Radiación Gravitacional

Juan Felipe Bravo Cárdenas

Universidad Nacional de Colombia
Facultad de Ciencias, Departamento de Física
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Juan Felipe Bravo Cárdenas

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Resumen

En la primera parte de este texto se presenta una introducción a la geometría diferencial como herramienta matemática para la Relatividad General. Se estudia la gravedad linealizada y el papel que esta desempeña en la radiación gravitacional, profundizando así en los conceptos de gauge, energía y contribución cuadrupolar. Seguidamente se presentan las ecuaciones de Einstein relajadas como una generalización para el estudio de la radiación gravitacional y se obtienen expresiones generales para la energía, el momentum lineal y angular. Posteriormente se muestra la relación entre las expresiones de flujo de energía, momentum lineal y angular con el tensor de Weyl.

Palabras clave: Radiación Gravitacional, Gravedad Linealizada, Ecuaciones de Einstein relajadas, Tensor de Weyl

Abstract

In the first part of this text an introduction to differential geometry is presented as a tool for General Relativity. The linearized gravity is studied and the role that this one plays in the gravitational radiation, deepening in the gauge, energy and quadrupolar contribution concepts. After this the relaxed Einstein equations are presented as a generalization for the study of gravitational radiation and general expression of energy, linear and angular momentum are obtained. Later it is shown the relation between the flux of energy, lineal and angular momentum with the Weyl tensor.

Keywords: Gravitational Radiation, Linearized Gravity, Relaxed Einstein field equations, Weyl tensor

Contenido

Introduction

In 1916 Albert Einstein predicted the existence of gravitational radiation, see [8] and [9], this under a weak field assumption also known as linearized gravity. This field then was vastly explored, even generalized to be able to work it with out a weak field assumption. Several of the ideas that were develop for the study of gravitational radiation were brought from the classical electrodynamics, but with the precaution of not confusing the concepts of electrodynamics and gravitation. Still the observational part was still missing.
The first observation of gravitational radiation was made by Hulse and Taylor in 1975 with the discovery of the pulsar PSR B1913+16, see [12]. It was such a discovery that in 1993 they earned the Nobel Price of physics, but this was an indirect observation. It wasn't until 2015 that LIGO and Virgo interferometers observed a transient gravitational-wave signal of the inspiral and merger of a pair of black holes, see [2]. With this finally was a direct observation of gravitational radiation, and it was not the only observation that these interferometers detected, see [3].
The purpose of this text is to introduce the basic mathematical and physical knowledge of gravitational radiation. For this, it has been divided in four chapters. Chapter one is an introduction to Differential Geometry and General Relativity, in the next chapter several mathematical concepts became really important. In chapter two is studied the relation between linearized gravity and gravitational radiation, this having in mind that this is only valid in a particular zone of radiation. The Einstein field equations are written as a wave equation in the zone of radiation where linearized gravity works. It is also study the gauge transformations, energy and multipole contribution. In Chapter 3 the Einstein field equations as a wave equation are shown again, but in this case with out any particular approximation over the gravitational filed, also expressions for the energy, lineal and angular momentum are obtained in general. Finally, in chapter four is shown the relation between gravitational radiation in linearized gravity and the Weyl tensor, but it is necessary to introduce first the tetrads, the Newman-Penrose formalism and the Weyl scalars. Once that this is introduced, the flux energy, lineal and angular momentum expressions are written in terms of a Weyl scalar. Two appendix are given, one is for the equivalence of the harmonic coordinates and the DeDonder gauge, and the other one is a deeper study of the relativistic angular momentum.

1. Chapter 1: Introduction to General Relativity

In this chapter we are going to give some of the main geometric definitions and properties used in General Relativity given by Differential Geometry. Then we introduce the Postulates of General Relativity and the obtaining of Einstein field equations from a variational principle. Main references of this chapter are [24, 10, 18, 19, 28].

1.1. Brief introduction to differential geometry

1.1.1. Differential Manifolds

Let $\mathcal{M}$ be a topological space, a **coordinate chart** $C_\alpha = (\varphi_\alpha, U_\alpha)$ over $\mathcal{M}$ is a homeomorphism

$$\varphi_\alpha : U_\alpha \subseteq \mathcal{M} \to \mathbb{R}^n,$$

where U_α is an open set over $\mathcal{M}$. We call a $\boldsymbol{C^r}$**-atlas** over $\mathcal{M}$ to a chart collection

$$\{C_\alpha = (\varphi_\alpha, U_\alpha)\}_{\alpha \in I}$$

such that

$$\mathcal{M} = \bigcup_{\alpha \in I} U_\alpha,$$

and if $U_\alpha \bigcap U_\beta \neq \emptyset$ then

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha \left(U_\alpha \bigcap U_\beta\right) \subseteq \mathbb{R}^n \to \varphi_\beta \left(U_\alpha \bigcap U_\beta\right) \subseteq \mathbb{R}^n$$

is a C^r diffeomorphism.
Two C^r atlas over a topological space $\mathcal{M}$ are **compatible** if the union of the atlas is a new atlas, then, the union of all compatible atlas over a topological space forms an equivalence class atlas, or a maximal atlas. Then, a C^r differentiable manifold $\mathcal{M}$ is a Hausdorff topological space with a maximal atlas.
A manifold is **orientable** if there is an atlas $\{\phi_\alpha, U_\alpha\}_{\alpha \in I}$ such that, in every no-empty intersection $U_\alpha \bigcap U_\beta$ of open sets, the determinant of the matrix $\partial x^i / \partial x'^j$ is greater than zero, where x^i are U_α coordinates and x'^i are U_β coordinates. An atlas is **locally finite**, if every point in the manifold has an open neighborhood that intercepts only a finite number

of neighborhood U_β. A manifold $\mathcal{M}$ is **paracompact** if for every atlas $\{\phi_\alpha, U_\alpha\}_{\alpha\in I}$, exist a a locally finite atlas $\{\psi_\beta, V_\beta\}_{\beta\in I}$ with each V_β contained in some U_α.

1.1.2. Tangent space , Dual space and Tensors

Tangent space

Let us define the set of all real value functions over a manifold $\mathcal{M}$ as

$$\mathcal{F}(\mathcal{M},\mathbb{R}) := \{f : \mathcal{M} \to \mathbb{R}\},$$

where for $f, g \in \mathcal{F}(\mathcal{M},\mathbb{R})$ we have a vector space structure and if f is differentiable respect to a chart (ϕ_α, U_α), it is also differentiable respect to (ϕ_β, U_β).
A **tangent vector** v_p to the manifold in a point p is a function

$$\begin{array}{rccc} v_p : & \mathcal{F}(\mathcal{M},\mathbb{R}) & \to & \mathbb{R} \\ & f & \mapsto & v_p(f) \end{array},$$

such that v_p is lineal in $\mathbb{R}$ and meets the Leibniz product property. The space of all tangent vectors at the point p, denoted by $T_p\mathcal{M}$, is a vector real space. The partial derivatives, denoted as ∂_α where $i = 1, 2, ..., n$ and n is the manifolds dimension, are tangent vectors in $T_p\mathcal{M}$ and form a base for the tangent space. Then, for $v_p \in T_p\mathcal{M}$ and introducing the Einstein sum convention

$$v_p = \sum_\alpha v_p^\alpha \partial_\alpha \bigg|_p \equiv v_p^\alpha \partial_\alpha\big|_p .$$

Dual space

A **one-form** $\boldsymbol{\omega}$ in the point p is a real function over $T_p\mathcal{M}$

$$\begin{array}{rccc} \boldsymbol{\omega} : & T_p\mathcal{M} & \to & \mathbb{R} \\ & \boldsymbol{v} & \mapsto & \boldsymbol{\omega}(\boldsymbol{v}) \equiv \langle \boldsymbol{\omega}, \boldsymbol{v} \rangle \end{array},$$

such that $\boldsymbol{\omega}$ is linear in $\mathbb{R}$. The space of all one-forms, denoted by $T_p^*\mathcal{M}$, is called the dual vector space.
Every function $f \in \mathcal{F}(\mathcal{M},\mathbb{R})$ defines a one-form df in p, this one-form is called the **differential** of f in p. For the local coordinates $\phi_\alpha(p) = (x^1, x^2, ..., x^n)$ we have that the set of differentials

$$\left\{dx^1, dx^2, ..., dx^n\right\}$$

in p forms a base for the dual vector space, then

$$\boldsymbol{\omega} = \omega_\alpha dx^\alpha .$$

This space meets the condition

$$\langle dx^\alpha, \partial_\beta \rangle = \delta^\alpha_\beta,$$

where $\{\partial_\alpha\}$ is a base of $T_p\mathcal{M}$.

Tensors

Let Π_r^s be defined as

$$\Pi_r^s := \left\{ \left(\boldsymbol{\eta}^1, ..., \boldsymbol{\eta}^r, \boldsymbol{Y}_1, ..., \boldsymbol{Y}_s \right) — \boldsymbol{\eta}^i \in T_p^*\mathcal{M}, \boldsymbol{Y}_j \in T_p\mathcal{M} \right\},$$

then a **tensor $\boldsymbol{T}$ of type $(\boldsymbol{r}, \boldsymbol{s})$** is a multilinear function over Π_r^s

$$\begin{array}{cccc} \boldsymbol{T}: & \Pi_r^s & \rightarrow & \mathbb{R} \\ & \left(\boldsymbol{\eta}^1, ..., \boldsymbol{\eta}^r, \boldsymbol{Y}_1, ..., \boldsymbol{Y}_s \right) & \mapsto & \boldsymbol{T}\left(\boldsymbol{\eta}^1, ..., \boldsymbol{\eta}^r, \boldsymbol{Y}_1, ..., \boldsymbol{Y}_s \right) \end{array}.$$

The space of all tensors, denoted as T_s^r, is called the tensor product

$$T_s^r := \underbrace{T_p\mathcal{M} \otimes \cdots \otimes T_p\mathcal{M}}_{r\text{-times}} \otimes \underbrace{T_p^*\mathcal{M} \otimes \cdots \otimes T_p^*\mathcal{M}}_{s\text{-times}},$$

this is a vector real space of dimension $r + s$. Given $\boldsymbol{T} \in T_s^r$ it can be written as

$$\boldsymbol{T} = T^{\alpha_1 \cdots \alpha_r}_{\beta_1 \cdots \beta_s} \partial_{\alpha_1} \otimes \cdots \otimes \partial_{\alpha_r} \otimes dx^{\beta_1} \otimes \cdots \otimes dx^{\beta_s},$$

where $T^{\alpha_1 \cdots \alpha_r}_{\beta_1 \cdots \beta_s}$ are the components of $\boldsymbol{T}$, we call the index $\alpha_1 \cdots \alpha_r$ contravariant and $\beta_1 \cdots \beta_s$ covariant, then the components of the tensor $\boldsymbol{T}$ related to to the contravariant index are called contravariant components and the ones related to the covariant index are called covariant components.

Under a change of coordinates the tensor components, the vector and the one-foms components as well, transform according to

$$T'^{\gamma_1 \cdots \gamma_r}_{\sigma_1 \cdots \sigma_s} = \frac{\partial x'^{\gamma_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial x'^{\gamma_r}}{\partial x^{\alpha_r}} \frac{\partial x^{\beta_1}}{\partial x'^{\sigma_1}} \cdots \frac{\partial x^{\beta_s}}{\partial x'^{\sigma_s}} T^{\alpha_1 \cdots \alpha_r}_{\beta_1 \cdots \beta_s}. \tag{1-1}$$

1.1.3. Transformation between manifolds

Let $\mathcal{M}$ and $\mathcal{N}$ be two manifolds. We can define a diffeomorfism between these manifolds

$$\phi : \mathcal{M} \rightarrow \mathcal{N}.$$

From this, we can induce a new function

$$\begin{array}{cccc} \tilde{\phi}: & \mathcal{F}(\mathcal{M}) & \rightarrow & \mathcal{F}(\mathcal{M}) \\ & f & \mapsto & \tilde{\phi}(f) \end{array}$$

defined as $\tilde{\phi}(f) = f(\phi(p))$, where $p \in \mathcal{M}$. We induce now the function

$$\begin{array}{cccc} \phi_*: & T_p\mathcal{M} & \rightarrow & T_{\phi(p)}\mathcal{N} \\ & \boldsymbol{X} & \mapsto & \phi_*(\boldsymbol{X}) \end{array}$$

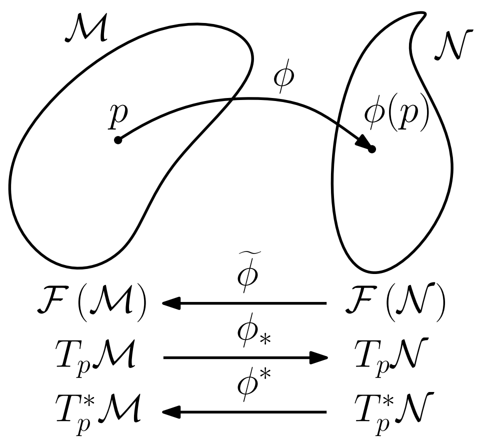

Figura 1-1.: Diffeomorfism ϕ between the manifolds $\mathcal{M}$ and $\mathcal{N}$ with the induced functions $\tilde{\phi}$, ϕ_* and ϕ^*. Traced image from [24].

such that

$$\phi_* \left(\boldsymbol{X}(f)\right)\big|_{\phi(p)} := \boldsymbol{X}\left(f \circ \phi\right)\big|_p ,$$

the function ϕ_* is known as **push forward**. Now, from the definition of ϕ_*, we define the function

$$\begin{array}{rccc} \phi^* : & T^*_{\phi(p)}\mathcal{N} & \rightarrow & T^*_p\mathcal{M} \\ & \boldsymbol{\omega} & \mapsto & \phi^*\left(\boldsymbol{\omega}\right) \end{array}$$

given by

$$\left\langle \phi^*\left(\boldsymbol{\omega}\right), \boldsymbol{X} \right\rangle = \left\langle \boldsymbol{\omega}, \phi_*\left(\boldsymbol{X}\right) \right\rangle ,$$

the function ϕ^* is known as **pullback**. These functions extends to tensors of type $(0, s)$ and $(r, 0)$, then extends to (r, s) tensors. Figure **1-1** shows a scheme of how this diffeomorphisms work.

1.2. Curvature

1.2.1. Lie derivative, connection and covariant derivative

Let $\lambda(t)$ be a curve over a manifold $\mathcal{M}$, there exist only one maximal curve $\lambda(t)$ over $\mathcal{M}$ that goes through each $p \in \mathcal{M}$ such that $\lambda(0) = p$ and its tangent vector in the point $\lambda(t)$ is the vector $\boldsymbol{X}|_{\lambda(t)}$. The flux of a vector field $\boldsymbol{X}$ over $\mathcal{M}$ is a transformation

$$\begin{gathered} \phi : \mathcal{M} \times \mathbb{R} \rightarrow \mathcal{M} \\ (p, t) \rightarrow \phi(p, t) := \lambda_t(p) \end{gathered}$$

if we fix t then we define a diffeomorphism that sends a point p in $\mathcal{M}$ to a point $\phi_t(p)$. With this we can define the **Lie derivative** of a tensor $\boldsymbol{T}$ with respect to a vector field $\boldsymbol{X}$ in the point p is

$$L_{\mathbf{X}}\,\boldsymbol{T}\big|_p = \operatorname*{Lim}_{t\to 0}\frac{1}{t}\left(\boldsymbol{T}\big|_p - (\phi_t)_*\,\boldsymbol{T}\big|_p\right).$$

Given the fact that under the change of coordinates de partial derivative is not invariant, we need to generalize this concept over a manifold. This generalization is given by the **covariant derivative** $\nabla_{\boldsymbol{X}}\boldsymbol{Y}$, where $\boldsymbol{X},\boldsymbol{Y}\in T_p\mathcal{M}$. Because $\nabla_{\boldsymbol{X}}\boldsymbol{Y}$ is a tensor, we can write it using the bases $\{\partial_\alpha\}$ and $\{dx^\alpha\}$, then the components of this tensor, denoted as $\nabla_\beta Y^\alpha$, are

$$\nabla_\beta Y^\alpha = \partial_\beta Y^\alpha + \Gamma^\alpha_{\beta\gamma}Y^\gamma.$$

The terms $\Gamma^\alpha_{\beta\gamma}$ are called **Christoffel symbols of the second kind**, these symbols are given by

$$\Gamma^\alpha_{\beta\gamma} = \left\langle dx^\alpha, \nabla_{\partial_\beta}\partial_\alpha\right\rangle.$$

Just like the partial derivative, it is linear and meets the Leibniz product property.
We extend the covariant derivative to arbitrary tensor. If $\boldsymbol{T}\in T^s_r$, then $\nabla\boldsymbol{T}\in T^s_{r+1}$, where the components of $\nabla\boldsymbol{T}$ are

$$\begin{aligned}\nabla_\gamma T^{\alpha_1\cdots\alpha_r}_{\beta_1\cdots\beta_s} = {}& \partial_\gamma T^{\alpha_1\cdots\alpha_r}_{\beta_1\cdots\beta_s} + \Gamma^{\alpha_1}_{\gamma\sigma}T^{\sigma\alpha_2\cdots\alpha_r}_{\beta_1\cdots\beta_s} + \cdots + \Gamma^{\alpha_r}_{\gamma\sigma}T^{\sigma\alpha_2\cdots\alpha_{r-1}\sigma}_{\beta_1\cdots\beta_s} \\ & - \Gamma^\sigma_{\gamma\beta_1}T^{\alpha_1\cdots\alpha_r}_{\sigma\beta_2\cdots\beta_s} - \cdots - \Gamma^\sigma_{\gamma\beta_s}T^{\alpha_1\cdots\alpha_r}_{\beta_1\cdots\sigma\beta_{s-1}},\end{aligned} \tag{1-2}$$

where it is still linear and meets the Leibniz product property. Given that is a tensor this have to transform like in (1-1), from this we can see that the transformation rule for the Christoffel symbols is given by

$$\Gamma'^\alpha_{\beta\gamma} = \frac{\partial x'^\alpha}{\partial x^\sigma}\frac{\partial x^\rho}{\partial x'^\beta}\frac{\partial x^\sigma}{\partial x'^\rho\partial x'^\gamma} + \frac{\partial x'^\alpha}{\partial x^\sigma}\frac{\partial x^\rho}{\partial x'^\beta}\frac{\partial x^\tau}{\partial x'^\gamma}\Gamma^\sigma_{\rho\tau}.$$

With the covariant derivative and the Lie derivative defined, given $\boldsymbol{T}\in T^r_s(\mathcal{M})$, we can write the components of $L_{\mathbf{X}}\boldsymbol{T}$ in terms of partial derivatives

$$\begin{aligned}(L_{\mathbf{X}}\boldsymbol{T})^{\alpha_1\cdots\alpha_r}_{\beta_1\cdots\beta_s} = {}& X^\sigma\partial_\sigma\boldsymbol{T}^{\alpha_1\cdots\alpha_r}_{\beta_1\cdots\beta_s} - \boldsymbol{T}^{\sigma\alpha_2\cdots\alpha_r}_{\beta_1\cdots\beta_s}\partial_\sigma X^{\alpha_1} - \cdots\boldsymbol{T}^{\alpha_1\cdots\alpha_{r-1}\sigma}_{\beta_1\cdots\beta_s}\partial_\sigma X^{\alpha_r} \\ & + \boldsymbol{T}^{\alpha_1\cdots\alpha_r}_{\sigma\beta_2\cdots\beta_s}\partial_{\beta_1}X^\sigma + \cdots + \boldsymbol{T}^{\alpha_1\cdots\alpha_r}_{\beta_1\cdots\beta_{s-1}\sigma}\partial_{\beta_s}X^\sigma\end{aligned}$$

and in terms of covariant derivatives

$$\begin{aligned}(L_{\mathbf{X}}\boldsymbol{T})^{\alpha_1\cdots\alpha_r}_{\beta_1\cdots\beta_s} = {}& X^\sigma\nabla_\sigma\boldsymbol{T}^{\alpha_1\cdots\alpha_r}_{\beta_1\cdots\beta_s} - \boldsymbol{T}^{\sigma\alpha_2\cdots\alpha_r}_{\beta_1\cdots\beta_s}\nabla_\sigma X^{\alpha_1} - \cdots\boldsymbol{T}^{\alpha_1\cdots\alpha_{r-1}\sigma}_{\beta_1\cdots\beta_s}\nabla_\sigma X^{\alpha_r} \\ & + \boldsymbol{T}^{\alpha_1\cdots\alpha_r}_{\sigma\beta_2\cdots\beta_s}\nabla_{\beta_1}X^\sigma + \cdots + \boldsymbol{T}^{\alpha_1\cdots\alpha_r}_{\beta_1\cdots\beta_{s-1}\sigma}\nabla_{\beta_s}X^\sigma.\end{aligned}$$

