Harmonic Balance Method for Contact Problem in Lipschitz Continuous Form

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1 Introduction

There are numerous research and examples of using harmonic balance methods for studying the periodic response to harmonic excitations for nonlinear system. A classical example is the duffing equations which yield a coupling between the first order vibration and the third order vibration. However, for contact problems which also yield quasi-periodic phenomenon in many machineries such as turbo machineries, the nonlinear term cannot be projected onto the frequency domain easily. Because the contact scenario can only be described in Cartesian coordinates. (Von Groll and Ewins, 2001) (Laxalde et al., 2007) In computational applications, the idea of the AFT method is to use Discrete Fourier Transformation (DFT) to derive the Fourier components of the nonlinear forces for given displacements in the frequency domain. However, numerous problems also arise while applying the AFT method. First, it is computational intense. The time domain simulation can be very costly. And the convergence is not guaranteed.

Therefore, we are seeking for an alternative to the AFT which cope with the problem easily. Instead of transforming between time-domain and frequency-domain, we proposes to solve the problem in a fixed domain i.e. either in the finite time space or the finite Fourier space.

2 Signorini Contact Treatment

A standard description of the (static/dynamic) unilateral contact problem is the Signorini problem. It uses inequality to treat contact constraints.

$$u_i \leq 0, \lambda_i \leq 0, u_i\lambda_i = 0 \tag{1}$$

Fischer (1992) gave an equivalent expression to the contact constraints.

$$s(t) = \lambda(t) + \max\{0, c(u(t) - g) - \lambda(t)\} \equiv 0 \tag{2}$$

where $c > 0$. The new function $s(t)$ is a Lipschitz continuous, which allows for Newton-like algorithm. We would like to solve contact problem in this expression.

In this project, Galerkin-like methods were used to approximate the problem and to find the periodic solution. Two base functions are used – hat function and Fourier function.

3 Single-point Contact

The dynamics of the one-DOF mass-spring system $p(t) = 0$ and its equivalent contact constraint are in Lipschitz function $s(t) = 0$

$$p(t) = m\ddot{u}(t) + d\dot{u}(t) + ku(t) - \lambda(t) - f(t) \equiv 0 \tag{3}$$

$$s(t) = \lambda(t) + \max\{0, c(u(t) - g) - \lambda(t)\} \equiv 0 \tag{4}$$

where $c > 0$ is an arbitrary positive constant. $u(t)$ is the displacement, $\lambda(t)$ is the contact force. $f(t)$ is the prescribed harmonic external force. (t) is omitted hereafter.

3.1 Hat Functional space

Approximate the displacement using the hat function.

$$u(t) \approx \sum_{i=1}^{N} N_i(t)\bar{u}_i = \mathbf{N}(t)\bar{\mathbf{u}} \tag{5}$$

Where N is the time mesh size of one period. the upper bar of $\bar{\mathbf{u}}$ means it is the projection vector on the hat functional space. Contact force can be projected onto the hat function space in the same way.

$$\lambda(t) \approx \sum_{i=1}^{N} N_i(t)\bar{u}_i = \mathbf{N}(t)\bar{\boldsymbol{\lambda}} \tag{6}$$

Detail calculating processes of the Galerkin approach can be found in Meingast et al. (2014). There we only give the result of the discretization using the Hat base functions.

$$\bar{\mathbf{p}}(t) = \mathbf{M}\ddot{\mathbf{u}} + \mathbf{D}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} - \boldsymbol{\lambda} - \mathbf{f} \equiv 0 \tag{7}$$

$$\bar{\mathbf{s}}(t) = \boldsymbol{\lambda} + \max\{0, c(\mathbf{u} - \mathbf{0}) - \boldsymbol{\lambda}\} \equiv \mathbf{0} \tag{8}$$

It is in the form of linear complementary problem (LCP) which has multiple feasible solutions.

