About the restricted three-body problem with Schwarszchild de Sitter's potential

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1 Introduction

2 Preliminaries

2.1 Schwarszchild de Sitter's Potential

A De Sitter Universe is a cosmological solution to Einstein field equations of General Relativity which is named after Willem de Sitter. It models the universe as spatially flat and neglects ordinary matter, so the dynamics of the universe are dominated by the cosmological constant, thought to correspond to dark energy in our universe or the inflaton field in the early universe. According to the models of inflation and current observations of the accelerating universe, the concordance models of physical cosmology are converging on a consistent model where our universe was best described as a de Sitter universe at about a time $t = 10^{-33}$ seconds after the fiducial Big Bang singularity, and far into the future.

De Sitter space is the simplest solution of Einstein's equation with a positive cosmological constant. It is spherically symmetric and it has a cosmological horizon surrounding any observer, and describes an inflating universe. The Schwarszchild solution is the simplest spherically symmetric solution of the Einstein equations with zero cosmological constant, and it describes a black hole event horizon in otherwise empty space. A de Sitter-Schwarzschild is a combination of the two, and describes a black hole horizon spherically centered in an otherwise de Sitter universe. An observer which hasn't fallen into the black hole, and which can still see the black hole despite the inflation, is sandwiched between the two horizons.

The Schwarszchild de Sitter metric, is given by

$$ds^2 = c^2\left(1 - \frac{2GM}{c^2r} - \frac{\Lambda}{3}r^2\right)dt^2 - \left(1 - \frac{2GM}{c^2r} - -\frac{\Lambda}{3}r^2\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1}$$

where G is the universal gravitational constant, M is the mass of the filed source, c is the speed of light and Λ is the cosmological constant. Is known that the associated potential to this metric is given by

$$U(r) = \frac{k}{r} + \frac{B}{r^3} + Cr^2 \tag{2}$$

where $k = GM$, $B = \frac{\Lambda c^2}{6}$ and $C = \frac{GML^2}{c^2}$

2.2 Relative equilibrium in celestial mechanics

A body is in relative equilibrium respect to others when it is in a constant distance between them. In celestial mechanics, only is known a case where is presented this condition; the restricted three body problem. In this phenomena, two bodies are orbiting between themselves and, when is placed another object in certain points of space (Euler-Lagrange points), the object remains invariant respect rotations; i.e. its position is constant in a rotating system with an axis crossing the two bodies orbiting.

2.3 Linear stability

Let us consider a continuous system with ℓ degrees of freedom, described by a set of differential equations of the type

$$\dot{x} = f(x), \tag{3}$$

where $x \in \mathbb{R}^\ell$ and $f = (f_1, \ldots, f_\ell)$ is a vector function from $\mathbb{R}^\ell$ to itself. Then, be J the Jacobian matrix defined as

$$J = J(x) = Df(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_\ell} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_\ell}{\partial x_1} & \cdots & \frac{\partial f_\ell}{\partial x_\ell} \end{pmatrix}. \tag{4}$$

Now, consider an arbitrary point $x_0 \in cod(f)$. We say that x_0 is linearly stable if the eigenvalues $\lambda_j, j = 1, \ldots, \ell$ associated to its Jacobian matrix ((4) evaluated in x_0) satisfy $|\lambda_j| < 1$ for all $j = 1, \ldots, \ell$; and it is linearly unstable if for some j it is $|\lambda_j| > 1$

3 Approach to the restricted problem

Let us consider two bodies, m_1 and m_2, that interact mutually under Schwarszchild de Sitter, describing a circular orbit, and be m_3 the mass of a body with spherical symmetry such that $m_1, m_2 >> m_3$. Also, we assume that the center of mass of m_1, m_2 is fixed at the origin. As we consider m_1 and m_2 source of potential of type (2), that we rewrite as

$$U(r) = G\frac{m_1 m_2}{r}\left(1+\frac{B_1+B_2}{r^2}+(C_1+C_2)r^3\right), \quad (5)$$

interaction among masses m_1 and m_2 is given by the equation:

$$\left(\frac{m_1 m_2}{m_1+m_2}\right)\ddot{R} = -\frac{dU(R)}{dR}$$

$$= -\frac{d}{dR}\left(\frac{Gm_1 m_2}{R}\left(1+\frac{B_1+B_2}{R^2}+(C_1+C_2)R^3\right)\right),$$

i.e.