We need to keep in mind that in general relativity we work with a free torsion connection, a consequence of this is that the Christoffel symbols are symmetric in its index, i.e.

$$\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}$$

1.2.2. Parallel transport

Let $\boldsymbol{T}$ a tensor field and $\lambda(t)$ a curve over a manifold $\mathcal{M}$[1], let us define the covariant derivative along the curve as $\nabla_{\partial_t}\boldsymbol{T}$, then if $\boldsymbol{X}$ is the tangent vector to the curve $\lambda(t)$ then the components of the covariant derivative along the curve is $\nabla_\gamma T^{\alpha_1\cdots\alpha_r}_{\beta_1\cdots\beta_s}X^\gamma$. We said that a vector is parallel transport along a curve $\lambda(t)$ if $\nabla_{\partial_t}\boldsymbol{T}=0$.
In the particular case of a vector $\boldsymbol{Y}$ we choose a curve $\lambda(t)$ such that we have a coordinate system $x^\alpha(t)$ and $X^\alpha = \frac{dx^\alpha}{dt}$. We said that the curve is a **geodesic curve** if the tangent vector is parallel transported along the curve, this means that

$$\nabla_{\boldsymbol{X}}\boldsymbol{X} = 0.$$

For the basis $\{\partial_\alpha\}$ and $\{dx^\alpha\}$ we can write this condition in the following way

$$\frac{d^2x^\alpha}{dt^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{dt}\frac{dx^\gamma}{dt} = 0.$$

1.2.3. Riemman tensor and metric tensor

In a euclidian space we have that we can commute the derivatives without problems, but this does not happen in a curve space. A measure of this non-commutativity is given by the Riemann tensor, the components of this tensor are given by

$$R^\alpha_{\beta\gamma\delta} = \partial_\gamma\Gamma^\alpha_{\delta\beta} - \partial_\delta\Gamma^\alpha_{\gamma\beta} + \Gamma^\alpha_{\gamma\sigma}\Gamma^\sigma_{\delta\beta} + \Gamma^\alpha_{\delta\sigma}\Gamma^\sigma_{\gamma\beta},$$

these components have de following properties

$$\begin{aligned}
&R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = R_{\alpha\beta\delta\gamma} = R_{\gamma\delta\alpha\beta},\\
&R_{\alpha\beta\gamma\delta} + R_{\alpha\beta\gamma\delta} + R_{\alpha\beta\gamma\delta} = 0,\\
&\nabla_\eta R^\alpha_{\beta\gamma\delta} + \nabla_\delta R^\alpha_{\beta\eta\gamma} + \nabla_\gamma R^\alpha_{\beta\eta\delta} = 0.
\end{aligned}$$

From a contraction we define the Ricci tensor as

$$R_{\alpha\beta} = R^\sigma_{\alpha\sigma\beta}.$$

Over a manifold $\mathcal{M}$ we define the metric tensor as a symmetric tensor field of the type T^0_2. Given the basis $\{dx^\alpha\}$ we have that

$$\boldsymbol{g} = g_{\alpha\beta}dx^\alpha \otimes dx^\beta,$$

with the metric tensor we can define a norm, the cosine of an "angle" for two given vectors and the length of a path between two points, this allow us to write the distance along a curve of two infinitesimal close points as

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta.$$

[1]For a formal definition of a curve over a manifold see [24]

We said that the metric is not degenerated if the determinant of the metric is distinct from zero, $\det|g_{\alpha\beta}| \neq 0$. This condition over the metric allows to define T_0^2 tensor type $g^{\alpha\beta}$ such that

$$g^{\alpha\beta}g_{\beta\gamma} = \delta^{\alpha}_{\gamma}. \tag{1-3}$$

From this we can built an isomorphism such that we can relate the covariant and contravariant index components of the tensors, if we take a vector with components X^{α} then

$$X_{\alpha} = g_{\alpha\beta}X^{\beta}$$

and using (1-3)

$$X^{\alpha} = g^{\alpha\beta}X_{\beta}.$$

We can generalize this to tensor components, for example, for a tensor $\boldsymbol{T}$ with components $T^{\alpha\beta\gamma}$

$$\begin{aligned} T^{\alpha\beta}_{\gamma} &= g_{\gamma\sigma}T^{\alpha\beta\gamma} \\ T^{\alpha}_{\beta\gamma} &= g_{\gamma\sigma_1}g_{\beta\sigma_2}T^{\alpha\sigma_2\sigma_1} \\ T_{\alpha\beta\gamma} &= g_{\gamma\sigma_1}g_{\beta\sigma_2}g_{\alpha\sigma_3}T^{\sigma_3\sigma_2\sigma_1} \end{aligned}$$

in a similar way we do this for $T_{\alpha\beta\gamma}$.
Let us define the signature of of the metric tensor as the number of positive eigenvalues less the number of negative eigenvalues. A particular case is the signature $n-2$, this is the case of a Minkowskian or Lorenzian metric, from here we are going to assume a Lorenzian signature. A consequence of this is the values that can take the inner product of a vector defined by the metric tensor, we are going to divide this cases in three. For a vector $\boldsymbol{X}$ we said that this vector is

- Null if $\boldsymbol{g}(\boldsymbol{X},\boldsymbol{X}) = 0$
- Timelike if $\boldsymbol{g}(\boldsymbol{X},\boldsymbol{X}) < 0$
- Spacelike if $\boldsymbol{g}(\boldsymbol{X},\boldsymbol{X}) > 0$

To get a relation between the metric tensor components we use variational calculus[2], with this we get the geodesic equation but in terms of the components of the metric tensor, this gives the following relation between the Christoffel symbols and the metric tensor components

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}g^{\alpha\sigma}\left(\partial_{\beta}g_{\sigma\gamma} + \partial_{\gamma}g_{\sigma\beta} - \partial_{\sigma}g_{\gamma\beta}\right). \tag{1-4}$$

With this in mind we can write the equation (1-2) in terms of the metric tensor components, a consequence of this is that

$$\nabla_{\gamma}g_{\alpha\beta} = 0 \text{ and } \nabla_{\gamma}g^{\alpha\beta} = 0.$$

[2] For details see [19]

1.2.4. Killing vector field

The metric, or the metric tensor components, $g_{\alpha\beta}$ is a form invariant under a transformation from x^α to x'^α if $g'_{\alpha\beta}(x')$ is the same function of x'^α as $g_{\alpha\beta}(x)$ is of x^α. We know that $g_{\alpha\beta}$ transform as a tensor, then if the metric is a form invariant

$$g_{\alpha\beta}(x) = \frac{\partial x'^\rho}{\partial x^\alpha}\frac{\partial x'^\sigma}{\partial x^\beta} g_{\rho\sigma}(x'), \tag{1-5}$$

any transformation $x \to x'$ that satisfies (1-5) is called an isometry. Let us consider an infinitesimal coordinate transformation

$$x'^\alpha = x^\alpha + \epsilon\xi^\alpha \text{ with } |\epsilon| \ll 1,$$

to first order in ϵ we have that $g_{\alpha\beta}(x') \approx g_{\alpha\beta}(x) + \epsilon\xi^\lambda\partial_\lambda g_{\alpha\beta}(x)$, then we can write (1-5) as follows

$$g_{\alpha\sigma}\partial_\beta\xi^\sigma + g_{\rho\beta}\partial_\alpha\xi^\rho + \xi^\lambda\partial_\lambda g_{\alpha\beta} = 0.$$

This can be rewritten in terms of derivatives of the covariant components $\xi_\alpha = g_{\alpha\mu}\xi^\mu$, then

$$\begin{aligned} 0 &= g_{\alpha\sigma}\partial_\beta\xi^\sigma + g_{\rho\beta}\partial_\alpha\xi^\rho + \xi^\lambda\partial_\lambda g_{\alpha\beta} \\ &= \partial_\beta\xi_\alpha + \partial_\alpha\xi_\beta - \xi^\lambda\left(\partial_\beta g_{\alpha\lambda} + \partial_\alpha g_{\beta\lambda} - \partial_\lambda g_{\alpha\beta}\right) \\ &= \partial_\beta\xi_\alpha + \partial_\alpha\xi_\beta - 2\xi_\lambda\Gamma^\lambda_{\alpha\beta} \end{aligned}$$

therefore

$$\nabla_\beta\xi_\alpha + \nabla_\alpha\xi_\beta = 0.$$

This is the Killing equation, every vector that satisfies this equation is called a Killing vector. The problem of determining all infinitesimal isometries of a given metric is now reduce to determining the Killing vectors. Any linear combination of Killing vectors, with constant coefficients, is a Killing vector. For more details see [28].

1.3. General relativity

Here we are going to name the postulates of general relativity and see which are the equations that rules the dynamics of the spacetime.

1.3.1. Postulates of General Relativity theory

This postulates are a motivation of the geometrization of the gravity force in classical mechanics.

Postulate 1 The spacetime is the collection of all events, it is described by the pair $(\mathcal{M}, \boldsymbol{g})$, where $\mathcal{M}$ is a Hausdorff smooth four-dimensional manifold and $\boldsymbol{g}$ is a Lorenzian metric over $\mathcal{M}$.

Now let us introduce the postulates that involves the matter fields in the theory.

Postulate 2 The equations that satisfy the matter fields must fulfill that if, for $U \subset \mathcal{M}$ is convex and $p, q \in U$, then a signal can be send in U between p and q if and only if p and $q$can be join by a c^1-curve contained in U, which tangent vector everywhere is non-zero and timelike or null.

Postulate 3 There exist a symmetric tensor

$$T_{\alpha\beta} = T_{\beta\alpha} = T_{\alpha\beta}\left(\Psi_i, \nabla\Psi_i\right),$$

where Ψ_i are the matter fields and i index the different matter field, such that the depend of the matter fields is finite and

- $T_{\alpha\beta} = 0$ over $U \subset \mathcal{M}$ and open set, if and only if $\Psi_i = 0$ for every i sover U.
- $\nabla_\beta T^{\alpha\beta} = 0$

For a further discussion of the postulates see [10].

1.3.2. Einstein Field Equations

We can deduce the Einstein field equations in vacuum from an action, this action is called the Einstein-Hilbert action and is given by

$$S_{EH} = \frac{1}{16\pi G}\int R\sqrt{-g}d^4x,$$

making $\delta S_{EH} = 0$ leads to

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = 0.$$

If we want to include the matter contribution to the field equations, we must add the following term to the Einstein-Hilbert action

$$S_M = \int \mathcal{L}\left(\Psi_i\right)\sqrt{-g}d^4x,$$

where $\mathcal{L}$ is a Lagrangian density. Making $\delta\left(S_{EH} + S_M\right) = 0$ we obtain

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = \frac{8\pi G}{c^4}T_{\alpha\beta},$$

these are ten non-linear coupled partial differential equations. For details see [10, 18, 24].

2. Chapter 2: Linearized gravity and gravitational radiation

In this chapter we will introduce the concept of gravitational radiation in linearized gravity, which lead to the linearized gravitational field. A comparison of the radiation zone with electrodynamics is made. The main geometric contributions are calculated and a wave equation for gravitational radiation is obtained. A very important discussion of the nature of a background and a physical geometry is made, this became important for the energy contribution of the gravitational radiation. A energy-momentum tensor is obtained for the gravitational radiation, this allows to calculate the energy and momentum flux in terms of the linearized gravitational field. Finally an expression is obtained for the quadrupolar contribution of the gravitational radiation. Main references for this chapter are [14, 23, 4, 25, 16, 17, 11, 20, 13, 21]

2.1. Radiation zone

Here we discuss the role of the radiation zone in gravitational radiation. First we take a look of how the radiation zone is treated for far away distances in electrodynamics and then we discuss its analogue in gravitational radiation, which is the radiation zone in linearized gravity.

2.1.1. In Electrodynamics

To describe the field of distributions from a certain distance of a source, where the dimensions of the source are of order d, the wavelength is λ and $d \ll \lambda$, we split the space in three spatial regions of interest

The near (static) zone	$d \ll r \ll \lambda$
The intermediate (induction) zone	$d \ll r \sim \lambda$
The far (radiation) zone	$d \ll \lambda \ll r$

To study the fields in these regions we start from the Maxwell equations

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}, \qquad \nabla \cdot \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{2-1}$$

$$\nabla \cdot \boldsymbol{B} = 0, \qquad \nabla \cdot \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}, \tag{2-2}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and the magnetic field respectively, ε_0 is the vacuum permittivity, μ_0 is the vacuum permeability, ρ in the charge density and $\boldsymbol{J}$ is the current density. Now we can arrive at the wave equations

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial}{\partial t}\right)\boldsymbol{E} = -\frac{1}{\varepsilon_0}\left(-\nabla\rho - \frac{1}{c^2}\frac{\partial \boldsymbol{J}}{\partial t}\right), \tag{2-3}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial}{\partial t}\right)\boldsymbol{B} = -\mu_0 \nabla \times \boldsymbol{J}. \tag{2-4}$$

The solution to these equation is given by[1]

$$\boldsymbol{E}(t, \boldsymbol{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{R}\left[-\nabla'\rho - \frac{1}{c^2}\frac{\partial \boldsymbol{J}}{\partial t'}\right]_{\text{ret}}, \tag{2-5}$$

$$\boldsymbol{B}(t, \boldsymbol{x}) = \frac{\mu_0}{4\pi}\int \frac{1}{R}\left[\nabla' \times \boldsymbol{J}\right]_{\text{ret}}, \tag{2-6}$$

where $[\cdots]_{\text{ret}}$ means that the time t' is to be evaluated at the retarded time $t' = t - |\boldsymbol{x} - \boldsymbol{x}'|/c$ and $R = |\boldsymbol{R}| = |\boldsymbol{x} - \boldsymbol{x}'|$. In the equations (2-5) and (2-6), we can write the static limits plus corrections

$$\begin{aligned}\boldsymbol{E}(t, \boldsymbol{x}) = \frac{1}{4\pi\varepsilon_0}\int \Bigg\{ & \frac{\hat{\boldsymbol{R}}}{R^2}\left[\rho(t', \boldsymbol{x}')\right]_{\text{ret}} + \frac{\hat{\boldsymbol{R}}}{cR}\left[\frac{\partial \rho(t', \boldsymbol{x}')}{\partial t'}\right]_{\text{ret}} \\ & - \frac{\hat{\boldsymbol{R}}}{cR^2}\left[\frac{\partial \boldsymbol{J}(t', \boldsymbol{x}')}{\partial t'}\right]_{\text{ret}}\Bigg\} d^3x',\end{aligned} \tag{2-7}$$

$$\boldsymbol{B}(t, \boldsymbol{x}) = \frac{\mu_0}{4\pi}\int \left\{\left[\boldsymbol{J}(t', \boldsymbol{x}')\right]_{\text{ret}} \times \frac{\hat{\boldsymbol{R}}}{R^2} + \left[\frac{\partial \boldsymbol{J}(t', \boldsymbol{x}')}{\partial t'}\right]_{\text{ret}} \times \frac{\hat{\boldsymbol{R}}}{cR}\right\} d^3x', \tag{2-8}$$

where $\hat{\boldsymbol{R}} = \boldsymbol{R}/R$.

If the charge and current densities are time independent, the expressions reduce to the static expressions

$$\boldsymbol{E}(\boldsymbol{x}) = \frac{1}{4\pi\varepsilon_0}\int \rho(\boldsymbol{x}')\frac{\boldsymbol{x} - \boldsymbol{x}'}{|\boldsymbol{x} - \boldsymbol{x}'|}d^3x',$$

$$\boldsymbol{B}(\boldsymbol{x}) = \frac{\mu_0}{4\pi}\int \boldsymbol{J}(\boldsymbol{x}') \times \frac{\boldsymbol{x} - \boldsymbol{x}'}{|\boldsymbol{x} - \boldsymbol{x}'|}d^3x'.$$

[1] See JACKSON, Section 6.4

In the case of the radiation fields, is better if we write the expression (2-7) just as follows

$$\boldsymbol{E}\left(t,\boldsymbol{x}\right)=\frac{1}{4\pi\varepsilon_0}\int\left\{\frac{\left[\rho\left(t',\boldsymbol{x}'\right)\right]_{\mathrm{ret}}}{R^2}\hat{\boldsymbol{R}}+\frac{1}{cR^2}\left(\left[\boldsymbol{J}\left(t',\boldsymbol{x}'\right)\right]_{\mathrm{ret}}\cdot\boldsymbol{R}\right)\boldsymbol{R}+\frac{1}{cR^2}\left(\left[\boldsymbol{J}\left(t',\boldsymbol{x}'\right)\right]_{\mathrm{ret}}\times\boldsymbol{R}\right)\times\boldsymbol{R}\right.$$
$$\left.\frac{1}{c^2R}\left(\left[\frac{\partial\boldsymbol{J}\left(t',\boldsymbol{x}'\right)}{\partial t'}\right]_{\mathrm{ret}}\times\boldsymbol{R}\right)\times\boldsymbol{R}\right\}d^3x',$$

in the radiation zone the term $1/R^2$ does not contribute to the radiation field, then

$$\begin{aligned}\boldsymbol{E}(t,\boldsymbol{x})&=\frac{1}{4\pi\varepsilon_0}\int\frac{1}{c^2R}\left(\left[\frac{\partial\boldsymbol{J}\left(t',\boldsymbol{x}'\right)}{\partial t'}\right]_{\mathrm{ret}}\times\boldsymbol{R}\right)\times\boldsymbol{R}d^3x',\\ \boldsymbol{B}(t,\boldsymbol{x})&=\frac{\mu_0}{4\pi}\int\frac{1}{cR}\left[\frac{\partial\boldsymbol{J}\left(t',\boldsymbol{x}'\right)}{\partial t'}\right]_{\mathrm{ret}}\times\hat{\boldsymbol{R}}d^3x'.\end{aligned}$$

In the radiation zone we can write $\hat{\boldsymbol{R}}\approx\hat{\boldsymbol{x}}$, $1/R\approx 1/r$ and $R\approx r-\hat{\boldsymbol{x}}\cdot\boldsymbol{x}'$, where $r=|\boldsymbol{x}|$, therefore we obtain

$$\begin{aligned}\boldsymbol{E}(t,\boldsymbol{x})&\approx\frac{1}{4\pi\varepsilon_0}\int\frac{1}{c^2R}\left(\left[\frac{\partial}{\partial t'}\boldsymbol{J}\left(t'+\frac{\hat{\boldsymbol{x}}\cdot\boldsymbol{x}'}{c},\boldsymbol{x}'\right)\right]\times\boldsymbol{R}\right)\times\boldsymbol{R}d^3x',\\ \boldsymbol{B}(t,\boldsymbol{x})&\approx\frac{\mu_0}{4\pi}\int\frac{1}{cR}\left[\frac{\partial}{\partial t'}\boldsymbol{J}\left(t'+\frac{\hat{\boldsymbol{x}}\cdot\boldsymbol{x}'}{c},\boldsymbol{x}'\right)\right]\times\hat{\boldsymbol{R}}d^3x',\end{aligned}$$

a comparison of aboves equations leads to

$$\boldsymbol{E}\left(t,\boldsymbol{x}\right)=\boldsymbol{B}\left(t,\boldsymbol{x}\right)\times\hat{\boldsymbol{x}}.$$

Given the charge conservation, the monopole contribution can not radiate, but let us suppose that this can happend. If the monopole radiate we will have a monopole term at a given time t_0

$$\boldsymbol{E}_m=\frac{1}{4\pi\varepsilon_0 c}\frac{\dot{Q}(t_0)}{r}\hat{\boldsymbol{e}}_r,$$

where ε_0 is the vacuum permittivity. Then one could thing that a sphere with an oscillating radius should radiate, but according to Gauss's law, is exactly $(Q/4\pi\epsilon_0 r^2)\hat{\boldsymbol{r}}$, regardless of the fluctiations in size. Therefore we find that the radiation contributions starts from the dipole.