3.2 Fourier Functional space

With the same philosophy, approximate the displacement $\mathbf{u}$ and the contact force $\boldsymbol{\lambda}$ are approximated as truncated Fourier series.

$$u(t) \approx a_0/\sqrt{2} + \sum_{m=1}^{M} a_m \cos(mwt) + b_m \sin(mwt) = \boldsymbol{\phi}(t)\tilde{\mathbf{u}} \tag{9}$$

$$\lambda(t) \approx c_0/\sqrt{2} + \sum_{m=1}^{M} c_m \cos(mwt) + d_m \sin(mwt) = \boldsymbol{\phi}(t)\tilde{\boldsymbol{\lambda}} \tag{10}$$

where w is the base frequency. It is usually chosen as the frequency of the harmonic excitation force. M is the number of harmonics. And define the base function in the vector form.

$$\boldsymbol{\phi}(t) = [1/\sqrt{2}, \cos(wt), ..., \cos(Mwt), \sin(wt), ..., \sin(Mwt)] \tag{11}$$

$$\dot{\boldsymbol{\phi}}(t) = [0, -w\sin(wt), ..., -Mw\sin(Mwt), w\cos(wt), ..., Mw\cos(Mwt)] \tag{12}$$

$$\ddot{\boldsymbol{\phi}}(t) = [0, -w^2\cos(wt), ..., -(Mw)^2\cos(Mwt), -w^2\sin(wt), ..., -(Mw)^2\sin(Mwt)] \tag{13}$$

To maintain the equilibrium property of the equations in the finite Fourier space:

$$\int_0^{2\pi/w} \boldsymbol{\phi}^{\mathrm{T}}(t)\left(m\ddot{\boldsymbol{\phi}}(t)\tilde{\mathbf{u}} + d\dot{\boldsymbol{\phi}}(t)\tilde{\mathbf{u}} + k\boldsymbol{\phi}(t)\tilde{\mathbf{u}} - \boldsymbol{\phi}(t)\tilde{\boldsymbol{\lambda}} - f(t)\right)\mathrm{d}t = 0 \tag{14}$$

$$\int_0^{2\pi/w} \boldsymbol{\phi}(t)\left(\boldsymbol{\phi}(t)\tilde{\boldsymbol{\lambda}} + \max\{0, c(\boldsymbol{\phi}(t)\tilde{\mathbf{u}} - \mathbf{0}) - \boldsymbol{\phi}(t)\tilde{\boldsymbol{\lambda}}\}\right)\mathrm{d}t = 0 \tag{15}$$

Again, we omit the symbol (t) and $\mathrm{d}t$ hereafter.

(14) is equivalent to

$$\left(\int_0^{2\pi/w} \boldsymbol{\phi}^{\mathrm{T}}(m\ddot{\boldsymbol{\phi}} + d\dot{\boldsymbol{\phi}} + k\boldsymbol{\phi})\right)\tilde{\mathbf{u}} - \left(\int_0^{2\pi/w} \boldsymbol{\phi}^{\mathrm{T}}\boldsymbol{\phi}\right)\tilde{\boldsymbol{\lambda}} - \left(\int_0^{2\pi/w} \boldsymbol{\phi}^{\mathrm{T}}\boldsymbol{\phi}\right)\tilde{\mathbf{f}} = \mathbf{0} \tag{16}$$

Define

$$\mathbf{L}_K = \int_0^{2\pi/w} \boldsymbol{\phi}^{\mathrm{T}}(t)\boldsymbol{\phi}(t) = \pi/w \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} \tag{17}$$

$$\mathbf{L}_D = \int_0^{2\pi/w} \boldsymbol{\phi}^{\mathrm{T}}(t)\dot{\boldsymbol{\phi}}(t) = \pi \begin{bmatrix} 0 & & & & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & M \\ & -1 & & & & & \\ & & \ddots & & & & \\ & & & -M & & & \end{bmatrix} \tag{18}$$

$$\mathbf{L}_M = \int_0^{2\pi/w} \boldsymbol{\phi}^{\mathrm{T}}(t)\ddot{\boldsymbol{\phi}}(t) = \pi w \begin{bmatrix} 0 & & & & & & \\ & -1 & & & & & \\ & & \ddots & & & & \\ & & & -M^2 & & & \\ & & & & -1 & & \\ & & & & & \ddots & \\ & & & & & & -M^2 \end{bmatrix} \tag{19}$$

