Nonlinear Modal Analysis for Contact Problems

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What is a mode of a dynamic system?

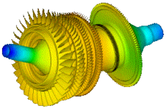

Figure: Rotor dynamics under periodic excitation.

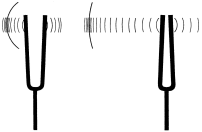

Figure: Tuning fork sympathy

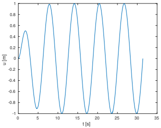

Figure: Displacement trajectory indicates periodicity

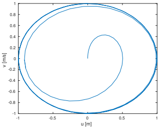

Figure: Phase plane. The trajectory merges into a closed orbit

Modal analysis for linear systems

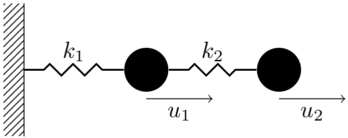

Figure: Linear mass-spring system

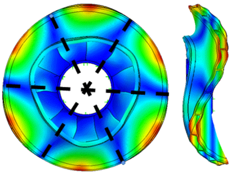

Figure: The amplitude field of standing waves in a shell

Linear modal analysis using finite element method

$$\begin{cases} \ddot{u}_1 = -k_1 u_1 + k_2(u_2 - u_1) \\ \ddot{u}_2 = -k_2(u_2 - u_1) \end{cases} \tag{1}$$

Given $k_1 = k_2 = 1$

$$\begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{pmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2}$$

Get analytical solutions by Laplace transform

$$(s^2\mathbf{I} + \mathbf{K})\mathbf{u} = \mathbf{0} \tag{3}$$

Solve the linear eigenvalue problem by plugging in $s = iw_n$, yielding the natural frequency where resonance happens

$$w_n = \sqrt{\mathrm{eig}(\mathbf{K})} = 0.618, 1.618 \tag{4}$$

Superposition principle of linear system

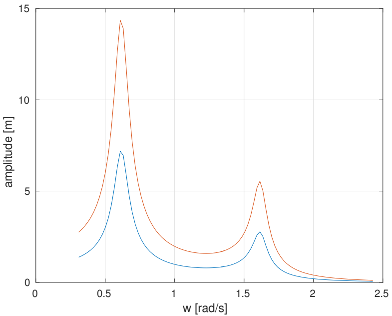

Figure: Amplitude - harmonic excitation frequency

Figure: Frequency response test rig based on superposition principle.

Superposition principle of linear systems

Smooth nonlinear system

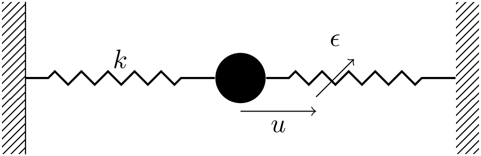

Figure: Duffing oscillator due to hardening effect

$$\ddot{u} + \delta \dot{u} + (k + \epsilon u^2) u = f \cos(wt) \tag{5}$$

$$w_n = \sqrt{\mathrm{eig}(k + \epsilon u^2)} = ? \tag{6}$$

The natural frequency CANNOT be calculated by solving the eigenvalue problem due to the nonlinearity.

Mode of nonlinear system

The steady state solution is calculated by time-stepping method e.g. Newmark scheme. `ode45` in Matlab.

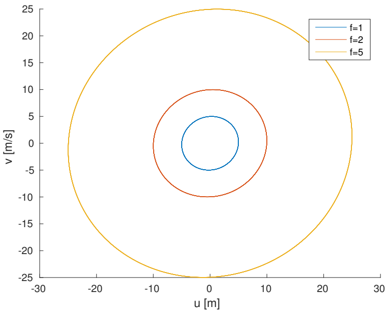

Figure: Closed orbit of $\ddot{u} + 0.2\dot{u} + u = f\cos(t)$ in phase plane

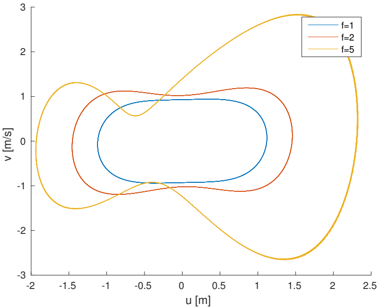

Figure: Closed orbit of $\ddot{u} + 0.2\dot{u} + u + u^3 = f\cos(t)$ in phase plane