2.1.2. In linearized gravity

For the work of gravitational radiation we have to keep in mind, henceforth, that gravitational waves propagates in a background curvature, notated here by $\overline{g}_{\alpha\beta}(x)$. Under a suitable gauge, we can split our metric tensor as follows

$$g_{\mu\nu}(x)=\overline{g}_{\alpha\beta}(x)+h_{\alpha\beta}(x), \tag{2-9}$$

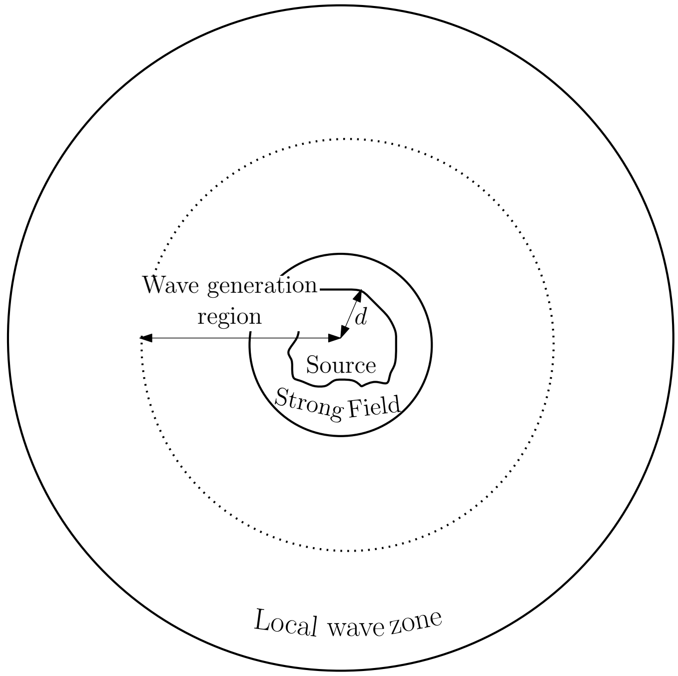

Figura 2-1.: Regions of spacetime surrounding a source. Traced image from [25].

where the term $h_{\alpha\beta}$ is known as the **linearized gravitational field**. Here we have not made any kind of aproximation over $\overline{g}_{\alpha\beta}(x)$ neather $h_{\alpha\beta}(x)$ yet.
For the case of linearized gravity, we are working in a region where the source's waves are weak, so we can say that we have outgoing ripples on a background spacetime where the effects of the background curvature on the wave propagation are totally negligible. This region is called the **local wave zone**, see figure **2-1**, this region is such that if d is the size of the source, $\bar{\lambda} = \lambda/2\pi$ is the reduced wavelength of the gravitational wave and r is the radial coordinate of our system, then

$$d \ll \bar{\lambda} \ll r.$$

This can be represented mathematically as

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \qquad |h_{\alpha\beta}| \ll 1, \tag{2-10}$$

where $\eta_{\alpha\beta}$ is the flat Minfkowski metric with

$$\eta_{\alpha\beta} = \mathrm{diag}\left(-1, 1, 1, 1\right),$$

then in the local wave zone the components of the metric only differ slightly from the Minkowski form, then it will be adequate to treat $h_{\alpha\beta}$ as a linearized field residing in flat spacetime. Therefore $h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$ and

$$g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta}.$$

2.2. Gauge transformations in linearized gravity

In this section we are going to deal with the gauge invariance issue. This issue arises in the fact that we can use another coordinate system where we can write $g_{\alpha\beta}$ just like in (2-10), but with a totally different $h_{\alpha\beta}$, then we have a non unique decomposition.

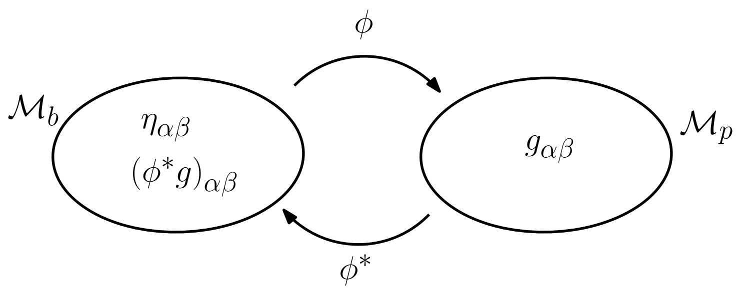

Figura 2-2.: A diffeomorphism relating the background spacetime $\mathcal{M}_b$, with flat spacetime $\eta_{\alpha\beta}$, to the physical spacetime $\mathcal{M}_p$. Traced image from [4].

We can formalize our concept of background in terms of a background spacetime $\mathcal{M}_b$, a physical spacetime $\mathcal{M}_p$ and a diffeomorphism between these manifolds, see figure **2-2**

$$\phi : \mathcal{M}_b \to \mathcal{M}_p.$$

Because the diffeomorphism ϕ, these manifolds describe the spacetime in the same way: we are assigning events from the background spacetime to the physic spacetime. In $\mathcal{M}_b$ we have defined the flat Minkowski metric $\eta_{\alpha\beta}$, while in $\mathcal{M}_p$ we have defined some metric $g_{\alpha\beta}$ that obey's Einstein field equations. Using the pullback function ϕ^* induced by ϕ, we have that $(\phi^* g)_{\alpha\beta}$ lives in $\mathcal{M}_b$, then we define the linearized gravitational field as

$$h_{\alpha\beta} = (\phi^* g)_{\alpha\beta} - \eta_{\alpha\beta}. \tag{2-11}$$

From this definition, there is no reason for the components of $h_{\alpha\beta}$ to be small. However, because we are working in the local wave zone, we limit our attention only to those diffeomorphisms for which $|h_{\alpha\beta}| \ll 1$ is true. Through ϕ, what we can said about $g_{\alpha\beta}$ and $h_{\alpha\beta}$ is that because $g_{\alpha\beta}$ obey's Einstein's field equations on the physical spacetime, then $h_{\alpha\beta}$ will obey the linearized equation on the background spacetime.
Now let us consider a vector field $\xi^\alpha(x)$ on $\mathcal{M}_b$. This vector field generates a one-parameter family of diffeomorphism, see figure **2-3**,

$$\psi_\epsilon : \mathcal{M}_b \to \mathcal{M}_p.$$

For ϵ sufficiently small, if ϕ is a diffeomorphism for which $h_{\alpha\beta}$ defined by (2-11) is small, then so will $(\phi \circ \psi_\epsilon)$ be, but the perturbation will have a different value. Then, we can define

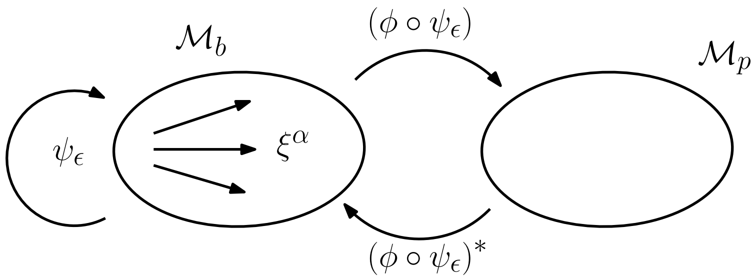

Figura 2-3.: A one-parameter family of diffeomorphism ψ_ϵ, generated by the vector field ξ^α on the background spacetime $\mathcal{M}_b$. Traced image from [4].

a family of perturbations parametrized by ϵ

$$\begin{aligned} h^{(\epsilon)}_{\alpha\beta} &= \left[(\phi \circ \psi_\epsilon)^* g\right]_{\alpha\beta} - \eta_{\alpha\beta} \\ &= \left[\psi^*_\epsilon (\phi g)^*\right]_{\alpha\beta} - \eta_{\alpha\beta} \\ &= \psi^*_\epsilon (h + \eta)_{\alpha\beta} - \eta_{\alpha\beta} \\ &= \psi^*_\epsilon (h_{\alpha\beta}) + \psi^*_\epsilon (\eta_{\alpha\beta}) - \eta_{\alpha\beta} \\ &= \psi^*_\epsilon (h_{\alpha\beta}) + \epsilon \left(\frac{\psi^*_\epsilon (\eta_{\alpha\beta}) - \eta_{\alpha\beta}}{\epsilon} \right). \end{aligned}$$

We use the fact that ϵ is small, then $\psi^*_\epsilon (h_{\alpha\beta}) = h_{\alpha\beta}$, now using the definition of a Lie derivative

$$h^{(\epsilon)}_{\alpha\beta} = h_{\alpha\beta} + \epsilon \mathcal{L}_\xi \eta_{\alpha\beta},$$

from the fact that the background metric is flat

$$h^{(\epsilon)}_{\alpha\beta} = h_{\alpha\beta} + \epsilon \left(\partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha\right). \tag{2-12}$$

This formula represents the change of the linearized gravitational field under an infinitesimal diffeomorphism along the vector field $\epsilon\xi^\alpha$. We will call this a **gauge transformation** in linearized theory. The interpretation of the physical situations through transformations is what is called active transformations, for a deeper analysis of this see [11].

The diffeomorphism ψ_ϵ provide a different representation of the same physical situation, while maintaining our requirement that the linearized gravitational field is small. Then the result (2-12) tells us that the linearized gravitational fields that denote physically equivalent spacetimes are related to each other by $\epsilon \left(\partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha\right)$ for some vector ξ_α. This gauge invariance can also be understood in the following way: the diffeomorphism ψ_ϵ can be thought of as the coordinate transformation

$$x^\alpha \to x^\alpha - \epsilon \xi^\alpha, \tag{2-13}$$

through the usual rules for transforming tensors under coordinate transformations we can obtain the expression (2-12). Typically the value of ϵ is set to one and the vector field ξ^α itself as small.

2.3. Linearized Einstein field equations

In this section we will write the Einstein field equations in terms of the linearized gravitational field, then we will see gravitational contribution of $h_{\alpha\beta}$. It is important to remark that here we are going to derivate with respect our background spacetime, then we have to use the backgrounds spacetime covariant derivative, which in this particular case is ∂_α.

2.3.1. General linearized expression of Einstein field equations

From the expression (1-4) and (2-10)

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}\eta^{\alpha\sigma}\left(\partial_\beta h_{\gamma\sigma} + \partial_\gamma h_{\sigma\beta} - \partial_\sigma h_{\beta\gamma}\right).$$

The contributions to the Riemann tensor will come only from the derivatives of the Γ's terms, then

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(\partial_\gamma\partial_\beta h_{\alpha\delta} + \partial_\delta\partial_\alpha h_{\beta\gamma} - \partial_\delta\partial_\beta h_{\alpha\gamma} - \partial_\gamma\partial_\alpha h_{\beta\delta}\right).$$

Contracting over α and δ we obtain the Ricci tensor

$$R_{\alpha\beta} = \frac{1}{2}\left(\partial_\sigma\partial_\beta h^\sigma_\alpha + \partial_\sigma\partial_\alpha h^\sigma_\beta - \partial_\alpha\partial_\beta h - \Box h_{\alpha\beta}\right),$$

where $h = \eta^{\alpha\beta}h_{\alpha\beta}$ and $\Box = \eta^{\alpha\beta}\partial_\alpha\partial_\beta$, the D'Alembert operator in flat spacetime. Contracting again we obtain the Ricci scalar

$$R = \partial_\alpha\partial_\beta h^{\alpha\beta} - \Box h,$$

and with this we can obtain the Einstein tensor

$$G_{\alpha\beta} = \frac{1}{2}\left(\partial_\sigma\partial_\beta h^\sigma_\alpha + \partial_\sigma\partial_\alpha h^\sigma_\beta - \partial_\alpha\partial_\beta h - \Box h_{\alpha\beta} - \eta_{\alpha\beta}\partial_\mu\partial_\nu h^{\mu\nu} + \eta_{\alpha\beta}\Box h\right),$$

from which the Einstein field equations are written as

$$\partial_\sigma\partial_\beta h^\sigma_\alpha + \partial_\sigma\partial_\alpha h^\sigma_\beta - \partial_\alpha\partial_\beta h - \Box h_{\alpha\beta} - \eta_{\alpha\beta}\partial_\mu\partial_\nu h^{\mu\nu} + \eta_{\alpha\beta}\Box h = -\frac{16\pi G}{c^4}T_{\alpha\beta}. \tag{2-14}$$

Because we are in the local wave zone we do not have any kind of local sources, then $T_{\alpha\beta} = 0$, therefore the equation (2-14) turns into

$$\partial_\sigma\partial_\beta h^\sigma_\alpha + \partial_\sigma\partial_\alpha h^\sigma_\beta - \partial_\alpha\partial_\beta h - \Box h_{\alpha\beta} - \eta_{\alpha\beta}\partial_\mu\partial_\nu h^{\mu\nu} + \eta_{\alpha\beta}\Box h = 0.$$

2.3.2. The DeDonder gauge in linearized gravity

First we introduce the **trace-reversed gravitational field** as[2]

$$\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h, \tag{2-15}$$

we observe that $\bar{h} \equiv \eta^{\alpha\beta}\bar{h}_{\alpha\beta} = -h$. With this gravitational field we can rewrite the equations (2-14) as

$$\Box\, \bar{h}_{\alpha\beta} + \eta_{\alpha\beta}\partial^{\rho}\partial^{\sigma}\bar{h}_{\rho\sigma} - \partial^{\rho}\partial_{\beta}\bar{h}_{\alpha\rho} - \partial^{\rho}\partial_{\alpha}\bar{h}_{\rho\beta} = -\frac{16\pi G}{c^4}T_{\alpha\beta}. \tag{2-16}$$

In the DeDonder gauge condition we set the coordinate system to fulfill the wave equation in the background spacetime, in this case

$$\Box x^{\alpha} = 0$$

where $\Box = \eta^{\alpha\beta}\partial_{\alpha}\partial_{\beta}$. This condition can be written as follows[3]

$$\partial^{\beta}\bar{h}_{\alpha\beta} = 0. \tag{2-17}$$

With this the equation (2-16) can be written as

$$\Box\, \bar{h}_{\alpha\beta} = -\frac{16\pi G}{c^4}T_{\alpha\beta}, \tag{2-18}$$

where, for consistency

$$\partial^{\beta}T_{\alpha\beta} = 0.$$

If we take $T_{\alpha\beta} = 0$, in local wave zone region, then equation (2-18) is

$$\Box\bar{h}_{\alpha\beta} = 0.$$

Returning for a moment to the gauge transformation problem we can get a gauge transformation expression, just like in equation (2-12). Using the equations (2-12) and (2-15) we have the gauge transformation

$$\bar{h}^{(\epsilon)}_{\alpha\beta} = \bar{h}_{\alpha\beta} - \epsilon\left(\partial_{\alpha}\xi_{\beta} + \partial_{\beta}\xi_{\alpha} - \eta_{\alpha\beta}\partial_{\sigma}\xi^{\sigma}\right).$$

From (2-17)

$$\partial^{\beta}\bar{h}^{(\epsilon)}_{\alpha\beta} = \partial^{\beta}\bar{h}_{\alpha\beta} - \Box\xi_{\alpha},$$

but we must set

$$\partial^{\beta}\bar{h}^{(\epsilon)}_{\alpha\beta} = 0.$$

[2] This is a particularization of the radiation field in the Relaxed Einstein field equations that we will see in 3.

[3] See Appendix A.

Let $f_\alpha(x) = \partial^\beta \bar{h}_{\alpha\beta}(x)$, therefore the vector field ξ_α must fulfill that

$$\Box \xi_\alpha(x) = f_\alpha(x).$$

Let us denote $G(x)$ the Green's function of the d'Alembert operator, so that

$$\Box_x G(x-y) = \delta^4(x-y),$$

then the corresponding solution is

$$\xi_\alpha = \int G(x-y) f_\alpha(y) d^4x.$$

For the DeDonder gauge to hold, we set $\xi_\alpha = 0$.

2.3.3. The Transverse-Traceless gauge

Here we are going to see the different contributions of $h_{\alpha\beta}$, with this we mean the scalar, vector and tensor contribution of the linearized gravitational field. This contributions comes from h_{00}, h_{0i} and h_{ij}, where $i, j = 1, 2, 3$. Under a well-behaved boundary conditions we can set

$$h^{0\mu} = 0, \quad h^\mu_\mu = 0, \quad \eta^{\sigma\nu}\partial_\nu h_{\mu\sigma} = 0. \tag{2-19}$$

Therefore we have a tensor, and only spatial, contribution from $h_{\mu\nu}$. This is known as the TT (Transverse-Traceless) gauge and will be noted as $h^{TT}_{\mu\nu}$. The equation of motion is then

$$\Box \, h^{TT}_{\alpha\beta} = 0 \tag{2-20}$$

and its solution is given by

$$h^{TT}_{\alpha\beta} = A_{\alpha\beta} e^{ik_\sigma x^\sigma}, \tag{2-21}$$

where $A_{\alpha\beta}$ is a constant, symmetric two rank tensor which is traceless and purely spatial

$$\begin{aligned} A_{0\alpha} &= 0 \quad , \\ \eta^{\alpha\beta} A_{\alpha\beta} &= 0 \quad , \end{aligned}$$

and k_α is the wave vector. Replacing the solution (2-21) in (2-20) we found that

$$k_\sigma k^\sigma = 0,$$

then (2-21) is a solution to the linearized equations if the wave vector is null, for hence the gravitational waves propagate at speed of light. We still need to ensure that the solution is transverse, from (2-19)

$$k_\beta A^{\beta\alpha} = 0, \tag{2-22}$$

we say then that the wave vector is orthogonal to $A^{\alpha\beta}$.

If we assume a particular direction of propagation, that is taking k^α such that

$$k^\alpha = \left(\frac{\omega}{c}, 0, 0, \frac{\omega}{c}\right), \tag{2-23}$$

from (2-22) and the fact that $A_{\alpha\beta}$ is purely spatial

$$A_{3\beta} = 0.$$

Therefore for $A^{\alpha\beta}$ we have the components

$$A_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{12} & -A_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

these non-zero components characterize completely the linearized field in this case. For these amplitudes following notation is used: $A_{11} = h_+$ and $A_{12} = h_\times$, then

$$A_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2-24}$$

These are called the "plus", h_+, and the "cross", $h_\times$, polarizations of the wave.
Now we are going to define a tensor that will allow us to find the form of the wave in the TT gauge. First we introduce the tensor

$$P_{ij}\left(\hat{\boldsymbol{n}}\right) = \delta_{ij} - n_i n_j,$$

where $\hat{\boldsymbol{n}}$ is $h_{\alpha\beta}$ propagating direction. With this tensor we construct

$$\Lambda_{ij,kl}\left(\hat{\boldsymbol{n}}\right) = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl},$$

this tensor is a projector, this means that

$$\Lambda_{ij,kl}\Lambda_{kl,mn} = \Lambda_{ij,mn}.$$

Furthermore, it is transverse on all its indices

$$n^i\Lambda_{ij,kl} = n^j\Lambda_{ij,kl} = n^k\Lambda_{ij,kl} = n^l\Lambda_{ij,kl} = 0,$$

it is traceless with respect to the (i, j) and (k, l) indices

$$\Lambda_{ii,kl} = \Lambda_{ij,kk} = 0,$$

and is symmetric under the simultaneous exchange $(i, j) \longleftrightarrow (k, l)$. In terms of $\hat{\boldsymbol{n}}$, its explicit form is

$$\Lambda_{ij,kl} = \delta_{ik}\delta_{jl} - \frac{1}{2}\delta_{ij}\delta_{kl} - n_j n_l \delta_{ik} - n_i n_k \delta_{jl} + \frac{1}{2}n_k n_l \delta_{ij} + \frac{1}{2}n_i n_j \delta_{kl} + \frac{1}{2}n_i n_j n_k n_l.$$

The gravitational wave in the TT gauge is given in terms of the spatial components h_{ij} of $h_{\alpha\beta}$ by

$$h_{ij}^{TT} = \Lambda_{ij,kl} h_{kl}$$

For the study of the interaction of gravitational waves in linearized gravity with test masses, see [16].