If $f(t) = \cos(wt)$,

$$\tilde{\mathbf{f}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{20}$$

In vector form, (14) is

$$(m\mathbf{L}_M + d\mathbf{L}_D + k\mathbf{L}_K)\,\tilde{\mathbf{u}} - \mathbf{L}_K\tilde{\boldsymbol{\lambda}} - \mathbf{L}_K\tilde{\mathbf{f}} = \mathbf{0} \tag{21}$$

Since the max operator is not linear in Fourier space, we'd better integrate the (15) with numerical integration method. Discretize the function $\boldsymbol{\phi}$ in time. Divide one period into N equal time intervals $\Delta t = 2\pi/w/N$.

$$\Delta t \sum_{n=1}^{N} \boldsymbol{\phi}^{\mathrm{T}}(t_n)\left(\boldsymbol{\phi}(t_n)\tilde{\mathbf{u}} + \max\{0, c(\boldsymbol{\phi}(t_n)\tilde{\mathbf{u}} - \mathbf{0}) - \boldsymbol{\phi}(t_n)\tilde{\boldsymbol{\lambda}}\}\right) = 0 \tag{22}$$

In vector form

$$\Delta t \boldsymbol{\Phi}^{\mathrm{T}}\left(\boldsymbol{\Phi}\tilde{\mathbf{u}} + \max\{0, c(\boldsymbol{\Phi}\tilde{\mathbf{u}} - \mathbf{0}) - \boldsymbol{\Phi}\tilde{\boldsymbol{\lambda}}\}\right) = 0 \tag{23}$$

where

$$\boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{\phi}(t_1) \\ \vdots \\ \boldsymbol{\phi}(t_N) \end{bmatrix} \tag{24}$$

4 Multiple point contact

Use the Kronecker product to transplant tools developed in the single point contact problem to multi-point contact problem.

$$\mathbf{p}(t) = \mathbf{M}\ddot{\mathbf{u}} + \mathbf{D}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} - \boldsymbol{\lambda} - \mathbf{f} \equiv \mathbf{0} \tag{25}$$

$$\mathbf{s}(t) = \boldsymbol{\lambda} + \max\{0, c(\mathbf{u} - \mathbf{0}) - \boldsymbol{\lambda}\} \equiv \mathbf{0} \tag{26}$$

where $\mathbf{u}, \boldsymbol{\lambda}, \mathbf{p}, \mathbf{s} \in \mathbb{R}^{N_d}$. N_d is the degree of freedom of the system discretized in space.

Discretize one period in finite Fourier space

$$\mathbf{u}(t) \approx (\mathbf{I}_{N_d} \otimes \boldsymbol{\phi}(t))\, \tilde{\mathbf{u}} \tag{27}$$

$$\boldsymbol{\lambda}(t) \approx (\mathbf{I}_{N_d} \otimes \boldsymbol{\phi}(t))\, \tilde{\boldsymbol{\lambda}} \tag{28}$$

where $\tilde{\mathbf{u}}, \tilde{\boldsymbol{\lambda}} \in \mathbb{R}^{N_d(2M+1)}$

Their projections onto the finite Fourier space are zeros

$$\int_0^{2\pi/w} (\mathbf{I}_{N_d} \otimes \boldsymbol{\phi}(t))^{\mathrm{T}}\, \mathbf{p}((\mathbf{I}_{N_d} \otimes \boldsymbol{\phi}(t))\, \tilde{\mathbf{u}}, (\mathbf{I}_{N_d} \otimes \boldsymbol{\phi}(t))\, \tilde{\boldsymbol{\lambda}}) = \mathbf{0} \tag{29}$$

$$\int_0^{2\pi/w} (\mathbf{I}_{N_d} \otimes \boldsymbol{\phi}(t))^{\mathrm{T}}\, \mathbf{s}((\mathbf{I}_{N_d} \otimes \boldsymbol{\phi}(t))\, \tilde{\mathbf{u}}, (\mathbf{I}_{N_d} \otimes \boldsymbol{\phi}(t))\, \tilde{\boldsymbol{\lambda}}) = \mathbf{0} \tag{30}$$

