Constructive Non-iterative Explicit Models of Turbofan Engines with Introduced Poles

Yulin Shi*, Xi Wang**, Chao Yang**, Bin Wang**, Zhipeng Li**

*McGill University, Montreal, Canada, **Beihang University, Beijing, China

***Abstract*—nonlinear implicit constraint equations and differential dynamic functions are the most accurate descriptions of open-loop model of turbofan engines. However, it's time-consuming to iterate the implicit constraint equations during time-domain simulation. In some literatures, those implicit constraint equations between components are replaced by explicit dynamic functions of volumes between components. However, there is no evidence to support the feasibility of the replacement, and solution of the volume dynamic model can diverge during large step simulation.**

In this article, the replacement is made by adding some differential terms to the implicit constraint equations. It's proved by the theory of secondary poles that the solution of the constructed new non-iterative dynamic model can converge to the solution of the original model if dynamic coefficients of new differential terms are much larger than then that of original differential terms. Since volume dynamic modeling methods are categorized into the proposed method, the feasibility condition of the proposed method also applies to volume dynamic modeling methods. We therefore proposed virtual volume dynamic modeling method to avoid step size limitation of simulation. Second, high-order linear space-state model of rotor speed dynamics and virtual volume dynamics are extracted from the constructed explicit dynamic functions. Third, low-order linear space-state model of rotor speed is reduced from the constructed high-order model by frequency decomposition method.

I. INTRODUCTION

The dynamic model of turbofan engine can be described as the combination of constraint equations, and dynamic differential functions. Wherein, the constraint equations describe the equilibrium relationship of flow rate and thermal energy; the dynamic differential functions describe the mass inertia effect of rotor acceleration. The rotor acceleration dynamic modalities, whose fuel rate step response time in the magnitude of one second and open-loop pole in the magnitude of 1 rad/s, are the major modalities of turbofan engines; on the contrary, the temperature and pressure dynamic modalities of volumes between cascaded engine components, whose fuel rate step response time in the magnitude of 0.1 second and open-loop pole in the magnitude of 10 rad/s, are the secondary modalities of turbofan engines[1]. Therefore, in our research, the volume dynamic modalities can be ignored, and our turbofan engine is a two-order system.

The unknown variables e.g. temperature and pressure of volumes in the constraint equations are all in implicit expression. Therefore, iterative methods e.g. Newton-Raphson method are usually used to solve them. However, it's time-consuming to compute iterative problems[2]. On the contrary, volume dynamic models of turbofan engines, wherein every function is in explicit form are applied in many research materials. The non-iterative model can be achieved by shortening step size, and by studying dynamic effect of volumes [3-5]. However, the introducing of volume dynamic effect greatly limits step size during simulations thus prolongs simulation time. Therefore, it's meaningful to find out a model that avoids iteration as well as avoids shortening simulation step size. One of these models is our proposed virtual dynamic model.

In our article, differential terms are added to the implicit constraint equations to construct explicit differential functions set. In the sense of frequency domain, the only difference between the original constraint equation problem and the new constructed differential function set problem is several secondary poles. Since there is no implicit equation, we do not have to iterate. The only thing to do is only solve the differential equation set by Euler or Runge-Kutta discrete integral methods. More importantly, one form of our constructed new non-iterative differential functions is similar to the so called "volume dynamic models" in many research materials. This proved that "volume dynamic models" should be categorized into our proposed construction methods, and the essential of "volume dynamic models" is to add in several secondary poles to the original low-order systems. Based on above analysis, we suppose that the size of volumes won't have apparent effect on the original system if the introduced secondary poles are large enough.

The paper is organized as follows: in Chapter 2, preliminary knowledge of dominant poles and secondary poles are introduced, and a sufficient condition of convergence of the solution of our constructed non-iterative algorithm is given. In Chapter 3, the component-level model of turbofan engine is introduced, which is a preparation for integrated model of turbofan engines in Chapter 4. In Chapter 4, a rotor speed dynamic model of turbofan engines with mass flow rate equilibrium constraints and thermal energy equilibrium constraints are build; the constructive algorithm of solving the dynamic model with equilibrium constraints are given. In Chapter 5, we introduced how to derive the linear model from our constructed high-order explicit function model of turbofan engines by discrete differential method. In Chapter 6, the introduced secondary poles are eliminated in the linear models by the singular value decomposition order-reduce method; It results in the linear model of real two-order model of turbofan engines.

II. ITERATIVE ALGORITHM FOR SOLVING CONSTRAINT FUNCTIONS

A. *Dominant Poles*

Definition 1: In Linear System Theory, close-loop poles (either plural or real) locating near to the imaginary axis (also far away from any zero points) have dominant influence on the dynamic property of a system. These poles are so-called dominant poles. Other poles are defined as secondary poles.

Moreover, in engineering practice, other poles whose real part is six times larger than that of dominant poles is usually neglected while evaluating the dynamic system. The effect of this neglect can be demonstrated by the following comparison.

For a simple transfer function having separate poles

$$T_0(s) = \frac{1}{(s+1)(0.2s+1)} \tag{1}$$

Eliminate the secondary pole, s=5, with residue decomposition method; we have

$$T_1(s) = \frac{1}{(s+1)} \tag{2}$$

It can be shown both from frequency aspect, and from time-domain response that the original high order system $T_0(s)$ and the reduce-order system $T_1(s)$ are similar in shape of curves.

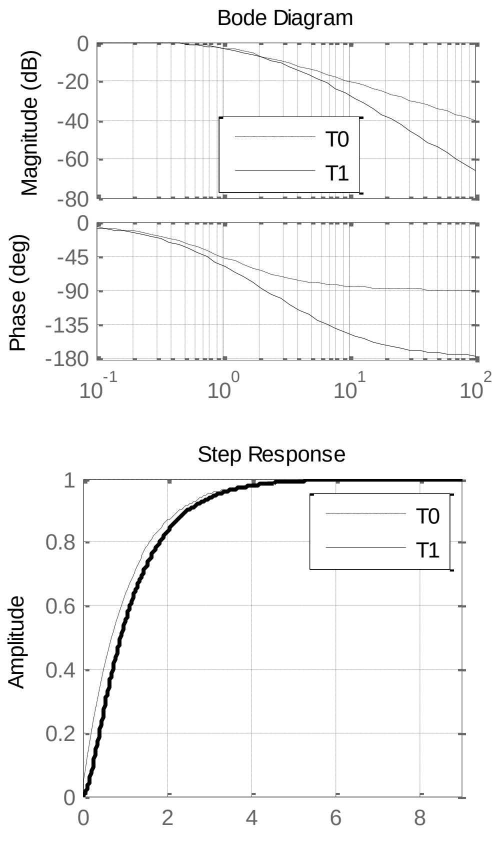


Figure 1. Comparison between high-order model and low-order model

Specifically, it's shown in the bode diagram that the two curves overlap at around 1 rad/s section where fuel rate feedback control concerns. In the time-domain response diagram, the effects of elimination 0.3 seconds of time delay, which is unimportant compared to the 3-second response time.

It's predictable that the difference will diminish if the two groups of poles locate even further. In fact, in industrial practice, we usually neglect secondary poles larger than dominant poles by six times.

B. *Solving constraint equations by constructing new differential equations*

From the view point of numerical methods, our proposed principle of solving dynamic equations with constraint equations can be explained as transforming a iterative problem into a non-iterative integral problem, meanwhile, to guarantee the convergence.

The generic form of dynamic equations with constraint equations in our research can be described as

$$\begin{aligned} 0 &= f_i(x, y, u) \\ \dot{y}_j &= g_j(x, y, u) \end{aligned} \tag{3}$$

Where, 0=fi are constraint equations;

g_i are differential equations related to dominant dynamic modal in the open-loop system. They refers to mass flow and thermal energy equilibrium equations in gas turbine engine models;

y_j are state variables whose open-loop pole is small in an absolute sense. They refer to rotor speed, and temperature of blade-case in heat transfer process;

x is chosen arbitrally wherein every element of xi is independent, as to make sure the dimension of x equals to the number of equilibrium function fi;

u is the input vector;

i=1, 2, …, n. j=1, 2, …, m. n is the number of constraint equations fi, m is the dimension of y.

This problem has a solution

$$y(t) = y\left(t, u(t), x_0(t=0), y_0(t=0)\right) \tag{4}$$

Where, x_0, y_0 are vectors of initial values.

In fact, they are just the generic expression of dynamic model of gas turbine engines.

Then, the steady state variables (x_s, y_s) of above system can be defined as

$$\begin{aligned} 0 &= f_i(x_s, y_s, u) \\ 0 &= g_j(x_s, y_s, u) \end{aligned} \tag{5}$$

Since the number of unknown variables of above equation set equals the number of above equations, x_s, y_s will are monodrome functions of input u. Moreover, it's easy to know that x_s, y_s do not include time parameter "t" explicitly.

The steady state variables (x_s, y_s) will be useful in the proof process.

Theorem 1: problem of dynamic equations with constraint equations can be solved by constructing the following dynamic equation (non-iterative) problem

$$\begin{aligned} V_i \dot{x}_{2i} &= f_i(x_2, y_2, u) \\ \dot{y}_{2i} &= g_j(x_2, y_2, u) \end{aligned} \tag{6}$$

Of course, we expect the problem to be internal stable. Therefore, without loss of generality, we assume that

1) V>0

2) $\partial f_i/\partial x_i<0$

3) $\partial g_i/\partial x_i<0$

4) $\partial f_i/\partial x_i$, $\partial f_i/\partial y_i$, $\partial g_i/\partial x_i$, $\partial g_i/\partial y_i$ are bounded. This prerequisition is reasonable for physical systems

5) $\dot{y}$, $\dot{u}$ are continuous and bounded. . This prerequisition is also reasonable for physical systems

The solution

$$y_2(t)=y_2\left(t,u\left(t\right),x_0,y_0\right) \tag{7}$$

will converge to the original solution if the constants V_i are small enough.

