Nonlinear modes of vibration of systems undergoing unilateral contact through the semi-smooth Newton approach

Oral defence of master's thesis

Yulin Shi   supervisor: Mathias Legrand

Structural Dynamics and Vibration Laboratory
McGill University

May 9, 2016

McGill UNIVERSITY

Modes of free conservative linear systems

- Governing equation

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{0}, \quad \forall t \tag{1}$$

- Assume a periodic solution $\mathbf{u}(t) = \bar{\mathbf{u}}e^{i\omega t}$ and plug into Eq. (1) to build an eigenvalue problem

$$\left(-\omega^2\mathbf{M} + \mathbf{K}\right)\bar{\mathbf{u}} = \mathbf{0} \tag{2}$$

- Example: for $k_1 = k_2 = 1$, and $m_1 = m_2 = 1$
  - Mode 1

$$\bar{\mathbf{u}}_1 = \begin{pmatrix} -0.53 \\ -0.85 \end{pmatrix}, \quad \omega_1 = 0.618$$

  - Mode 2

$$\bar{\mathbf{u}}_2 = \begin{pmatrix} -0.85 \\ 0.53 \end{pmatrix}, \quad \omega_2 = 1.618$$

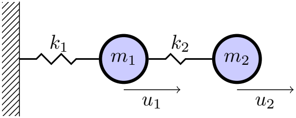

Figure 1: System of interest

Figure 2: Mode 1

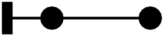

Figure 3: Mode 2

Forced damped linear systems and energy-frequency plots

Why looking for modes of vibration?

- in industry: slightly damped system
- excited by periodic forcing at ω
- energy response E at different ω

Figure 4: Automotive structure test rig

Because *modes* predict vibratory *resonances*

- two peaks at the two natural frequencies

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{D}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f}\cos(\omega t) \qquad (3)$$

- Backbone curves of free systems predict resonance curves of forced systems

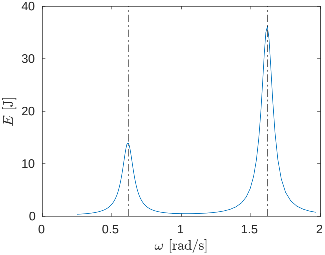

Figure 5: Energy-frequency plot of solution **u** to Eq. (3)

Modes of nonlinear smooth systems

- Example: free Duffing oscillator

$$\ddot{u} + (k + \epsilon u^2)u = 0 \qquad (4)$$

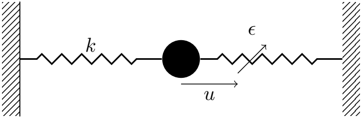

Figure 6: Simple Duffing oscillator

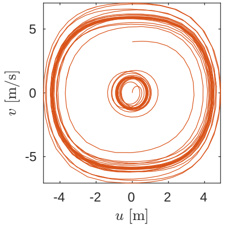

Figure 7: Limit cycles at $\omega = 1.5$ rad/s

- Natural frequency depends on u

$$\omega_n = \sqrt{\mathrm{eig}(k + \epsilon u^2)} = ? \qquad (5)$$

- Forced Duffing oscillator

$$\ddot{u} + 0.1\dot{u} + u + \epsilon u^3 = \cos(\omega t) \qquad (6)$$

  - *Limit cycles* $\rightarrow$ periodicity
  - new: not elliptic $\rightarrow$ non-harmonic
  - *resonance* in energy-frequency plot
  - new: bending of resonance: hardening

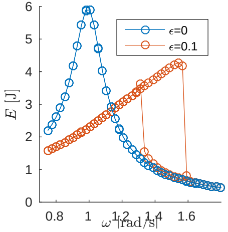

Figure 8: Energy-frequency plot

Periodic solutions of nonlinear systems: Harmonic balance method

- Governing equations of (discrete) mechanical systems

$$\text{free: } r(\ddot{u}, u) = 0, \quad \text{forced: } r(\ddot{u}, \dot{u}, u, t) = 0 \tag{7}$$