Solution to the linearized equation

Here we start from the equation (2-18), this can be solved by the method of Green's function: if $G(\boldsymbol{x} - \boldsymbol{x}')$ is a solution of the equation

$$\Box_{\boldsymbol{x}} G(\boldsymbol{x} - \boldsymbol{x}') = \delta^4(\boldsymbol{x} - \boldsymbol{x}'), \tag{2-25}$$

where $\Box_{\boldsymbol{x}}$ is the d'Alembertian operator with derivatives taken with respect to the variable $\boldsymbol{x}$, then the corresponding solution of equation (2-18) is

$$\bar{h}_{\alpha\beta} = -\frac{16\pi G}{c^4} \int G(\boldsymbol{x} - \boldsymbol{x}') T_{\alpha\beta}(\boldsymbol{x}') d^4\boldsymbol{x}'.$$

The solution of equation (2-25) depends of the boundary conditions. Just as in electromagnetism, for a radiation problem the appropriate solution is the retarded Green's function[4]

$$G(\boldsymbol{x} - \boldsymbol{x}') = -\frac{1}{4\pi |\boldsymbol{x} - \boldsymbol{x}'|} \delta\left(x_{\mathrm{ret}}^0 - x'^0\right),$$

where $x'^0 = ct'$, $x_{\mathrm{ret}}^0 = ct_{\mathrm{ret}}$ and

$$t_{\mathrm{ret}} = t - \frac{|\boldsymbol{x} - \boldsymbol{x}'|}{c}$$

is called retarded time. Then the solution of equation (2-18) is

$$\bar{h}_{\alpha\beta} = \frac{4G}{c^4} \int \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} T_{\alpha\beta}\left(t - \frac{|\boldsymbol{x} - \boldsymbol{x}'|}{c}, \boldsymbol{x}'\right).$$

Outside the source we can put this solution in the TT gauge using the $\Lambda_{ij,kl}$ tensor

$$h_{ij}^{TT} = \Lambda_{ij,kl} \bar{h}_{kl},$$

therefore outside the source

$$h_{ij}^{TT} = \frac{4G}{c^4} \Lambda_{ij,kl}(\hat{\boldsymbol{n}}) \int \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} T_{\alpha\beta}\left(t - \frac{|\boldsymbol{x} - \boldsymbol{x}'|}{c}, \boldsymbol{x}'\right).$$

[4] For details see Box 6.5 from [20]

2.3.4. Energy of gravitational waves in linearized gravity

The non-location of energy of the gravitational field

In spite of there exist a mathematical tool to work with the energy of the gravitational field[5], there is not such thing as a local gravity energy-momentum. It does not curve the spacetime or serve as a source term on the Einstein's field equations.
One can always find in any given locality a frame of reference in which all Christoffel symbols disappear, but this does not mean that there is no curvature. No Γ's means no local gravitational field and no local gravitational field means no local gravitational energy momentum. We want to describe this now in a more mathematical way, then we are going to introduce one example of a coordinate system in which Γ's are zero, this is the Riemann normal coordinates. We pick an event $\mathcal{P}_0$ and a set of basis vectors $\{\boldsymbol{e}_\alpha(\mathcal{P}_0)\}$ such that we have the following properties

$$\boldsymbol{e}_\alpha(\mathcal{P}_0) = \left(\frac{\partial}{\partial x^\alpha}\right)_{\mathcal{P}_0} \tag{2-26}$$

$$\Gamma^\alpha_{\beta\gamma}(\mathcal{P}_0) = 0 \tag{2-27}$$

$$\partial_\mu \Gamma^\alpha_{\beta\gamma}(\mathcal{P}_0) = -\frac{1}{3}\left(R^\alpha_{\beta\gamma\mu} + R^\alpha_{\gamma\beta\mu}\right). \tag{2-28}$$

Then, just like is shown in equation (2-28), the fact that the Γ's are zero does not mean that there is no curvature.

Separation of gravitational radiation and background

Here we take again the form of our metric tensor just like in equation (2-9) but with the following condition: $|h_{\alpha\beta}| \ll 1$. The idea of this subsection is to discuss how to decide which part of $g_{\alpha\beta}$ is a contribution given by $\overline{g}_{\alpha\beta}$ or by $h_{\alpha\beta}$. Just like the splitting between the spacetime background and the linearized gravitational field, we can make a separation of scales. We begging considering an exact plane-wave solution to the Einstein field equations, this will allow us to compare the magnitude of the values when we make the separation of scales. With this separation of scales we can determine the scale contribution of $h_{\alpha\beta}$.

An exact plane-wave solution

We are going to consider an exact solution of a plane wave first, this to understand how the background curvature and the gravitational radiation curvature can be distinguished. The solution that we are going to consider is

$$\begin{aligned} ds^2 &= L^2\left(e^{2\beta}dx^2 + e^{-2\beta}dy^2\right) + dz^2 - c^2dt^2 \\ &= L^2\left(e^{2\beta}dx^2 + e^{-2\beta}dy^2\right) - dudv, \end{aligned}$$

[5] See section 2.3.4

where

$$u = ct - z,\ v = ct + z,\ L = L\left(u\right),\ \beta = \beta\left(u\right).$$

The function L is known as the background factor and β the wave factor.
First we are going to calculate the non-vanishing components of the Riemann tensor, for this we are going to use the variational principle. Let us consider the equation

$$\begin{aligned} S &= \frac{1}{2}\int g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}d\lambda \\ &= \int\left[\frac{1}{2}L^{2}\left(e^{2\beta}\dot{x}^{2} + e^{-2\beta}\dot{y}^{2}\right) - \dot{u}\dot{v}\right]d\lambda \end{aligned}$$

where $\dot{x} = dx/d\lambda$ where λ is a parameter along a curve. First vary $x\left(\lambda\right)$ keeping fixed the functions $y\left(\lambda\right)$, $u\left(\lambda\right)$ and $v\left(\lambda\right)$

$$\delta S = \int\left(L^{2}e^{2\beta}\dot{x}\right)\delta\dot{x}d\lambda = \int\left(L^{2}e^{2\beta}\dot{x}\right)^{\cdot}\delta x d\lambda = 0,$$

then

$$0 = \left(L^{2}e^{2\beta}\dot{x}\right)^{\cdot} = L^{2}e^{2\beta}\ddot{x} + \dot{x}\dot{u}\frac{\partial}{\partial u}\left(L^{2}e^{2\beta}\right).$$

In the same way for y, u and v

$$\begin{aligned} 0 &= L^{2}e^{-2\beta}\ddot{y} + \dot{y}\dot{u}\frac{\partial}{\partial u}\left(L^{2}e^{2\beta}\right) && \text{for } y, \\ 0 &= \ddot{v} + \frac{1}{2}\dot{x}^{2}\frac{\partial}{\partial u}\left(L^{2}e^{2\beta}\right) + \frac{1}{2}\dot{y}^{2}\frac{\partial}{\partial u}\left(L^{2}e^{2\beta}\right) && \text{for } u, \\ 0 &= \ddot{u} && \text{for } v. \end{aligned}$$

Let us denote with a prime the derivative with respect to u, then we can rewrite above equations

$$\begin{aligned} 0 &= \ddot{x} + 2\left(L^{-1}L' + \beta'\right)\dot{x}\dot{u} \\ 0 &= \ddot{y} + 2\left(L^{-1}L' - \beta'\right)\dot{y}\dot{u} \\ 0 &= \ddot{v} + L^{2}e^{2\beta}\left(L^{-1}L' + \beta'\right)\dot{x}^{2} + L^{2}e^{-2\beta}\left(L^{-1}L' - \beta'\right)\dot{y}^{2} \\ 0 &= \ddot{u} \end{aligned}$$

then the equations are now in the standard form of the geodesics equations, then

$$\begin{aligned} \Gamma^{x}_{ux} = \Gamma^{x}_{xu} &= \left(L^{-1}L' + \beta'\right), \\ \Gamma^{y}_{uy} = \Gamma^{y}_{yu} &= \left(L^{-1}L' - \beta'\right), \\ \Gamma^{v}_{xx} &= L^{2}e^{2\beta}\left(L^{-1}L' + \beta'\right), \\ \Gamma^{v}_{yy} &= L^{2}e^{-2\beta}\left(L^{-1}L' - \beta'\right), \end{aligned}$$

now writing the non-vanishing components of the Riemann tensor

$$\begin{aligned} R^{v}_{xux} &= \left(\Gamma^{v}_{xx}\right)' - \Gamma^{v}_{xx}\Gamma^{x}_{xu} \\ &= \left(L^2 e^{2\beta}\right)\left(\frac{L''}{L} + \beta'' + 2\frac{L'}{L}\beta' + \beta'^2\right) \end{aligned} \tag{2-29}$$

$$\begin{aligned} R^{v}_{yuy} &= \left(\Gamma^{v}_{yy}\right)' - \Gamma^{v}_{yy}\Gamma^{y}_{yu} \\ &= \left(L^2 e^{-2\beta}\right)\left(\frac{L''}{L} - \beta'' - 2\frac{L'}{L}\beta' + \beta'^2\right) \end{aligned} \tag{2-30}$$

and with these ones we calculate the non-vanishing component of the Ricci tensor

$$R_{uu} = -2\left[L^{-1}L'' + \beta'^2\right].$$

If we are in vacuum

$$L'' + \beta'^2 L = 0, \tag{2-31}$$

the linearized version of this equation is $L'' = 0$, since β'^2 is a second order quantity. Because in linearized theory we have a flat background geometry then $L = 1$. The corresponding line element is

$$ds^2 = (1+2\beta)\,dx^2 + (1-2\beta)\,dy^2 + dz^2 - c^2dt^2,$$

from equation (2-24) we have that this is a plane wave of polarization "+".

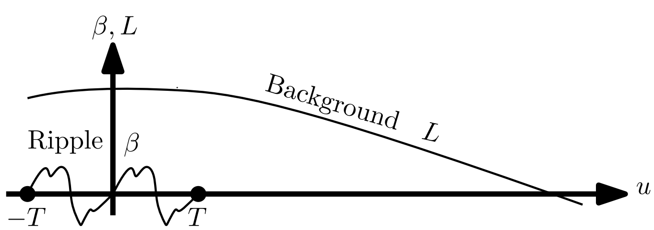

Figura 2-4.: Pulse profile for an exact plane wave solution to Einstein field equations. Traced figure from [17].

We are going to take the case where $\beta(u)$ is a pulse of duration $2T$, see figure **2-4**, and $|\beta'| \ll 1/T$ throughout the pulse. We are going to take three cases:

- For $u < -T$, we have a flat spacetime, then the pulse has not yet arrived. This means that

$$\beta = 0,\ \ L = 1$$

- For $-T < u < T$, here we are in the interior of the pulse. In this case we solve equation (2-31) and we use this solution recursively in such a way that

$$\beta = \beta(u) \ \text{ is arbitrary, except that } \ |\beta'| \ll 1/T$$
$$L(u) = 1 - \int_{-T}^{u}\left\{\int_{-T}^{\bar{u}} \left[\beta'\left(\bar{\bar{u}}\right)\right]^2 d\bar{\bar{u}}\right\} d\bar{u} + \mathcal{O}\left(\left[b'T\right]^4\right)$$

- For $u > T$, here the pulse has passed, then

$$\beta = 0,\ \ L = 1 - \frac{u}{a},\ \text{ where } a \equiv \frac{1 + \mathcal{O}\left([b'T]^2\right)}{\int_{-T}^{T} \left(\beta'\right)^2 du}$$

From equations (2-29) and (2-30) we have that these components vanish in any extended region where $\beta = 0$. Thus spacetime is completely flat in regions where the wave factor vanishes, which is outside the pulse. In particular, spacetime is flat near $u = a$, then here we have a coordinate singularity.

Let us consider now the line element

$$ds^2 = L^2\left(u\right)\left(dx^2 + dy^2\right) - dudv$$

which is always flat if satisfies the vacuum Einstein equations. In this metric the electromagnetic potential

$$\boldsymbol{A} = A_\alpha dx^\alpha = A\left(u\right)dx$$

satisfies Maxwell's equations for an arbitrary $A\left(u\right)$. The only nonzero field components of this wave are

$$F_{ux} = A'$$

or

$$F_{tx} = -F_{zx} = A',$$

then the electric vector oscillates back and forth in the x-direction, the magnetic vector oscillates in the y-direction. To calculate the momentum-energy tensor we use the following equation[6]

$$T^{\alpha\beta} = \frac{1}{4\pi}\left[F^{\alpha\mu}F^{\beta}_{\mu} - \frac{1}{4}\eta^{\alpha\beta}F_{\mu\nu}F^{\mu\nu}\right],$$

therefore in x, y, u, v coordinates the non-vanishing component of the momentum-energy tensor is

$$T_{uu} = \left(\frac{1}{4\pi L^2}\right)\left(A'\right)^2,$$

The Maxwell equations are satisfied by the potential A in the background metric. We impose the Einstein equations $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$, because always satisfies the vacuum Einstein equations the Einstein tensor is equal to the Ricci tensor. Because we imposed the Einstein equations then

$$R_{uu} = 8\pi\left(\frac{1}{4\pi L^2}\right)\left(A'\right)^2,$$

therefore in the vacuum

$$L'' + \left(4\pi T_{uu}\right)L = 0$$

[6] For a simpler comparison we are going to set the constant values $c = G = 1$, but later we are going to return to the values of c and G in the SI system.

which has exactly the form of the equation (2-31), then the electromagnetic plane wave and the gravitational plane wave produce the same gravitational attractions,

$$\left[\frac{(\beta')^2}{4\pi}\right]_{\text{Grav. Wave}} = [T_{uu}]_{\text{EM Wave}} = \frac{(A')^2}{4\pi L^2}.$$

We are going to hold the quantities $(\beta')^2/4\pi$ and $(A')^2/4\pi L^2$

$$\left\langle \frac{(\beta')^2}{4\pi} \right\rangle = \left\langle \frac{(A')^2}{4\pi L^2} \right\rangle = \text{const.}$$

with respect to this quantity we said that the reduce wavelength $\bar{\lambda}$ is small. With this is mind we are able to define the characteristic length of the background geometry, radius of curvature of background geometry, as

$$L_b \sim \left|\frac{L}{L''}\right|^{1/2}_{\substack{\text{Inside}\\ \text{the wave}}} \sim \frac{1}{|\beta'|},$$

from this we see that when there is no wave we return to the flat spacetime.

Scale contribution of linearized gravitational field

In a more general way, we find some coordinate system where we can write the metric tensor just like in (2-9), the term $\overline{g}_{\alpha\beta}$ has a typical scale of spatial variation L_b, on top of which small amplitude perturbations are superimposed, characterized by a reduced wavelength $\bar{\lambda}$ such that

$$\bar{\lambda} \ll L_b. \tag{2-32}$$

Then $h_{\alpha\beta}$ are, from a physical viewpoint, small ripples on a smooth background. This can be done also in the frequency space, if $\overline{g}_{\alpha\beta}$ has frequencies up to maximum value f_b, while $h_{\alpha\beta}$ is peaked around a frequency f such that

$$f \gg f_b. \tag{2-33}$$

In this case $h_{\alpha\beta}$ is a high-frequency perturbation of a static or slowly varying background. The scales L_b and f_b that characterize the background are unrelated, therefore we only need that one of (2-32) and (2-33) be satisfied.
Now we are going to see from the typical variation scales how the curvature is determined by gravitational waves and by matter[7]. First let us consider and expansion of the Ricci tensor to quadratic order in $h_{\alpha\beta}$[8],

$$R_{\alpha\beta} = \bar{R}_{\alpha\beta} + R^{(1)}_{\alpha\beta} + R^{(2)}_{\alpha\beta} + \cdots, \tag{2-34}$$

[7]Here we return to the usual values of c and G in the SI system
[8]For an extensive work of series expansion applied to tensors see HORTUA

where $\bar{R}_{\alpha\beta}$ is constructed with $\overline{g}_{\alpha\beta}$ only, $R^{(1)}_{\alpha\beta}$ is linear in $h_{\alpha\beta}$, $R^{(2)}_{\alpha\beta}$ is quadratic in $h_{\alpha\beta}$ and so on. We can calculate the expressions of $R^{(1)}_{\alpha\beta}$ and $R^{(2)}_{\alpha\beta}$ in terms of $h_{\alpha\beta}$, $\overline{g}_{\alpha\beta}$ and $\bar{\nabla}$, the covariant derivative with respect to the background[9]

$$R^{(1)}_{\alpha\beta} = \frac{1}{2}\left(\bar{\nabla}^{\mu}\bar{\nabla}_{\alpha}h_{\beta\mu} + \bar{\nabla}^{\mu}\bar{\nabla}_{\beta}h_{\alpha\mu} - \bar{\nabla}^{\mu}\bar{\nabla}_{\mu}h_{\alpha\beta} - \bar{\nabla}_{\beta}\bar{\nabla}_{\alpha}h\right) \tag{2-35}$$

$$\begin{aligned} R^{(2)}_{\alpha\beta} = \frac{1}{2}\overline{g}^{\rho\sigma}\overline{g}^{\mu\nu}\Bigg[&\frac{1}{2}\bar{\nabla}_{\alpha}h_{\rho\mu}\bar{\nabla}_{\beta}h_{\sigma\nu} + \left(\bar{\nabla}_{\rho}h_{\beta\mu}\right)\left(\bar{\nabla}_{\sigma}h_{\alpha\nu} - \bar{\nabla}_{\nu}h_{\alpha\sigma}\right) \\ &h_{\rho\mu}\left(\bar{\nabla}_{\beta}\bar{\nabla}_{\alpha}h_{\sigma\nu} + \bar{\nabla}_{\nu}\bar{\nabla}_{\sigma}h_{\alpha\beta} - \bar{\nabla}_{\nu}\bar{\nabla}_{\beta}h_{\alpha\sigma} - \bar{\nabla}_{\nu}\bar{\nabla}_{\alpha}h_{\beta\sigma}\right) \\ &\left(\frac{1}{2}\bar{\nabla}_{\mu}h_{\rho\sigma} - \bar{\nabla}_{\rho}h_{\mu\sigma}\right)\left(\bar{\nabla}_{\beta}h_{\alpha\nu} + \bar{\nabla}_{\alpha}h_{\beta\nu} - \bar{\nabla}_{\nu}h_{\alpha\beta}\right)\end{aligned} \tag{2-36}$$

and with this one finds that the order of magnitude of $\bar{R}_{\alpha\beta}$ in the absence of matter fields reads

$$\bar{R}_{\alpha\beta} \sim (\partial h)^2\,,$$

this means that the derivatives of $h_{\alpha\beta}$ affect the curvature of the background metric $\overline{g}_{\alpha\beta}$. The scale of variation of $\overline{g}_{\alpha\beta}$ is L_b and of $h_{\alpha\beta}$ is $\bar{\lambda}$, then

$$\partial\overline{g}_{\alpha\beta} \sim \frac{(\text{typical value of } \overline{g}_{\alpha\beta})}{L_b},$$

while

$$\partial h \sim \frac{(\text{typical value of } h_{\alpha\beta})}{\bar{\lambda}}. \tag{2-37}$$

The background expression $\bar{R}_{\alpha\beta}$ is constructed from the second derivatives of the background metric, then, if we set the typical values of $\overline{g}_{\alpha\beta}$ to one rescaling our coordinates

$$\bar{R}_{\alpha\beta} \sim \partial^2\overline{g}_{\alpha\beta} \sim \frac{1}{L_b^2},$$

while from (2-37)

$$(\partial h)^2 \sim \left(\frac{h}{\bar{\lambda}}\right)^2,$$

therefore we have the relation

$$\frac{1}{L_b^2} \sim \left(\frac{h}{\bar{\lambda}}\right)^2,$$

that is

$$h \sim \frac{\bar{\lambda}}{L_b},$$

[9] For details see MAGGIORE

which is the curvature determined by gravitational radiation. If we consider a non-vanishing energy-momentum tensor, the contribution of gravitational waves to the background curvature is negligible compared to the contribution of matter sources. Then the background curvature will be much bigger than the contribution of gravitational waves,

$$h \ll \frac{\bar{\lambda}}{L_b}.$$

How gravitational waves in linearized gravity curve the background

We take a coordinate system in which we can split the metric tensor, in a metric background and the linearized gravitational field. We can split the metric in this way because we have a clear distinction of scales, then one of the conditions, (2-32) or (2-33), applies. From the Einstein equations written in the form

$$R_{\alpha\beta} = \frac{8\pi G}{c^4}\left(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T\right),$$

where T is the trace of the energy-momentum tensor $T_{\alpha\beta}$. We take again the expansion of the Ricci tensor like is shown in the expression (2-34). We know the differences of frequencies between gravitational waves and background, then this allows to split the Einsteins field equations into two separate parts, the low and the high frequency parts,

$$\bar{R}_{\alpha\beta} = -\left[R^{(2)}_{\alpha\beta}\right]^{\text{Low}} + \frac{8\pi G}{c^4}\left(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T\right)^{\text{Low}}_{,} \tag{2-38}$$

$$R^{(1)}_{\alpha\beta} = -\left[R^{(2)}_{\alpha\beta}\right]^{\text{High}} + \frac{8\pi G}{c^4}\left(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T\right)^{\text{High}}_{,} \tag{2-39}$$

where “Low” denotes the projection on the low frequencies and “High” for the high ones. We can also work with the length scales if we want to.

