POSGRADUATE PROJECT

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Abstract.

Introduction

1. Section 1

A manifold without boundary M of dimension m is said to be of class C^∞ if the differentiability class of the transition maps is C^∞. In this case, for every $p \in M$ one can define the tangent space to p, T_pM, as the set of all derivations of the algebra of germs of smooth functions from M to $\mathbb{R}$, based on p. This definition of tangent space is equivalent (when the differentiability class of M is ∞) to the equivalence class of germs of curves that passes through p. If a manifold M is C^r, with $r < \infty$, the definition for tangent space is the latter, so we use that definition for the tangent space throughout this text, independently of the differentiability class.

Joining the tangent spaces of every point of the manifold induces the tangent bundle,

$$TM = \bigsqcup_{p \in M} T_pM,$$

equipped with manifold structure given by the one of M. It is called a bundle because the triple $(TM, M, \pi : TM \to M)$ is in fact a bundle, where $\pi : TM \to M$ is the projection function, that assigns each tangent vector to the point where it is based. Each fiber of this bundle is a tangent space based on a point, and the tangent bundle is in particular a vector bundle because each fiber has structure of finite dimensional real vector space.

For an arbitrary bundle $(E, B, \pi : E \to B)$, a subbundle of this bundle $(E', B', \pi' : E' \to B')$ is just a bundle that satisfies $E' \subset E$, $B' \subset B$ and $\pi|_{E'} = \pi'$. A distribution $\mathcal{H}$ is a vector subbundle of the tangent bundle, with a fiber inner-product, and the pair of a manifold and a distribution over it defines a subriemannian geometry, as is stated in the following definition.

Definition 1 (Subriemannian Geometry)**.** A subriemannian geometry over a manifold M consists of a distribution $\mathcal{H} \subset TM$ (referred as horizontal distribution), *i.e.*, a vector subbundle of the tangent bundle of M, together with a fiber inner-product $< \cdot, \cdot >$ on this subbundle.

Some examples of this definition are given below.

Example 1 (Riemannian Geometry). Every riemannian geometry is in particular a subriemannian geometry, where the distribution is all the tangent bundle.

Example 2 (Heisenberg Group). This is a non-trivial case of subriemannian geometry. Take $\mathbb{R}^3$ as the manifold, and define over it the distribution $\mathcal{H} \subset T\mathbb{R}^3$, whose fiber at an arbitrary $(x, y, z) \in \mathbb{R}^3$ is

$$\mathcal{H}_{(x,y,z)} = \left\{ (v_1, v_2, v_3) \in T_{(x,y,z)}\mathbb{R}^3 : v_3 - \frac{1}{2}\left(xv_2 - yv_1\right) = 0 \right\}.$$

The inner product over $\mathcal{H}_{(x,y,z)}$ is given by $< \cdot, \cdot >: \mathcal{H}_{(x,y,z)} \times \mathcal{H}_{(x,y,z)} \to \mathbb{R} : (v, w) \mapsto v_1w_1 + v_2w_2$, where $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$.

This last example is of particular interest because of its connection with the isoperimetrical problem

References

[1] Richard Montgomery. *A tour of subriemannian geometries, their geodesics and applications.* Number 91. American Mathematical Soc., 2002.