- Periodic solutions are of interest in vibration analysis (either forced or free)
- Harmonic Balance Method: seeks solutions in the form of truncated Fourier series
  - *Truncated* Fourier series

$$u(t) \approx \bar{u}_0 + \sum_{m=1}^{M} \cos(m\omega t)\bar{u}_{2m-1} + \sin(m\omega t)\bar{u}_{2m} = \phi(t)\bar{\mathbf{u}} \tag{8}$$

  where $\omega = 2\pi/T$ is the frequency, and M is the number of harmonics.
  - *Galerkin projection* $\rightarrow$ nonlinear equations in $\bar{\mathbf{u}}$ (and ω when free)

$$\bar{\mathbf{r}}(\bar{\mathbf{u}}) = \int_0^T \phi^\top(t) r(\ddot{\phi}(t)\bar{\mathbf{u}}, \dot{\phi}(t)\bar{\mathbf{u}}, \phi(t)\bar{\mathbf{u}}, t)\mathrm{d}t = \mathbf{0} \tag{9}$$

  - *Newton-Raphson* solution procedure can be used if $\bar{\mathbf{r}}(\bar{\mathbf{u}})$ is *smooth*.

$$\bar{\mathbf{u}}^{(k+1)} \leftarrow \bar{\mathbf{u}}^{(k)} - (\nabla\bar{\mathbf{r}}(\bar{\mathbf{u}}^{(k)}))^{-1}\bar{\mathbf{r}}(\bar{\mathbf{u}}^{(k)}) \tag{10}$$

Unilateral contact conditions

- Unilateral mass-spring system
  - governing equation (statics)

$$r(u) = ku - \lambda - f = 0 \tag{11}$$

  - contact condition

$$\begin{aligned} \text{Open gap: } & u - g \leq 0, \;\; \lambda = 0 \\ \text{Closed gap: } & u - g = 0, \;\; \lambda \leq 0 \end{aligned} \tag{12}$$

  where g is the initial gap and λ, the contact force.

- Newton method does not work because $\lambda(u)$ is multi-valued.

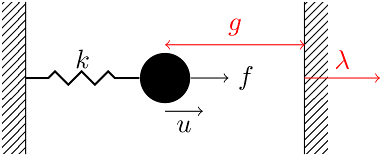

Figure 9: Vibro-impact oscillator

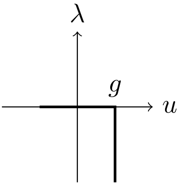

Figure 10: Non-differentiable contact constitutive law

Regularizations of the non-differentiable contact constitutive law

- Allow penetration: $\max\{0, u - g\}$
- contact force is proportional to penetration: $\lambda = -\epsilon \max\{0, u - g\}$
- $u^{(k)} \to u^{(\infty)}$ when $\epsilon \to \infty$

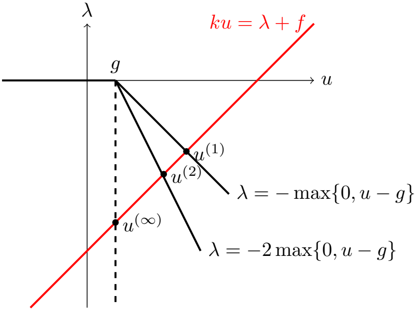

Figure 11: Penalty method

- Other treatments: Augmented Lagrange method, linear complementarity problem, active set method, Nitsche method

Implicit form of complementarity conditions

- Contact condition seen as the root set of the semismooth function

$$s(u, \lambda) = \lambda + \max\{0, c(u-g) - \lambda\} = 0 \quad (13)$$

  - plot in Figure 12
  - nonlinear function (actually piecewise linear)
  - semi-smooth function only (Newton method does not apply)