Computation of averages

Given that the gravitational energy can not be located, an average is made over the tensor fields. Integrating a tensor field does not give a tensor in a curved space, because tensors at different points have different transformation properties. To be able to sum tensors in different points we must transport in parallel to add in a common point.
Let us introduce the bivector of geodesic parallel displacement, denoted by $\gamma_{\alpha}^{\beta'}(\boldsymbol{x}, \boldsymbol{x}')$. This transforms as a vector with respect to coordinate transformations at either $\boldsymbol{x}$ or $\boldsymbol{x}'$, and, assuming that $\boldsymbol{x}$ and $\boldsymbol{x}'$ are sufficiently close together to insure the existence of a unique geodesic of the metric $\bar{g}_{\alpha\beta}$, where $\bar{g}_{\alpha\beta}$ is the background geometry, between them, given the vector components $A_{\alpha'}$ at $\boldsymbol{x}'$, then $A_\alpha = \gamma_{\alpha}^{\alpha'} A_{\alpha'}$ is the unique vector at x which can be obtained by transporting in parallel $A_{\alpha'}$ from $\boldsymbol{x}'$ back to $\boldsymbol{x}$ along the geodesic.

Let $T_{\alpha\beta}$ be a tensor and let us assume that have high frequency components of wavelength λ and a background geometry $\bar{g}_{\alpha\beta}$ containing only low frequency components of wavelength $L \gg \lambda$, then we define the average of $T_{\alpha\beta}$ to be the tensor

$$\langle T_{\alpha\beta}(\boldsymbol{x})\rangle = \int_{\text{all space}} g_{\alpha}^{\alpha'}(\boldsymbol{x}, \boldsymbol{x}')\, g_{\beta}(\boldsymbol{x}, \boldsymbol{x}')\, T_{\alpha'\beta'}(\boldsymbol{x}')\, f(\boldsymbol{x}, \boldsymbol{x}')\, d^4x'$$

where $f(\boldsymbol{x}, \boldsymbol{x}')$ is a weighting function which falls smoothly to zero when $\boldsymbol{x}$ and $\boldsymbol{x}'$ differ by a distance d such that $\bar{\lambda} \ll d \ll L$ and

$$\int_{\text{all space}} f(\boldsymbol{x}, \boldsymbol{x}')\, d^4x' = 1$$

What we are doing here, in a more physical way, is to average over several wavelengths if (2-32) applies, or a temporal average over several periods of the gravitational waves if (2-33) applies. An advantage of this average is that the derivatives will average to zero,

$$\langle \partial_\mu A\rangle = 0,$$

then

$$\langle A(\partial_\mu B)\rangle = -\langle(\partial_\mu A)\, B\rangle .$$

With this in mind, we write equation (2-38) as

$$\bar{R}_{\alpha\beta} = -\left\langle R^{(2)}_{\alpha\beta}\right\rangle + \frac{8\pi G}{c^4}\left\langle T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T\right\rangle, \tag{2-40}$$

let us define the tensor $t_{\alpha\beta}$ such that

$$-\left\langle R^{(2)}_{\alpha\beta}\right\rangle = \frac{8\pi G}{c^4}\left(t_{\alpha\beta} - \frac{1}{2}\bar{g}_{\alpha\beta} t\right),$$

where $t = \bar{g}^{\alpha\beta} t_{\alpha\beta}$. Because $\bar{g}_{\alpha\beta}$ is a pure low-frequency quantity then $\left\langle \bar{g}^{\alpha\beta} R^{(2)}_{\alpha\beta}\right\rangle = \bar{g}^{\alpha\beta}\left\langle R^{(2)}_{\alpha\beta}\right\rangle$ and $\bar{g}^{\alpha\beta}\bar{g}_{\alpha\beta} = 4$, this allow us to find an expression for $t_{\alpha\beta}$

$$t_{\alpha\beta} = -\frac{c^4}{8\pi G}\left\langle R^{(2)}_{\alpha\beta} - \frac{1}{2}\bar{g}_{\alpha\beta} R^{(2)}\right\rangle \tag{2-41}$$

where $R^{(2)} = \bar{g}^{\alpha\beta} R^{(2)}_{\alpha\beta}$, the equation (2-41) is the energy-momentum tensor of gravitational waves in linearized gravity. We can rewrite equations (2-40) as

$$\bar{R}_{\alpha\beta} = \frac{8\pi G}{c^4}\left(t_{\alpha\beta} - \frac{1}{2}\bar{g}_{\alpha\beta} t\right) + \frac{8\pi G}{c^4}\left(T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T\right),$$

or in an equivalent way

$$\bar{R}_{\alpha\beta} - \frac{1}{2}\bar{g}_{\alpha\beta}\bar{R} = \frac{8\pi G}{c^4}\left(\bar{T}_{\alpha\beta} + t_{\alpha\beta}\right).$$

Due to the Bianchi identity we have

$$\bar{\nabla}^\beta\left(T_{\beta\alpha} + t_{\beta\alpha}\right) = 0,$$

and if $T_{\beta\alpha} = 0$ then

$$\bar{\nabla}^\beta t_{\beta\alpha} = 0. \tag{2-42}$$

The energy-momentum tensor of gravitational waves and the flux of energy and momentum

In above section we obtain an expression for the energy-momentum tensor of gravitational waves in linearized gravity, equation (2-41). Now we are going to return to the flat background metric splitting, equation (2-10), then in our equations (2-35) and (2-36) we replace $\bar{\nabla}_\alpha \to \partial_\alpha$. Applying the DeDonder gauge (2-17) and the TT gauge (2-19), the equation (2-36) is written as follows

$$\left\langle R^{(2)}_{\alpha\beta} \right\rangle = -\frac{1}{4} \left\langle \partial_\alpha h_{\mu\nu} \partial_\beta h^{\mu\nu} \right\rangle ,$$

therefore the equation (2-41) is now

$$t_{\alpha\beta} = \frac{c^4}{32\pi G} \left\langle \partial_\alpha h_{\mu\nu} \partial_\beta h^{\mu\nu} \right\rangle .$$

In particular the t_{00} component is then

$$t^{00} = \frac{c^2}{32\pi G} \left\langle \dot{h}^{TT}_{ij} \dot{h}^{TT}_{ij} \right\rangle ,$$

if we assume again the direction of the gravitational wave, just like in (2-23), we can write t_{00} in terms of h_+ and $h_\times$,

$$t^{00} = \frac{c^2}{16\pi G} \left\langle \dot{h}^2_+ + \dot{h}^2_\times \right\rangle .$$

Now we want to obtain an expression for the radiated power of the gravitational radiation. The gravitational wave energy inside the volume V is[10]

$$E_V = \int_V t^{00} d^3x,$$

from equation (2-42)

$$\int_V \left(\partial_0 t^{00} + \partial_i t^{i0} \right) d^3x = 0,$$

we have

$$\frac{1}{c}\frac{dE_v}{dt} = -\int_V \partial_i t^{i0} d^3x = -\int_S n_i t^{0i} dA$$

where n^i is the outer normal to the surface S and dA is the surface element. Taking its normal in the unitary radial direction

$$\frac{dE_v}{dt} = -\,c \int_S t^{0r} dA,$$

$$t^{0r} = \frac{c^2}{32\pi G} \left\langle \partial^0 h^{TT}_{ij} \partial_r h^{TT}_{ij} \right\rangle .$$

[10]see Chapter 3 to check the expression of energy and momentum

At sufficiently large distances r, just like in electrodynamics, the linearized gravitational field has the general form

$$h_{ij}^{TT}(t,r) = \frac{1}{r} f_{ij}\left(t - \frac{r}{c}\right)$$

then

$$\partial_r h_{ij}^{TT}(t,r) = -\frac{1}{r^2} f_{ij}\left(t - \frac{r}{c}\right) + \frac{1}{r}\partial_r f_{ij}\left(t - \frac{r}{c}\right).$$

On a function of the combination $t - r/c$ we have

$$\partial_r f_{ij}\left(t - \frac{r}{c}\right) = -c^{-1}\partial_t f_{ij}\left(t - \frac{r}{c}\right),$$

therefore

$$\begin{aligned}\partial_r h_{ij}^{TT}(t,r) &= -\partial_0 h_{ij}^{TT}(t,r) + O\left(1/r^2\right),\\ &= \partial^0 h_{ij}^{TT}(t,r) + O\left(1/r^2\right).\end{aligned}$$

From above equation, at large distances $t^{0r} = t^{00}$, therefore the energy inside a volume V satisfies

$$\frac{dE_v}{dt} = -c\int_S t^{00} dA,$$

the fact that E_V decreases means that the outward-propagating gravitational wave carries away an energy flux. Writing the surface element $dA = r^2 d\Omega$

$$\frac{dE}{dt} = \frac{c^3 r^2}{32\pi G}\int \left\langle \partial_t h_{ij}^{TT} \partial_t h_{ij}^{TT} \right\rangle d\Omega,$$

in terms of h^+ and $h^\times$

$$\frac{dE}{dt} = \frac{c^3 r^2}{16\pi G}\int \left\langle \left(\partial_t h^+\right)^2 + \left(\partial_t h^\times\right)^2 \right\rangle d\Omega. \tag{2-43}$$

In the same way we can compute the flux of momentum. The momentum of the gravitational wave inside a spherical shell V at large distances from the source is

$$P^i = \frac{1}{c}\int_V t^{0i} dx^3,$$

we consider, again, a propagating radially outward gravitational wave, in a similar way that we did with the energy flux

$$\begin{aligned}c\partial_0 P^i &= \int_V \partial_0 t^{0i} d^3x\\ &= -\int t^{0i} dA,\end{aligned}$$

therefore the momentum flux carried away by the gravitational wave along the radial direction is

$$\frac{dP^k}{dt} = -\frac{c^3 r^2}{32\pi G}\int \left\langle \partial_t h_{ij}^{TT} \partial^k h_{ij}^{TT} \right\rangle d\Omega.$$

2.3.5. Quadrupolar contribution of linearized gravity

In section 1 was discussed the dipolar contribution of radiation in electrodynamics. Here we will see from which contribution starts to radiate. It is a fact given the positive value nature of masses is not able to produce a dipole oscillating system just like in electrodynamics, then we can expect that the main contribution of radiation is the quadrupolar contribution. The idea is to see in an analytical way that we have such contribution.
Let us assume an isolated source that is far away from an observer and is slowly moving, this means that most of the radiation emitted will be at frequencies ω sufficiently low that if d is the source size then $d \lll \omega^{-1}$, or $d \lll \bar{\lambda}$. To perform the multipole expansion we start from the expression

$$h_{ij}^{TT} = \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij,kl}\left(\hat{\boldsymbol{n}}\right)\int T_{kl}\left(t-\frac{r}{c}+\frac{\boldsymbol{x}'\cdot\hat{\boldsymbol{n}}}{c},\boldsymbol{x}'\right)d^3x'$$

we write T_{kl} in terms of its Fourier transformation

$$T_{kl}\left(t-\frac{r}{c}+\frac{\boldsymbol{x}'\cdot\hat{\boldsymbol{n}}}{c},\boldsymbol{x}'\right) = \frac{1}{(2\pi)^4}\int \widetilde{T}_{kl}\left(\omega,\boldsymbol{k}\right)e^{-i\omega(t-r/c+\boldsymbol{x}'\cdot\hat{\boldsymbol{n}}/c)+i\boldsymbol{k}\cdot\boldsymbol{x}'}d^4k \tag{2-44}$$

given the radiation conditions, $\widetilde{T}_{kl}\left(\omega,\boldsymbol{k}\right)$ is peaked around a typical frequency ω with $\omega d \lll c$. On the other hand, the energy momentum tensor is non-vanishing only inside the source, so we restrict ourselves to $|\boldsymbol{x}'| \leq d$. Then the dominant contribution to h_{ij}^{TT} comes from frequencies ω that satisfy

$$\frac{\omega}{c}\boldsymbol{x}'\cdot\boldsymbol{n} \lesssim \frac{\omega d}{c} \lll 1,$$

and therefore we can expand the exponential in equation (2-44)

$$e^{-i\omega(t-r/c+\boldsymbol{x}'\cdot\hat{\boldsymbol{n}}/c)+i\boldsymbol{k}\cdot\boldsymbol{x}'} = e^{-i\omega(t-r/c)}\left[1-i\frac{\omega}{c}x'^i n^i+\frac{1}{2}\left(-i\frac{\omega}{c}\right)^2 x'^i x'^j n^i n^j+\cdots\right],$$

therefore

$$T_{kl}\left(t-\frac{r}{c}+\frac{\boldsymbol{x}'\cdot\hat{\boldsymbol{n}}}{c},\boldsymbol{x}'\right) \approx T_{kl}\left(t-\frac{r}{c},\boldsymbol{x}'\right)+\frac{x'^i n^i}{c}\partial_0 T_{kl}+\frac{1}{2c^2}x'^i x'^j n^i n^j \partial_0^2 T_{kl}+\cdots$$

where all derivatives are evaluated at the point $(t-r/c,\boldsymbol{x}')$. Let us define the momenta stress tensor of T^{ij} as

$$\begin{aligned}
S^{ij} &= \int T^{ij}\left(t,\boldsymbol{x}\right)d^3x \\
S^{ij,k} &= \int T^{ij}\left(t,\boldsymbol{x}\right)x^k d^3x \\
S^{ij,kl} &= \int T^{ij}\left(t,\boldsymbol{x}\right)x^k x^l d^3x \\
&\vdots
\end{aligned}$$

where the comma separates the spatial indices, which originates from T^{ij}, from the indices coming from x^i. Given the symmetry of T^{ij}, then $S^{ij,k} = S^{ji,k}$, but not necessarily is symmetric under the change of indices of different type, $S^{ij,k} \neq S^{ik,j}$.

$$h_{ij}^{TT} = \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij,kl}\left(\hat{\boldsymbol{n}}\right)\int T_{kl}\left(t-\frac{r}{c}+\frac{\boldsymbol{x}'\cdot\hat{\boldsymbol{n}}}{c},\boldsymbol{x}'\right)d^3x' \tag{2-45}$$

$$\approx \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij,kl}\left(\hat{\boldsymbol{n}}\right)\left[\int T^{kl}d^3x' + \frac{1}{c}n_m\partial_t\int T^{kl}x'^m d^3x' + \frac{1}{2c^2}n_m n_p\partial_t\partial_t\int T^{kl}x'^m x'^p d^3x' + \cdots\right]_{\text{ret}} \tag{2-46}$$

$$= \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij,kl}\left(\hat{\boldsymbol{n}}\right)\left[S^{kl} + \frac{1}{c}n_m\partial_t S^{kl,m} + \frac{1}{2c^2}n_m n_p\partial_t\partial_t S^{kl,mp} + \cdots\right]_{\text{ret}}, \tag{2-47}$$

where the subscript "ret" means that the quantities S^{kl}, $S^{kl,m}$, $S^{kl,ml}$, etc. are evaluated at the retarded time. Let us define the mass density as[11]

$$\begin{aligned}
M &= \frac{1}{c^2}\int T^{00}\left(t,\boldsymbol{x}\right)d^3x\\
M^i &= \frac{1}{c^2}\int T^{00}\left(t,\boldsymbol{x}\right)x^i d^3x\\
M^{ij} &= \frac{1}{c^2}\int T^{00}\left(t,\boldsymbol{x}\right)x^i x^j d^3x\\
M^{ijk} &= \frac{1}{c^2}\int T^{00}\left(t,\boldsymbol{x}\right)x^i x^j x^k d^3x\\
&\vdots
\end{aligned}$$

while the momentum density is given by[12]

$$\begin{aligned}
P^i &= \frac{1}{c}\int T^{0i}\left(t,\boldsymbol{x}\right)d^3x\\
P^{i,j} &= \frac{1}{c}\int T^{0i}\left(t,\boldsymbol{x}\right)x^j d^3x\\
P^{i,jk} &= \frac{1}{c}\int T^{0i}\left(t,\boldsymbol{x}\right)x^j x^k d^3x.\\
&\vdots
\end{aligned}$$

We are going to obtain identities to obtain the quadrupolar contribution. Given that $\partial_\beta T^{\alpha\beta} = 0$

$$\partial_0 T^{00} = -\partial_i T^{0i}$$

[11] This is a consequence of the expression of energy

[12] Just like it was mention before, the obtaining of this expression is obtained forward

we integrate in the hole space

$$c\partial_t M = \int \partial_0 T^{00} d^3x = -\int T^{0i} dS^i = -\int T^{0i} dS^i = 0,$$

this means that we are taking the boundary where there is no matter field, then in this boundary T^{0i} vanishes. In a similar way

$$\begin{aligned} c\partial_t M^i &= \int \partial_0 T^{00} x^i d^3x = -\int \partial_j T^{0j} x^i d^3x = \int T^{0j} \partial_j x^i d^3x = \int T^{0j} \delta^i_j d^3x \\ &= \int T^{0i} d^3x = cP^i, \end{aligned}$$

this process can be done for the higher order momentum, therefore

$$\partial_t M = 0 \tag{2-48}$$
$$\partial_t M^i = P^i \tag{2-49}$$
$$\partial_t M^{ij} = P^{i,j} + P^{j,i} \tag{2-50}$$
$$\partial_t M^{ijk} = P^{i,jk} + P^{j,ki} + P^{k,ij} \tag{2-51}$$

now for the case of the stress

$$c\partial_t P^i = \int \partial_0 T^{0i} d^3x = -\int T^{ji} dS^j = 0$$

just like the mass density, and

$$\begin{aligned} c\partial_t P^{i,j} &= \int \partial_0 T^{0i} x^j d^3x = -\int \partial_k T^{ki} x^j d^3x = \int T^{ki} \partial_k x^j d^3x = \int T^{ki} \delta^j_k d^3x \\ &= \int T^{ij} d^3x = S^{ij}, \end{aligned}$$

therefore

$$\partial_t P^i = 0 \tag{2-52}$$
$$\partial_t P^{i,j} = S^{ij} \tag{2-53}$$
$$\partial_t P^{i,jk} = S^{ij,k} + S^{ik,j}. \tag{2-54}$$

The equations $\partial_t M = 0$ and $\partial_t P^i = 0$ are the conservation of the mass and the conservation of the total momentum of the source. Derivating (2-50) with respect t, using (2-53) and the fact that $S^{ij} = S^{ji}$ then

$$S^{ij} = \frac{1}{2}\ddot{M}^{ij}. \tag{2-55}$$

We are going to use (refeq:sm) as the leading term in the expansion (2-47),

$$h^{TT}_{ij}(t, \boldsymbol{x}) \approx \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}(\hat{\boldsymbol{n}})\ddot{M}_{ij},$$

we decompose $\ddot{M}^{ij}$ in the following way

$$\ddot{M}^{ij} = \left(\ddot{M}^{ij} - \frac{1}{3}\delta^{ij} M_{kk}\right) + \frac{1}{3}\delta^{ij} M_{kk}, \tag{2-56}$$

where M_{kk} is the trace of M_{ij}. Because $\Lambda_{ij,kl}\delta^{kl} = \Lambda_{ij,kk} = 0$, we have contribution only from the traceless term of (2-56), let us denote this term as follows

$$Q^{ij} = \ddot{M}^{ij} - \frac{1}{3}\delta^{ij} M_{kk},$$

therefore

$$\begin{aligned} h_{ij}^{TT}(t, \boldsymbol{x}) &\approx \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}(\hat{\boldsymbol{n}})\, Q_{ij}\left(t - \frac{r}{c}\right) \\ &= \frac{1}{r}\frac{2G}{c^4} Q_{ij}^{TT}\left(t - \frac{r}{c}\right) \end{aligned}$$

which is the quadrupolar contribution.