$$\left(\frac{m_1 m_2}{m_1+m_2}\right)\ddot{R} = -\frac{Gm_1 m_2}{R}\left(1+\frac{3(B_1+B_2)}{R^3} - 2R^2(C_1+C_2)\right).$$

As is supposed that m_1, m_2 are in an orbit with uniform circular movement, we have (R_0, ω). This is equivalent to find the equilibrium points of increased potential or effective potential.
Doing a rescaling, we consider $Gm_1 m_2 = 1$; then, the increased potential will be defined by

$$U_{aug}(R) = -\frac{1}{r}\left(1+\frac{B_1+B_2}{r^2}+(C_1+C_2)r^3\right)+\frac{r^2\omega^2}{2} \quad (6)$$

and effective potential as

$$U_{eff}(r) = -\frac{1}{r}\left(1+\frac{B_1+B_2}{r}+(C_1+C_2)r^3\right)+\frac{L^2}{2r^2} \quad (7)$$

Remember that equilibrium points are critical ones in the effective potential. So, operating and making $R = 1$ we have

$$\omega = \sqrt{1+3(B_1+B_2)-2(C_1+C_2)}. \quad (8)$$

Now, to guarantee orbit's stability, we use the fact that a critical point is further a minimal potential, namely, $U''_{eff}(R)|_{R=1} > 0$.

$$U''_{eff}(R)|_{R=1} = \Big[-\frac{2}{R^3}-12\frac{B_1+B_2}{R^4} - 2(C_1+C_2)+\frac{3L^2}{R^4}\Big]_{R=1} > 0 \quad (9)$$

and replacing (8) in (9) we get

$$-2-12(B_1+B_2)-2(C_1+C_2) + 3(1+3(B_1+B_2)-2(C_1+C_2)) > 0.$$

$$1 > 3(B_1+B_2)+8(C_1+C_2) \quad (10)$$

In the other way, the expression inside the root of (8) must be positive. So, another constraint for the coefficients is

$$1+3(B_1+B_2) \geq 2(C_1+C_2) \quad (11)$$

With (10) and (11), is possible to uncouple one pair of the coefficients:

$$\frac{1}{5} > C_1+C_2. \quad (12)$$

Also, in (10), since C_1 and C_2 are always non-negative, the other pair of coefficients is uncoupled:

$$\frac{1}{3} > B_1+B_2. \quad (13)$$

A particle's Hamiltonian in a central field is given by $H(r,\dot{r}) = \frac{1}{2}m\dot{r}^2 - U(r)$, then the Hamiltonian of m_3 in the inertial reference system is

$$H(r,\dot{r}) = \frac{1}{2}m\dot{r}^2 - \frac{(1-\mu)}{l_1}\left(1+\frac{B_1}{l_1^2}+C_1 l_1^3\right) - \frac{\mu}{l_2}\left(1+\frac{B_2}{l_2^2}+C_2 l_2^3\right). \quad (14)$$

Where

$$l_1 = \sqrt{(\xi+\mu)^2+\eta^2} \quad (15)$$

and

$$l_2 = \sqrt{(\xi+\mu-1)^2+\eta^2} \quad (16)$$

are the distances from the masses m_1, m_2 to the mass m_3, respectively.
Now, we name $m_1 = \mu$, located on ξ_1; and $m_2 = 1-\mu$, located on ξ_2. In this order, $\mu \leq \frac{1}{2}$, $\xi_1 - \xi_2 = 1$ and $\mu\xi_2 + (1-\mu)\xi_1 = 0$. So, $\xi_1 = -\mu$ and $\xi_2 = 1-\mu$. Also,

$$m_1 = \begin{cases} x = -\mu\cos(\omega t) \\ y = -\mu\sin(\omega t) \end{cases} \quad (17)$$

and

$$m_2 = \begin{cases} x = (1-\mu)\cos(\omega t) \\ y = (1-\mu)\sin(\omega t), \end{cases} \quad (18)$$

as seen in the figure 1.
Lets consider (ξ,η) as the coordinates of m_3 in the non-inertial system; therefore, the interaction between the mases m_1 and m_2 with m_3 is given by the next potential:

$$U_{m_3}(\xi,\eta) = \frac{(1-\mu)}{l_1}\left(1+\frac{B_1}{l_1^2}+C_1 l_1^3\right)+\frac{\mu}{l_2}\left(1+\frac{B_2}{l_2^2}+C_2 l_2^3\right). \quad (19)$$

and the Hamiltonian for m_3 in the non-inertial system is

$$H(\xi,\eta,P_\xi,P_\eta) = \frac{1}{2}(P_\xi^2+P_\eta^2)+\omega(P_\xi\eta - P_\eta\xi) - U_{m_3}(\xi,\eta). \quad (20)$$