Frequency response and bifurcations

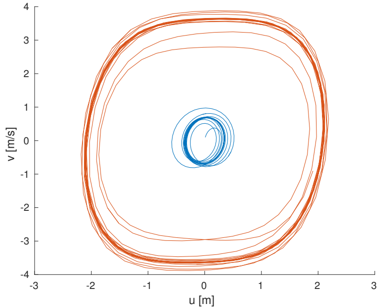

Figure: At $w = 2$rad/s, there are two steady orbits.

The trajectories converge to two distinct periodic orbits.

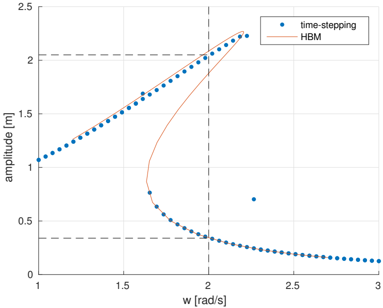

Figure: Frequency response of $\ddot{u} + 0.2\dot{u} + u + u^3 = \cos(wt)$.

The harmonic balance method can be used for solving unstable solutions.

Harmonic balance method for nonlinear system

For example, for second-order nonlinear dynamic equation

$$f(\ddot{x}, x, t) = 0 \tag{7}$$

Approximate the periodic solution in the truncated Fourier series

$$x(t) \approx \sum_{i=-M}^{M} \mathrm{e}^{iwt} \bar{x}_i = \phi(t)\bar{\mathbf{x}} \tag{8}$$

Plug it into the system equaiton, and use Galerkin method to preserve the equilibrium of the equation in the truncated base space.

$$\mathbf{F}(\bar{\mathbf{x}}) = \int_0^T \phi^\top(t) f(\ddot{\phi}(t)\bar{\mathbf{x}}, \phi(t)\bar{\mathbf{x}}, t) = \mathbf{0} \tag{9}$$

Time and differential term is not included in the nonlinear equation which can be solved by Newton iteration method.

$$\bar{\mathbf{x}} \leftarrow \bar{\mathbf{x}} - (\nabla \mathbf{F}(\bar{\mathbf{x}}))^{-1} F(\bar{\mathbf{x}}) \tag{10}$$

Contact condition

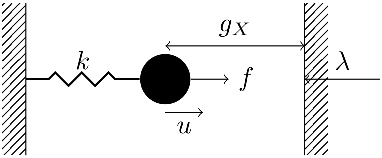

Figure: Contact problem of a mass-spring system. g_X is the initial gap. λ is the contact force

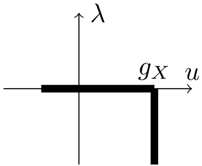

Figure: The contact force as a non-differentiable function of the gap.

$$\ddot{u} + ku = \lambda(u) + f \tag{11}$$

$$\begin{cases} u - g_X \leq 0 \\ \lambda \leq 0 \\ (u - g_X)\lambda = 0 \end{cases} \tag{12}$$

$\nabla\lambda(u)$ does not exist at $u = g_X$. Newton method does not work!

Auxiliary surface and Semismooth Newton method

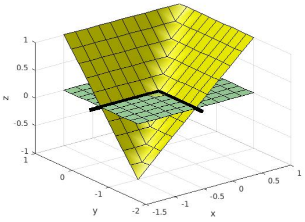

Figure: Semismooth surface $\mathcal{C}(u - g_X, \lambda) = \lambda + \max\{0, c(u - g_X) - \lambda\} = 0$.

