Chapter 1

HBM

A way of using the express the contact problem in a linear complimentary problem. [Ulbrich(2011)]

Popularly, methods such as harmonic balance based [8] or shooting methods [9] are employed to find solutions of smooth non- linear systems in an efficient manner. the Harmonic Balance Method (HBM) is known to produce poor approximations of non-smooth functions with a finite number of harmonics, producing artefacts such as the Gibbs phenomenon [10]. Finally, the use of a Fourier basis is discussed, which represents a High Dimension HBM (HDHBM) as proposed by [21] and presented in a contact frame- work by [22], specifically considering non-regularized contact conditions.

1.1 Lipschitz Continuity

Fischer function.

1.1.1 Nonsmooth Reformulation of Complementarity Conditions

Example of semismooth functions are

- Fischer-Burmeister function
- $\phi_{\max} : \mathbb{R}^2 \to \mathbb{R}$

$$u_i \leq 0, \lambda_i \leq 0, u_i \lambda_i = 0 \Leftrightarrow \boldsymbol{\lambda} + \max\{0, c(\mathbf{u} - \mathbf{0}) - \boldsymbol{\lambda}\} \tag{1.1}$$

where $c > 0$.

A reduce-order description of the dynamic contact problem discretized in space can be expressed as

$$p(t) = \mathbf{M}\ddot{\mathbf{u}}_r + \mathbf{D}\dot{\mathbf{u}}_r + \mathbf{K}\mathbf{u}_r - \boldsymbol{\lambda} - \mathbf{f}_r \equiv 0 \tag{1.2}$$

$$s(t) = \boldsymbol{\lambda} + \max\{0, c(\mathbf{u}_r - \mathbf{0}) - \boldsymbol{\lambda}\} \equiv \mathbf{0} \tag{1.3}$$

For single point contact problem, in the harmonic balance method, the displacement $\mathbf{u}_r$ and the contact force $\boldsymbol{\lambda}$ are approximated as truncated Fourier series.

$$u_r(t) \approx a_0/2 + \sum_{m=1}^{M} a_m \cos(mwt) + b_m \sin(mwt) = \boldsymbol{\phi}(t)\mathbf{a} \tag{1.4}$$

$$\lambda(t) \approx c_0/2 + \sum_{m=1}^{M} c_m \cos(mwt) + d_m \sin(mwt) = \boldsymbol{\phi}(t)\mathbf{c} \tag{1.5}$$

where w is the base frequency. It is usually chosen as the frequency of the harmonic excitation force.

To maintain the equilibrium equations in the Fourier space:

$$\int_0^{2\pi/w} \boldsymbol{\phi}^{\mathrm{T}}(t) \left(m\ddot{\boldsymbol{\phi}}(t)\mathbf{a} + d\dot{\boldsymbol{\phi}}(t)\mathbf{a} + k\boldsymbol{\phi}(t)\mathbf{a} - \boldsymbol{\phi}(t)\mathbf{c} - \mathbf{f}(t)_r \right) = 0 \tag{1.6}$$

$$\int_0^{2\pi/w} \boldsymbol{\phi}(t) \left(\boldsymbol{\phi}\mathbf{c} + \max\{0, c(\boldsymbol{\phi}\mathbf{a} - \mathbf{0}) - \boldsymbol{\phi}\mathbf{c}\} \right) = 0 \tag{1.7}$$

$$\left(\int_0^{2\pi/w} \boldsymbol{\phi}^{\mathrm{T}}(t)(m\ddot{\boldsymbol{\phi}}(t) + d\dot{\boldsymbol{\phi}}(t) + k\boldsymbol{\phi}(t)) \right) \mathbf{a} - \left(\int_0^{2\pi/w} \boldsymbol{\phi}^{\mathrm{T}}(t)\boldsymbol{\phi}(t) \right) \mathbf{c} - \left(\int_0^{2\pi/w} \boldsymbol{\phi}^{\mathrm{T}}(t)\mathbf{f}(t)_r \right) = 0 \tag{1.8}$$

$$\int_0^{2\pi/w} \boldsymbol{\phi}^{\mathrm{T}}(t) \left(\boldsymbol{\phi}(t)\mathbf{c} + \max\{0, c(\boldsymbol{\phi}(t)\mathbf{a} - \mathbf{0}) - \boldsymbol{\phi}(t)\mathbf{c}\} \right) = 0 \tag{1.9}$$

Since the max operator is not linear in Fourier space, we'd better integrate the second equation with numerical integration method. Discretize the function $\boldsymbol{\phi}$ in time. Divide one period into N equal time intervals $\Delta t = 2\pi/w/N$.