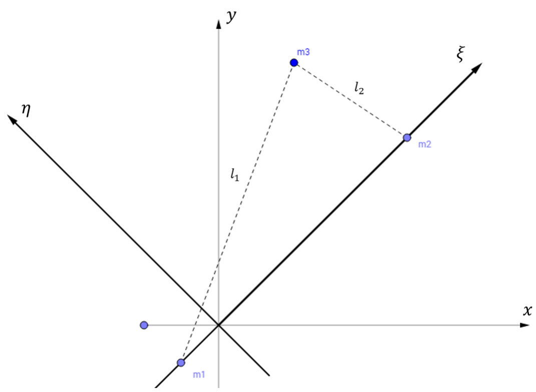

Figure 1: Representation of the restricted three body problem in the non-inertial system.

Applying the Hamilton's movement equations:

$$\boxed{\begin{aligned} \frac{\partial H}{\partial P_\xi} &= P_\xi + \omega\eta = \omega\dot{\xi}. && (21a)\\ \frac{\partial H}{\partial P_\eta} &= P_\eta - \omega\xi = \omega\dot{\eta}. && (21b)\end{aligned}}$$

Multiplying the equation (21a) by ω and deriving it respect to time, knowing that $\dot{\omega} = 0$, since the circular movement is uniform,

$$\omega\dot{P_\xi} = \omega^2(\ddot{\xi} - \dot{\eta})$$

$$\boxed{\dot{P_\xi} = \omega(\ddot{\xi} - \dot{\eta}).} \tag{22}$$

In an analogous way, multiplying the equation (21b) by ω and deriving it with respect to time,

$$\boxed{\dot{P_\eta} = \omega(\ddot{\eta} + \dot{\xi})} \tag{23}$$

Before continuing, the partial derivatives of U_{m_3} are going to be calculated, in order to facilitate the calculus of the other two Hamilton's movement equations:

$$\begin{aligned} \frac{\partial U_{m_3}(\xi,\eta)}{\partial \xi} &= (1-\mu)\frac{\partial l_1}{\partial \xi}\Big(-\frac{1}{l_1^2} - \frac{3B_1}{l_1^4} + 2C_1 l_1\Big) \\ &\quad + \mu\frac{\partial l_2}{\partial \xi}\Big(-\frac{1}{l_2^2} - \frac{3B_2}{l_2^4} + 2C_2 l_2\Big) \\ &= -\frac{(1-\mu)(\xi+\mu)}{l_1^3}\Big(1 + \frac{3B_1}{l_1^2} - 2C_1 l_1^3\Big) \\ &\quad - \frac{\mu(\xi+\mu-1)}{l_2^3}\Big(1 + \frac{3B_2}{l_2^2} - 2C_2 l_2^3\Big), \end{aligned} \tag{24}$$

in the other hand,

$$\begin{aligned} \frac{\partial U_{m_3}(\xi,\eta)}{\partial \eta} &= -\eta\Big[\frac{(1-\mu)}{l_1^3}\Big(1 + \frac{3B_1}{l_1^2} - 2C_1 l_1^3\Big) \\ &\quad + \frac{\mu}{l_2^3}\Big(1 + \frac{3B_2}{l_2^2} - 2C_2 l_2^3\Big)\Big]. \end{aligned} \tag{25}$$

By the last two Hamilton's movement equations:

$$\boxed{\begin{aligned} \frac{\partial H}{\partial \xi} &= -\omega\dot{P_\xi}. && (26a)\\ \frac{\partial H}{\partial \eta} &= -\omega\dot{P_\eta}. && (26b)\end{aligned}}$$

Replacing (20) in these equations gets,

$$-\omega P_\eta - \frac{\partial U_{m_3}}{\partial \xi} = -\omega\dot{P_\xi}. \tag{27a}$$

$$\omega P_\xi - \frac{\partial U_{m_3}}{\partial \eta} = -\omega\dot{P_\eta}. \tag{27b}$$