Prove:

Let's begin with solving an one-dimensional case i.e. n=1, m=1, and the problem is simplified to

$$\begin{aligned} 0&=f\left(x,y,u\right) \\ \dot{y}&=g\left(x,y,u\right) \end{aligned} \tag{8}$$

For any u=u(t), if there is a solution $(x(t),y(t))=(x(t_1),y(t_1))$ at $t=t_1$, then, within a very short period of time after $t=t_1$, we can linearize the above equations by expanding them with Taylor method separately.

$$\begin{aligned} 0&=f_x\Delta x+f_y\Delta y+f_u\Delta u \\ \Delta\dot{y}&=g_x\Delta x+g_y\Delta y+g_u\Delta u \end{aligned} \tag{9}$$

Wherein, constant $f_x=\dfrac{\partial f}{\partial x},f_y=\dfrac{\partial f}{\partial y},f_u=\dfrac{\partial f}{\partial u}$,

$g_x=\dfrac{\partial g}{\partial x},g_y=\dfrac{\partial g}{\partial y},g_u=\dfrac{\partial g}{\partial u}$

$$\begin{aligned} \Delta x\left(t\right)&=x\left(t\right)-x\left(t_1\right) \\ \Delta y\left(t\right)&=y\left(t\right)-y\left(t_1\right) \\ \Delta u\left(t\right)&=u\left(t\right)-u\left(t_1\right) \end{aligned} \tag{10}$$

Transfer them with Laplace transformation methods

$$\begin{aligned} 0&=f_xX\left(s\right)+f_yY\left(s\right)+f_uU(s) \\ sY\left(s\right)&=g_xX\left(s\right)+g_yY\left(s\right)+g_uU(s) \end{aligned} \tag{11}$$

The solution of Y(s) can be expressed in the form of transfer function.

$$Y=\frac{-g_xf_{u1}+g_uf_x}{sf_x+g_xf_y-g_yf_x}U \tag{12}$$

We learn therefore that the dominate pole of our problem is $s=-f_x^{-1}g_xf_y+f_x^{-1}g_yf_x$

Then, solving our constructed differential functions with Taylor expansion.

$$\begin{aligned} V\Delta\dot{x}&=f_x\Delta x+f_y\Delta y+f_u\Delta u \\ \Delta\dot{y}&=g_x\Delta x+g_y\Delta y+g_u\Delta u \end{aligned} \tag{13}$$

Transfer them with Laplace transformation methods

$$\begin{aligned} sX\left(s\right)&=f_xX\left(s\right)+f_yY\left(s\right)+f_uU(s) \\ sY\left(s\right)&=g_xX\left(s\right)+g_yY\left(s\right)+g_uU(s) \end{aligned} \tag{14}$$

Then, the solution of Y(s) is

$$Y=\frac{Vg_us+g_xf_u-g_uf_x}{Vs^2-\left(f_x+Vg_y\right)s-g_xf_y+g_yf_x}U \tag{15}$$

The explicit expression of poles of above system can be obtained by solving quadratic equation of variable s

$$Vs^2-\left(f_x+Vg_y\right)s-g_xf_y+g_yf_x=0 \tag{16}$$

And we get the solution:

$$\begin{aligned} s_1&\approx V^{-1}f_x+f_x^{-1}g_xf_y \\ s_2&\approx f_x^{-1}g_yf_x-f_x^{-1}g_xf_y \end{aligned} \tag{17}$$

Wherein, we used the following Taylor expansion and approximation.

$$\sqrt{1+\varepsilon}=1+\frac{1}{2}\varepsilon+o\left(\varepsilon\right)\approx1+\frac{1}{2}\varepsilon \tag{18}$$

Wherein，$0<|\varepsilon|\ll1$.

Given $f_x<0$, we have, when V approaches 0,

$$Y=\frac{-g_xf_{u1}+g_uf_x}{sf_x+g_xf_y-g_yf_x}U \tag{19}$$

And poles becomes

$$\begin{aligned} s_1&\to-\infty \\ s_2&\to-f_x^{-1}g_xf_y+f_x^{-1}g_yf_x \end{aligned} \tag{20}$$

Now, we can see from the comparison of Y(s) of the original problem and of our constructed differential equations that:

1) When constant V is small enough, the effect of adding differential item to the constraint equation is to add a large pole to the original system. This new pole changed the order of the system from one to two, but it does not change the solution of Y(s) very much.

2) if the transfer function expression of Y(s) of the constructed system can converge to that of the original system, and the variable y(t) of the two system starts from the same initial value $y(t_1)$, then, within a very short period of time starting from $t=t_1$, the solution of y(t) of the constructed system will converge to the solution of the original system.

Then, generalize above proof process to higher dimensional cases.

$$\begin{aligned} 0&=f_i\left(x,y,u\right) \\ \dot{y}_j&=g_j\left(x,y,u\right) \end{aligned} \tag{21}$$

Wherein, i=1, 2, …, n; j=1, 2, …, m; f_i is equation set with n equilibrium equations. gj is equation set with m differential equations. x is a vector of n dimension, y is output vector of m dimension, u is input vector of L dimension.

Exam its solution within a very short period of time from t_1 to $t_1+\delta t$, $\delta t\to0$.

Expand by Taylor method.

$$
\begin{aligned}
0 &= \Sigma_{r=1}^{n} \frac{\partial f_i}{\partial x_r} \Delta x_r + \Sigma_{s=1}^{m} \frac{\partial f_i}{\partial y_s} \Delta y_s + \Sigma_{t=1}^{l} \frac{\partial f_i}{\partial u_t} \Delta u_t \\
\dot{y}_j &= \Sigma_{r=1}^{n} \frac{\partial g_j}{\partial x_r} \Delta x_r + \Sigma_{s=1}^{m} \frac{\partial g_j}{\partial y_s} \Delta y_s + \Sigma_{t=1}^{l} \frac{\partial g_j}{\partial u_t} \Delta u_t
\end{aligned} \tag{22}
$$

Wherein, $\Delta x_r(t) = x_r(t) - x_r(t_1)$,

$\Delta y_s(t) = y_s(t) - y_s(t_1)$

$\Delta u_t(t) = u_t(t) - u_t(t_1)$

Express these linear equations in lumped forms,

$$
\begin{aligned}
0 &= J_{fx}\Delta x + J_{fy}\Delta y + J_{fu}\Delta u \\
\dot{y} &= J_{gx}\Delta x + J_{gy}\Delta y + J_{gu}\Delta u
\end{aligned} \tag{23}
$$

wherein coefficient expressions of partial differentials are expressed in to form of Jaccobi matrixes, Δx, Δy, Δu are vectors.

Transform them by Laplace transformation

$$
\begin{aligned}
0 &= J_{fx}X(s) + J_{fy}Y(s) + J_{fu}U(s) \\
sY(s) &= J_{gx}X(s) + J_{gy}Y(s) + J_{gu}U(s)
\end{aligned} \tag{24}
$$

Then, we get the matrix transfer function solution

$$
Y = \left(sI - \left(J_{gy} - J_{gx}J_{fx}^{-1}J_{fy}\right)\right)^{-1}\left(-J_{gx}J_{fx}^{-1}J_{fu} + J_{gu}\right)U \tag{25}
$$

It's easy to see that the poles of our iterative model is the eigenvalues of matrix $J_{gy} - J_{gx}J_{fx}^{-1}J_{fy}$

On the other hand, for the high-dimensional non-iterative problem

$$
\begin{aligned}
V_i\dot{x}_i &= f_i(x, y, u) \\
\dot{y}_j &= g_j(x, y, u)
\end{aligned} \tag{26}
$$

Expand by Taylor method

$$
\begin{aligned}
V_i\dot{x}_i &= \Sigma_{r=1}^{n} \frac{\partial f_i}{\partial x_r} \Delta x_r + \Sigma_{s=1}^{m} \frac{\partial f_i}{\partial y_s} \Delta y_s + \Sigma_{t=1}^{l} \frac{\partial f_i}{\partial u_t} \Delta u_t \\
\dot{y}_j &= \Sigma_{r=1}^{n} \frac{\partial g_j}{\partial x_r} \Delta x_r + \Sigma_{s=1}^{m} \frac{\partial g_j}{\partial y_s} \Delta y_s + \Sigma_{t=1}^{l} \frac{\partial g_j}{\partial u_t} \Delta u_t
\end{aligned} \tag{27}
$$

And write in the form of lumped Jaccobi matrixes

$$
\begin{aligned}
\Lambda_V\dot{x} &= J_{fx}\Delta x + J_{fy}\Delta y + J_{fu}\Delta u \\
\dot{y} &= J_{gx}\Delta x + J_{gy}\Delta y + J_{gu}\Delta u
\end{aligned} \tag{28}
$$

Wherein, Λ_v is a diagonal matrix whose diagonal elements are a_i.

Transform is by Laplace transformation

$$
\begin{aligned}
s\Lambda_V X &= J_{fx}X + J_{fy}Y + J_{fu}U \\
sY &= J_{gx}X + J_{gy}Y + J_{gu}U
\end{aligned} \tag{29}
$$

We finally get

$$
\begin{aligned}
Y = &\left(sI - \left(J_{gy} - J_{gx}\left(J_{fx} - s\Lambda_V\right)^{-1}J_{fy}\right)\right)^{-1} \\
&\left(-J_{gx}\left(J_{fx} - s\Lambda_V\right)^{-1}J_{fu} + J_{gu}\right)U
\end{aligned} \tag{30}
$$

It's difficult to give an explicit expression of poles of the new system, but it's obvious by studying the above transfer function that it will approach the original function when $\Lambda_v \rightarrow 0$.

III. COMPONENT LEVEL INPUT-OUTPUT MODEL

A. Input-Output Model of compressors/turbines

The flow rate output W of compressor/turbine component can be obtained by map characteristics data and similarity transformation as follows:

$$
n_{cor} = n\sqrt{\frac{288.15}{T_{in}}}
$$

$$
\pi = \frac{p_{out}}{p_{in}}
$$

$$
W_{cor} = f_1(n_{cor}, \pi)
$$

$$
\eta = f_2(n_{cor}, \pi)
$$

$$
W = W_{cor}\frac{p_{in}}{101325}\sqrt{\frac{288.15}{T_{in}}}
$$

The enthalpy output h_{out} can be calculated by

$$
h = h\left(\Psi_{in}(T_{in}) + \lg\pi\right)
$$

$$
h_{out} = h_{in} + \eta\cdot\left[h_{out,ideal}\left(T_{out,ideal}\right) - h_{in}\right]
$$

Wherein, h, Ψ are monochrome functions of temperature T.

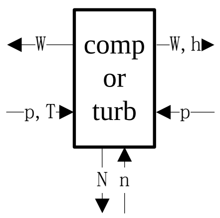
The parameter dependency relationship can be written in the following form

$$
W = W\left(n, p_{in}, p_{out}, T_{in}\right)
$$

$$
h_{out} = h_{out}\left(n, p_{in}, p_{out}, T_{in}\right)
$$

It means, variable W can be influenced by variables n, pin, pout, Tin lying in the bracket. Other structural parameters such as cross area A, inertia J, and volume size V which are constant would not appear in the brackets. These expression rules also apply to h_{out}, and other variables in this article.