- Augmented system equations

$$\begin{aligned} ku - \lambda - f &= 0 \\ \lambda + \max\{0, c(u-g) - \lambda\} &= 0 \end{aligned} \quad (14)$$

to be solved in displacement u and contact force λ using an appropriate solver: *addressed next*

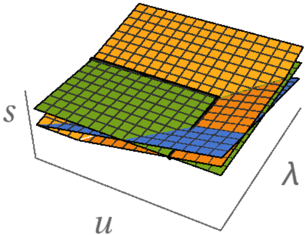

Figure 12: (Yellow) Semismooth surface $s(u, \lambda)$

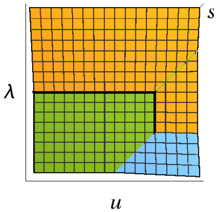

Figure 13: Above view: (—) root set

Semismooth Newton method

Issues arise when implementing the Newton-Raphson method for nonsmooth systems

- Example of semismooth function

$$r(u) = \max\{-u, u\} \tag{15}$$

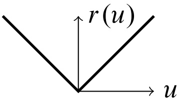

Figure 14: Simple semismooth function

- Concept of sub-differential

$$\partial r(0) := \left[\lim_{u \to 0^-} \frac{r(u) - r(0)}{u}, \lim_{u \to 0^+} \frac{r(u) - r(0)}{u} \right] = [-1, 1] \tag{16}$$

Attributes	Smooth Newton method	Semismooth Newton method
Function $r(u)$	smooth	semismooth
Iteration	$\Delta u = -(G(u))^{-1} r(u)$	$\Delta u = -(G(u))^{-1} r(u)$
Jacobian matrix	$G(u) = \nabla r(u)$	$G(u) \in \partial r(u)$
Convergence speed	quadratic	linear

Harmonic balance method for contact problems

- For the reformulated system equation of a contact problem

$$\begin{aligned} r(\ddot{u}(t), \dot{u}(t), u(t), \lambda(t), t) &= m\ddot{u}(t) + \delta\dot{u}(t) + ku(t) - \lambda(t) - f\cos(\omega t) = 0, \quad \forall t \\ s(u(t), \lambda(t)) &= \lambda(t) + \max\{0, c(u(t) - g) - \lambda(t)\} = 0, \quad \forall t \end{aligned} \tag{17}$$

- Solutions approximated by truncated Fourier series

$$u(t) \approx \sum_{i=1}^{M} \phi_i(t)\bar{u}_i = \boldsymbol{\phi}(t)\bar{\mathbf{u}} \quad \text{and} \quad \lambda(t) \approx \sum_{i=1}^{M} \phi_i(t)\bar{\lambda}_i = \boldsymbol{\phi}(t)\bar{\boldsymbol{\lambda}} \tag{18}$$

- *Galerkin projection* $\rightarrow$ nonlinear equations w.r.t. $\bar{\mathbf{u}}$, $\bar{\boldsymbol{\lambda}}$ and ω (when unknown)

$$\int_0^T \boldsymbol{\phi}(t)^\top r(\ddot{\boldsymbol{\phi}}(t)\bar{\mathbf{u}}, \dot{\boldsymbol{\phi}}(t)\bar{\mathbf{u}}, \boldsymbol{\phi}(t)\bar{\mathbf{u}}, \boldsymbol{\phi}(t)\bar{\boldsymbol{\lambda}}, t)\mathrm{d}t = 0 \tag{19}$$

$$\int_0^T \boldsymbol{\phi}(t)^\top s(\boldsymbol{\phi}(t)\bar{\mathbf{u}}, \boldsymbol{\phi}(t)\bar{\boldsymbol{\lambda}})\mathrm{d}t = 0 \tag{20}$$

- Equation (20) is semismooth only: can be solved by semismooth Newton method.
- Advantages over penalty method etc. where contact is enforced in time domain.