Therefore, using (22) and (23) in the last couple of equations, is obtained that

$$\omega^2(\ddot{\xi} - \dot{\eta}) = \omega P_\eta + \frac{\partial U_{m_3}}{\partial \xi}. \tag{28a}$$

$$\omega^2(\ddot{\eta} + \dot{\xi}) = -\omega P_\xi + \frac{\partial U_{m_3}}{\partial \eta}. \tag{28b}$$

Now, with the centrifuge potential

$$\boxed{\Omega(\xi,\eta) = \frac{\omega^2}{2}(\xi^2 + \eta^2) + U_{m_3}(\xi,\eta),} \tag{29}$$

one can use it to find the critical points of m_3 by deriving it with respect to ξ and μ and making it equal to zero. Before to do that, it should be considered the next equations:

$$\frac{\partial \Omega}{\partial \xi} = \omega^2(\ddot{\xi} - 2\dot{\eta}). \tag{30a}$$

$$\frac{\partial \Omega}{\partial \eta} = \omega^2(\ddot{\eta} + 2\dot{\xi}). \tag{30b}$$

obtained summing (28a)$-\omega\cdot$(21b) and $\omega\cdot$(21a)+(28b), respectively. With this pair of equations, is possible to deduce that the components (ξ,η) are orthogonal between them, but this is already known because of the nature of the problem and the coordinate axis. Consequently, the relation that is going to be used to find the critical points is

$$\frac{\partial \Omega}{\partial \xi} = \frac{\partial \Omega}{\partial \eta} = 0.$$

3.1 Collinear Stability Points

In order to obtain the collinear stability points, the partial derivative of Ω with respect to ξ is done, and all the η are replaced by zero. This gives the stability points that are in the ξ axis. After some algebra, one obtains that

$$\begin{aligned} &-\mu x^4[3B_2 + (x-1)^2] - (x-1)^4[3B_1(\mu-1) \\ &\quad + 2C_1 x^5(\mu-1) - 2C_2\mu x^4(x-1) \\ &\quad + \omega^2 x^4(\mu - x) - x^2(\mu-1)] = 0. \end{aligned} \tag{31}$$

Where $x = \xi + \mu$. Since (31) is a ninth grade polynom, it has at least a real solution.

3.2 Non-Collinear Stability Points ($\eta \neq 0$)

In this case, both partial derivatives of Ω are zero, but $\eta \neq 0$, so one has two equations, the derivative respect to ξ and η of (29). These two equations can be written as, respectively

$$\begin{aligned} 0 = & \frac{(1-\mu)(\xi+\mu)}{l_1^3}\Big(1+\frac{3B_1}{l_1^2}-2C_1l_1^3-\omega^2l_1^3\Big) \\ & + \frac{\mu(\xi+\mu-1)}{l_2^3}\Big(1+\frac{3B_2}{l_2^2}-2C_2l_2^3-\omega^2l_2^3\Big) \end{aligned} \tag{32}$$

and

$$\begin{aligned} 0 = \eta\Big[& \frac{(1-\mu)}{l_1^3}\Big(1+\frac{3B_1}{l_1^2}-2C_1l_1^3-\omega^2l_1^3\Big) \\ & + \frac{\mu}{l_2^3}\Big(1+\frac{3B_2}{l_2^2}-2C_2l_2^3-\omega^2l_2^3\Big)\Big] \end{aligned} \tag{33}$$

due to the fact that

$$(1-\mu)(\xi+\mu)+\mu(\xi+\mu-1)=\xi.$$

Considering l_1, l_2 as an independent system of variables, last two equations hold if and only if

$$(\omega^2+2C_i)l_i^5 - l_i^2 - 3B_i = 0, \tag{34}$$

for $i = 1,2$. Since (34) has a single change of sign, by the Descartes's rule of signs, each equation has exactly one positive root. The next proposition shows that these roots satisfy the triangle inequalities.