$$\Delta t \sum_{n=1}^{N} \boldsymbol{\phi}^{\mathrm{T}}(t_n) \left(\boldsymbol{\phi}(t_n)\mathbf{c} + \max\{0, c(\boldsymbol{\phi}(t_n)\mathbf{a} - \mathbf{0}) - \boldsymbol{\phi}(t_n)\mathbf{c}\} \right) = 0 \tag{1.10}$$

1.1.2 Multiple contact

A reduce-order description of the dynamic contact problem discretized in space can be expressed as

$$\mathbf{p}(t) = \mathbf{M}\ddot{\mathbf{u}}_r + \mathbf{D}\dot{\mathbf{u}}_r + \mathbf{K}\mathbf{u}_r - \boldsymbol{\lambda} - \mathbf{f}_r \equiv \mathbf{0} \tag{1.11}$$

$$\mathbf{s}(t) = \boldsymbol{\lambda} + \max\{0, c(\mathbf{u}_r - \mathbf{0}) - \boldsymbol{\lambda}\} \equiv \mathbf{0} \tag{1.12}$$

where $\mathbf{u}_r, \boldsymbol{\lambda}, \mathbf{p}, \mathbf{s} \in \mathbb{R}^{N_d}$. N_d is the degree of freedom of the reduce-order system.

In finite Fourier space, discretize

$$\mathbf{u}_r(t) \approx (\mathbf{I}_{N_d} \otimes \boldsymbol{\phi}(t))\,\mathbf{U} \tag{1.13}$$

$$\boldsymbol{\lambda}(t) \approx (\mathbf{I}_{N_d} \otimes \boldsymbol{\phi}(t))\,\boldsymbol{\Lambda} \tag{1.14}$$

$$\boldsymbol{\phi}(t) = [1/2, \cos(wt), \sin(wt), ..., \cos(Mwt), \sin(Mwt)] \tag{1.15}$$

where $\mathbf{U}, \boldsymbol{\Lambda} \in \mathbb{R}^{N_d(2M+1)}$

Their projections onto the finite Fourier space are zeros

$$\int_0^{2\pi/w} \left(\mathbf{I}_{N_d} \otimes \boldsymbol{\phi}(t)\right)^{\mathrm{T}} \mathbf{p}(\left(\mathbf{I}_{N_d} \otimes \boldsymbol{\phi}(t)\right) \mathbf{U}, \left(\mathbf{I}_{N_d} \otimes \boldsymbol{\phi}(t)\right) \boldsymbol{\Lambda}) = \mathbf{0} \tag{1.16}$$

$$\int_0^{2\pi/w} \left(\mathbf{I}_{N_d} \otimes \boldsymbol{\phi}(t)\right)^{\mathrm{T}} \mathbf{s}(\left(\mathbf{I}_{N_d} \otimes \boldsymbol{\phi}(t)\right) \mathbf{U}, \left(\mathbf{I}_{N_d} \otimes \boldsymbol{\phi}(t)\right) \boldsymbol{\Lambda}) = \mathbf{0} \tag{1.17}$$

Integrate the second function over time by discrete integration

$$\Delta t \sum_0^N \left(\mathbf{I}_{N_d} \otimes \boldsymbol{\phi}(t_n)\right)^{\mathrm{T}} \mathbf{s}(\left(\mathbf{I}_{N_d} \otimes \boldsymbol{\phi}(t_n)\right) \mathbf{U}, \left(\mathbf{I}_{N_d} \otimes \boldsymbol{\phi}(t_n)\right) \boldsymbol{\Lambda}) = \mathbf{0} \tag{1.18}$$

where $\Delta t = 2\pi/w/N$. N is the mesh size over one period.

In vector form:

$$\Delta t \left(\mathbf{I}_{N_d} \otimes \boldsymbol{\Phi}\right)^{\mathrm{T}} \mathbf{s}(\left(\mathbf{I}_{N_d} \otimes \boldsymbol{\Phi}\right) \mathbf{U}, \left(\mathbf{I}_{N_d} \otimes \boldsymbol{\Phi}\right) \boldsymbol{\Lambda}) = \mathbf{0} \tag{1.19}$$

$$\boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{\Phi}(t_1) \\ \cdots \\ \boldsymbol{\Phi}(t_N) \end{bmatrix} \tag{1.20}$$

Bibliography

[Ulbrich(2011)] Michael Ulbrich. *Semismooth Newton methods for variational inequalities and constrained optimization problems in function spaces*, volume 11. SIAM, 2011.