/G \rightarrow M$ , we can construct the  $G$ -invariant vertical vector field  $X : Q \rightarrow V$ ,  $X_q = w_q$ , where  $\phi(m) = [w_q]$ .

The isomorphism between sections of  $\text{Ad}(Q)$  and  $V/G$  will be denoted as  $\sigma : \Gamma(\text{Ad}(Q)) \rightarrow \Gamma(V)_G$ ,  $\xi \mapsto \sigma\xi$ , where  $\Gamma$  denotes the space of sections.

**Definition 43** ([4], p. 212). *The gauge group of  $\pi : Q \rightarrow M$  consists of all the  $G$ -maps  $\Phi : Q \rightarrow Q$  which induce the identity on  $M$ . That is,  $\Phi$  is a diffeomorphism that makes the following diagram commutative,*

$$\begin{array}{ccc} Q/G & \xrightarrow{\tilde{\Phi}} & Q/G \\ \tilde{\pi}^{-1} \uparrow & & \downarrow \tilde{\pi} \\ M & \xrightarrow{1_M} & M \end{array}$$

where  $\tilde{\pi} : Q/G \rightarrow M$  is given by  $[q] \mapsto \pi(q)$ , and  $\tilde{\Phi} : Q/G \rightarrow Q/G$ ,  $[q] \mapsto [\Phi(q)]$  is well defined because  $\Phi$  is a  $G$ -map ( $\Phi(q \cdot g) = \Phi(q)$ ).

**The Lie algebra of the gauge group is the space of equivariant vector fields.** Therefore, sections of the adjoint bundle form the Lie algebra of the gauge group via the isomorphism  $\sigma : \Gamma(\text{Ad}(Q)) \rightarrow \Gamma(V)_G$ .

**Definition 44** ([4], p. 212). Let  $\pi : Q \rightarrow M$  be a principal  $G$ -bundle with connection  $\mathcal{H}$ , that is,  $(d\alpha_g)_q(\mathcal{H}_q) = \mathcal{H}_{q \cdot g}$ . *The connection one-form on  $Q$ , denoted as  $A$ , is a Lie-algebra-valued one-form on  $Q$ , that is,  $A_q : T_q Q \rightarrow \mathfrak{g}$ , which satisfies the following conditions:*

- (1)  $\text{Ker } A_q = \mathcal{H}_q$ ,  $\forall q \in Q$ .
- (2)  $A_q \circ \eta_q = 1_{\mathfrak{g}}$ ,  $\forall q \in Q$ .

The second condition of previous definition is a normalization condition: as the image of  $\eta_q$  is  $V_q$ , when restricting to the vertical  $A_q$  is the canonical identification between  $V_q$  and  $\mathfrak{g}$ . Both conditions guarantee that the connection one-form is unique, and that it satisfies the following equivariance condition:

$$\text{Ad}_{g^{-1}} \circ A_q = A_{q \cdot g} \circ (d\alpha_g)_q \circ \eta_q \circ A_q.$$

As seen in Proposition 15, differentiating the group action on  $Q$  yields the infinitesimal generator map  $\sigma : \text{Ad}(Q) \rightarrow (TQ)/G$ , whose image is  $V/G$ .  $(TQ)/G \rightarrow M$  is a vector bundle over  $M$ , with projection  $[q, w] \mapsto \pi(q)$ . Similarly,  $d\pi : TQ \rightarrow TM$  is equivariant with respect to the  $G$ -action, where  $G$  acts trivially on  $M$ , which means that  $d\pi : (TQ)/G \rightarrow M$  is well-defined. As  $\text{Ker } d\pi = \text{Im } \sigma$ , these maps fit together in the following exact sequence

$$\text{Ad}(Q) \xrightarrow{\sigma} (TQ)/G \xrightarrow{d\pi} TM$$

of vector bundles over  $M$ . This sequence is called the *Atiyah sequence*.