The input and output relationship of variables related to compressor/turbine can be expressed in the following diagram.



B. Input-Output Model of Nozzles

When friction, flow loss, and unsteady phenomena are ignored, the nozzle can be modeled as a one-dimensional tube that has the following inputs and outputs

$$
\pi = \frac{p_{in}}{p_{out}}
$$

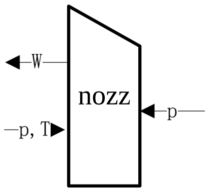
$$W = K\frac{p_{in}q(\pi)A}{\sqrt{T_{in}}}$$

Wherein, K is function of T. p and T refers to stagnation pressure, and stagnation temperature.

Therefore, if expressed in input-output form, we have the parameterized model of nozzle

$$W = W(p_{in}, p_{out}, T_{in})$$

Also, the relationship of input and output variables of nozzles can be expressed in the following diagram.



C. Input-Output Model of volumes

The differential form mass m=ρV is

$$\frac{\dot{m}}{V} = \dot{\rho} = \left(\frac{p}{RT}\right)' = \frac{1}{R}\left(\frac{\dot{p}}{T} - \frac{p\dot{T}}{T^2}\right) = \frac{p}{RT}\left(\frac{\dot{p}}{p} - \frac{\dot{T}}{T}\right)$$

The differential form of intrinsic energy U=um=(h-RT)m is

$$\dot{U} = \left[(h-RT)m\right]' = \left(C_p\dot{T} - R\dot{T}\right)m + (h-RT)\dot{m}$$

Therefore, the dynamic function of pressure $\dot{p}$ and temperature $\dot{T}$ can be expressed as

$$T' = \frac{RT}{Vp}\frac{1}{C_p - R}\left((RT-h)m' + U'\right)$$

$$\begin{aligned} p' &= \frac{RT}{V}m' + \frac{p}{T}T' \\ &= \frac{RT}{V}m' + \frac{1}{(C_p - R)}\frac{R}{V}\left(U' - m'(h-RT)\right) \\ &= \frac{R}{V}\frac{1}{C_p - R}\left((C_pT - h)m' + U'\right) \end{aligned}$$

Then, put in mass conservation function and thermal energy conservation function

$$m' = W_{up} + W_f - W_{down}$$

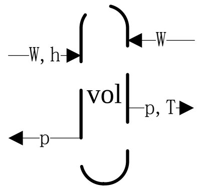
$$U' = h_{up}W_{up} + \eta_b H_u W_f - h_{down}W_{down}$$

We have

$$\begin{aligned} T' = &-\frac{RT}{Vp}\frac{h-RT}{(C_p - R)}\left(W_{up} + W_f - W_{down}\right) \\ &+\frac{RT}{Vp}\frac{1}{(C_p - R)}\left(h_{up}W_{up} + \eta_b H_u W_f - h_{down}W_{down}\right) \end{aligned}$$

$$\begin{aligned} p' = &\frac{R}{V}\left(T - \frac{h-RT}{C_p - R}\right)\left(W_{up} + W_f - W_{down}\right) \\ &+\frac{R}{V}\frac{1}{(C_p - R)}\left(h_{up}W_{up} + \eta_b H_u W_f - h_{down}W_{down}\right) \end{aligned}$$

It means, the output T and p of a volume are function of flow rate and inlet enthalpy.



D. Input-Output Model of rotors

According to the differential form of kinetic energy definition $E_k = \frac{1}{2}J\omega^2$

$$\dot{E}_k = J\omega\dot{\omega} = \left(\frac{\pi}{30}\right)^2 Jn\dot{n}$$

Wherein, the rotor speed of n=30ω/π

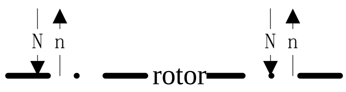
According to kinetic energy conservation equation

$$\dot{E}_k = \eta_m N_t - N_c$$

Wherein η_m is mechanical efficiency, N_t is turbine power, N_c is compressor power.

Then,

$$\dot{n} = \left(\frac{30}{\pi}\right)^2 \frac{\eta_m N_t - N_c}{Jn}$$



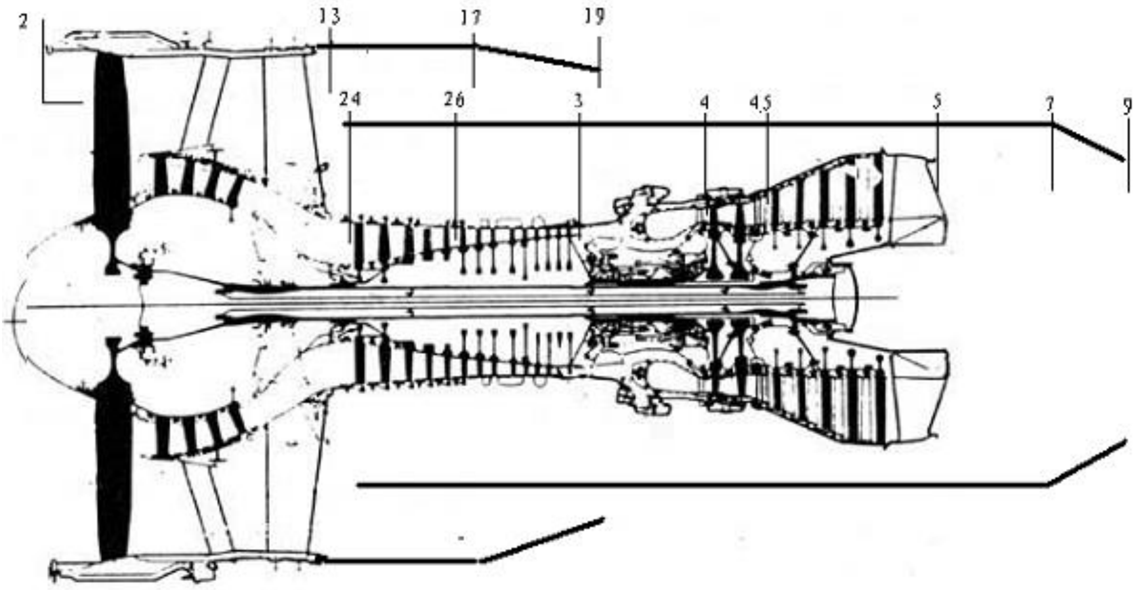


Figure 2. Annotation – cross numbers

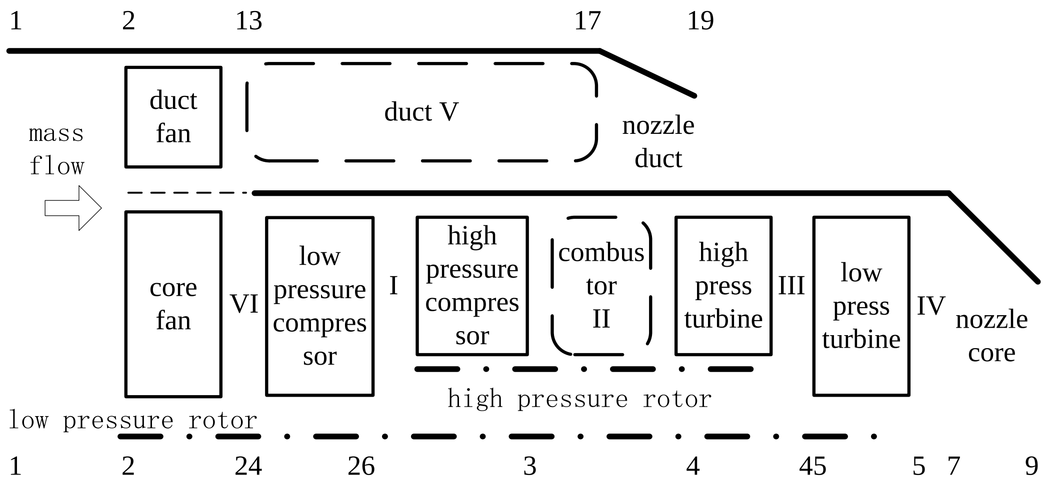


Figure 3. Component level model of a turbofan engine

IV. THE OVERALL MODEL OF A TURBOFAN ENGINE

A. The constructed non-iterative model for turbofan engines

Our modeling object is a turbofan engine, as is shown in Figure 2. It's assumed that the outer fan and the inner fan are irrelative. That is to say, there won't be any leakage from the inner passage to the outer passage, or vise verse at the outlet of fan component. Therefore, the above object can be modeled into 10 parts, as is shown in Figure 3.

It's also assumed that the transient model of this turbofan engine has only two major dynamic modal i.e. low pressure rotor speed, and high pressure rotor speed. Any other physical parameters like pressure and temperature of the flow can be adjusted immediately. Therefore, we assume that both the mass flow constraint and heat flow constraint are satisfied. Based on these assumptions, we have equilibrium functions between components, and transient function of rotor as follows:

Duct passage V

$$W_{fnd}(p_V, T_V, n_1) = W_{nzd}(p_V, T_V, n_1)$$

$$h_{fnd}(p_V, T_V, n_1)W_{fnd} = h_V(T_V)W_{nzd}$$

Cross section VI

$$W_{fnc}(p_{VI}, n_1) = W_{lpc}(p_{VI}, p_I, T_{VI}, n_1)$$

$$h_{fnc}(p_{VI}, n_1)W_{fnc} = h_{VI}(T_{VI})W_{lpc}$$

Cross section I

$$W_{lpc} = W_{hpc}(p_I, p_{II}, T_I, n_2)$$

$$h_{lpc}(p_I, p_{II}, T_{VI}, n_1)W_{lpc} = h_I(T_I)W_{hpc}$$

Combustor II

$$W_{hpc} + W_f = W_{hpt}(p_{II}, p_{III}, T_{II}, n_2)$$

$$h_{hpc}(p_I, p_{II}, T_I, n_2)W_{hpc} + \eta_b H_u W_f = h_{II}(T_{II})W_{hpt}$$

Cross section III

$$W_{hpt} = W_{lpt}(p_{III}, p_{IV}, T_{III}, n_1)$$
$$h_{hpt}(p_{II}, p_{III}, T_{II}, n_2)W_{hpt} = h_{IV}(T_{IV})W_{lpt}$$

Cross section IV

$$W_{lpt} = W_{znc}(p_{IV}, T_{IV})$$
$$h_{lpt}(p_{III}, T_{III}, n_1)W_{lpt} = h_{IV}(T_{IV})W_{znc}$$