3. Chapter 3: Relaxed Einsteins equations

In chapter 2 we discuss the nature of a background and a physical manifold, now we are going to write the Einsteins equations in such a way that are going to be helpful for the analysis of the measure of energy, linear and angular momentum carried by gravitational waves. The expressions obtained in this chapter are general expressions, this means that there is no assumptions over the gravitational field. Then a wave equation is obtained for the general case, which can be reduced, in the case of the linearized gravitational field, to the wave equation obtained in Chapter 2. Main references of this chapter [15, 20, 6, 26]

3.1. Landau-Lifshitz form of Einstein field equations

The rewriting of the Einsteins field equations arises from the fact that the equations

$$\nabla_\beta T^\beta_\alpha = \frac{1}{\sqrt{-g}}\partial_\beta\left(T^\beta_\alpha\sqrt{-g}\right) - \frac{1}{2}\left(\partial_\alpha g_{\beta\gamma}\right)T^{\beta\gamma} = 0 \tag{3-1}$$

do not generally express any conservation law whatever, we must include the conservation of the gravitational field. We choose a coordinate system such that at some particular point in spacetime all the first derivatives of $g_{\alpha\beta}$ vanish, so the equation (3-1) is written as

$$\partial_\beta T^{\alpha\beta} = 0.$$

We are going to show that we ca write $T^{\alpha\beta}$ in the following form

$$T^{\alpha\beta} = \partial_\gamma h^{\alpha\beta\gamma}, \tag{3-2}$$

where $h^{\alpha\beta\gamma}$ is such that

$$h^{\alpha\beta\gamma} = -h^{\alpha\gamma\beta}.$$

In order to do this we start from

$$T^{\alpha\beta} = \frac{c^4}{8\pi G}\left(R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R\right), \tag{3-3}$$

because we are in a particular coordinate system we write the tensor $R^{\alpha\beta}$ as

$$R^{\alpha\beta} = \frac{1}{2}g^{\alpha\sigma}g^{\beta\rho}g^{\delta\gamma}\left(\partial_\sigma\partial_\gamma g_{\delta\rho} + \partial_\delta\partial_\rho g_{\sigma\gamma} - \partial_\sigma\partial_\rho g_{\delta\gamma} - \partial_\delta\partial_\delta g_{\sigma\rho}\right),$$

therefore the equation (3-3) takes the form

$$T^{\alpha\beta} = \partial_\sigma \left\{ \frac{c^4}{16\pi G\,(-g)} \partial_\mu \left[(-g) \left(g^{\alpha\beta} g^{\sigma\mu} - g^{\alpha\sigma} g^{\beta\mu} \right) \right] \right\}.$$

Let us define

$$H^{\alpha\beta\mu\nu} = \frac{c^4}{16\pi G} (-g) \left(g^{\alpha\beta} g^{\mu\nu} - g^{\alpha\mu} g^{\beta\nu} \right), \tag{3-4}$$

therefore according to (3-2)

$$h^{\alpha\beta\mu} = \partial_\nu H^{\alpha\beta\mu\nu},$$

then we write our field equations as

$$\partial_\mu \partial_\nu H^{\alpha\beta\mu\nu} = (-g)\, T^{\alpha\beta}.$$

This relations face a problem, is that not allways we are in a coordinate system such that all the first derivatives of $g_{\alpha\beta}$ vanish, then in the general case

$$\partial_\mu \partial_\nu H^{\alpha\beta\mu\nu} - (-g)\, T^{\alpha\beta} \neq 0.$$

Let us denote this difference as $(-g)\, t_{LL}^{\alpha\beta}$, where $t_{LL}^{\alpha\beta}$ is symmetric in α and β, then

$$\partial_\mu \partial_\nu H^{\alpha\mu\beta\nu} = (-g) \left(T^{\alpha\beta} + t_{LL}^{\alpha\beta} \right), \tag{3-5}$$

from (3-3) and (3-4) we obtain an expression of $t^{\alpha\beta}$ in terms of $g^{\alpha\beta}$ and $\mathfrak{g}^{\alpha\beta}$, where

$$\mathfrak{g}^{\alpha\beta} = \sqrt{-g} g^{\alpha\beta}, \tag{3-6}$$

then

$$\begin{aligned} (-g)\, t_{LL}^{\alpha\beta} = \frac{c^4}{16\pi G} \Big\{ & \partial_\lambda \mathfrak{g}^{\alpha\beta} \partial_\mu \mathfrak{g}^{\lambda\mu} - \partial_\lambda \mathfrak{g}^{\alpha\lambda} \partial_\mu \mathfrak{g}^{\beta\mu} + \frac{1}{2} g^{\alpha\beta} g_{\lambda\mu} \partial_\rho \mathfrak{g}^{\lambda\nu} \partial_\nu \mathfrak{g}^{\mu\rho} \\ & g^{\alpha\lambda} g_{\mu\nu} \partial_\rho \mathfrak{g}^{\beta\nu} \partial_\lambda \mathfrak{g}^{\mu\rho} - g^{\beta\lambda} g_{\mu\nu} \partial_\rho \mathfrak{g}^{\alpha\nu} \partial_\lambda \mathfrak{g}^{\mu\rho} + g_{\lambda\mu} g^{\nu\rho} \partial_\nu \mathfrak{g}^{\alpha\lambda} \partial_\rho \mathfrak{g}^{\mu\beta} \\ & \frac{1}{8} \left(2 g^{\alpha\lambda} g^{\beta\mu} - g^{\alpha\beta} g^{\lambda\mu} \right) \left(2 g_{\nu\rho} g_{\sigma\tau} - g_{\rho\sigma} g_{\nu\tau} \right) \partial_\lambda \mathfrak{g}^{\nu\tau} \partial_\mu \mathfrak{g}^{\rho\sigma} \Big\}. \end{aligned}$$

This is known as the Landau-Lifshitz pseudotensor, and the expression (3-5) is known as the Landau-Lifshitz form of the Einsteins field equations. From equation (3-5) we have, finally, our conservation equation

$$\partial_\beta \left[(-g) \left(T^{\alpha\beta} + t^{\alpha\beta} \right) \right] = 0. \tag{3-7}$$

In the above analysis we supposed that we had a Minkowskian background metric, thats why appeared partial derivatives. In the case where we have a background metric $\bar{g}_{\alpha\beta}$ we must change the derivative respect to the flat background by the covariant derivative, $\partial_\mu \rightarrow \bar{\nabla}_\mu$.

3.2. Linear and angular momentum

3.2.1. In Flat spacetime

Let ξ^α be a Killing vector , therefore satisfies the Killing equation

$$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0,$$

we can build conserved quantities out of $T^{\alpha\beta}$, then

$$\begin{aligned}
\nabla_\beta \left(\xi_\alpha T^{\alpha\beta}\right) &= T^{\alpha\beta} \nabla_\beta \xi_\alpha + \xi_\alpha \nabla_\beta T^{\alpha\beta} \\
&= \frac{1}{2} T^{\alpha\beta} \nabla_\beta \xi_\alpha + \frac{1}{2} T^{\beta\alpha} \nabla_\alpha \xi_\beta \\
&= \frac{1}{2} \left(\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha\right) T^{\alpha\beta} = 0,
\end{aligned}$$

implying the conservation of

$$\int \xi_\alpha T^{\alpha\beta} dS_\beta \tag{3-8}$$

where we are integrating over any hypersurface. If we do this for the flat spacetime, we have the Killing equation

$$\partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha = 0,$$

which has the general solution

$$\xi_\alpha = a_\alpha + l_{\alpha\beta} x^\beta \tag{3-9}$$

where a_α and $l_{\alpha\beta}$ are constants with $l_{\alpha\beta} = -l_{\beta\alpha}$. From equation (3-8) the corresponding conserved quantity for ξ_α is

$$\int \left(a_\alpha + l_{\alpha\beta} x^\beta\right) T^{\alpha\sigma} dS_\sigma = a_\alpha P^\alpha - \frac{1}{2} l_{\alpha\beta} J^{\alpha\beta}$$

given that a_α and $l_{\alpha\beta}$ are constants, we set them in such a way that

$$\begin{aligned}
P^\alpha &= \frac{1}{c} \int T^{\alpha\beta} dS_\beta, \\
J^{\alpha\beta} &= \frac{1}{c} \int \left(x^\alpha T^{\beta\sigma} - x^\alpha T^{\beta\sigma}\right) dS_\sigma,
\end{aligned}$$

where P^α the total energy momentum four vector and $J^{\alpha\beta}$ the total angular momentum tensors, we see that these agrees with the expression obtained in Appendix B

3.2.2. General expressions

Equation (3-7) shows us a conservation law equation, from this we are going to obtain expressions for the linear and angular momentum. Let us define the quantities

$$P^\alpha = \frac{1}{c} \int (-g) \left(T^{\alpha\beta} + t^{\alpha\beta}\right) dS_\beta,$$

where S_β is an hypersurface, P^α this is known as the total four-momentum of matter plus gravitational field. This integration can be taken over any infinite hypersurface, including all of the three-dimensional space. If we choose for this the hypersurface x^0 = const, then P^α can be written in the form of a three-dimensional space integral

$$P^\alpha = \frac{1}{c}\int (-g)\left(T^{\alpha 0} + t^{\alpha 0}\right) dV. \tag{3-10}$$

The fact that $T^{\alpha\beta} + t^{\alpha\beta}$ has symmetric index implies that there is a conservation law for the angular momentum [1], which is defined as

$$J^{\alpha\beta} = \frac{1}{c}\int (-g)\left[x^\alpha\left(T^{\beta\sigma} + t^{\beta\sigma}\right) - x^\beta\left(T^{\alpha\sigma} + t^{\alpha\sigma}\right)\right] dS_\sigma.$$

As we saw in Chapter 2, the expression (3-10) is used to obtain the energy flux and the momentum flux for the gravitational radiation, therefore there must be and expression for the flux of angular momentum for the gravitational waves in the local wave zone, this expression was obtained in [26]. In the TT gauge this expression is

$$\frac{dJ^i}{dt} = \frac{r^2}{32\pi c^2}\int \epsilon^{ijk}\left(x_j\partial_k h_{ab} + 2\delta_{aj}h_{bk}\right)\partial_r h^{ab} d\Omega, \tag{3-11}$$

where ϵ^{ijk} is the three-dimensional Levi-Civita antisymmetric tensor. This expression is going to be useful in the next chapter.

3.3. Relaxed Einstein equations

3.3.1. Harmonic coordinates and a wave equation

We want to reach a wave equation form for de Einstein field equations, just like in (2-18), but without any assumptions over the dimensions of $g_{\alpha\beta}$. We define the field $\mathrm{h}^{\alpha\beta}$ such that

$$\mathrm{h}^{\alpha\beta} = \eta^{\alpha\beta} - \mathfrak{g}^{\alpha\beta} \tag{3-12}$$

this is indeed a expression that generalize our linearized gravitational field[2], if we assume that $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} + \mathcal{O}\left(h^2\right)$ then, taking $h_{\alpha\beta}$ only to linear order

$$\begin{aligned}\mathrm{h}^{\alpha\beta} &\approx \eta^{\alpha\beta} - \sqrt{1+h}\left(\eta^{\alpha\beta} - h^{\alpha\beta}\right)\\ &= h^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta}h,\end{aligned}$$

then $\mathrm{h}^{\alpha\beta}$ reduce to $\bar{h}^{\alpha\beta}$.

[1]See Appendix B

[2]For details of this discussion see [27]

We will write our wave equation using the Landau-Lifshitz form of Einstein field equations. We now impose the DeDonder gauge

$$\partial_\beta \mathrm{h}^{\alpha\beta} = 0, \tag{3-13}$$

calculating $H^{\alpha\mu\beta\nu}$ using (3-4), (3-12) and (3-6)

$$H^{\alpha\mu\beta\nu} = \left(\eta^{\alpha\beta}\eta^{\mu\nu} - \eta^{\alpha\beta}\mathrm{h}^{\mu\nu} - \mathrm{h}^{\alpha\beta}\eta^{\mu\nu} + \mathrm{h}^{\alpha\beta}\mathrm{h}^{\mu\nu}\right) - \left(\eta^{\alpha\nu}\eta^{\beta\mu} - \eta^{\alpha\nu}\mathrm{h}^{\beta\mu} - \mathrm{h}^{\alpha\nu}\eta^{\beta\mu} + \mathrm{h}^{\alpha\nu}\mathrm{h}^{\beta\mu}\right),$$

derivating with respect to x^ν and x^μ and using (3-13)

$$\partial_\mu\partial_\nu H^{\alpha\mu\beta\nu} = -\Box\mathrm{h}^{\alpha\beta} + \mathrm{h}^{\mu\nu}\partial_\mu\partial_\nu\mathrm{h}^{\alpha\beta} - \partial_\mu\mathrm{h}^{\alpha\nu}\partial_\nu\mathrm{h}^{\beta\mu},$$

let us define the pseudotensor $t_H^{\alpha\beta}$ such that

$$(-g)\, t_H^{\alpha\beta} = \frac{c^4}{16\pi G}\left(\partial_\mu h^{\alpha\nu}\partial_\nu h^{\beta\mu} - h^{\mu\nu}\partial_\mu\partial_\nu h^{\alpha\beta}\right),$$

and the pseudotensor $\tau^{\alpha\beta}$ in the following way

$$\tau^{\alpha\beta} = (-g)\left(T^{\alpha\beta} + t_{LL}^{\alpha\beta} + t_H^{\alpha\beta}\right),$$

this tensor is known as the **effective energy-momentum pseudotensor**. With this we have the following wave equation

$$\Box\mathrm{h}^{\alpha\beta} = -\frac{16\pi G}{c^4}\tau^{\alpha\beta}. \tag{3-14}$$

The pseudotensor $t_H^{\alpha\beta}$ have the property that $\partial_\beta\left[(-g)\, t_H^{\alpha\beta}\right] = 0$, this because, using (3-13)

$$\begin{aligned}\partial_\beta t_H^{\alpha\beta} &= \partial_\beta\partial_\mu\mathrm{h}^{\alpha\nu}\cdot\partial_\nu\mathrm{h}^{\beta\mu} - \partial_\beta\mathrm{h}^{\mu\nu}\partial_\mu\partial_\nu\mathrm{h}^{\alpha\beta}\\ &= \partial_\nu\partial_\mu\mathrm{h}^{\alpha\beta}\cdot\partial_\beta\mathrm{h}^{\nu\mu} - \partial_\beta\mathrm{h}^{\mu\nu}\partial_\mu\partial_\nu\mathrm{h}^{\alpha\beta}\\ &= 0,\end{aligned}$$

therefore

$$\partial_\beta\tau^{\alpha\beta} = 0. \tag{3-15}$$

The name of relaxed equations is because the equation (3-14) can be solved without enforcing the gauge condition (3-13) or the conservation statement (3-15).

3.3.2. Formal solution to the wave equation

As the linearized equation, this can be solved by the method of Green's function: if $G(\boldsymbol{x}-\boldsymbol{x}')$ is a solution of the equation

$$\Box_{\boldsymbol{x}} G\left(\boldsymbol{x}-\boldsymbol{x}'\right) = \delta^4\left(\boldsymbol{x}-\boldsymbol{x}'\right),$$

where $\Box_{\boldsymbol{x}}$ is the d'Alembertian operator with derivatives taken with respect to the variable $\boldsymbol{x}$, then the corresponding solution of equation (3-14) is

$$\mathrm{h}^{\alpha\beta} = \frac{4G}{c^4} \int G\left(\boldsymbol{x} - \boldsymbol{x}'\right) \tau^{\alpha\beta}\left(\boldsymbol{x}'\right) d^4\boldsymbol{x}'.$$

Just like in the linearized case, the appropriate solution is the retarded Green's function

$$G\left(\boldsymbol{x} - \boldsymbol{x}'\right) = -\frac{1}{4\pi\left|\boldsymbol{x} - \boldsymbol{x}'\right|}\delta\left(x^0_{\mathrm{ret}} - x'^0\right).$$

For details of the calculation of the Green's function see Box 6.5 of [20].

4. Chapter 4: Weyl tensor and gravitational radiation

In this chapter we return to the linearized gravitational field, but in this case we want to obtain an expression that depends of the origin of curvature which contributes to the linearized gravitational field. For this the tetrad concept is introduce, together with the Newman-Penrose formalism and the Weyl scalars. The final flux equations are obtained in terms of a particular Weyl scalar, which is the geometric origin of curvature. Main references for this chapter are [5, 1, 22]

4.1. Tetrads and Newman-Penrose formalism

4.1.1. Tetrads

Here we are going to define four linearly independent vectors on a local coordinate system as basis and name some properties of this vectors. We set up at each point of space-time a basis of four vectors

$$\boldsymbol{e}_{(a)} = e^{\alpha}_{(a)}\partial_{\alpha},$$

where $a = 0, 1, 2, 3$. This vectors are known as **tetrad basis**, the enclosure in parentheses distinguishes the tetrad indices from the tensor indices. Just like any other vector we have that

$$e_{(a)\alpha} = g_{\alpha\beta}e^{\beta}_{(a)}.$$

To define its inverse we have the following relations

$$e^{\alpha}_{(a)}e^{(b)}_{\alpha} = \delta^{(b)}_{(a)}, \quad e^{\alpha}_{(a)}e^{(a)}_{\beta} = \delta^{\alpha}_{\beta},$$

and we will also assume that

$$e^{\alpha}_{(a)}e_{(b)\alpha} = \eta_{(a)(b)}$$

where $\eta_{(a)(b)}$ is constant and symmetric.
We will not define yet the values of $\eta_{(a)(b)}$, we will only say that are constant. The inverse of $\eta_{(a)(b)}$ is defined in such a way that

$$\eta_{(a)(b)}\eta^{(b)(c)} = \delta^{(b)}_{(a)},$$

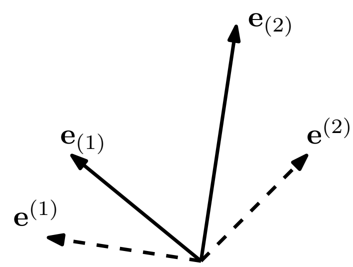

Figura 4-1.: Starting from the two vectors $\left(\boldsymbol{e}_{(1)}, \boldsymbol{e}_{(2)}\right)$, we construct a new vectors $\left(\boldsymbol{e}^{(1)}, \boldsymbol{e}^{(2)}\right)$ such that $\boldsymbol{e}_{(1)} \cdot \boldsymbol{e}^{(2)} = 0$, $\boldsymbol{e}_{(2)} \cdot \boldsymbol{e}^{(1)} = 0$, $\boldsymbol{e}_{(1)} \cdot \boldsymbol{e}^{(1)} = 0$ and $\boldsymbol{e}_{(2)} \cdot \boldsymbol{e}^{(2)} = 0$. Traced image from [1]

and as a consequence of the various definitions

$$\eta_{(a)(b)} e^{(b)}_{\alpha} = e_{(b)\alpha}, \quad \eta^{(a)(b)} e_{(b)\alpha} = e^{(a)}_{\alpha} \tag{4-1}$$

and

$$e_{(a)\alpha} e^{(a)}_{\beta} = g_{\alpha\beta}. \tag{4-2}$$

A graphical representation of the relation between tetrads is given in figure **4-1**.
Given any tensor field, we project it onto the tetrad frame to obtain its tetrad components. Thus

$$\begin{aligned}
A_{(a)} &= e_{(a)\alpha} A^{\alpha} = e^{\alpha}_{(a)} A_{\alpha}, \\
A^{(a)} &= \eta^{(a)(b)} A_{(b)} = e^{(a)}_{\alpha} A^{\alpha} = e^{(a)\alpha} A_{\alpha}, \\
A^{\alpha} &= e^{\alpha}_{(a)} A^{(a)} = e^{(a)\alpha} A_{(a)},
\end{aligned}$$

in a more general way we have

$$\begin{aligned}
T_{(a)(b)} &= e^{\alpha}_{(a)} e^{\beta}_{(b)} T_{\alpha\beta} = e^{\alpha}_{(a)} T_{\alpha(b)}, \\
T_{\alpha\beta} &= e^{(a)}_{\alpha} e^{(b)}_{\beta} T_{(a)(b)} = e^{(a)}_{\alpha} T_{(a)\beta}.
\end{aligned}$$

4.1.2. Newman-Penrose formalism

Here we have a special choice of basis vectors, these vectors are a tetrad $\boldsymbol{l}$, $\boldsymbol{n}$, $\boldsymbol{m}$ and $\boldsymbol{m}^*$, where $\boldsymbol{l}$ and $\boldsymbol{n}$ are real and $\boldsymbol{m}$ and $\boldsymbol{m}^*$ are complex conjugates of one another. These vectors are null and they satisfy the following orthogonality conditions

$$\boldsymbol{l} \cdot \boldsymbol{m} = \boldsymbol{l} \cdot \boldsymbol{m}^* = \boldsymbol{n} \cdot \boldsymbol{m} = \boldsymbol{n} \cdot \boldsymbol{m}^* = 0,$$

besides the requirements

$$\boldsymbol{l} \cdot \boldsymbol{l} = \boldsymbol{n} \cdot \boldsymbol{n} = \boldsymbol{m} \cdot \boldsymbol{m} = \boldsymbol{m}^* \cdot \boldsymbol{m}^* = 0.$$

The vectors fulfill the following normalizations conditions

$$\boldsymbol{l}\cdot\boldsymbol{m} = 1,$$
$$\boldsymbol{m}\cdot\boldsymbol{m}^* = -1.$$