Definition 3.1 (Parameter domain)**.** *The set of all possible combinations of the non-negative parameters* (B_1,B_2,C_1,C_2) *that satisfies the constraints* (10)*,* (11)*,* (12) *and* (13) *will be called D.*

Proposition 3.1. *For every combination in D, there exists a unique non-collinear rotating equilibrium.*

Proof. There will be shown that every possible combination of D gives positive solutions in (34) that satisfy the triangle inequalities. It can be seen that l_1 and l_2 depend on the values of the constants in D, and moreover,

$$l_1 = l_1(B_1,B_2,C_1,C_2) = l_2(B_2,B_1,C_2,C_1) = l_2 \tag{35}$$

taking advantage of the symmetry in (34). Define $\bar{D}$ as the set D with its frontier, *i.e.*

$$\bar{D} = D \cup \delta D.$$

It is known that a differentiable real-valued function whose domain is closed and bounded attains its extreme values either at a critical point or on the boundary. In this context, the functions

$$\begin{array}{cccc} l_i: & \bar{D} & \to & \mathbb{R} \\ & (B_1,B_2,C_1,C_2) & \to & l_i = l_i(B_1,B_2,C_1,C_2) \end{array}$$

despite of being implicitly defined, are differentiable. A direct calculation proves that l_i doesn't accept critical points inside $\bar{D}$, so the extreme values of it must be in the frontier. All cases are shown below.

1. For $B_1 = 0$,
$$l_1 = \frac{1}{\sqrt[3]{1+3B_2-2C_2}}.$$
Given the constraints for the sum of two constants, it follows that $l_1^{\min} = \sqrt[3]{\frac{1}{2}} \approx 0.79$.

2. For $B_2 = 0$, the equation (34) becomes
$$l_1^5 - \frac{1}{(1+3B_1-2C_2)}l_1^2 - \frac{3B_1}{(1+3B_1-2C_2)} = 0.$$
To find a minimum bound, notice that the last polynomial can be rearranged as
$$l_1^2\Big(l_1^3 - \frac{1}{1+3B_1-2C_2}\Big) = \frac{3B_1}{1+3B_1-2C_2},$$
from where is deduced that
$$l_1 \geq \frac{1}{\sqrt[3]{1+3B_1-2C_2}} \geq \sqrt[3]{\frac{1}{2}} = l_1^{\min}.$$

3. For $C_1 = 0$, the minimum value for l_1 is given by the same arguments shown in the last case, so
$$l_1^{\min} = \sqrt[3]{\frac{1}{2}}.$$

4. For $C_2 = 0$, by similar reasons to the previous cases, it follows that
$$l_1^{\min} = \sqrt[3]{\frac{1}{2}}.$$

5. For $C_1 + C_2 = \frac{1}{5}$, equation (34) can be written as
$$\begin{aligned} 0 &= (1+3(B_1+B_2)-2(C_1+C_2)+2C_1)l_1^5 - l_1^2 - 3B_1 \\ &= (\frac{3}{5}+3(B_1+B_2)+2C_1)l_1^5 - l_1^2 - 3B_1. \end{aligned}$$
Calculating the derivative of the last polynomial expression with respect to C_1 and clearing dl_1/dC_1 yields to
$$\begin{aligned} \frac{dl_1}{dC_1} &= \frac{-6l_1^5}{5(3/5+3(B_1+B_2)+2C_1)l_1^4 - 2l_1} \\ &= \frac{-6l_1^6}{3l_1^2+15B_1} \\ &< 0, \end{aligned}$$
since $5(3/5+3(B_1+B_2)+2C_1)l_1^5 = 5l_2^2 + 15B_1$. This implies that the function $l_1(C_1)$ with its other variables fixed is decreasing on $C_1 + C_2 = 1/5$. Then, its minimum

is reached when C_1 is maximum. Therefore, if $C_1 = 1/5$, notice that the polynomial equation can be rearranged as

$$l_1^2\Big(l_1^3 - \frac{1}{1+3(B_1+B_2)}\Big) = \frac{3B_1}{1+3(B_1+B_2)},$$

from where is deduced that

$$l_1 \geq \frac{1}{\sqrt[3]{1+3(B_1+B_2)}} \geq \frac{1}{\sqrt[3]{2}} = l_1^{\min}.$$