Rotors

$$\dot{n}_1\left(\frac{30}{\pi}\right)^2 J_1 n_1 + N_{fnd} + N_{fnc} + N_{lpc} = \eta_{m1}N_{hpt}$$
$$\dot{n}_2\left(\frac{30}{\pi}\right)^2 J_2 n_2 + N_{hpc} = \eta_{m2}N_{lpt}$$

Then, construct an equivalent transformation of above equations. This can be done by moving all items to the right side of these equations, and timing them with several time-varying variables, as the following.

$$0 = \frac{T_V}{p_V}\frac{R}{C_p - R}\left[(RT_V - h_V)(W_{fnd} - W_{nzd}) + (h_{fnd}W_{fnd} - h_V W_{nzd})\right]$$
$$0 = \frac{R}{C_p - R}\left[\left(C_p T_V - h_V\right)(W_{fnd} - W_{nzd}) + (h_{fnd}W_{fnd} - h_V W_{nzd})\right]$$
$$0 = \frac{T_{VI}}{p_{VI}}\frac{R}{C_p - R}\left[(RT_{VI} - h_{VI})\left(W_{fnc} - W_{lpc}\right) + \left(h_{fnc}W_{fnc} - h_{VI}W_{lpc}\right)\right]$$
$$0 = \frac{R}{C_p - R}\left[\left(C_p T_{VI} - h_{VI}\right)\left(W_{fnc} - W_{lpc}\right) + \left(h_{fnc}W_{fnc} - h_{VI}W_{lpc}\right)\right]$$
$$0 = \frac{T_I}{p_I}\frac{R}{C_p - R}\left[(RT_I - h_I)\left(W_{lpc} - W_{hpc}\right) + \left(h_{lpc}W_{lpc} - h_I W_{hpc}\right)\right]$$
$$0 = \frac{R}{C_p - R}\left[\left(C_p T_I - h_I\right)\left(W_{lpc} - W_{hpc}\right) + \left(h_{lpc}W_{lpc} - h_I W_{hpc}\right)\right]$$
$$0 = \frac{T_{II}}{p_{II}}\frac{R}{C_p - R}\left[(RT_{II} - h_{II})\left(W_{hpc} + W_f - W_{hpt}\right) + \left(h_{hpc}W_{hpc} + \eta_b H_u W_f - h_{II}W_{hpt}\right)\right]$$
$$0 = \frac{R}{C_p - R}\left[\left(C_p T_{II} - h_{II}\right)\left(W_{hpc} + W_f - W_{hpt}\right) + \left(h_{hpc}W_{hpc} + \eta_b H_u W_f - h_{II}W_{hpt}\right)\right]$$
$$0 = \frac{T_{III}}{p_{III}}\frac{R}{C_p - R}\left[(RT_{III} - h_{III})\left(W_{hpt} - W_{lpt}\right) + \left(h_{hpt}W_{hpt} - h_{IV}W_{lpt}\right)\right]$$
$$0 = \frac{R}{C_p - R}\left[\left(C_p T_{III} - h_{III}\right)\left(W_{hpt} - W_{lpt}\right) + \left(h_{hpt}W_{hpt} - h_{IV}W_{lpt}\right)\right]$$
$$0 = \frac{T_{IV}}{p_{IV}}\frac{R}{C_p - R}\left[(RT_{IV} - h_{IV})\left(W_{lpt} - W_{znc}\right) + \left(h_{lpt}W_{lpt} - h_{IV}W_{znc}\right)\right]$$
$$0 = \frac{R}{C_p - R}\left[\left(C_p T_{IV} - h_{IV}\right)\left(W_{lpt} - W_{znc}\right) + \left(h_{lpt}W_{lpt} - h_{IV}W_{znc}\right)\right]$$
$$\dot{n}_1 = \left(\frac{30}{\pi}\right)^2 \frac{\eta_{m1}N_{hpt} - N_{fnd} - N_{fnc} - N_{lpc}}{J_1 n_1}$$
$$\dot{n}_2 = \left(\frac{30}{\pi}\right)^2 \frac{\eta_{m2}N_{lpt} - N_{hpc}}{J_2 n_2}$$

It's amazing to notice that the above equations constitute a standard problem of dynamic system with constraint equations. Therefore, it's possible to apply the theorem 1 in this article, and to construct a non-iterative dynamic equation set problem by substituting 0 in the left side of each equation with differential items.

$$V_V\dot{T}_V = \frac{T_V}{p_V}\frac{R}{C_p - R}\left[(RT_V - h_V)(W_{fnd} - W_{nzd}) + (h_{fnd}W_{fnd} - h_V W_{nzd})\right]$$
$$V_V\dot{p}_V = \frac{R}{C_p - R}\left[\left(C_p T_V - h_V\right)(W_{fnd} - W_{nzd}) + (h_{fnd}W_{fnd} - h_V W_{nzd})\right]$$
$$V_{VI}\dot{T}_{VI} = \frac{T_{VI}}{p_{VI}}\frac{R}{C_p - R}\left[(RT_{VI} - h_{VI})\left(W_{fnc} - W_{lpc}\right) + \left(h_{fnc}W_{fnc} - h_{VI}W_{lpc}\right)\right]$$
$$V_{VI}\dot{p}_{VI} = \frac{R}{C_p - R}\left[\left(C_p T_{VI} - h_{VI}\right)\left(W_{fnc} - W_{lpc}\right) + \left(h_{fnc}W_{fnc} - h_{VI}W_{lpc}\right)\right]$$
$$V_I\dot{T}_I = \frac{T_I}{p_I}\frac{R}{C_p - R}\left[(RT_I - h_I)\left(W_{lpc} - W_{hpc}\right) + \left(h_{lpc}W_{lpc} - h_I W_{hpc}\right)\right]$$
$$V_I\dot{p}_I = \frac{R}{C_p - R}\left[\left(C_p T_I - h_I\right)\left(W_{lpc} - W_{hpc}\right) + \left(h_{lpc}W_{lpc} - h_I W_{hpc}\right)\right]$$
$$V_{II}\dot{T}_{II} = \frac{T_{II}}{p_{II}}\frac{R}{C_p - R}\left[(RT_{II} - h_{II})\left(W_{hpc} + W_f - W_{hpt}\right) + \left(h_{hpc}W_{hpc} + \eta_b H_u W_f - h_{II}W_{hpt}\right)\right]$$
$$V_{II}\dot{p}_{II} = \frac{R}{C_p - R}\left[\left(C_p T_{II} - h_{II}\right)\left(W_{hpc} + W_f - W_{hpt}\right) + \left(h_{hpc}W_{hpc} + \eta_b H_u W_f - h_{II}W_{hpt}\right)\right]$$
$$V_{III}\dot{T}_{III} = \frac{T_{III}}{p_{III}}\frac{R}{C_p - R}\left[(RT_{III} - h_{III})\left(W_{hpt} - W_{lpt}\right) + \left(h_{hpt}W_{hpt} - h_{IV}W_{lpt}\right)\right]$$
$$V_{III}\dot{p}_{III} = \frac{R}{C_p - R}\left[\left(C_p T_{III} - h_{III}\right)\left(W_{hpt} - W_{lpt}\right) + \left(h_{hpt}W_{hpt} - h_{IV}W_{lpt}\right)\right]$$
$$V_{IV}\dot{T}_{IV} = \frac{T_{IV}}{p_{IV}}\frac{R}{C_p - R}\left[(RT_{IV} - h_{IV})\left(W_{lpt} - W_{znc}\right) + \left(h_{lpt}W_{lpt} - h_{IV}W_{znc}\right)\right]$$
$$V_{IV}\dot{p}_{IV} = \frac{R}{C_p - R}\left[\left(C_p T_{IV} - h_{IV}\right)\left(W_{lpt} - W_{znc}\right) + \left(h_{lpt}W_{lpt} - h_{IV}W_{znc}\right)\right]$$
$$\dot{n}_1 = \left(\frac{30}{\pi}\right)^2 \frac{\eta_{m1}N_{hpt} - N_{fnd} - N_{fnc} - N_{lpc}}{J_1 n_1}$$
$$\dot{n}_2 = \left(\frac{30}{\pi}\right)^2 \frac{\eta_{m2}N_{lpt} - N_{hpc}}{J_2 n_2}$$

