Harmonic balance method for Duffing oscillators

Yulin Shi, Structure dynamics and vibration laboratory, McGill University

May 11, 2016

1 Harmonic balance method

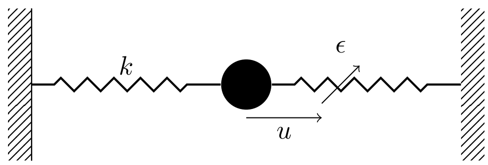

Figure 1: Duffing oscillator

We are interested in finding the steady solutions of the one degree-of freedom Duffing oscillator under periodic forcing in Figure 1

$$r(t) := m\ddot{u}(t) + \delta\dot{u}(t) + ku(t) + \epsilon u(t)^3 - f(t) \tag{1}$$

where $u(t)$ is the unknown displacement variable, t is the time, $f(t)$ is the periodic external forcing. Others are constant coefficients.

Assume the solutions takes the form of *truncated Fourier series*

$$u(t) \approx \frac{a_0}{\sqrt{2}} + \sum_{m=1}^{M} \cos(m\omega t)a_m + \sin(m\omega t)b_m = \boldsymbol{\phi}(t)\bar{\mathbf{u}} \tag{2}$$

where ω is the external forcing frequency, and M is the number of harmonics. Array $\bar{\mathbf{u}}$ stores the unknown coefficients a_m and b_m, array $\boldsymbol{\phi}(t)$ stores the base functions $\cos(m\omega t)$ and $\sin(m\omega t)$. Then

$$\dot{u}(t) \approx \dot{\boldsymbol{\phi}}(t)\bar{\mathbf{u}}, \quad \ddot{u}(t) \approx \ddot{\boldsymbol{\phi}}(t)\bar{\mathbf{u}} \tag{3}$$

The *Galerkin projection* creates nonlinear equations with respect to $\bar{\mathbf{u}}$

$$\bar{\mathbf{r}}(\bar{\mathbf{u}}) = T^{-1}\int_0^T \boldsymbol{\phi}^\top(t) r(\ddot{\boldsymbol{\phi}}(t)\bar{\mathbf{u}}, \dot{\boldsymbol{\phi}}(t)\bar{\mathbf{u}}, \boldsymbol{\phi}(t)\bar{\mathbf{u}}, t)\mathrm{d}t = \mathbf{0} \tag{4}$$

where $T = \frac{2\pi}{\omega}$ is the period. It can be solved by the Newton-Raphson method

$$\bar{\mathbf{u}}^{(k+1)} \leftarrow \bar{\mathbf{u}}^{(k)} - (\nabla\bar{\mathbf{r}}(\bar{\mathbf{u}}^{(k)}))^{-1}\bar{\mathbf{r}}(\bar{\mathbf{u}}^{(k)}) \tag{5}$$

For Fourier base functions, the integration of the time-variant part in (4) yields the closed form

$$(m\mathbf{L}_M + \delta\mathbf{L}_D + k\mathbf{L}_K)\bar{\mathbf{u}} + \frac{1}{N}\sum_{n=1}^{N}\boldsymbol{\phi}^\top(t_n)\epsilon(\boldsymbol{\phi}(t_n)\bar{\mathbf{u}})^3 - \mathbf{L}_f = \mathbf{0} \tag{6}$$

where N is the number of discretized time intervals of one period, $\mathbb{R}^{2M+1} \times R^{2M+1}$ matrices $\mathbf{L}_D$ and $\mathbf{L}_K$ denote respectively

$$\mathbf{L}_K := T^{-1}\int_0^T \boldsymbol{\phi}(t)^\top\boldsymbol{\phi}(t)\mathrm{d}t \quad \text{and} \quad \mathbf{L}_D := T^{-1}\int_0^T \boldsymbol{\phi}(t)^\top\dot{\boldsymbol{\phi}}(t)\mathrm{d}t \tag{7}$$