6. For $B_1+B_2 = \frac{1}{3}$, equation (34) becomes

$$(2-2C_2)l_1^5 - l_1^2 - 3B_1 = 0.$$

Differentiating it respect to B_1 and clearing dl_1/dB_1 leads to

$$\begin{aligned}\frac{dl_1}{dB_1} &= \frac{3}{5(2-2C_2)l_1^4 - 2l_1} \\ &= \frac{3l_1}{5(2-2C_2)l_1^5 - 2l_1^2} \\ &= \frac{3l_1}{3l_1^2 + 15B_1} \\ &> 0,\end{aligned}$$

since $5(2-2C_2)l_1^5 = 5l_1^2 + 15B_1$. This implies that the function $l_1(B_1)$ with its other variables fixed is increasing on $B_1+B_2 = 1/3$. Then, its minimum value is reached when B_1 minimum. Therefore, when $B_1 = 0$,

$$l_1^{\min} = \sqrt[3]{\frac{1}{2}}.$$

7. For $1 = 3(B_1+B_2) + 8(C_1+C_2)$, one writes equation (34) as

$$l_1^3\Big(l_1^3 - \frac{1}{2-8C_1-10C_2}\Big) = \frac{3B_1}{2-8C_1-10C_2},$$

replacing $3(B_1+B_2)$ with $1-8(C_1+C_2)$. Using the same argument of the previous cases,

$$l_1 > \sqrt[3]{\frac{1}{2-8C_1-10C_2}} \geq \sqrt[3]{\frac{1}{2}} = l_1^{\min}.$$

Testing the triangular inequalities with $l_1^{\min} = \sqrt[3]{\frac{1}{2}}$, one gets that if $l_1^{\max}$ is in the vicinity

$$1 - \sqrt[3]{\frac{1}{2}} \leq l_1^{\max} \leq 1 + \sqrt[3]{\frac{1}{2}},$$

$l_1^{\max}$ and $l_1^{\min}$ satisfy the triangular inequalities. Therefore, a candidate to be an upper bound is $l_1^{\max} = 1 + \sqrt[3]{\frac{1}{2}}$. To show that is, in fact, a valid bound, notice that replacing $l_1 = l_1^{\max}$ in (34) yields to

$$\begin{aligned}&(\omega^2 + 2C_1)(l_1^{\max})^5 - (l_1^{\max})^2 - 3B_1 \\ &\geq \frac{3}{5}\Big(1 + \sqrt[3]{\frac{1}{2}}\Big)^5 - \Big(1 + \sqrt[3]{\frac{1}{2}}\Big)^2 - 1 \\ &> 0.\end{aligned}$$

Since the result is positive, independently of the constants's value, $l_1^{\max}$ is effectively an upper bound for the real root of (34), because the polynomial is positive only after the root.
By (35), l_1 and l_2 share the same minimum and maximum values, so every combination of constants

$$(B_1, B_2, C_1, C_2) \in D$$

raises solutions of (34) for l_1 and l_2 that satisfy the triangular inequalities, since their bounds satisfy them. $\square$

3.2.1 Isosceles Cases

The distances between the primaries was normalized to be one. Thus, a possible isosceles solution is when $l_i = 1$, and for that (34) raises the next condition:

$$3B_i = 2C_i, \tag{36}$$

and with this, equation (34) for $j \neq i$ becomes

$$(\omega^2 + 2C_j)l_j^5 - l_j^2 - 2C_j = 0.$$

Therefore, if (36) is hold, $l_i = 1$ and l_j is given by the last polynomial equation, that can be numerically solved in terms of ω^2 and C_j (see figure 2).
Another possible case is when $l_1 = l_2$; for where a sufficient condition of this happening is the trivial case when the bodies m_1 and m_2 have the same constants, and the same mass.

4 Stability

To study how is the movement near the equilibrium points in this problem, the Hamiltonian (20) is expanded through Taylor series around these points, the linear terms in this are omitted because the equilibrium points are zeroes in the potential and the constant term doesn't affect the form of the movement equation, so it is not taken into account. The Hamiltonian function rises the Hamiltonian matrix

$$\begin{pmatrix} 0 & \omega & 1 & 0 \\ -\omega & 0 & 0 & 1 \\ \dfrac{\partial^2 U_{m3}}{\partial \xi^2} & \dfrac{\partial^2 U_{m3}}{\partial \xi \partial \eta} & 0 & \omega \\ \dfrac{\partial^2 U_{m3}}{\partial \xi \partial \eta} & \dfrac{\partial^2 U_{m3}}{\partial \eta^2} & -\omega & 0 \end{pmatrix}, \tag{37}$$

whose eigenvalues determine the behavior of the linearized system. The characteristic equation reads