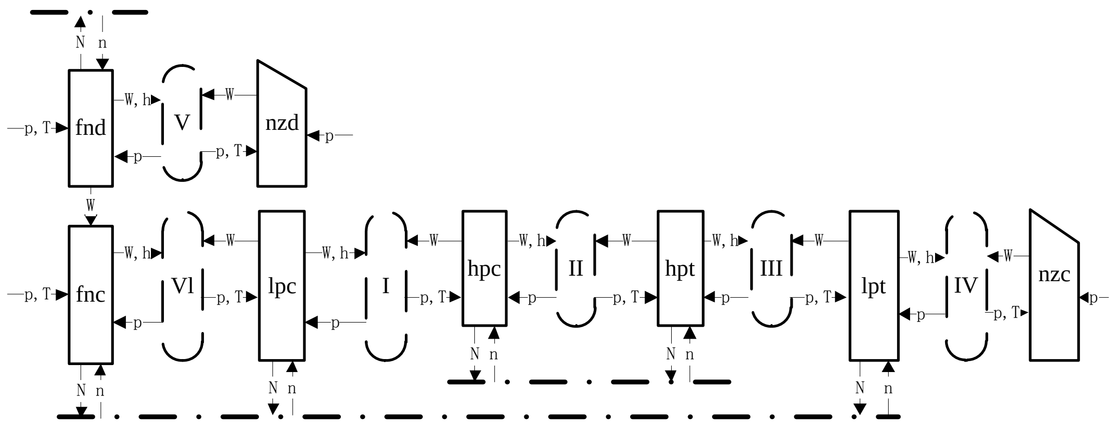


Figure 4. Parameter transmission model of a turbofan engine

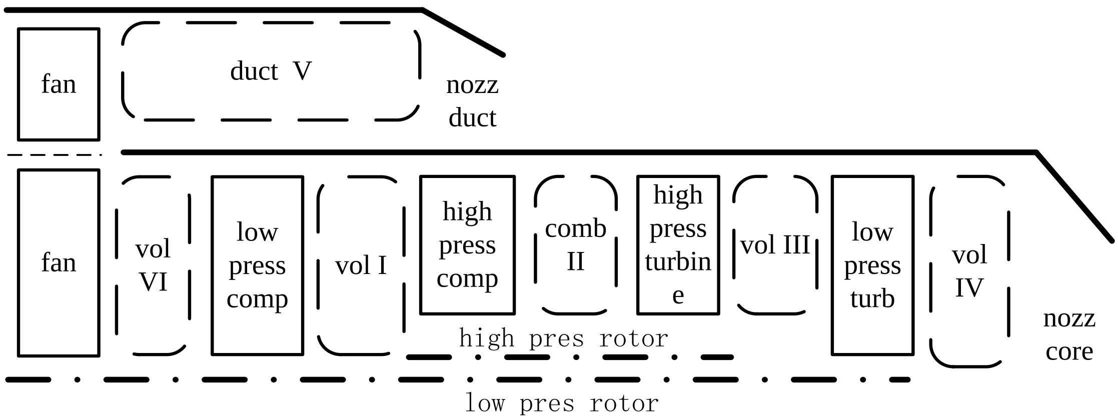


Figure 5. Component – volume model of an ideal turbofan engine

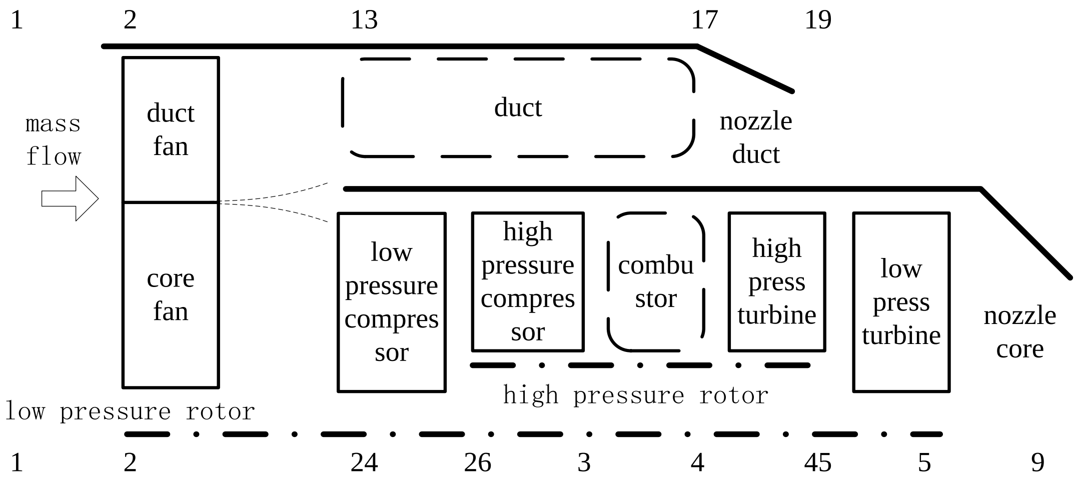


Figure 6. Component level model of a real turbofan engine

Then, the above dynamic equation set of the overall turbofan engine system can be solved with Euler, Runge-Kutta discrete integral methods, which avoided time-consuming iterations.

Furthermore, in simulations, if we express the above equations in input-output form, and combine them together, we will have a overall parameter flow diagram Figure 4.

By comparison between the above equation set and the original volume dynamic model of this kind of turbofan[6], as is shown in Figure 5.

It's amazing to find that they are identical in form. The only difference is that in the constructed equations of an unconstrained dynamic problem, the "V" is an arbitrarily small positive constant; on the contrary, the "V" in a volume dynamic model is the exact size of volumes.

This is a significant finding which derives the following conclusion: the theoretical explanation of inserting virtual volume is constructing non-iterative dynamic equation problems. Since the volumes are comparatively small, and volume dynamics are usually ten times faster to rotor dynamics, the impact of virtual volumes can also be explained as adding several large (in norm sense) poles to the original system, and keeping the original rotor dynamics unchanged.

If building volume dynamic models can be equivalent to adding large poles, we have the following corollaries:

1) If volumes are small enough, changing the size of volumes won't change the time-domain response pattern of the rotor dynamic system. Volumes with amplified size or diminished size shall be called virtual volumes.

2) The smaller the size of virtual volumes are, the little time-domain response difference there is between volume dynamic model and rotor dynamic model.

B. Modeling of Real Turbofan Engine with Fan Leakage

In real turbofan engines, there is flow leakage between the duct fan and core fan. That is to say, at the outlet of fan, there is mass flow leakage either from the duct passage to the core passage, or from the core passage to the duct passage. This can be shown as Figure 6.

Therefore, the original flow rate equilibrium at the outlet of fan disappears, thus turned the original constraint dynamic problem to an under-constraint problem. That is to say:

According to the no leakage assumption, the equilibrium equations of volume V are:

$$W_{fnd}(p_V, n_1) = W_{nzd}(p_V, T_V)$$
$$h_{fnd}(p_V, n_1)W_{fnd} = h_V(T_V)W_{nzd}$$

The equilibrium equations of volume VI are

$$W_{fnc}(p_{VI}, n_1) = W_{lpc}(p_{VI}, p_I, T_{VI}, n_1)$$
$$h_{fnc}(p_{VI}, n_1)W_{fnc} = h_{VI}(T_{VI})W_{lpc}$$

Then, there are 14 independent variables for 14 unrelated equilibrium/dynamic functions. Therefore, there is only one certain solution for this problem.

But for real engines, the above equations were reduced to two equations as the following:

$$W_{fnd} + W_{fnc} = W_{nzd} + W_{lpc}$$
$$h_{fnd}W_{fnd} = h_V W_{nzd}$$

This means, the problem of 14 independent variables has only 12 unrelated constraint/dynamic equations. Therefore, there are no certain solutions to this problem.

To overcome it, one of our strategies is to cut down the number of independent variables. We have to use a dependent equation of flow rate between duct fan and core fun:

$$W_{fnc} = \frac{1}{BPR}W_{fnd}$$

Wherein, the bypass ratio between the flow rate of duct fan and core fan is constant.

Then, while the outlet air parameter p_{VI}, and T_{VI} can be calculated with flow rate and inlet temperature as the following

$$p_{VI} = p_{VI}\left(\frac{1}{BPR}W_{fnd}, n_1\right)$$

$$T_{VI} = T_{VI}\left(\frac{1}{BPR}W_{fnd}, n_1\right)$$

We can build the relationship between p_{VI}, T_{VI}, and p_V, T_{VI}. Buy this way, the number of variables is reduced from 14 to 12. This accurately matches the 12 constraint/dynamic function, making the problem solvable. The new overall volume dynamic model of turbofan engines becomes:

$$\begin{aligned}&W_{fnd}(p_V, n_1) + \frac{1}{BPR}W_{fnd}\\ &= W_{nzd}(p_V, T_V, n_1)\\ &+ W_{lpc}\left(p_{VI}\left(\frac{1}{BPR}W_{fnd}, n_1\right), T_{VI}\left(\frac{1}{BPR}W_{fnd}, n_1\right), n_1\right)\end{aligned}$$

$$\begin{aligned}h_{fnd}(p_V, T_V, n_1)W_{fnd} + h_{fnc}(p_V, T_V, n_1)\frac{1}{BPR}W_{fnd}\\ = h_V(T_V)W_{nzd} + h_{fnc}W_{lpc}\end{aligned}$$

The above equations stands for the new volume equilibrium function of volume V. And, according to the parameter transfer relationship among all equations, there will be no need to construct volume VI, therefore, we cut it.

Other equations remain unchanged.

$$W_{lpc} = W_{hpc}(p_I, T_I, n_2)$$
$$h_{lpc}(p_I, T_0, n_1)W_{lpc} = h_I(T_I)W_{hpc}$$
$$W_{hpc} + W_f = W_{hpt}(p_{II}, T_{II}, n_2)$$
$$h_{hpc}(p_I, T_I, n_2)W_{hpc} + \eta_b H_u W_f = h_{II}(T_{II})W_{hpt}$$
$$W_{hpt} = W_{lpt}(p_{III}, T_{III}, n_1)$$
$$h_{hpt}(p_{II}, T_{II}, n_2)W_{hpt} = h_{IV}(T_{IV})W_{lpt}$$
$$W_{lpt} = W_{znc}(p_{IV}, T_{IV})$$
$$h_{lpt}(p_{III}, T_{III}, n_1)W_{lpt} = h_{IV}(T_{IV})W_{znc}$$
$$N_{fnd} + N_{fnc} + N_{lpc} = \eta_{m1}N_{hpt}$$
$$N_{hpc} = \eta_{m2}N_{lpt}$$

Again, using the same method of Part A, we can construct the following non-iterative differential function set.

$$\begin{aligned}T'_V = \frac{RT_V}{V_V p}\frac{1}{C_p - R}\Big[&(RT_V - h_V)\Big(W_{fnd} + \frac{1}{BPR}W_{fnd}\\ &- W_{nzd} - W_{lpc}\Big)\\ &+ \Big(h_{fnd}\ W_{fnd} + h_{fnc}\frac{1}{BPR}W_{fnd}\\ &- h_V W_{nzd} + h_{fnc}W_{lpc}\Big)\Big]\end{aligned}$$

$$\begin{aligned}p'_V = \frac{R}{V_V}\frac{1}{C_p - R}\Big[&(C_p T_V - h_V)\Big(W_{fnd} + \frac{1}{BPR}W_{fnd}\\ &- W_{nzd} - W_{lpc}\Big)\\ &+ \Big(h_{fnd}\ W_{fnd} + h_{fnc}\frac{1}{BPR}W_{fnd}\\ &- h_V W_{nzd} + h_{fnc}W_{lpc}\Big)\Big]\end{aligned}$$

$$\begin{aligned}\dot{T}_I = \frac{RT_I}{V_I p_I}\frac{1}{C_p - R}\big[&(RT_I - h_I)(W_{lpc} - W_{hpc})\\ &+ (h_{lpc}W_{lpc} - h_I W_{hpc})\big]\end{aligned}$$

$$\begin{aligned}\dot{p}_I = \frac{R}{V_I}\frac{1}{C_p - R}\big[&(C_p T_I - h_I)(W_{lpc} - W_{hpc})\\ &+ (h_{lpc}W_{lpc} - h_I W_{hpc})\big]\end{aligned}$$