Also, according to the Green equation, $\mathbf{L}_M$ simplifies as

$$\mathbf{L}_M := T^{-1}\int_0^T \boldsymbol{\phi}(t)^\top \ddot{\boldsymbol{\phi}}(t)\mathrm{d}t = -T^{-1}\int_0^T \dot{\boldsymbol{\phi}}(t)^\top \dot{\boldsymbol{\phi}}(t)\mathrm{d}t \tag{8}$$

The corresponding vector of external forces is

$$\mathbf{L}_f := T^{-1}\int_0^T \boldsymbol{\phi}(t)^\top f(t)\mathrm{d}t \tag{9}$$

2 Example

For example, for system

$$\ddot{u} + 0.1\dot{u} + u + 0.1u^3 = \cos(\omega t) \tag{10}$$

Assume

$$\boldsymbol{\phi}(t) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \cos(\omega t) & \sin(\omega t) \end{pmatrix} \tag{11}$$

Then, all the coefficients can be computed as

$$\mathbf{L}_K = T^{-1}\int_0^T \begin{bmatrix} \frac{1}{2} & & \\ & \cos^2(\omega t) & \\ & & \sin^2(\omega t) \end{bmatrix} \mathrm{d}t = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{12}$$

In similar ways, compute

$$\mathbf{L}_D = \frac{w}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \tag{13}$$

$$\mathbf{L}_M = \frac{w^2}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{14}$$

$$\mathbf{L}_f = \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{15}$$

Plug these coefficient matrices to (**??**) to solve $\bar{\mathbf{u}}$, and plug resulted $\bar{\mathbf{u}}$ to (**??**) to get the approximate solutions.

3 Appendix: arclength continuation

If we solve (10) at different excitation frequencies, and plot the energy of solutions with respect to frequencies, we get a curve looks like Figure 2. It requires the use of arclength continuation method illustrated

The idea of arclength continuation is quite illustrative by comparing the two figures in Figure 3. In the left figure, a implicit function with respect to u and w denotes the continuation curve. On the continuation curve u_1 is known and u_2 is to be solved. Δw is the shift of parameter w. The idea of continuation is to solve u_2 given $w_2 = w_1 + \Delta w$. It demonstrates that the parametric continuation methods is not capable of finding u_3 given u_2 and Δw. It is not capable of tracing the curve through the turning point (also called the fold). In the right figure, vector $\mathbf{x}$ denotes $(u, w)^\top$. On the continuation curve $\mathbf{x}_1$ is known and $\mathbf{x}_2$ is to be solved. Δs is the arc-length, $\dot{\mathbf{x}}_1$ is the normal direction of node $\mathbf{x}_1$. The idea of solving the node $\mathbf{x}_2$ is to solve the intersection between the continuation curve and the line $\mathbf{x}_2^0$-$\mathbf{x}_2$ which is orthogonal to the $\dot{\mathbf{x}}_1$. It demonstrates that the arc-length continuation methods is capable of overcoming the fold.

Given the solution $\mathbf{u}_0 \in \mathbb{R}^N$ and w_0 of a nonlinear (no differential term) equation $\mathbf{f}(\mathbf{u}, w) : \mathbb{R}^{N+1} \to \mathbb{R}^N$, as well as the direction vector $\dot{\mathbf{u}}_0$ and $\dot{w}$, instead of moving the parameter variable w by $w = w_0 + \Delta w$ which does not work of folders, a pseudo arc length Δs is introduced, and move the solution along the curve for Δs. It still work when crossing a smooth folder where $\dot{w} = 0$.