Since

$$\int_0^{2\pi/w} (\mathbf{I}_{N_d} \otimes \boldsymbol{\phi}(t))^{\mathrm{T}}\, \mathbf{M} \left(\mathbf{I}_{N_d} \otimes \ddot{\boldsymbol{\phi}}(t)\right) = \mathbf{M} \otimes \int_0^{2\pi/w} \boldsymbol{\phi}(t)^{\mathrm{T}} \ddot{\boldsymbol{\phi}}(t) = \mathbf{M} \otimes \mathbf{L}_M \tag{31}$$

$$\int_0^{2\pi/w} (\mathbf{I}_{N_d} \otimes \boldsymbol{\phi}(t))^{\mathrm{T}}\, \mathbf{D} \left(\mathbf{I}_{N_d} \otimes \dot{\boldsymbol{\phi}}(t)\right) = \mathbf{D} \otimes \int_0^{2\pi/w} \boldsymbol{\phi}(t)^{\mathrm{T}} \dot{\boldsymbol{\phi}}(t) = \mathbf{D} \otimes \mathbf{L}_D \tag{32}$$

$$\int_0^{2\pi/w} (\mathbf{I}_{N_d} \otimes \boldsymbol{\phi}(t))^{\mathrm{T}}\, \mathbf{K} \left(\mathbf{I}_{N_d} \otimes \boldsymbol{\phi}(t)\right) = \mathbf{K} \otimes \int_0^{2\pi/w} \boldsymbol{\phi}(t)^{\mathrm{T}} \boldsymbol{\phi}(t) = \mathbf{K} \otimes \mathbf{L}_K \tag{33}$$

(29) can be written as

$$[\mathbf{M} \otimes \mathbf{L}_M + \mathbf{D} \otimes \mathbf{L}_D + \mathbf{K} \otimes \mathbf{L}_K]\, \tilde{\mathbf{u}} - (\mathbf{I}_{N_d} \otimes \mathbf{L}_K)\tilde{\boldsymbol{\lambda}} - (\mathbf{I}_{N_d} \otimes \mathbf{L}_K)\tilde{\mathbf{f}} = \mathbf{0} \tag{34}$$

Integrate the second function over time by discrete integration

$$\Delta t \sum_0^N (\mathbf{I}_{N_d} \otimes \boldsymbol{\phi}(t_n))^{\mathrm{T}}\, \mathbf{s}((\mathbf{I}_{N_d} \otimes \boldsymbol{\phi}(t_n))\, \tilde{\mathbf{u}}, (\mathbf{I}_{N_d} \otimes \boldsymbol{\phi}(t_n))\, \tilde{\boldsymbol{\lambda}}) = \mathbf{0} \tag{35}$$

where $\Delta t = 2\pi/w/N$. N is the mesh size over one period.

In vector form:

$$\Delta t\, (\mathbf{I}_{N_d} \otimes \boldsymbol{\Phi})^{\mathrm{T}}\, \mathbf{s}((\mathbf{I}_{N_d} \otimes \boldsymbol{\Phi})\, \tilde{\mathbf{u}}, (\mathbf{I}_{N_d} \otimes \boldsymbol{\Phi})\, \tilde{\boldsymbol{\lambda}}) = \mathbf{0} \tag{36}$$

5 Reduction

For finite element problem with large space mesh size, the number of degree of freedom of the spatial discretized system can be very large. But one good thing is that nonlinear part is only at the contact border. The rest spatial nodes are in a linear relationship. Therefore, it is possible to divide all of the nodes into two sets: contact nodes and extra nodes, use the contact border nodes to calculate extra nodes linearly, and reduce the order of the system.