The constructed vectors are given by

$$\boldsymbol{l} = \frac{1}{\sqrt{2}}\left(\boldsymbol{e}_{(0)} + \boldsymbol{e}_{(1)}\right) \tag{4-3}$$
$$\boldsymbol{n} = \frac{1}{\sqrt{2}}\left(\boldsymbol{e}_{(0)} - \boldsymbol{e}_{(1)}\right) \tag{4-4}$$
$$\boldsymbol{m} = \frac{1}{\sqrt{2}}\left(\boldsymbol{e}_{(2)} + i\boldsymbol{e}_{(3)}\right) \tag{4-5}$$
$$\boldsymbol{m}^* = \frac{1}{\sqrt{2}}\left(\boldsymbol{e}_{(2)} - i\boldsymbol{e}_{(3)}\right) \tag{4-6}$$

For $\eta_{(a)(b)}$, is defined in such a way that

$$\eta_{(a)(b)} = \eta^{(a)(b)} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The four vectors $(\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{m}, \boldsymbol{m}^*)$ form what is known as a null tetrad. Using (4-2)

$$g_{\alpha\beta} = -l_\alpha n_\beta - n_\alpha l_\beta + m_\alpha m^*_\beta + m^*_\alpha m_\beta.$$

4.2. The Weyl tensor

The Weyl tensor for a 4 dimensional manifold is defined as

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - \frac{1}{2}\left(g_{\alpha\gamma}R_{\beta\delta} + g_{\beta\delta}R_{\alpha\gamma} - g_{\beta\gamma}R_{\alpha\delta} - g_{\alpha\delta}R_{\beta\gamma}\right) + \frac{1}{6}R\left(g_{\alpha\gamma}g_{\beta\delta} - g_{\beta\gamma}g_{\alpha\delta}\right).$$

All contractions of the Weyl tensor vanish identically,

$$C^{\alpha}_{\beta\alpha\delta} = 0,$$

while its index symmetries are identically to those of the Riemann tensor,

$$C_{\alpha\beta\gamma\delta} = -C_{\beta\alpha\gamma\delta} = C_{\alpha\beta\delta\gamma} = C_{\gamma\delta\alpha\beta},$$
$$C_{\alpha\beta\gamma\delta} + C_{\alpha\beta\gamma\delta} + C_{\alpha\beta\gamma\delta} = 0.$$

We can write it in its tetrad components

$$C_{(a)(b)(c)(d)} = R_{(a)(b)(c)(d)} + \frac{1}{2}\left(\eta_{(a)(c)}R_{(b)(d)} - \eta_{(b)(c)}R_{(a)(d)} - \eta_{(a)(d)}R_{(b)(c)} + \eta_{(b)(d)}R_{(a)(c)}\right)$$
$$+ \frac{1}{6}\left(\eta_{(a)(c)}\eta_{(b)(d)} - \eta_{(a)(d)}\eta_{(b)(c)}\right),$$

in the Newman-Penrose formalism, the ten independent components of the Weyl tensor are represented by the five complex scalars

$$\Psi_0 = C_{(0)(2)(0)(2)} = C_{\alpha\beta\gamma\delta} l^\alpha m^\beta l^\gamma m^\delta \tag{4-7}$$
$$\Psi_1 = C_{(0)(1)(0)(2)} = C_{\alpha\beta\gamma\delta} l^\alpha n^\beta l^\gamma m^\delta \tag{4-8}$$
$$\Psi_2 = C_{(0)(2)(3)(1)} = C_{\alpha\beta\gamma\delta} l^\alpha m^\beta m^{*\gamma} n^\delta \tag{4-9}$$
$$\Psi_3 = C_{(1)(2)(3)(2)} = C_{\alpha\beta\gamma\delta} l^\alpha n^\beta m^{*\gamma} n^\delta \tag{4-10}$$
$$\Psi_4 = C_{(1)(3)(1)(3)} = C_{\alpha\beta\gamma\delta} n^\alpha m^{*\beta} n^\gamma m^{*\delta}, \tag{4-11}$$

this scalars are useful, in particular Ψ_4, for the radiated energy and momentum expressions.

4.2.1. Radiated energy and momentum

Let us define the complex quantity H such that

$$H = h^+ - ih^\times,$$

we can write the average

$$\left\langle \left(\partial_t h^+\right)^2 + \left(\partial_t h^\times\right)^2 \right\rangle$$

in terms of H. We have that

$$\partial_\alpha H \partial_\beta H^* = \left[\partial_\alpha h^+ \partial_\beta h^+ + \partial_\alpha h^\times \partial_\beta h^\times\right] + i\left[\partial_\alpha h^+ \partial_\beta h^\times - \partial_\alpha h^\times \partial_\beta h^+\right],$$

we take the following case

$$\partial_t H \partial_r H^* = \left[\partial_t h^+ \partial_r h^+ + \partial_t h^\times \partial_r h^\times\right] + i\left[\partial_t h^+ \partial_r h^\times - \partial_t h^\times \partial_r h^+\right],$$

then

$$\mathrm{Re}\left(\partial_t H \partial_r H^*\right) = \partial_t h^+ \partial_r h^+ + \partial_t h^\times \partial_r h^\times, \tag{4-12}$$

where $\mathrm{Re}\left(\cdots\right)$ is the real part of a complex number. In the second chapter we show that $\partial_r h_{ij}^{TT}\left(t,r\right) \approx -\partial_0 h_{ij}^{TT}\left(t,r\right)$ in the radiation zone, therefore

$$\begin{aligned} \left\langle \partial_t H \partial_r H^* \right\rangle &= \left\langle \partial_t H \partial_t H^* \right\rangle \\ &= \left\langle \left|\partial_t H\right|^2 \right\rangle \end{aligned}$$

and from (4-12)

$$\left\langle \left|\partial_t H\right|^2 \right\rangle = \left\langle \left(\partial_t h^+\right)^2 + \left(\partial_t h^\times\right)^2 \right\rangle.$$

This allow us to write equation (2-43) as

$$\frac{dE}{dt} = \frac{c^3 r^2}{16\pi G} \int \left\langle \left|\partial_t H\right|^2 \right\rangle d\Omega.$$

We construct a null tetrad from the orthonormal spherical basis using the equations (4-3), (4-4), (4-5) and (4-6)

$$
\begin{aligned}
l^{\alpha} &:= \frac{1}{\sqrt{2}}\left(e_t^{\alpha} + e_r^{\alpha}\right), \\
n^{\alpha} &:= \frac{1}{\sqrt{2}}\left(e_t^{\alpha} - e_r^{\alpha}\right), \\
m^{\alpha} &:= \frac{1}{\sqrt{2}}\left(e_\theta^{\alpha} + i e_\phi^{\alpha}\right), \\
m^{*\alpha} &:= \frac{1}{\sqrt{2}}\left(e_\theta^{\alpha} - i e_\phi^{\alpha}\right),
\end{aligned}
$$

where the vectors $\boldsymbol{e}_t$, $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_\phi$ are the usual orthonormal basis induced by the spherical coordinates. Calculating Ψ_4 from equation (4-11) and the null tetrad constructed above, knowing that there is no explicit dependence of t, θ and ϕ we obtain that

$$\Psi_4 = -\frac{1}{4}\left(\partial_t^2 h^{+} - 2\partial_t\partial_r h^{+} + \partial_r^2 h^{+}\right) + \frac{i}{4}\left(\partial_t^2 h^{\times} - 2\partial_t\partial_r h^{\times} + \partial_r^2 h^{\times}\right),$$

given that we are in the local wave zone then $\partial_r h = -\partial_0 h$, therefore

$$\Psi_4 = -\partial_t^2 h^{+} + i\partial_t^2 h^{\times} = -\partial_t^2 H.$$

This implies that for outgoing gravitational waves we can write

$$H = -\int_{-\infty}^{t}\int_{-\infty}^{t'} \Psi_4 dt'' dt', \tag{4-13}$$

then the total flux energy leaving the system is

$$\frac{dE}{dt} = \frac{c^3 r^2}{16\pi G}\int \left|\int_{-\infty}^{t} \Psi_4 dt'\right|^2 d\Omega.$$

Now we are going to write the flux of momentum along the radial direction, this expression is given in Chapter 2, writing it in term of H we have that

$$\frac{dP_i}{dt} = \frac{c^3 r^2}{16\pi G}\int k_i \left|\partial_t H\right|^2 d\Omega$$

where

$$k_i = \frac{\boldsymbol{x}}{r} = \left(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta\right),$$

for this it was used the fact that in the wave zone the angular dependence can be neglected, then $\partial_i H = \left(x_i/r\right)\partial_r H$ and that $\partial_r h = -\partial_0 h$. Then, from (4-13) we finally got

$$\frac{dP_i}{dt} = \frac{c^3 r^2}{16\pi G}\int k_i \left|\int_{-\infty}^{t} \Psi_4 dt'\right|^2 d\Omega.$$

Now we are going to calculate the flux of angular momentum, we are going to calculate it from the expression (3-11). We are going to rewrite this expression introducing the angular vectors $\boldsymbol{\xi}_i$ associated to rotations around the three coordinate axis. These vectors are Killing fields of the flat metric, and in Cartesian coordinates have components given by $\xi_i^k = \epsilon_i^{jk} x_j$[1], where ξ_i^k represents the k component of the vector $\boldsymbol{\xi}_i$. Let us write the flux of angular momentum in terms of the Lie derivative, for this we are going to use the expression

$$\left(\mathcal{L}_{\boldsymbol{\xi}_i}\boldsymbol{h}\right)_{\alpha\beta} = \xi_i^\sigma \partial_\sigma h_{\alpha\beta} + h_{\sigma\beta}\partial_\alpha \xi_i^\sigma + h_{\alpha\sigma}\partial_\beta \xi_i^\sigma, \tag{4-14}$$

given that we are in the local wave zone we restrict ourselves to the index $a, b, c, ...$ where these run from 1 to 2. If we multiply (4-14) by $\partial_r h^{ab}$ and sum over a and b, because $\boldsymbol{\xi}_i$ is a Killing vector then

$$h_{cb}\partial_a \xi_i^c \partial_r h^{ab} + h_{ac}\partial_b \xi_i^c \partial_r h^{ab} = 0,$$

and because we are in the gauge TT the term $\left(2\delta_{aj}\epsilon_i^{jk} h_{bj}\right)\partial_r h^{ab} = 0$. Then we can write the flux of angular momentum as

$$\frac{dJ_i}{dt} = -\frac{c^3 r^2}{32\pi G}\int \left(\mathcal{L}_{\boldsymbol{\xi}_i}\boldsymbol{h}\right)_{ab}\partial_r h^{ab} d\Omega. \tag{4-15}$$

Now we want to write (4-15) in terms of H, for this we are going to introduce spherical coordinates (r, θ, φ), this implies that

$$\begin{aligned}
\boldsymbol{\xi}_x &= (0, -\sin\varphi, -\cos\varphi\cot\theta) \\
\boldsymbol{\xi}_y &= (0, \cos\varphi, -\sin\varphi\cot\theta) \\
\boldsymbol{\xi}_z &= (0, 0, 1)\,,
\end{aligned}$$

because along $\boldsymbol{\xi}_z$ the Lie derivatives reduce to partial derivatives, we are going to introduce two angular vectors to calculate the Lie derivatives in the other directions

$$\begin{aligned}
\boldsymbol{\xi}_\pm &= \boldsymbol{\xi}_x \pm i\boldsymbol{\xi}_y \\
&= e^{\pm i\varphi}\left(0, \pm i, -\cot\theta\right).
\end{aligned}$$

Let us also introduce an orthonormal spherical basis $(\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\varphi)$ and two unit complex vectors

$$\begin{aligned}
\boldsymbol{e}_\pm &= \frac{1}{\sqrt{2}}\left(\boldsymbol{e}_\theta \pm i\boldsymbol{e}_\varphi\right) \\
&= \frac{1}{r\sqrt{2}}\left(0, 1, \mp i\csc\theta\right).
\end{aligned}$$

[1]This can be view from the general solution of the Killing equation in flat spacetime, from eq. (3-9) we take $a_\alpha = 0$ and particular values for $l_{\alpha\beta}$ such that we obtain this form of the vector.

Let us calculate the Lie derivative of $\boldsymbol{e}_\pm$ along $\boldsymbol{\xi}_\pm$, because these are vectors we calculate the Lie derivative with the Lie brackets

$$\begin{aligned}
\mathcal{L}_{\boldsymbol{\xi}_\pm}\boldsymbol{e}_\pm =& \left[\boldsymbol{\xi}_\pm, \boldsymbol{e}_\pm\right] \\
=& \left[\pm ie^{\pm i\varphi}\partial_\theta - e^{\pm i\varphi}\cot\theta\partial_\varphi\right]\left(\frac{1}{r\sqrt{2}}\partial_\theta \mp \frac{i}{r\sqrt{2}}\csc\theta\right) \\
&- \left[\frac{1}{r\sqrt{2}}\partial_\theta \mp \frac{i}{r\sqrt{2}}\csc\theta\partial_\varphi\right]\left(\pm ie^{\pm i\varphi}\partial_\theta - e^{\pm i\varphi}\cot\theta\partial_\varphi\right),
\end{aligned}$$

calculating the derivatives and factoring we have

$$\left(\mathcal{L}_{\boldsymbol{\xi}_\pm}\boldsymbol{e}_\pm\right)^a = \mp\left(ie^{\pm i\varphi}\csc\theta\right)\left(\boldsymbol{e}_\pm\right)^a.$$

Now we write the linearized field h_{ab} in the TT gauge in terms of the orthonormal basis

$$\begin{aligned}
h_{ab} &= h^+\left[\left(\boldsymbol{e}_\theta\right)_a\left(\boldsymbol{e}_\theta\right)_b - \left(\boldsymbol{e}_\varphi\right)_a\left(\boldsymbol{e}_\varphi\right)_b\right] + h^\times\left[\left(\boldsymbol{e}_\theta\right)_a\left(\boldsymbol{e}_\varphi\right)_b - \left(\boldsymbol{e}_\varphi\right)_a\left(\boldsymbol{e}_\theta\right)_b\right] \\
&= \left(h^+ - ih^\times\right)\left(\boldsymbol{e}_-\right)_a\left(\boldsymbol{e}_-\right)_b + \left(h^+ + ih^\times\right)\left(\boldsymbol{e}_+\right)_a\left(\boldsymbol{e}_+\right)_b \\
&= H\left(\boldsymbol{e}_-\right)_a\left(\boldsymbol{e}_-\right)_b + \bar{H}\left(\boldsymbol{e}_+\right)_a\left(\boldsymbol{e}_+\right)_b,
\end{aligned}$$

with this we have

$$\begin{aligned}
\left(\mathcal{L}_{\boldsymbol{\xi}_\pm}\boldsymbol{h}\right)_{ab} =& H\left[\left(\mathcal{L}_{\boldsymbol{\xi}_\pm}\boldsymbol{e}_-\right)_a\left(\boldsymbol{e}_-\right)_b + \left(\boldsymbol{e}_-\right)_a\left(\mathcal{L}_{\boldsymbol{\xi}_\pm}\boldsymbol{e}_-\right)_b\right] + \bar{H}\left[\left(\mathcal{L}_{\boldsymbol{\xi}_\pm}\boldsymbol{e}_+\right)_a\left(\boldsymbol{e}_+\right)_b + \left(\boldsymbol{e}_+\right)_a\left(\mathcal{L}_{\boldsymbol{\xi}_\pm}\boldsymbol{e}_+\right)_b\right] \\
=& -2ie^{\pm i\varphi}\csc\theta\left(\boldsymbol{e}_-\right)_a\left(\boldsymbol{e}_-\right)_b H + 2ie^{\pm i\varphi}\csc\theta\left(\boldsymbol{e}_-\right)_a\left(\boldsymbol{e}_-\right)_b\bar{H}.
\end{aligned}$$

Let us define the following operators

$$\hat{J}_\pm = \xi^a_\pm\partial_a - ise^{\pm i\varphi}\csc\theta = e^{\pm i\varphi}\left(\pm i\partial_\theta - \cot\theta\partial_\varphi - is\csc\theta\right),$$

where s is known as the spin weight of the function[2], $s = -2$ for H and $s = 2$ for $\bar{H}$, then

$$\left(\mathcal{L}_{\boldsymbol{\xi}_\pm}\boldsymbol{h}\right)_{ab} = \left(\boldsymbol{e}_-\right)_a\left(\boldsymbol{e}_-\right)_b\hat{J}_\pm H + \left(\boldsymbol{e}_+\right)_a\left(\boldsymbol{e}_+\right)_b\hat{J}_\pm\bar{H}.$$

Calculating $\left(\mathcal{L}_{\boldsymbol{\xi}_\pm}\boldsymbol{h}\right)_{ab}\partial_t h^{ab}$

$$\left(\mathcal{L}_{\boldsymbol{\xi}_\pm}\boldsymbol{h}\right)_{ab}\partial_t h^{ab} = \left[\left(\boldsymbol{e}_-\right)_a\left(\boldsymbol{e}_-\right)_b\hat{J}_\pm H + \left(\boldsymbol{e}_+\right)_a\left(\boldsymbol{e}_+\right)_b\hat{J}_\pm\bar{H}.\right]\partial_t\left[H\left(\boldsymbol{e}_-\right)_a\left(\boldsymbol{e}_-\right)_b + \bar{H}\left(\boldsymbol{e}_+\right)_a\left(\boldsymbol{e}_+\right)_b\right],$$

because $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_\varphi$ are orthogonal $\left(\boldsymbol{e}_-\right)_a\left(\boldsymbol{e}_-\right)^a = \left(\boldsymbol{e}_+\right)_b\left(\boldsymbol{e}_+\right)^b = 0$ and $\left(\boldsymbol{e}_-\right)_a\left(\boldsymbol{e}_+\right)^a = \left(\boldsymbol{e}_+\right)_b\left(\boldsymbol{e}_-\right)^b = 1$, therefore

$$\left(\mathcal{L}_{\boldsymbol{\xi}_\pm}\boldsymbol{h}\right)_{ab}\partial_t h^{ab} = \hat{J}_\pm H\partial_t\bar{H} + \hat{J}_\pm\bar{H}\partial_t H = 2\mathrm{Re}\left\{\hat{J}_\pm H\partial_t\bar{H}\right\}. \tag{4-16}$$

From (4-16) and the properties of the Lie derivative

$$\hat{J}_+ H\partial_t\bar{H} + \hat{J}_+\bar{H}\partial_t H = \left(\mathcal{L}_{\boldsymbol{\xi}_x}\boldsymbol{h} + i\mathcal{L}_{\boldsymbol{\xi}_y}\boldsymbol{h}\right)_{ab}\partial_t h_{ab} \tag{4-17}$$

$$\hat{J}_- H\partial_t\bar{H} + \hat{J}_-\bar{H}\partial_t H = \left(\mathcal{L}_{\boldsymbol{\xi}_x}\boldsymbol{h} - i\mathcal{L}_{\boldsymbol{\xi}_y}\boldsymbol{h}\right)_{ab}\partial_t h_{ab} \tag{4-18}$$

[2] For details of the general theory where the spin weight of a function is used see [1] Appendix D

then making (4-17)+(4-18) and (4-17)-(4-18), and defining the operators

$$\hat{J}_x = \frac{\hat{J}_+ + \hat{J}_-}{2} \qquad\qquad \hat{J}_y = \frac{\hat{J}_+ - \hat{J}_-}{2}$$

we have

$$\left(\mathcal{L}_{\boldsymbol{\xi}_x}\boldsymbol{h}\right)_{ab}\partial_t h_{ab} = \hat{J}_x H\partial_t\bar{H} + \hat{J}_x\bar{H}\partial_t H = 2\mathrm{Re}\left\{\hat{J}_x H\partial_t\bar{H}\right\} \tag{4-19}$$

$$\left(\mathcal{L}_{\boldsymbol{\xi}_y}\boldsymbol{h}\right)_{ab}\partial_t h_{ab} = \hat{J}_y H\partial_t\bar{H} + \hat{J}_y\bar{H}\partial_t H = 2\mathrm{Re}\left\{\hat{J}_y H\partial_t\bar{H}\right\}. \tag{4-20}$$

From (4-19) and (4-20) we write (4-15) as

$$\frac{dJ_i}{dt} = \frac{c^3r^2}{16\pi G}\mathrm{Re}\left\{\int \hat{J}_i H\partial_t\bar{H}d\Omega\right\},$$

because (4-13)

$$\frac{dJ_i}{dt} = -\frac{c^3r^2}{16\pi G}\mathrm{Re}\left\{\int\left[\left(\int_{-\infty}^{t}\bar{\Psi}_4 dt'\right)\hat{J}_i\left(\int_{-\infty}^{t''}\int_{-\infty}^{t'}\Psi_4 dt''dt'\right)\right]d\Omega\right\}.$$

Therefore we obtained the flux of energy, linear and angular momentum for the gravitational radiation in terms of the Weyl scalar Ψ_4. The importance of the Weyl scalar is seen in Figure

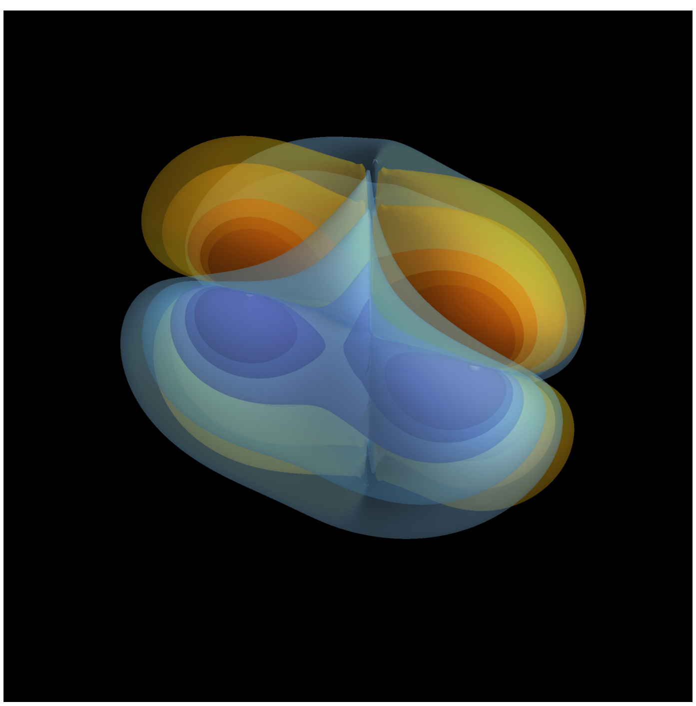

Figura 4-2.: Real part of Ψ_4 representing outgoing gravitational radiation. Image taken from *https://einsteintoolkit.org*

4-2. This is a simulation of *Einstein toolkit* code. Is made for a merger 36 + 29 solar mass binary black hole system. It is the Ψ_4 real part contribution of outgoing radiation in the wave zone. From this scalar we can characterize somehow the astrophysical systems and help to identify some aspects of them, for example, in gravitational collapse there is no gravitational radiation, this is because the Weyl tensor in this system is zero, therefore $\Psi_4 = 0$. For details see [17].