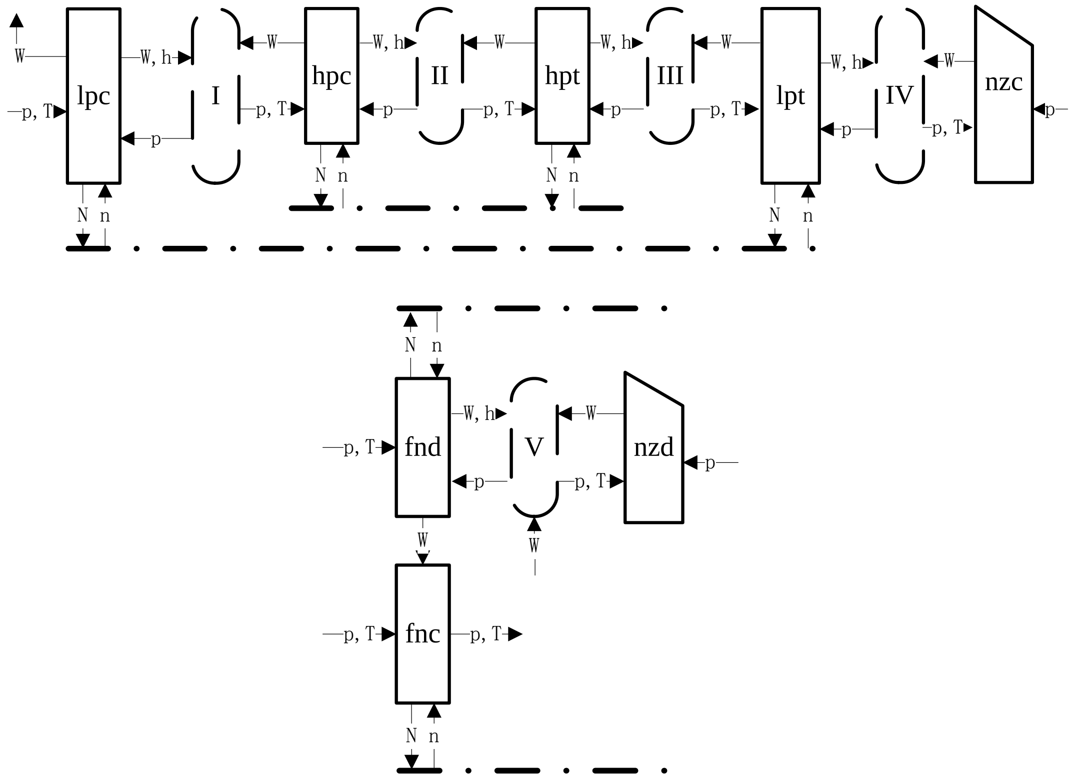


Figure 7. Parameter transmission model of a real turbofan engine

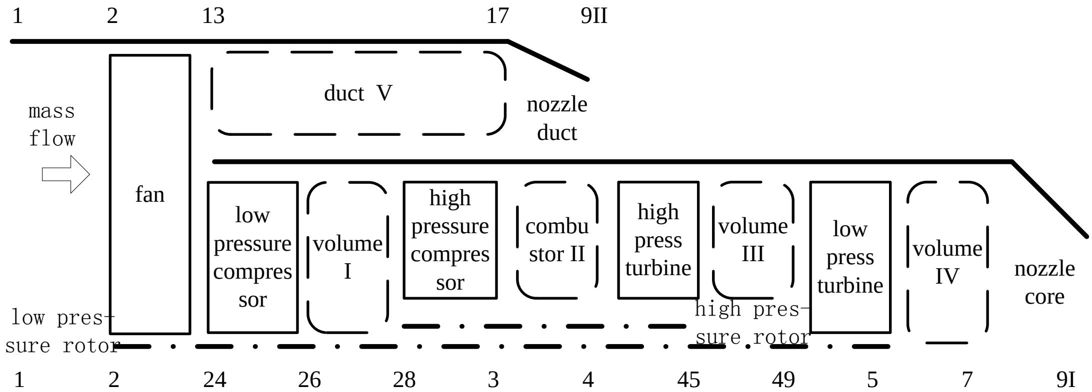


Figure 8. Component - volume model of a real turbofan engine

$$\dot{T}_{II} = \frac{RT_{II}}{V_{II}p_{II}}\frac{1}{C_p - R}\left[(RT_{II} - h_{II})(W_{hpc} + W_f - W_{hpt}) + (h_{hpc}W_{hpc} + \eta_b H_u W_f - h_{II}W_{hpt})\right]$$

$$\dot{p}_{II} = \frac{R}{V_{II}}\frac{1}{C_p - R}\left[(C_p T_{II} - h_{II})(W_{hpc} + W_f - W_{hpt}) + (h_{hpc}W_{hpc} + \eta_b H_u W_f - h_{II}W_{hpt})\right]$$

$$\dot{T}_{III} = \frac{RT_{III}}{V_{III}p_{III}}\frac{1}{C_p - R}\left[(RT_{III} - h_{III})(W_{hpt} - W_{lpt}) + (h_{hpt}W_{hpt} - h_{IV}W_{lpt})\right]$$

$$\dot{p}_{III} = \frac{R}{V_{III}}\frac{1}{C_p - R}\left[(C_p T_{III} - h_{III})(W_{hpt} - W_{lpt}) + (h_{hpt}W_{hpt} - h_{IV}W_{lpt})\right]$$

$$\dot{T}_{IV} = \frac{RT_{IV}}{V_{IV}p_{IV}}\frac{1}{C_p - R}\left[(RT_{IV} - h_{IV})(W_{lpt} - W_{znc}) + (h_{lpt}W_{lpt} - h_{IV}W_{znc})\right]$$

$$\dot{p}_{IV} = \frac{R}{V_{IV}}\frac{1}{C_p - R}\left[(C_p T_{IV} - h_{IV})(W_{lpt} - W_{znc}) + (h_{lpt}W_{lpt} - h_{IV}W_{znc})\right]$$

$$\dot{n}_1 = \left(\frac{30}{\pi}\right)^2 \frac{\eta_{m1}N_{hpt} - N_{fnd} - N_{fnc} - N_{lpc}}{J_1 n_1}$$

$$\dot{n}_2 = \left(\frac{30}{\pi}\right)^2 \frac{\eta_{m2}N_{lpt} - N_{hpc}}{J_2 n_2}$$

Again, build the overall parameter flow diagram as the Figure 7.

In fact, parameter flow model can also help us to construct volume dynamic models. Therefore, the parameter flow relationship shown in the above diagram can help us building a volume dynamic model, as is shown in Figure 8.

I. MODEL LINEARIZATION FOR VOLUME DYNAMIC MODELS

Even if our 12 independent variable model can operate in a 12 dimension parameter space, real engines only work near a narrow steady state line. Therefore, in this part, our linearization work only deal with the working points at the steady state line by differential methods as the following

Let's consider this problem

$$\dot{x} = f(x, u) \tag{31}$$

Where,

x=[P17,T17,P26,T26,P4,T4,P45,T45,P7,T7,N1,N2]T;

u=[H,Ma, Wf]T.

For a certain point (x_{st},u_{st}) at operation line,

$$0 = f(x_{st}, u_{st}) \tag{32}$$

There is

$$\begin{aligned}\dot{x} &= f(x,u) - f(x_s,u) + f(x_s,u) - f(x_s,u_s) \\ &\approx J_{fx}\Delta x + J_{fu}\Delta u = A\Delta x + B\Delta u\end{aligned} \tag{33}$$

Wherein, Jaccobi matrix is

$$J_{fx} = \left\{\frac{\partial f_i}{\partial x_j}\right\}_{n\times n}, J_{fu} = \left\{\frac{\partial f_i}{\partial u_j}\right\}_{n\times l}$$

Wherein, n is the dimension of state variable vector; L is the dimension of output vector.

Therefore, the task of linearization is to calculate coefficient elements

$$a_{ij} = \frac{\partial f_i}{\partial x_j}, u_{ij} = \frac{\partial f_i}{\partial u_j}$$

Since it's almost impossible to deduce analytical formulae of partial functions, we choose perturbation method. That is to say, according to theory of partial difference, partial expression can be approximate to discrete expression

$$a_{ij} = \frac{\partial f_i}{\partial x_j} \approx \frac{f_i(x_{j,st}+\Delta x_{j,st}) - f_i(x_{j,st}-\Delta x_{j,st})}{2\Delta x_{j,st}} \tag{34}$$

In fact, it is not difficult to calculate $f_i(x_{j,st} \pm \Delta x_{j,st})$ under any small perturbation with our Matlab codes of engine.

$$f_i(x_{j,st} \pm \Delta x_{j,st}) = f(u, x_1, x_2, ..., x_{j,st} \pm \Delta x_{j,st}, ..., x_{14,st}) \tag{35}$$

II. STATE SPACE ORDER-REDUCTION BY FREQUENCY DECOMPOSITION

It's assumed in the introduction section of this article that there are only two rotor dynamics for turbofan engines, or there is no volume dynamic effect in real turbofan engines. Therefore, we need to reduce the order of our constructed high-order volume dynamic.

Let's start with a typical state space expression

$$\begin{cases}\dot{x} = A\tilde{x} + Bu \\ y = C\tilde{x} + Du\end{cases} \tag{36}$$

Transform it into a diagonal form,

$$\begin{cases}\dot{\tilde{x}} = \Lambda\tilde{x} + \tilde{B}u \\ y = \tilde{C}\tilde{x} + \tilde{D}u\end{cases} \tag{37}$$

wherein, $\tilde{x}$ has no physical meaning, $x=T\tilde{x}$;

$\Lambda=T^{-1}AT$ is a diagonal matrix with n×n dimension. Its diagonal elements are eigenvalues of matrix A. Without loss of generality, let's assume diagonal elements of Λ are ranked in descendent order. Then, divide above model into a reduced state coordinate $\tilde{x}1$ of r dimension, and the other state coordinate $\tilde{x}2$ of (n-4) dimension.

$$\begin{cases}\begin{bmatrix}\dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2\end{bmatrix} = \begin{bmatrix}\Lambda_1 & 0 \\ 0 & \Lambda_2\end{bmatrix}\begin{bmatrix}\tilde{x}_1 \\ \tilde{x}_2\end{bmatrix} + \begin{bmatrix}\tilde{B}_1 \\ \tilde{B}_2\end{bmatrix}u \\ y = \begin{bmatrix}\tilde{C}_1 & \tilde{C}_2\end{bmatrix}\begin{bmatrix}\tilde{x}_1 \\ \tilde{x}_2\end{bmatrix} + \tilde{D}u\end{cases} \tag{38}$$

In above equation, Λ_1 is a diagonal matrix of r×r size. Λ_2 is a diagonal matrix of (n-r)×(n-r) size.

