Generalized FEM for Two-Dimensional Periodic Contact Problem

Yulin Shi
Department of Mechanical Engineering
McGill University

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1 Spatial Discretization

Consider the small-deformation elastic-contact problem. The spatial discretization yields

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{D}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{B}^{\mathrm{T}}\boldsymbol{\lambda}(t) + \mathbf{f}(t) \tag{1}$$

$$\mathbf{B}\mathbf{u} - \mathbf{g}_0 \leq \mathbf{0}, \quad \boldsymbol{\lambda} \leq \mathbf{0}, \quad (\mathbf{B}\mathbf{u} - \mathbf{g}_0)_i \lambda_i = 0, \quad \forall i = 1, 2, ..., N_d \tag{2}$$

where $\mathbf{u}(t) \in \mathbb{R}^{N_d}$ is the small displacement vector. $\boldsymbol{\lambda}(t) \in \mathbb{R}^{N_{dc}}$ is the contact force vector of contact nodes in normal direction. $\mathbf{f}(t) \in \mathbb{R}^{N_d}$ is the external force vector. $\mathbf{B}$ is the normal matrices that maps the global coordinates to the normal directions of contact nodes. $\mathbf{g}_0 \in \mathbb{R}^{N_{dc}}$ is the constant vector of upper displacement limits of the contact nodes in normal directions. N_d is the number of degree-of-freedom, N_{dc} is the number of contact nodes on the dynamic contact border.

2 Temporal Approximation

Approximate one period of the displacement of one degree-of-freedom as the sum of truncated function series

$$u_i(t) \approx \hat{\mathbf{u}}_i(t) = \sum_{m=1}^{M} \phi_m(t)\tilde{u}_{i,m} = \boldsymbol{\phi}(t)\tilde{\mathbf{u}}_i \tag{3}$$

where the base function vector $\boldsymbol{\phi}(t)$ are consist of C^1 functions such as Fourier functions.

Displacement vector can be discretized as

$$\mathbf{u}(t) = (\mathbf{I}_{N_d} \otimes \boldsymbol{\phi}(t))\tilde{\mathbf{u}} \tag{4}$$

where

$$\tilde{\mathbf{u}} = \begin{bmatrix} \vdots \\ \tilde{\mathbf{u}}_i \\ \vdots \end{bmatrix} \tag{5}$$

Do it in the same way for contact force and external force

$$\boldsymbol{\lambda}(t) = (\mathbf{I}_{N_{dc}} \otimes \boldsymbol{\phi}(t))\,\tilde{\boldsymbol{\lambda}} \tag{6}$$

$$\mathbf{f}(t) = (\mathbf{I}_{N_d} \otimes \boldsymbol{\phi}(t))\,\tilde{\mathbf{f}} \tag{7}$$

The conservation of the equilibrium of (1) in the finite functional space yields

$$T^{-1}\int_0^T (\mathbf{I}_{N_d} \otimes \boldsymbol{\phi}(t))^{\mathrm{T}} \Big(\mathbf{M}(\mathbf{I}_{N_d} \otimes \ddot{\boldsymbol{\phi}}(t))\tilde{\mathbf{u}} + \mathbf{D}(\mathbf{I}_{N_d} \otimes \dot{\boldsymbol{\phi}}(t))\tilde{\mathbf{u}} + \mathbf{K}(\mathbf{I}_{N_d} \otimes \boldsymbol{\phi}(t))\tilde{\mathbf{u}} - \mathbf{B}^{\mathrm{T}}(\mathbf{I}_{N_{dc}} \otimes \boldsymbol{\phi}(t))\tilde{\boldsymbol{\lambda}} - (\mathbf{I}_{N_d} \otimes \boldsymbol{\phi}(t))\tilde{\mathbf{f}}\Big) = \mathbf{0} \tag{8}$$

Where the T is the period. The differential operator $\mathrm{d}t$ is omitted for simplification.

Simplify the equation according to the laws of the Kronecker operator

$$T^{-1}\int_0^T \Big([\mathbf{M}\otimes(\boldsymbol{\phi}(t)^{\mathrm{T}}\ddot{\boldsymbol{\phi}}(t))]\tilde{\mathbf{u}}+[\mathbf{D}\otimes(\boldsymbol{\phi}(t)^{\mathrm{T}}\dot{\boldsymbol{\phi}}(t))]\tilde{\mathbf{u}}+[\mathbf{K}\otimes(\boldsymbol{\phi}(t)^{\mathrm{T}}\boldsymbol{\phi}(t))]\tilde{\mathbf{u}}-[\mathbf{B}^{\mathrm{T}}\otimes(\boldsymbol{\phi}(t)^{\mathrm{T}}\boldsymbol{\phi}(t))]\tilde{\boldsymbol{\lambda}}-[\mathbf{I}_{N_d}\otimes(\boldsymbol{\phi}(t)^{\mathrm{T}}\boldsymbol{\phi}(t))]\tilde{\mathbf{f}}\Big) = \tag{9}$$

Define

$$\mathbf{L}_K = T^{-1}\int_0^T \boldsymbol{\phi}(t)^{\mathrm{T}}\ddot{\boldsymbol{\phi}}(t) \tag{10}$$

$$\mathbf{L}_D = T^{-1}\int_0^T \boldsymbol{\phi}(t)^{\mathrm{T}}\dot{\boldsymbol{\phi}}(t) \tag{11}$$

And, according to the Green equation

$$\mathbf{L}_M = T^{-1}\int_0^T \boldsymbol{\phi}(t)^{\mathrm{T}}\ddot{\boldsymbol{\phi}}(t) = -T^{-1}\int_0^T \dot{\boldsymbol{\phi}}(t)^{\mathrm{T}}\dot{\boldsymbol{\phi}}(t) \tag{12}$$