5. Conclusions and recommendations

5.1. Conclusions

It was understood the radiation conditions for the linearized gravity, this includes the comparison of the radiation zone in electrodynamics and in linearized gravity, the role of the background and physical manifold, how gauge transformations works and the energy contribution of gravitational radiation and the study of the characteristic length of the background and the radiation. The function of manifolds in gravitational radiation is usually omitted, just like why the energy has to be average and how it can be average. Gauge transformations are not usually study from the point of view of the diffeomorphism between manifolds and the characteristic length is usually just mention. The general equation for the study of gravitational radiation was deduced and the expressions of energy, lineal and angular momentum where obtained. Finally the contribution of the Weyl tensor in gravitational radiation was highlighted, this role is really important for the generation of gravitational radiation for astrophysical systems.

5.2. Recommendations

The idea is to apply this theory into an astrophysical system, be able to calculate the energy contribution of this system, and even try to simulate it. Make a comparison between a simulation made from linearized gravity point of view and the relaxed Einstein field equations, so it can be check at which point linearized gravity is a good approximation and study the Weyl tensor contribution for this system. There exists codes which helps in the analysis of this systems, a particular one is *Einstein Toolkit*[1], which use numerical relativity for its simulations. The study of this kind of codes could improve the knowledge in gravitational radiation for astrophysical systems.

[1]Einstein Toolkit website: https//einsteintoolkit.org

A. Appendix A: Harmonic coordinates

In this appendix first we will proof some properties of the Christoffel symbols and the metric tensor components. Then these properties are going to be use to arrived, from the harmonic coordinates, to the DeDonder gauge. Main reference for this appendix is [7].

A.1. Useful properties

The following properties are going to be useful for this appendix

Property 1

$$\partial_\sigma g^{\alpha\beta} = -g^{\alpha\mu} g^{\beta\nu} \partial_\sigma g_{\mu\nu}, \tag{A-1}$$

Proof: We start from

$$\begin{aligned}\partial_\sigma g^{\alpha\mu} \cdot g_{\mu\nu} + g^{\alpha\mu} \cdot \partial_\sigma g_{\mu\nu} &= \partial_\sigma \left(g^{\alpha\mu} g_{\mu\nu}\right) \\ &= \partial_\sigma \delta^\alpha_\nu \\ &= 0\end{aligned}$$

then

$$\partial_\sigma g^{\alpha\mu} \cdot g_{\mu\nu} = -g^{\alpha\mu} \cdot \partial_\sigma g_{\mu\nu},$$

multiplying by $g^{\beta\nu}$ and adding in ν, because $g^{\beta\nu} g_{\mu\nu} = \delta^\beta_\mu$, we obtain (A-1).

Property 2

$$\Gamma_{\alpha\beta\sigma} + \Gamma_{\beta\alpha\sigma} = \partial_\sigma g_{\alpha\beta}, \tag{A-2}$$

Proof: From the fact that

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\sigma} \left(\partial_\beta g_{\sigma\gamma} + \partial_\gamma g_{\sigma\beta} - \partial_\sigma g_{\beta\gamma}\right)$$

we have that

$$\Gamma_{\alpha\beta\sigma} = \frac{1}{2} \left(\partial_\beta g_{\alpha\sigma} + \partial_\sigma g_{\alpha\beta} - \partial_\alpha g_{\beta\sigma}\right)$$

and

$$\Gamma_{\beta\alpha\sigma} = \frac{1}{2}\left(\partial_\alpha g_{\beta\sigma} + \partial_\sigma g_{\beta\alpha} - \partial_\beta g_{\alpha\sigma}\right),$$

if we make $\Gamma_{\alpha\beta\sigma} + \Gamma_{\beta\alpha\sigma}$ we obtain (A-2).

Property 3

$$\Gamma^\beta_{\alpha\beta}\sqrt{-g} = \partial_\alpha\sqrt{-g}, \tag{A-3}$$

Proof: First we have to differentiate the determinant of the metric tensor g, we must differentiate each element $g_{\lambda\mu}$ in it and then multiply by the cofactor $gg^{\lambda u}$. Thus

$$\partial_\nu g = gg^{\lambda\mu}\partial_\nu g_{\lambda\mu}. \tag{A-4}$$

Now we are going to calculate $\Gamma^\mu_{\nu\mu}$

$$\begin{aligned}
\Gamma^\mu_{\nu\mu} &= \frac{1}{2}g^{\mu\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\nu\mu}\right) && \text{(A-5)}\\
&= \frac{1}{2}\left(g^{\mu\sigma}\partial_\mu g_{\sigma\nu} - g^{\mu\sigma}\partial_\sigma g_{\nu\mu} + g^{\mu\sigma}\partial_\nu g_{\sigma\mu}\right) && \text{(A-6)}\\
&= \frac{1}{2}g^{\mu\sigma}\partial_\nu g_{\sigma\mu}. && \text{(A-7)}
\end{aligned}$$

Let us write $g^{-1}\partial_\nu g$ in the following way

$$\begin{aligned}
g^{-1}\partial_\nu g &= g^{-1}\partial_\nu\left(\left[\sqrt{-g}\right]^2\right)\\
&= 2\left(\sqrt{-g}\right)^{-1}\partial_\nu\sqrt{-g}, && \text{(A-8)}
\end{aligned}$$

from (A-4) and (A-8)

$$\partial_\nu\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\lambda\mu}\partial_\nu g_{\lambda\mu}. \tag{A-9}$$

Using (A-9) in (A-7) we obtain (A-3).

A.2. The DeDonder Gauge

To understand harmonic coordinates we have to start from the d'Alembert equation for a scalar V, namely $\square V = 0$, in a curved spacetime

$$\square V = g^{\alpha\beta}\nabla_\alpha\nabla_\beta V,$$

then

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta V = g^{\alpha\beta}\nabla_\alpha\left(\partial_\beta V\right) = g^{\alpha\beta}\left(\partial_\alpha\partial_\beta V - \Gamma^\sigma_{\alpha\beta}\partial_\sigma V\right) = 0,$$

if we are using rectilinear axes in flat space, each of the four coordinates x^α satisfies $\Box x^\alpha = 0$. This impose a restriction over the coordinates, because x^α is not an scalar like V, so it holds only in certain coordinate system.
If we substitute x^α for V

$$\begin{aligned} g^{\alpha\beta}\left(\partial_\alpha \partial_\beta x^\lambda - \Gamma^\sigma_{\alpha\beta}\partial_\sigma x^\lambda\right) &= g^{\alpha\beta}\left(\partial_\alpha \delta^\lambda_\beta - \Gamma^\sigma_{\alpha\beta}\delta^\lambda_\sigma\right) \\ &= -\, g^{\alpha\beta}\Gamma^\lambda_{\alpha\beta} = 0, \end{aligned}$$

the coordinates that satisfies this condition are called **harmonic coordinates**. They provide the closest approximation to rectilinear coordinates that we can have in curved spacetime. Now, we are going to prove that the harmonic coordinate system is equivalent to set the DeDonder gauge $\partial_\beta\left(g^{\alpha\beta}\sqrt{-g}\right) = 0$, from equations (A-1) and (A-2)

$$\begin{aligned} \partial_\sigma g^{\alpha\beta} &= -\, g^{\alpha\mu}g^{\beta\nu}\left(\Gamma_{\mu\nu\sigma} + \Gamma_{\nu\mu\sigma}\right) \\ &= -\, g^{\alpha\mu}\Gamma^\beta_{\mu\sigma} - g^{\beta\nu}\Gamma^\alpha_{\nu\sigma}, \end{aligned}$$

from equation (A-2) and (A-3)

$$\begin{aligned} \partial_\sigma\left(g^{\alpha\beta}\sqrt{-g}\right) &= \sqrt{-g}\partial_\sigma g^{\alpha\beta} + g^{\alpha\beta}\partial_\sigma\sqrt{-g} \\ &= \sqrt{-g}\left(-g^{\alpha\mu}\Gamma^\beta_{\mu\sigma} - g^{\beta\nu}\Gamma^\alpha_{\nu\sigma} + g^{\alpha\beta}\Gamma^\mu_{\sigma\mu}\right), \end{aligned}$$

contracting σ and β

$$\begin{aligned} \partial_\beta\left(g^{\alpha\beta}\sqrt{-g}\right) &= \sqrt{-g}\left(-g^{\alpha\mu}\Gamma^\beta_{\mu\beta} - g^{\beta\nu}\Gamma^\alpha_{\nu\beta} + g^{\alpha\beta}\Gamma^\mu_{\beta\mu}\right) \\ &= -\, \sqrt{-g}g^{\beta\nu}\Gamma^\alpha_{\nu\beta}, \end{aligned}$$

because $g^{\alpha\beta}\Gamma^\lambda_{\alpha\beta} = 0$, then

$$\partial_\beta\left(g^{\alpha\beta}\sqrt{-g}\right) = 0$$

which is what we wanted to proof.

B. Appendix B: Relativistic Angular Momentum

In this appendix will be obtain the expression for the relativistic angular momentum for a closed particle system. Then a general system is consider with a general action integral, a momentum and angular momentum expression is obtained and the role of the angular momentum conservation in the symmetry of the energy-momentum tensor. Main reference is [15, 6].

B.1. For a closed particle system

Here we are going to obtain a relativistic expression for the angular momentum. We need to take into account that there is conservation of angular momentum in classical mechanics, so here we are goint to take a rotation and proof that there is an expression that is conserved under rotations in the four-dimensional space and this expression is the angular momentum. We also verify that this correspond to the angular momentum expression when we return to a classical system.

Let us take a system of several particles, let x^α be the coordinates of one of the particles of the system. We make an infinitesimal rotation in the four-dimensional space. Under a transformation, the coordinates x^α take on new values x'^α such that the differences $x'^\alpha - x^\alpha$ are linear functions

$$x'^\alpha - x^\alpha = x_\beta \delta\Omega^{\alpha\beta}, \tag{B-1}$$

where $\delta\Omega^{\alpha\beta}$ are infinitesimal coefficients. Under rotation, the length of the four-position vector must remain unchaged

$$x'_\alpha x'^\alpha = x_\alpha x^\alpha, \tag{B-2}$$

from equation (B-1) $x'^\alpha = x^\alpha + x_\beta \delta\Omega^{\alpha\beta}$, then

$$\begin{aligned} x'_\alpha x'^\alpha = &\left(x_\alpha + x_\beta \delta\Omega^\beta_\alpha\right)\left(x^\alpha + x_\beta \delta\Omega^{\alpha\beta}\right) \\ &= x_\alpha x^\alpha + x_\alpha x_\beta \delta\Omega^{\alpha\beta} + x^\alpha x_\beta \delta\Omega^\beta_\alpha + \mathcal{O}\left(\delta\Omega^2\right), \end{aligned}$$

because

$$x_\alpha x_\beta \delta\Omega^{\alpha\beta} = x^\alpha x_\beta \delta\Omega^\beta_\alpha = x^\alpha x^\beta \delta\Omega_{\alpha\beta}$$

we have that

$$x'_\alpha x'^\alpha = x_\alpha x^\alpha + 2x^\alpha x^\beta \delta\Omega_{\alpha\beta} + \mathcal{O}\left(\delta\Omega^2\right),$$

taking only the first orden contribution $\mathcal{O}\left(\delta\Omega^2\right) = 0$. From equation (B-2)

$$x^\alpha x^\beta \delta\Omega_{\alpha\beta} = 0,$$

since $x^\alpha x^\beta$ is a symmetric tensor $\delta\Omega_{\alpha\beta}$ must be antisymmetric, $\delta\Omega_{\alpha\beta} = -\delta\Omega_{\beta\alpha}$.
Now, we have that the action for a free material point is, (see [15])

$$S = -mc\int_a^b ds$$

then $\delta S = -mc\int_a^b ds = 0$, it can be shown that, (see [15])

$$\delta S = -p^\alpha \delta x_\alpha,$$

where p^α is the four-momentum. Adding over all particles in the system

$$\delta S = -\sum p^\alpha \delta x_\alpha\big|_a^b,$$

which goes from the event a to the event b, in the case of a rotation $\delta x_\alpha = \delta\Omega_{\alpha\beta}x^\beta$, therefore

$$\delta S = -\delta\Omega_{\alpha\beta}\sum p^\alpha x^\beta\big|_a^b.$$

We split $\sum p^\alpha x^\beta\big|_a^b$ into its symmetric and antisymmetric part

$$\begin{aligned}\delta S =& -\delta\Omega_{\alpha\beta}\left[\left(\sum p^\alpha x^\beta\big|_a^b\right)_{\text{symmetric}} + \left(\sum p^\alpha x^\beta\big|_a^b\right)_{\text{antisymmetric}}\right]\\ =& -\delta\Omega_{\alpha\beta}\left(\sum p^\alpha x^\beta\big|_a^b\right)_{\text{antisymmetric}},\end{aligned}$$

then

$$\delta\Omega_{\alpha\beta}\left[\frac{1}{2}\sum\left(p^\alpha x^\beta - p^\beta x^\alpha\right)\big|_a^b\right] = 0.$$

For a closed system, the action is not changed by a rotation in the four-space, this means that the coefficients $\delta\Omega_{\alpha\beta}$ must vanish, according to this

$$\sum\left(p^\alpha x^\beta - p^\beta x^\alpha\right)\big|_a = \sum\left(p^\alpha x^\beta - p^\beta x^\alpha\right)\big|_b.$$

Consequently, for a closed system the antisymmetric tensor of angular momentum $J^{\alpha\beta}$ is defined as

$$J^{\alpha\beta} = \sum\left(p^\alpha x^\beta - p^\beta x^\alpha\right).$$

We check that the spatial components of $J^{\alpha\beta}$ correspond to the classical angular momentum

$$J^{23} = J_x,\ J^{31} = J_y,\ J^{12} = J_z.$$

B.2. For a general system

Here we consider a general system whose action integral has the form

$$S = \int \Lambda \left(q, \partial_\alpha q\right) dV dt = \frac{1}{c} \int \Lambda \left(q, \partial_\alpha q\right) dx^4,$$

where Λ is some function of the quantities q, describing the state of the system, and of their first derivatives with respect to coordinates and time, we also have that

$$\int \Lambda dV$$

is the lagrangian of the system and Λ its lagrangian density. The equations of motion for the lagrangian density are given by

$$\frac{\partial}{\partial x^\alpha}\left(\frac{\partial \Lambda}{\partial\left(\partial_\alpha q\right)}\right) - \frac{\partial \Lambda}{\partial q} = 0, \tag{B-3}$$

we are going to use the fact that

$$\frac{\partial \Lambda}{\partial x^\alpha} = \frac{\partial \Lambda}{\partial q}\frac{\partial q}{\partial x^\alpha} + \frac{\partial \Lambda}{\partial\left(\partial_\beta q\right)}\frac{\partial\left(\partial_\beta q\right)}{\partial x^\alpha}, \tag{B-4}$$

if we substitute the term $\partial\Lambda/\partial q$ given by equation (B-3) in (B-4)

$$\frac{\partial \Lambda}{\partial x^\alpha} = \frac{\partial}{\partial x^\beta}\left(\frac{\partial \Lambda}{\partial\left(\partial_\beta q\right)}\right)\partial_\alpha q + \frac{\partial \Lambda}{\partial\left(\partial_\beta q\right)}\frac{\partial\left(\partial_\alpha q\right)}{\partial x^\beta} = \frac{\partial}{\partial x^\beta}\left(\partial_\alpha q \frac{\partial \Lambda}{\partial\left(\partial_\beta q\right)}\right). \tag{B-5}$$

On the other hand, we can write

$$\frac{\partial \Lambda}{\partial x^\alpha} = \delta_\alpha^\beta \frac{\partial \Lambda}{\partial x^\beta},$$

then we write equation (B-5) as

$$\frac{\partial}{\partial x^\beta}\left(\partial_\alpha q \frac{\partial \Lambda}{\partial\left(\partial_\beta q\right)} - \delta_\alpha^\beta \Lambda\right) = 0,$$

let us introduce the tensor T_α^β definde as

$$T_\alpha^\beta = \partial_\alpha q \frac{\partial \Lambda}{\partial\left(\partial_\beta q\right)} - \delta_\alpha^\beta \Lambda,$$

then

$$\frac{\partial T_\alpha^\beta}{\partial x^\beta} = 0$$

this is satisfied for several quantities $q^{(l)}$, then

$$T_\alpha^\beta = \sum_l \partial_\alpha q^{(l)} \frac{\partial \Lambda}{\partial\left(\partial_\beta q^{(l)}\right)} - \delta_\alpha^\beta \Lambda.$$

We have the four divergence of a vector equal to zero, then the integral over a hypersurface which contains all of three-dimensional space is conserved. Given the units of the lagrangian density T^{00} must be considered the energy density of the system, then $\int T^{00}dV$ is the total energy o the system. We can multiply our integral by a constant and the integral is still conserved, we set this constant to $1/c$, so P^0 is equal to the energy of the system multiplied by $1/c$, therefore we get four momentum of the system expression

$$P^{\alpha} = \frac{1}{c}\int T^{\alpha\beta}dS_{\beta},$$

which agrees with the expression obtained solving the Killing equation, see [6].
From the above section we have that we can define the angular momentum as

$$J^{\alpha\beta} = \frac{1}{c}\int \left(x^{\alpha}T^{\beta\sigma} - x^{\alpha}T^{\beta\sigma}\right)dS_{\sigma},$$

lets see that the law of conservation of angular momentum implies symmetric index in $T^{\alpha\beta}$, this law of conservation is given by

$$\partial_{\sigma}\left(x^{\alpha}T^{\beta\sigma} - x^{\alpha}T^{\beta\sigma}\right) = 0,$$

then

$$\begin{aligned}
\delta^{\alpha}_{\sigma}T^{\beta\sigma} + x^{\alpha}\partial_{\sigma}T^{\beta\sigma} - \delta^{\beta}_{\sigma}T^{\alpha\sigma} - x^{\beta}\partial_{\sigma}T^{\alpha\sigma} &= \delta^{\alpha}_{\sigma}T^{\beta\sigma} - \delta^{\beta}_{\sigma}T^{\alpha\sigma}\\
&= T^{\alpha\beta} - T^{\beta\alpha}\\
&=0,
\end{aligned}$$

therefore $T^{\alpha\beta} = T^{\beta\alpha}$.

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