We also define

$$\begin{bmatrix}\dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2\end{bmatrix} = \begin{bmatrix}T_{11} & T_{12} \\ T_{21} & T_{22}\end{bmatrix}\begin{bmatrix}\tilde{x}_1 \\ \tilde{x}_2\end{bmatrix} \tag{39}$$

According to our assumption, dynamic process related to $\tilde{x}2$ is greatly larger than $\tilde{x}1$. That means $\dot{\tilde{x}}_2 \approx 0$. Therefore,

$$\dot{\tilde{x}}_1 = \Lambda_1\tilde{x}_1 + \tilde{B}_1 u \tag{40}$$

finally, the reduced state space model can be expressed as follows

$$\begin{aligned}&\dot{x}_1 = A_r x_1 + B_r u \\ &\begin{bmatrix}x_2 \\ y\end{bmatrix} = \begin{bmatrix}\tilde{C}_r \\ C_r\end{bmatrix}x_1 + \begin{bmatrix}\tilde{D}_r \\ D_r\end{bmatrix}u\end{aligned} \tag{41}$$

wherein,

$$\begin{aligned}
A_r &= T_{11}\Lambda T_{11}^{-1} \\
B_r &= T_{11}\left(\Lambda_1 T_{11}^{-1} T_{12}\Lambda_2^{-1}\tilde{B}_2 + \tilde{B}_1\right) \\
\tilde{C}_r &= T_{21}T_{11}^{-1} \\
C_r &= C_1 + C_2 T_{21}T_{11}^{-1} = C_1 + C_2\tilde{C}_r \\
\tilde{D}_r &= \left(T_{21}T_{11}^{-1}T_{12} - T_{22}\right)\Lambda_2^{-1}B_2 \\
D_r &= D + C_2\left[\left(T_{21}T_{11}^{-1}T_{12} - T_{22}\right)\Lambda_2^{-1}\tilde{B}\right] = D + C_2\tilde{D}_r
\end{aligned}$$

$$A = \begin{bmatrix} -13.70 & -0.02 & -156.41 & 0.00 & -8.07 & 0.47 & 0.01 & 0.00 & 63.49 & 13.56 & -96.55 & 0.00 \\ 0.00 & -3.63 & 0.00 & 0.00 & -60.86 & 42.52 & 7.87 & 4.83 & -10.73 & 0.00 & 0.00 & 0.00 \\ 10.11 & 0.00 & -297.88 & -32.04 & 12.75 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 6.02 & 0.00 & -131.45 & -114.94 & 7.57 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 66.41 & -18.03 & 1342.68 & 0.00 & -872.24 & 90.70 & 6.85 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 36.18 & -8.31 & 713.89 & 0.00 & -323.40 & -318.64 & 3.16 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 43.45 & 0.00 & 0.00 & 956.84 & -464.21 & -139.12 & -129.54 & 4.60 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.67 & 0.00 & 0.00 & -207.78 & 244.88 & -10.70 & -118.84 & 0.57 & 0.00 & 0.00 & 0.00 \\ 20.40 & -39.02 & 0.00 & 0.00 & -121.19 & 83.08 & 802.16 & 1054.01 & -3483.26 & -1234.6 & 937.44 & 0.00 \\ 7.97 & -54.15 & 0.00 & 0.00 & -195.60 & 134.09 & 123.14 & 2751.01 & -576.99 & -3423.2 & 366.41 & 0.00 \\ -1.54 & -0.12 & 0.00 & 0.00 & -3.79 & 2.59 & 0.04 & 0.00 & 96.17 & 39.33 & -654.0 & -45.7 \\ -5.66 & -0.52 & 0.00 & 0.00 & -16.79 & 11.51 & 0.20 & 0.00 & 48.77 & 329.06 & -520.51 & -419.15 \end{bmatrix}$$

$$B = [61.25 \quad 27.49 \quad 0 \quad 0 \quad 0 \quad 0 \quad 65674.04 \quad 33978.08 \quad 10797.96 \quad 17428.34 \quad 337.26 \quad 1495.93]^T$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0.01 \\ 0 & 0 & 0 & 0 & 0 & 2646.63 \\ 0 & 0 & 0 & 0 & 0 & -1.04 \\ 0 & 0 & 0 & 0 & 0 & 0.33 \\ 0 & 0 & 0 & 0 & 0 & -0.23 \\ 0 & 0 & 0.956 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 927.22 \\ 0 & 0 & 0 & 0 & 0 & -0.61 \end{bmatrix}^T, \quad D = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -29.43]^T$$

Box 1. Linear model of a turbofan engine

III. VALIDATION

Based on the above, a nonlinear model of turbofan engine can be built according to the data shown in GPS11. Without the consideration of real gas effects and variable geometries, a simplified model can be obtain since its accuracy would not be the best, which has no influence on the topic in this paper.

Table 1 shows the results of model solving the implicit nonlinear constraint equations and the differential dynamic functions. The flight conditions (altitude and velocity) are defined in the table. Besides, fuel flow is the only input for the model because variable geometries are ignored in this topic. It is obvious that it is an iteration model if implicit nonlinear constraint equations are used in the calculation. Therefore, it has no reflection of the relationship between states of engine and time it works. However, if the differential dynamic functions are applied to get the results of this model, the value will be stable if the simulation time is long enough. Then this value can be compared with the one calculated by the model with iteration equations.

Table 1. Comparison between the iterative model and non-iterative model under standard atmospheric condition

$PH(kPa)$	101.3250		$TH(K)$	288.1500
State Point	Model with Iteration	Model without Iteration	Model with Iteration	Model without Iteration
$W_f(kg/s)$	2.4912	2.4912	0.4912	0.4912
$N_1(rpm)$	3400.6665	3400.6945	1587.1787	1586.7747
$N_2(rpm)$	10392.3168	10392.9663	6995.2203	6994.5018
$P_3(kPa)$	2904.9892	2904.4628	811.0407	811.1239
$T_3(K)$	826.4746	826.5078	578.1026	578.1458
$P_4(kPa)$	2904.9892	2904.4628	811.0407	811.1239
$T_4(K)$	1461.3105	1461.5333	980.8553	980.7689
$P_{45}(kPa)$	711.2051	711.0820	229.1750	229.1868
$T_{45}(K)$	1080.6877	1080.8649	747.0911	747.0232
$P_5(kPa)$	160.9515	160.9319	108.0679	108.0689
$T_5(K)$	781.5014	781.6451	635.1354	635.0660
$F(N)$	262728.4599	262724.9776	41787.1177	41764.4943

Table 2 gives another group of results between the models with iteration and without iteration.

Table 2. . Comparison between the iterative model and non-iterative model under nonstandard atmospheric condition

$PH(kPa)$	22.6105		$TH(K)$	216.6500
State Point	Model with Iteration	Model without Iteration	Model with Iteration	Model without Iteration
$W_f(kg/s)$	0.8200	0.8200	0.4200	0.4200
$N_1(rpm)$	3237.5825	3237.5778	2630.6916	2631.1742
$N_2(rpm)$	9615.8195	9616.7247	8366.3062	8369.7464
$P_3(kPa)$	1050.7553	1050.6588	640.4588	640.5691
$T_3(K)$	719.0494	719.0891	621.5733	621.6169
$P_4(kPa)$	1050.7553	1050.6588	640.4588	640.5691
$T_4(K)$	1278.0302	1278.1211	1068.3169	1068.4890
$P_{45}(kPa)$	256.1140	256.0844	155.8589	155.8800
$T_{45}(K)$	938.9905	939.0489	781.9117	781.9989
$P_5(kPa)$	55.9498	55.9434	35.1107	35.1141
$T_5(K)$	666.5491	666.5962	555.4251	555.4827
$F(N)$	51955.5975	51946.8444	24293.1467	24311.6095

From the tables shown above, it can be seen that the proposed model using differential dynamic functions has the same response as the model with the implicit nonlinear constraint equations.

The comparisons between the nonlinear no iteration model and linear small perturbation model are shown in figure 10. The set point of above linear model is given in Table 3.

Table 3. Steady states for linearization in this paper

$H(km)$	$N_1(rpm$	$N_2(rpm$	$P_{17}(kP$	$T_{17}(K)$	$P_{26}(kPa$	$T_{26}(K)$	$P_4(kPa)$
0	3257.2794	10231.837	161.6260	336.1303	230.9457	374.4402	2668.1416
$T_4(K)$	$P_{45}(kPa$	$T_{45}(K)$	$P_7(kPa$	$T_7(K)$	$P_{s3}(kPa$	$F(N)$	$W_f(kg/s)$
1427.2922	653.5763	1054.8986	149.3467	766.6013	2550.7434	238138.6096	2.2412

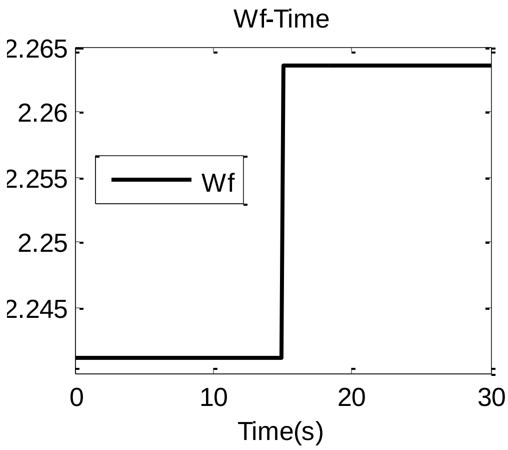
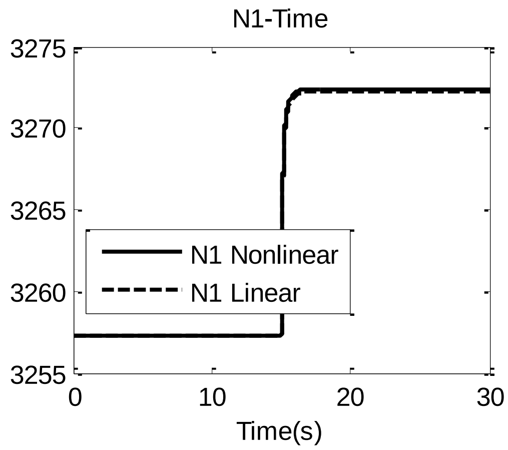
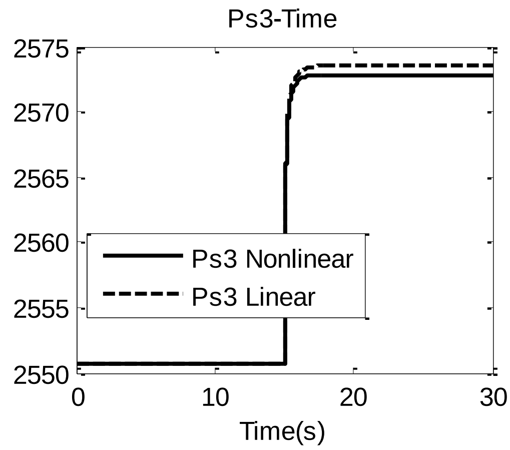