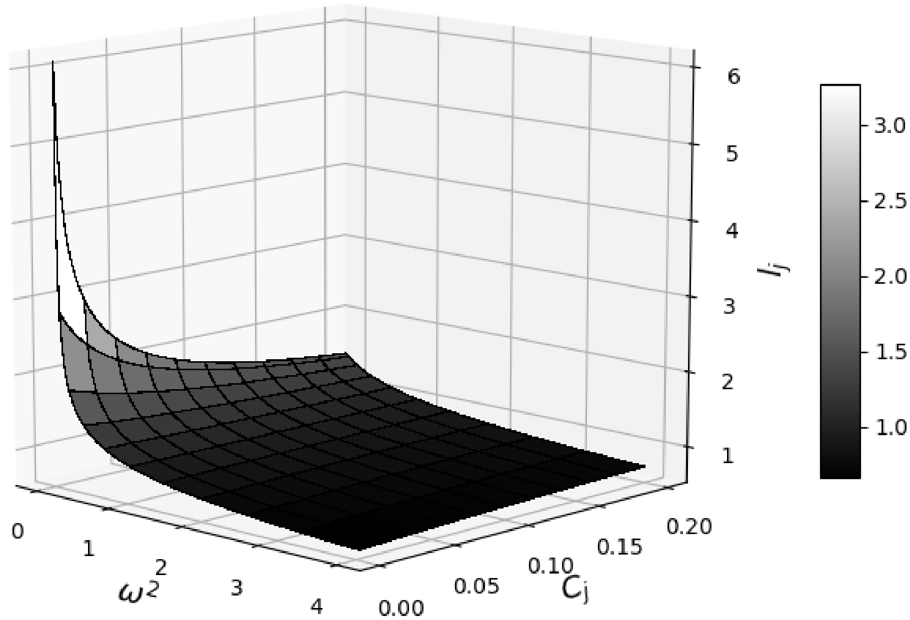

Figure 2: l_j in function as C_j and ω^2 when $l_i = 1$.

$$\lambda^4 + \Big(2\omega^2 - \frac{\partial^2 U_{m3}}{\partial \xi^2} - \frac{\partial^2 U_{m3}}{\partial \eta^2}\Big)\lambda^2 + \Big(\frac{\partial^2 U_{m3}}{\partial \xi^2} + \frac{\partial^2 U_{m3}}{\partial \eta^2}\Big)\omega^2 + \omega^4 - \Big(\frac{\partial^2 U_{m3}}{\partial \xi \eta}\Big)^2 + \frac{\partial^2 U_{m3}}{\partial \xi^2}\frac{\partial^2 U_{m3}}{\partial \eta^2} = 0. \tag{38}$$

The conditions that insure linear stability are given by the root of the quadratic formula,

$$G_1(B_1,B_2,C_1,C_2,\mu) \equiv \Big(2\omega^2 - \frac{\partial^2 U_{m3}}{\partial \xi^2} - \frac{\partial^2 U_{m3}}{\partial \eta^2}\Big)^2 - 4\Big(\Big(\frac{\partial^2 U_{m3}}{\partial \xi^2} + \frac{\partial^2 U_{m3}}{\partial \eta^2}\Big)\omega^2 + \omega^4 - \Big(\frac{\partial^2 U_{m3}}{\partial \xi \eta}\Big)^2 + \frac{\partial^2 U_{m3}}{\partial \xi^2}\frac{\partial^2 U_{m3}}{\partial \eta^2}\Big) > 0 \tag{39}$$

and by the sign of the part outside the root

$$G_1(B_1,B_2,C_1,C_2,\mu) \equiv 2\omega^2 - \frac{\partial^2 U_{m3}}{\partial \xi^2} - \frac{\partial^2 U_{m3}}{\partial \eta^2} > 0. \tag{40}$$

Both conditions must be fulfilled in order to have spectral stability. Five dimentions are needed to visualize the regions of the parameter domain and the values of μ for which exists spectral stability. One way to display the data in three dimentions is to make projections: fix B_1 and B_2 and graph μ_{crit} (the maximum value of μ that satisfies both conditions) as function of B_2 and C_2 (see figures 3 and 4).