A connection one-form is a splitting of this exact sequence:

$$\text{Ad}(Q) \xleftarrow{A} (TQ)/G \xleftarrow{h} TM,$$

where  $h$  is the collection of horizontal lifts  $h_q : T_m M \rightarrow T_q Q$  composed with the canonical map  $TQ \rightarrow (TQ)/G$ , and  $A$  results when we consider the connection one form  $A : TQ \rightarrow Q \times \mathfrak{g}$ ,  $w_q \mapsto (q, A_q(w_q))$ , and take its quotient by the  $G$ -action. These maps satisfy the following algebraic relations:

$$\begin{aligned} \text{Im } h_q &= \mathcal{H}_q = \text{Ker } A_q, \\ \text{Im } \eta_q &= V_q = \text{Ker } d\pi_q, \\ A_q \circ \eta_q &= 1_{\mathfrak{g}}, \quad d\pi_q \circ h_q = 1_{T_m M}. \end{aligned}$$

**Proposition 16.** *The vector bundles  $(TQ)/G$  and  $TM \oplus \text{Ad}(Q)$  are isomorphic.*

*Proof.* Take an element  $v \oplus [q, w] \in T_m M \oplus \text{Ad}(Q)$  to  $[h_q(v) + \sigma_w(q)] \in (TQ)/G$ . The inverse sends  $[w_q] \in (TQ)/G$  to  $d\pi_q(w_q) \oplus [q, A_q(w_q)]$ . The proof that these maps are inverses relies on the algebraic relations stated above, and the commutativity of the diagram

$$\begin{array}{ccccc} TM \oplus \text{Ad}(Q) & \longrightarrow & (TQ)/G & \longrightarrow & TM \oplus \text{Ad}(Q) \\ & & \downarrow & & \\ & & M & & \end{array}$$

is immediate from the definition. □

**Definition 45** ([4], definition 4.1.1). *The curvature of a distribution  $\mathcal{H}$  is the linear bundle map  $F : \Lambda^2 \mathcal{H} \rightarrow TQ/\mathcal{H}$  defined by  $F(X, Y) = -[X, Y] \bmod \mathcal{H}$ .*

The connection one-form  $A$  provides an identification  $(TQ)/\mathcal{H} \rightarrow \mathfrak{g}$ , given that the kernel of  $A_q$  is precisely  $\mathcal{H}_q$ .

**Definition 46** ([4], p. 214). *The curvature form* is a  $G$ -equivariant horizontal two-form  $\Omega$  defined by the assignment  $\Omega(X, Y) = -A([X, Y])$ , if  $X, Y$  are horizontal, and  $\Omega(X, Y) = 0$  if  $X$  or  $Y$  are vertical vectors.

**Definition 47** ([4], p. 214). Let  $\rho(Q)$  be an associated bundle of  $\pi : Q \rightarrow M$ . If  $\psi : M \rightarrow \rho(Q)$  is a section of this bundle, and if  $X \in \mathfrak{X}(M)$ , we define the *covariant derivative*  $D_X\psi$  of  $\psi$  in the direction of  $X$  as follows. The section  $\psi$  corresponds to an equivariant map  $F : Q \rightarrow E$ , and  $hX$  is a  $G$ -equivariant horizontal vector field on  $Q$ , so that  $dF(hX)$  is an equivariant map  $Q \rightarrow E$ , which corresponds to a section of  $\rho(Q)$ ,  $D_X\psi$ .

**Definition 48** ([4], p. 214). Let  $c : I \rightarrow M$  be a curve. A curve  $\psi : \text{Im}(c) \rightarrow \rho(Q)$  lying over  $c$  is said to be *covariantly constant* if  $D_{\dot{c}}\psi = 0$ .

The  $G$ -invariant vector fields on  $Q$  can be identified with sections of  $(TQ)/G$ , and thanks to Proposition 16 we can split them up into horizontal and vertical vector fields, according to  $(TQ)/G = TM \oplus \text{Ad}(Q)$ . Then, if  $X \in \mathfrak{X}(M)$  and  $\xi : M \rightarrow \rho(Q)$ , we have the following relations:

$$\begin{aligned} [hX, hY] &= h([X, Y]) \oplus -\eta(\Omega(X, Y)), & (\text{horizontal-horizontal}) \\ [\sigma\xi, \sigma\chi] &= -\sigma([\xi, \chi]), & (\text{vertical-vertical}) \\ [hX, \sigma\xi] &= \eta(D_X\xi). & (\text{horizontal-vertical}) \end{aligned}$$

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