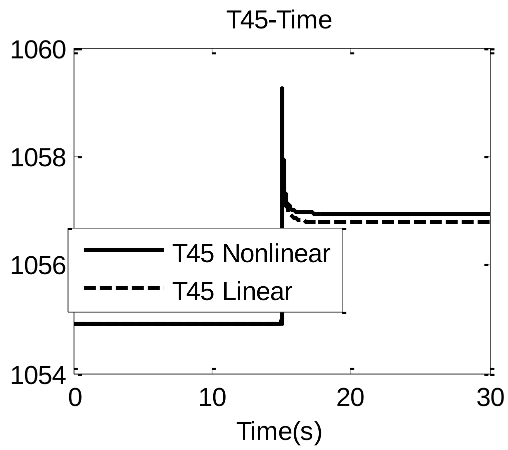
Figure 9. Fuel flow input to the engine



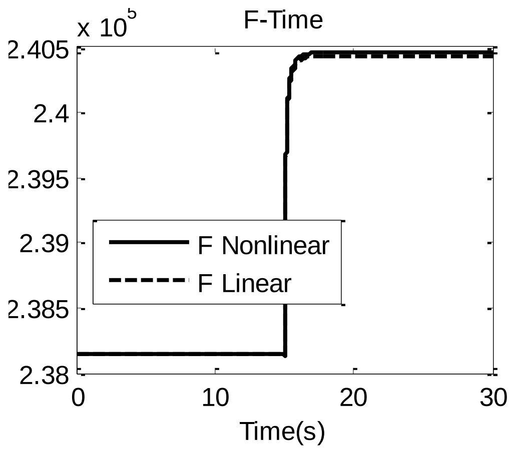
(a) Rotor speed of low pressure shaft



(b) Static pressure at the outlet of LPC

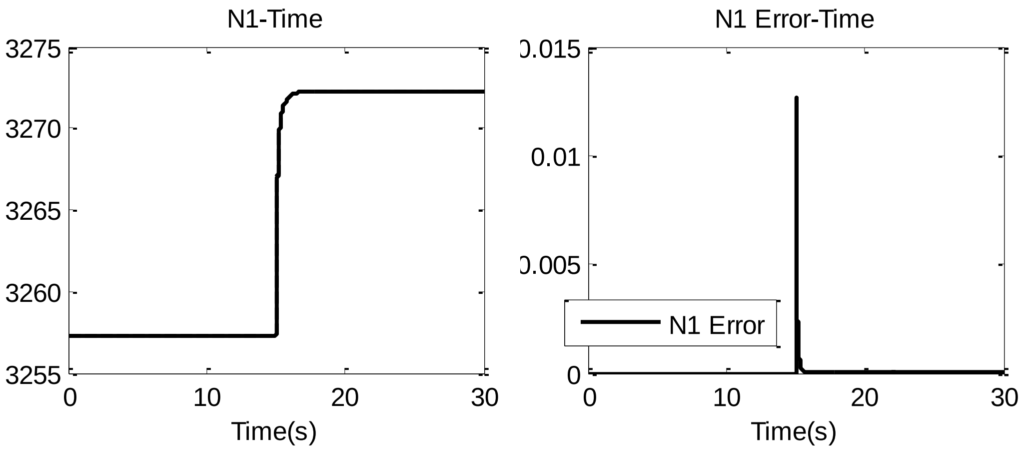


(c) Total temperature before LPT

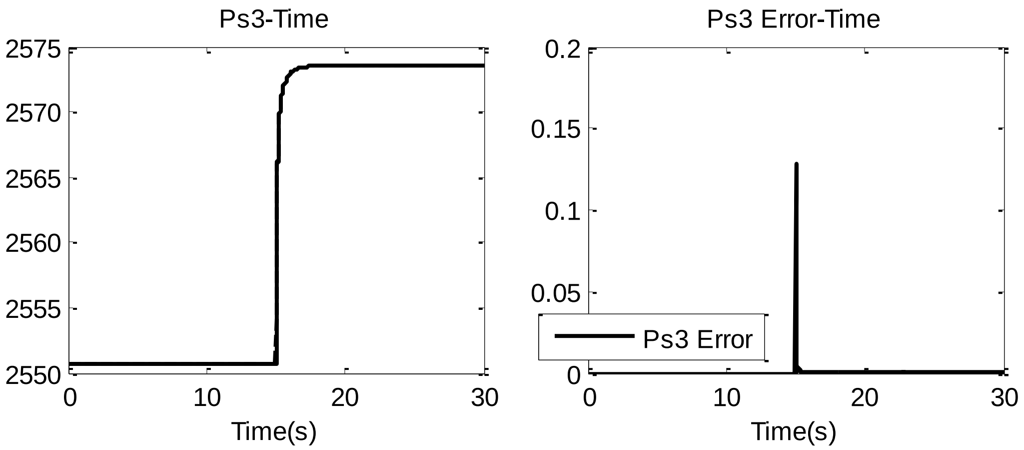


(d) Thrust response

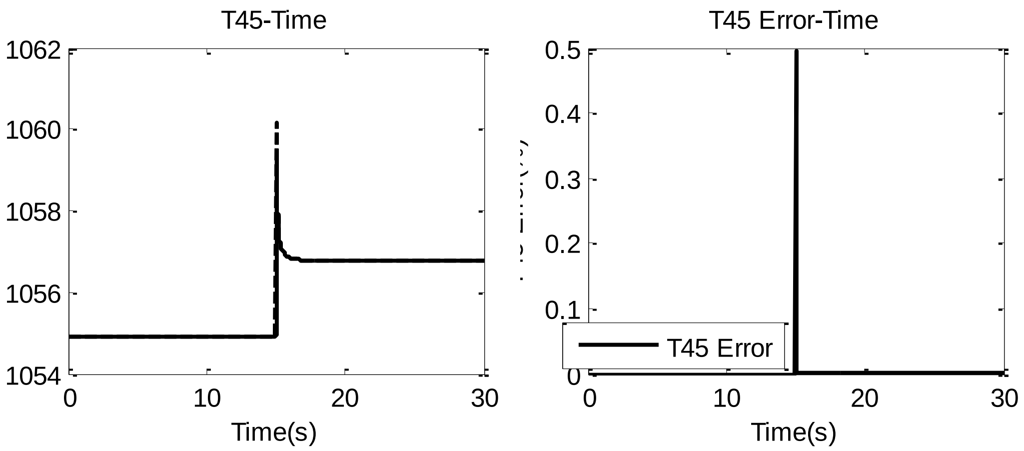
Figure 10. Simulation comparison between the nonlinear model and linear model



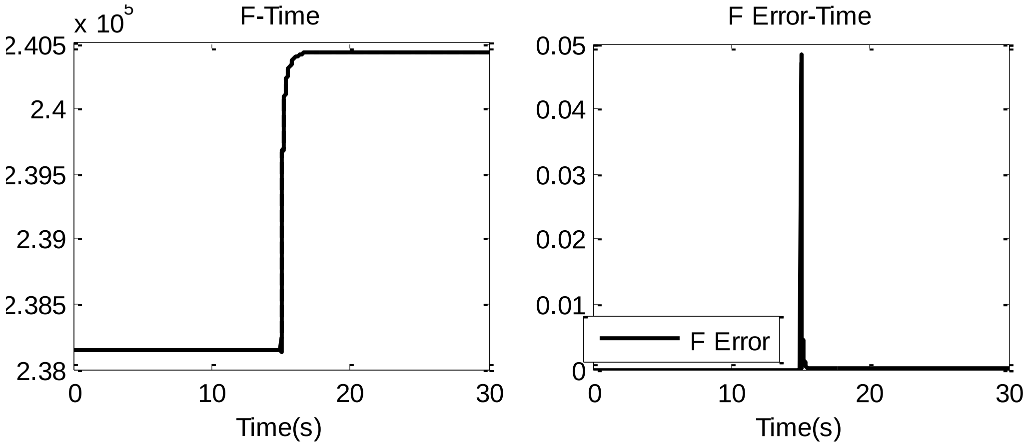
(a) Rotor speed of low pressure shaft



(b) Static pressure at the outlet of LPC



(c) Total temperature before LPT



(d) Thrust response

Figure 11. Simulation comparison between the 14-order model and 2-order model

Here a small step of fuel flow is given in this validation, as is shown in Figure 9. The results can be seen in figure 10.

From the figures 10, it can be concluded that the high-order linear model has a good accuracy compared with the nonlinear dynamic model. Although the results of nonlinear model and linear one are not the same strictly, the accuracy of this linear model based on the method of small perturbations is good enough to build a right LPV over the flight envelope of this engine. However, the simulation step time of this high-order linear model may be the same as the nonlinear model because its high-order and sickness of equations. So the order reduction method mentioned is this paper can be used in this linear model. The linear model above can be reduced to only 2 orders while the matrixes in this model could be shown as following equations.

$$A = \begin{bmatrix} -9.75292609 & 1.80305909 \\ -1.75511125 & -1.57418790 \end{bmatrix}$$

$$B = \begin{bmatrix} 5377.11166800 \\ 2186.35185697 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1.17170481 & 0.13226449 \\ -0.27993244 & 0.02637924 \\ -0.23063328 & 0.00732620 \\ 145.08261806 & 0.68175569 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 \\ 0 \\ 146.20739332 \\ 285.55601666 \\ 233.97362280 \\ 4982.51041164 \end{bmatrix}$$

Figure 11 give the comparisons between the results of high-order linear model and the lower one. The input of fuel is the same as figure 9.

From the figures above, it can be concluded that the reduced linear model are fully consistent with the high-ordered linear model and easier to solve. The maximum difference between this two kinds of linear model is about 0.5%. As a result, it can accept a longer simulation step.

IV. SUMMARY

In this article, a constructive non-iterative algorithm for solving implicit constraint equation set is introduced, and it's sufficient condition is given. Based on this algorithm, we constructed non-iterative dynamic models of turbofan engines. Then, a high-order linear model of the turbofan engine is derived from the nonlinear dynamic model. Finally, the real second-order linear model of this turbofan engine is derived by singular value decomposition algorithm.

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