Reformulate the spatial discretization

$$\begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix} \begin{pmatrix} \ddot{\mathbf{u}}_1 \\ \ddot{\mathbf{u}}_2 \end{pmatrix} (t) + \begin{bmatrix} \mathbf{D}_{11} & \mathbf{D}_{12} \\ \mathbf{D}_{21} & \mathbf{D}_{22} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{u}}_1 \\ \dot{\mathbf{u}}_2 \end{pmatrix} (t) + \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix} (t) = \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 + \boldsymbol{\lambda} \end{pmatrix} (t) \tag{37}$$

where subscript 2 denotes the nodes on the contact border, subscript 1 denotes the extra nodes. It can be seen from the (37) that the relationship between the $\mathbf{u}_1$ and $\mathbf{u}_2$ are linear;

$$\mathbf{u}_2(t) = (\mathbf{I}_{N_{d2}} \otimes \boldsymbol{\phi})\tilde{\mathbf{u}}_2 \tag{38}$$

Other time-variant variables can be projected onto the finite Fourier functional space in the same manner.

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{pmatrix} \tilde{\mathbf{u}}_1 \\ \tilde{\mathbf{u}}_2 \end{pmatrix} = \begin{pmatrix} (\mathbf{I}_{N_{d1}} \otimes \mathbf{L}_K)\tilde{\mathbf{f}}_1 \\ (\mathbf{I}_{N_{d2}} \otimes \mathbf{L}_K)(\tilde{\mathbf{f}}_2 + \tilde{\boldsymbol{\lambda}}) \end{pmatrix} \tag{39}$$

where

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}_{N_d M \times N_d M} = \begin{bmatrix} \mathbf{M}_{11} \otimes \mathbf{L}_M & \mathbf{M}_{12} \otimes \mathbf{L}_M \\ \mathbf{M}_{21} \otimes \mathbf{L}_M & \mathbf{M}_{22} \otimes \mathbf{L}_M \end{bmatrix} + \begin{bmatrix} \mathbf{D}_{11} \otimes \mathbf{L}_D & \mathbf{D}_{12} \otimes \mathbf{L}_D \\ \mathbf{D}_{21} \otimes \mathbf{L}_D & \mathbf{D}_{22} \otimes \mathbf{L}_D \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{11} \otimes \mathbf{L}_K & \mathbf{K}_{12} \otimes \mathbf{L}_K \\ \mathbf{K}_{21} \otimes \mathbf{L}_K & \mathbf{K}_{22} \otimes \mathbf{I} \end{bmatrix} \tag{40}$$

(39) is equivalent to

$$\bar{\mathbf{u}}_1 = -\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\bar{\mathbf{u}}_2 + \mathbf{A}_{11}^{-1}\bar{\mathbf{f}}_1 \tag{41}$$

$$(-\mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12} + \mathbf{A}_{22})\tilde{\mathbf{u}}_2 = (\mathbf{I}_{N_{d2}} \otimes \mathbf{L}_K)\tilde{\boldsymbol{\lambda}} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}(\mathbf{I}_{N_{d1}} \otimes \mathbf{L}_K)\tilde{\mathbf{f}}_1 + (\mathbf{I}_{N_{d2}} \otimes \mathbf{L}_K)\tilde{\mathbf{f}}_2 \tag{42}$$

If define

$$\mathbf{A}_r = (\mathbf{I}_{N_{d2}} \otimes \mathbf{L}_K)^{-1}(-\mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12} + \mathbf{A}_{22}) \tag{43}$$

$$\mathbf{f}_r = (\mathbf{I}_{N_{d2}} \otimes \mathbf{L}_K)^{-1}[-\mathbf{A}_{21}\mathbf{A}_{11}^{-1}(\mathbf{I}_{N_{d1}} \otimes \mathbf{L}_K)\tilde{\mathbf{f}}_1 + (\mathbf{I}_{N_{d2}} \otimes \mathbf{L}_K)\tilde{\mathbf{f}}_2] \tag{44}$$

(42) can be expressed as

$$\mathbf{A}_r\tilde{\mathbf{u}}_2 = \tilde{\boldsymbol{\lambda}} + \mathbf{f}_r \tag{45}$$

which can be solved with tools developed in Section 5.

6 Penalty Contact Treatment

As an alternative to the Lipschitz mapping, the HBM can also be applied to the penalty contact treatment. Augmented Lagrangian contact treatment Wriggers and Laursen (2006) is a good approach which combines the merit of the former two, which is not discussed in the report.