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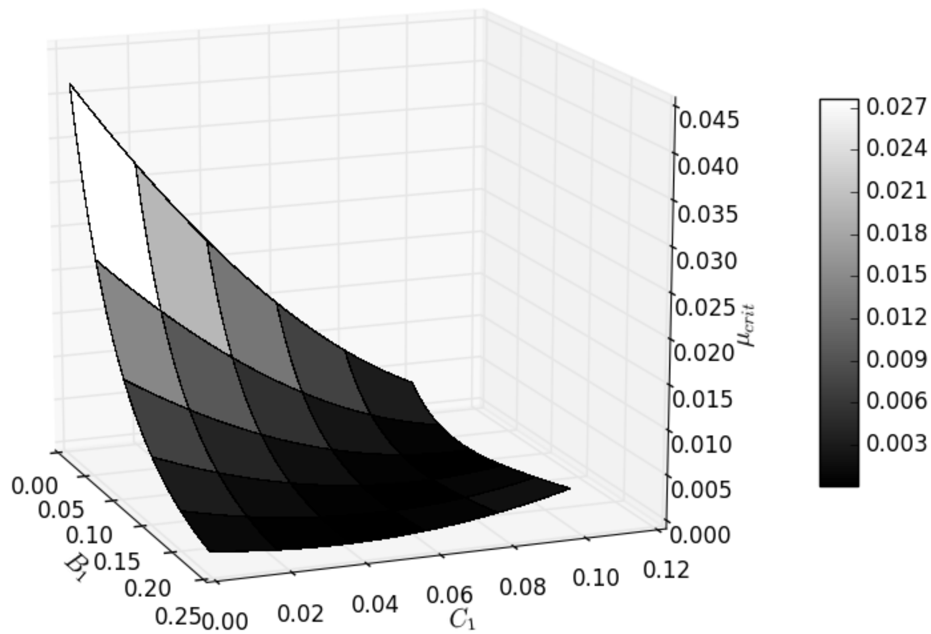

Figure 3: μ_{crit} as function of B_1 and C_1 when $B_2 = C_2 = 0.1$.

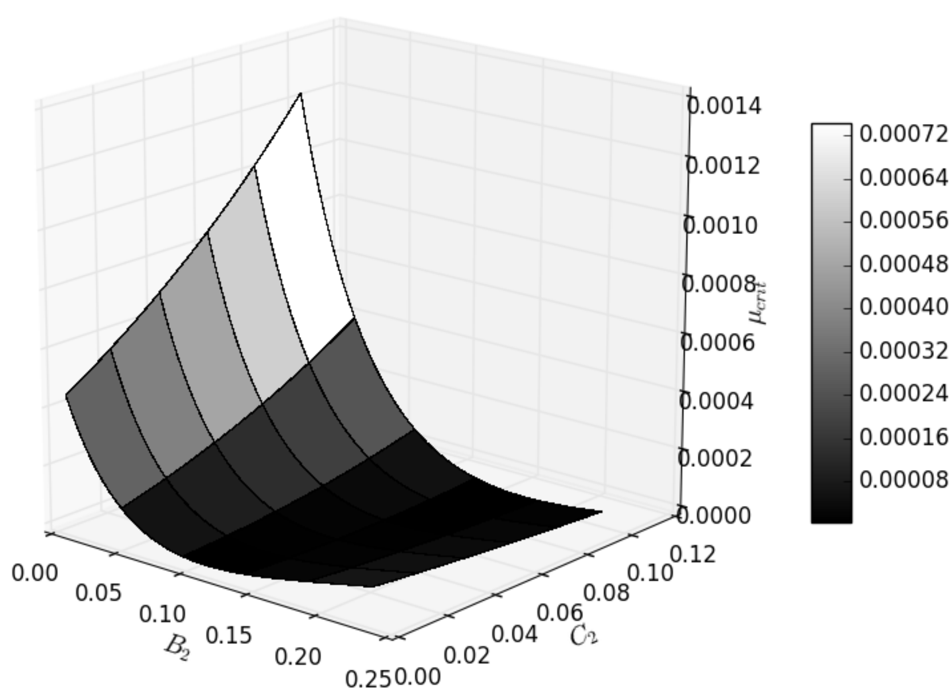

Figure 4: μ_{crit} as function of B_2 and C_2 when $B_1 = C_1 = 0.1$.