Harmonic balance method for contact problems

Example: one degree-of-freedom mass-spring contact system

$$\ddot{u} + 0.1\dot{u} + u = \lambda + 0.2\cos(\omega t), \quad u - 1 \leq 0, \quad \lambda \leq 0, \quad (u-1)\lambda = 0 \tag{21}$$

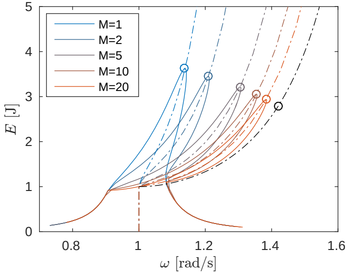

Figure 15: Energy-frequency curves of autonomous system [dashed] and forced system [solid]

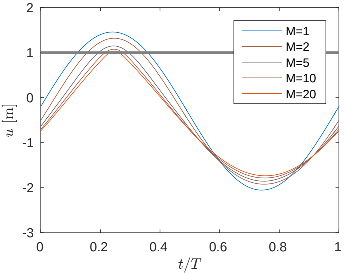

Figure 16: Displacements

Advantages:

- resonance
- energy-frequency dependency properties are reflected

Drawbacks:

- slow in convergence
- unavoidable residual penetrations.

Resonance of one degree-of-freedom contact problem

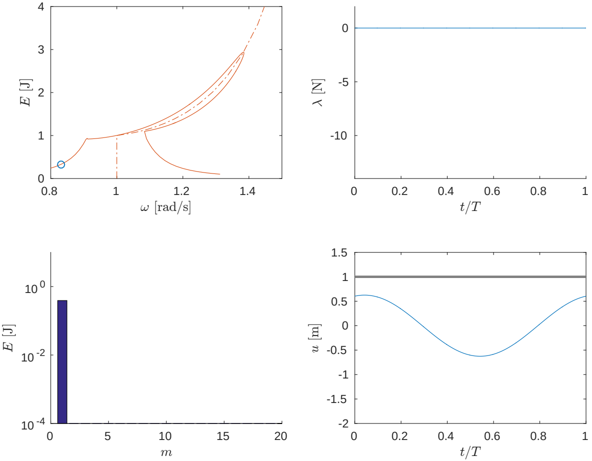

Figure 17: Travelling along resonance

Resonance of two degree-of-freedom contact problem

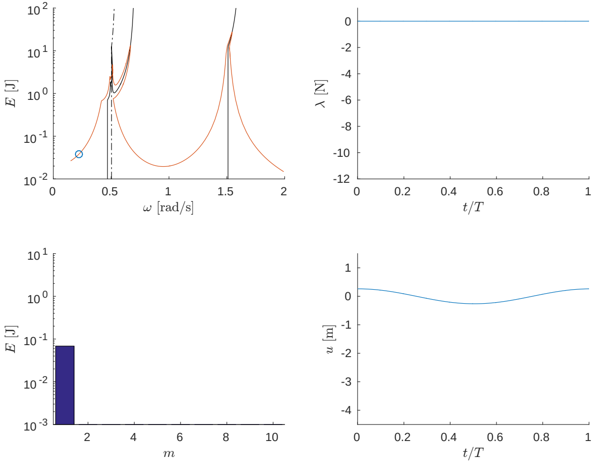

Figure 18: Travelling along resonance

Summary of the research

- Treatment of contact condition
  - *complementarity contact conditions* $\Rightarrow$ *semismooth implicit functions*
  - Augmented system
  - *Semismooth Newton methods* for static contact problems
- Solving *nonlinear modes* of dynamic contact problems
  - *Harmonic balance methods*
  - Root of the semismooth functions $\bar{\mathbf{s}}(\bar{\mathbf{u}}, \bar{\boldsymbol{\lambda}})$ can be solved by the *semismooth Newton method*
  - Advantages over penalty methods and augmented Lagrange methods
- *Properties* of the modes of contact problems
  - Resonance
  - Energy-frequency dependence
  - Sub-harmonic resonance
  - Generalization to multiple degree-of-freedom systems