Use the single point contact benchmark in Section 3 as example to introduce the penalty treatment.

$$p(t) = m\ddot{u}(t) + d\dot{u}(t) + ku(t) - \lambda(t) - f(t) \equiv 0 \tag{46}$$

where $\lambda(t)$ is explicit as a function of displacement $u(t)$

$$\lambda(t) = -\epsilon \max\{0, u(t) - g\} \tag{47}$$

$\epsilon \to \infty$

We show here the way to project $p(t)$ onto Fourier functional space. The method of projecting it onto the hat functional space is similar, and it's therefore not discussed in this report.

We use the same approximation of displacement as that in Section 3.

$$u(t) \approx \sum_{i=1}^{N} N_i(t)\bar{u}_i = \mathbf{N}(t)\bar{\mathbf{u}} \tag{48}$$

$$\boldsymbol{\phi}(t) = [1/\sqrt{2}, \cos(wt), ..., \cos(Mwt), \sin(wt), ..., \sin(Mwt)] \tag{49}$$

However, contact force is

$$\lambda(t) = -\epsilon \max\{0, \mathbf{N}(t)\bar{\mathbf{u}} - g\} \tag{50}$$

The projection of equation $p(t) = 0$ yields

$$\int_0^{2\pi/w} \boldsymbol{\phi}^{\mathrm{T}}(t)\left(m\ddot{\boldsymbol{\phi}}(t)\tilde{\mathbf{u}} + d\dot{\boldsymbol{\phi}}(t)\tilde{\mathbf{u}} + k\boldsymbol{\phi}(t)\tilde{\mathbf{u}} + \epsilon \max\{0, \boldsymbol{\phi}(t)\bar{\mathbf{u}} - g\} - f(t)\right) \mathrm{d}t = 0 \tag{51}$$

In vector form, it is

$$(m\mathbf{L}_M + d\mathbf{L}_D + k\mathbf{L}_K)\,\tilde{\mathbf{u}} + \int_0^{2\pi/w} \boldsymbol{\phi}^{\mathrm{T}}(t)\epsilon \max\{0, \boldsymbol{\phi}(t)\tilde{\mathbf{u}} - g\} - \mathbf{L}_K\tilde{\mathbf{f}} = \mathbf{0} \tag{52}$$

where in the integral can be calculated numerically

$$\int_0^{2\pi/w} \boldsymbol{\phi}^{\mathrm{T}}(t)\epsilon \max\{0, \boldsymbol{\phi}(t)\tilde{\mathbf{u}} - g\} = \Delta t \boldsymbol{\Phi}^{\mathrm{T}}\epsilon \max\{0, \boldsymbol{\Phi}\tilde{\mathbf{u}} - g\} \tag{53}$$

Define

$$\mathbf{A} = m\mathbf{L}_M + d\mathbf{L}_D + k\mathbf{L}_K \tag{54}$$

According to Fischer (1992), the Lipschitz continuous equation (52)

$$\mathbf{A}\tilde{\mathbf{u}} + \Delta t \boldsymbol{\Phi}^{\mathrm{T}}\epsilon \max\{0, \boldsymbol{\Phi}\tilde{\mathbf{u}} - g\} - \mathbf{L}_K\tilde{\mathbf{f}} = \mathbf{0} \tag{55}$$

can be solved using Newton-like iteration methods.

References

Andreas Fischer. A special newton-type optimization method. *Optimization*, 24(3-4):269–284, 1992.

Denis Laxalde, Fabrice Thouverez, J-J Sinou, and J-P Lombard. Qualitative analysis of forced response of blisks with friction ring dampers. *European Journal of Mechanics-A/Solids*, 26(4):676–687, 2007.

Markus B Meingast, Mathias Legrand, and Christophe Pierre. A linear complementarity problem formulation for periodic solutions to unilateral contact problems. *International Journal of Non-Linear Mechanics*, 66: 18–27, 2014.

Gotz Von Groll and David J Ewins. The harmonic balance method with arc-length continuation in rotor/stator contact problems. *Journal of sound and vibration*, 241(2):223–233, 2001.

Peter Wriggers and Tod A Laursen. *Computational contact mechanics*, volume 30167. Springer, 2006.