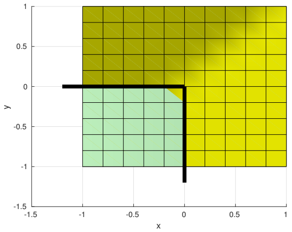

Figure: The intersection (—) of the surface $\mathcal{C}(u - g_X, \lambda)$ and the plane $z = 0$ is the non-differentiable function $\lambda(u)$.

Attributes	Smooth Newton method	Semismooth Newton method
Function $f(x)$	smooth	semi-smooth
Iteration	$\Delta x = -(\nabla f(x))^{-1} F(x)$	$\Delta x = -(\partial f(x))^{-1} f(x)$
Jacobian matrix	differential $\nabla f(x)$	sub-differential $\partial f(x)$
Convergence speed	Q-quadratic	Q-linear

Harmonic balance method for contact problem

The reformulated system equation of a contact problem

$$\begin{cases} \mathcal{R}(u, \lambda) = \ddot{u} + u - \lambda - \cos(wt) = 0 \\ \mathcal{C}(u, \lambda) = \lambda + \max\{0, c(u - g_X) - \lambda\} = 0 \end{cases} \tag{13}$$

Plugging in with the approximated solutions

$$\begin{cases} u(t) \approx \sum_{i=1}^{M} \phi_i(t)\bar{u}_i = \boldsymbol{\phi}(t)\bar{\mathbf{u}} \\ \lambda(t) \approx \sum_{i=1}^{M} \phi_i(t)\bar{\lambda}_i = \boldsymbol{\phi}(t)\bar{\boldsymbol{\lambda}} \end{cases} \tag{14}$$

yields nonlinear equations w.r.t. $\bar{\mathbf{u}}$, $\bar{\boldsymbol{\lambda}}$ and w

$$\begin{cases} \int_0^T \boldsymbol{\phi}(t)^\top \mathcal{R}(\boldsymbol{\phi}(t)\bar{\mathbf{u}}, \boldsymbol{\phi}(t)\bar{\boldsymbol{\lambda}})\mathrm{d}t = 0 \\ \int_0^T \boldsymbol{\phi}(t)^\top \mathcal{C}(\boldsymbol{\phi}(t)\bar{\mathbf{u}}, \boldsymbol{\phi}(t)\bar{\boldsymbol{\lambda}})\mathrm{d}t = 0 \end{cases} \tag{15}$$

The second nonlinear equation is semismooth which can be solved by semismooth Newton method.

Compare of time-stepping method and harmonic balance method.

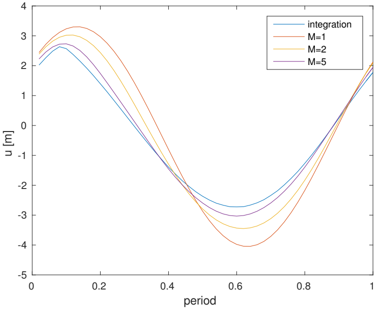

Figure: Compare of solutions of time-stepping method and HBM at $w = 1$ rad/s.

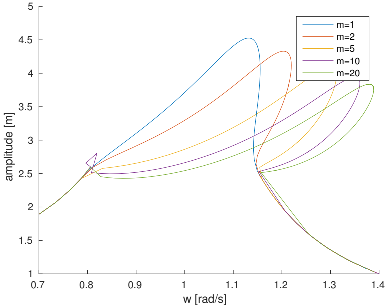

Figure: Energy-frequency solved by harmonic balance method with different number of harmonics.

$$\begin{cases} \ddot{u} + 0.2\dot{u} + u = \lambda + \cos(wt) \\ \mathcal{C}\{u - 2.5, \lambda\} = 0 \end{cases} \tag{16}$$

Conclusion

- **Linear mode** can be solved by solving eigenvalue problem.
- **Smooth nonlinear mode** can be solved by time-stepping method and harmonic balance method.
- **Contact system** is non-differentiable. We used the auxiliary equation to reformulate it as an augmented semismooth system before applying the semismooth Newton method.
- The **nonlinear mode** of the reformulated contact system can be solved by the time-stepping method or the harmonic balance method.