Then (8) is

$$(\mathbf{M}\otimes\mathbf{L}_M)\tilde{\mathbf{u}}+(\mathbf{D}\otimes\mathbf{L}_D)\tilde{\mathbf{u}}+(\mathbf{K}\otimes\mathbf{L}_K)\tilde{\mathbf{u}}-(\mathbf{B}^{\mathrm{T}}\otimes\mathbf{L}_K)\tilde{\boldsymbol{\lambda}}-(\mathbf{I}_{N_d}\otimes\mathbf{L}_K)\tilde{\mathbf{f}}=\mathbf{0} \tag{13}$$

Left divide it by an nonsingular matrix $\mathbf{I}_{N_d}\otimes\mathbf{L}_K$ yields

$$(\mathbf{M}\otimes\mathbf{L}_K^{-1}\mathbf{L}_M)\tilde{\mathbf{u}}+(\mathbf{D}\otimes\mathbf{L}_K^{-1}\mathbf{L}_D)\tilde{\mathbf{u}}+(\mathbf{K}\otimes\mathbf{I}_M)\tilde{\mathbf{u}}-(\mathbf{B}^{\mathrm{T}}\otimes\mathbf{I}_M)\tilde{\boldsymbol{\lambda}}-(\mathbf{I}_{N_d}\otimes\mathbf{I}_M)\tilde{\mathbf{f}}=\mathbf{0} \tag{14}$$

Define

$$\mathbf{A} = (\mathbf{M}\otimes\mathbf{L}_K^{-1}\mathbf{L}_M)+(\mathbf{D}\otimes\mathbf{L}_K^{-1}\mathbf{L}_D)+(\mathbf{K}\otimes\mathbf{I}_M) \tag{15}$$

(8) is

$$\mathbf{A}\tilde{\mathbf{u}}-(\mathbf{B}^{\mathrm{T}}\otimes\mathbf{I}_M)\tilde{\boldsymbol{\lambda}}-\tilde{\mathbf{f}}=\mathbf{0} \tag{16}$$

3 Order Reduction

For small-deformation problem, define the linear gap function

$$\mathbf{g} = \mathbf{B}\mathbf{u}-\mathbf{g}_0 \tag{17}$$

Use the same methods: approximate variables using the same base function series, and integral it over one period yields

$$\tilde{\mathbf{g}} = (\mathbf{B}\otimes\mathbf{I}_M)\tilde{\mathbf{u}}-\tilde{\mathbf{g}}_0 \tag{18}$$

Times both side of (16) by $\mathbf{B}\otimes\mathbf{I}_M$

$$(\mathbf{B}\otimes\mathbf{I}_M)\tilde{\mathbf{u}}-(\mathbf{B}\otimes\mathbf{I}_M)\mathbf{A}^{-1}(\mathbf{B}^{\mathrm{T}}\otimes\mathbf{I}_M)\tilde{\boldsymbol{\lambda}}-(\mathbf{B}\otimes\mathbf{I}_M)\mathbf{A}^{-1}\tilde{\mathbf{f}}=\mathbf{0} \tag{19}$$

Plug (18) into (20) yields

$$\tilde{\mathbf{g}}+\tilde{\mathbf{g}}_0-(\mathbf{B}\otimes\mathbf{I}_M)\mathbf{A}^{-1}(\mathbf{B}^{\mathrm{T}}\otimes\mathbf{I}_M)\tilde{\boldsymbol{\lambda}}-(\mathbf{B}\otimes\mathbf{I}_M)\mathbf{A}^{-1}\tilde{\mathbf{f}}=\mathbf{0} \tag{20}$$

The new equation is of order $N_{dc}M$ which is greatly reduced from N_dM.

4 Linear Complementarity Reformulation

An equivalent expression to the KKT condition (2) is

$$\boldsymbol{\lambda}(t) + \max\{0, c\mathbf{g}(t) - \boldsymbol{\lambda}(t)\} = \mathbf{0} \tag{21}$$

The projection to the finite functional space is

$$T^{-1}\int_0^T (\mathbf{I}_{N_d} \otimes \boldsymbol{\phi}(t))^{\mathrm{T}} \left((\mathbf{I}_{N_d} \otimes \boldsymbol{\phi}(t))\tilde{\boldsymbol{\lambda}} + \max\{0, (\mathbf{I}_{N_d} \otimes \boldsymbol{\phi}(t))(c\tilde{\mathbf{g}} - \tilde{\boldsymbol{\lambda}})\}\right) = \mathbf{0} \tag{22}$$

Mesh one period into N equivalent segments $\Delta t = T/N$. And integrate the nonlinear equation using integration method

$$(\mathbf{I}_{N_d} \otimes \mathbf{L}_K)\tilde{\boldsymbol{\lambda}} + \Delta t T^{-1} \sum_{n=1}^{N} (\mathbf{I}_{N_d} \otimes \boldsymbol{\phi}(t_n))^{\mathrm{T}} \max\{0, (\mathbf{I}_{N_d} \otimes \boldsymbol{\phi}(t_n))(c\tilde{\mathbf{g}} - \tilde{\boldsymbol{\lambda}})\} = \mathbf{0} \tag{23}$$

Solve the (20) and (23) together for the unknown vector $\mathbf{g}, \boldsymbol{\lambda} \in \mathbb{R}^{N_{dc}M}$
$\tilde{\mathbf{u}}$ is calculated by

$$\tilde{\mathbf{u}} = \mathbf{A}^{-1}\left[(\mathbf{B}^{\mathrm{T}} \otimes \mathbf{I}_M)\tilde{\boldsymbol{\lambda}} + \tilde{\mathbf{f}}\right] \tag{24}$$