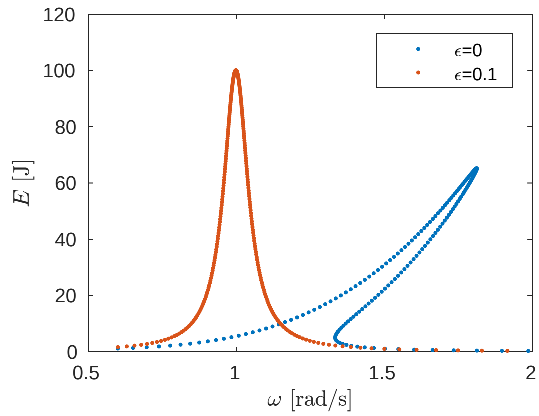

Figure 2: Energy-frequency plot

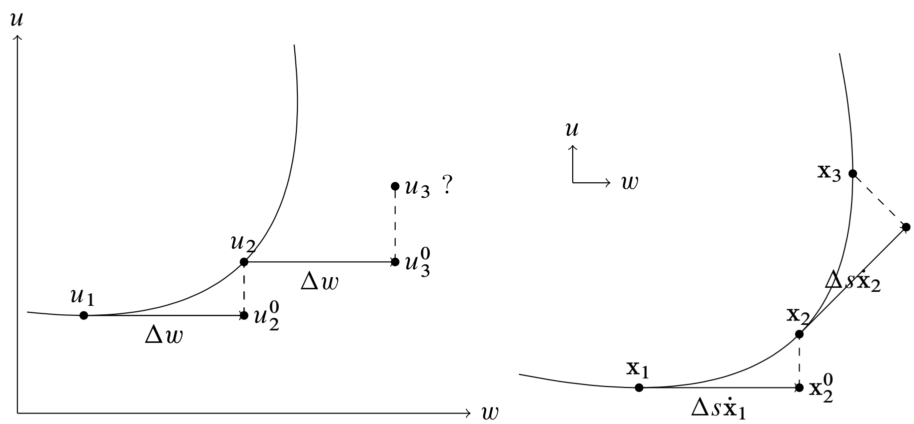

Figure 3: [Left] Demo of parametric continuation methods. [Right] Demo of arc-length continuation method.

Assume the curve is smooth, in the neighbour of $(\mathbf{u}_0, w_0)$, the curve can be approximated by Taylor expansion, and the initial guess of $(\mathbf{u}_1, w_1)$ can be predicted by

$$\begin{aligned} \mathbf{u}_1^{(0)} &= \mathbf{u}_0 + \Delta s \dot{\mathbf{u}}_0 \\ w_1^{(0)} &= w_0 + \Delta s \dot{w}_0 \end{aligned} \tag{16}$$

Instead of solving nonlinear equation $f(\mathbf{u}_1) = 0$ given a fixed w_1, we solve the augmented system where $f(\mathbf{u}_1, w_1) = 0$ intersects with the line $(\mathbf{u}_1^{(0)}, w_1)$. Given Δs small enough, and $f(\mathbf{u}_1) = 0$ smooth enough, the solution is unique.

The orthogonal line is formulated as

$$\begin{pmatrix} \dot{\mathbf{u}}_0 & \dot{w}_0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 - \mathbf{u}_1^{(0)} \\ w_1 - w_1^{(0)} \end{pmatrix} = 0 \tag{17}$$

For normalized direction vector $\begin{pmatrix} \dot{\mathbf{u}}_0 & \dot{w}_0 \end{pmatrix}$,

$$\begin{pmatrix} \dot{\mathbf{u}}_0 & \dot{w}_0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 - \mathbf{u}_0 - \Delta s \dot{\mathbf{u}}_0 \\ w_1 - w_0 - \Delta s \dot{w}_0 \end{pmatrix} = \begin{pmatrix} \dot{\mathbf{u}}_0 & \dot{w}_0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 - \mathbf{u}_0 \\ w_1 - w_0 \end{pmatrix} - \Delta s \begin{pmatrix} \dot{\mathbf{u}}_0 & \dot{w}_0 \end{pmatrix} \begin{pmatrix} \dot{\mathbf{u}}_0 \dot{w}_0 \end{pmatrix} \tag{18}$$

$$\begin{pmatrix} \dot{\mathbf{u}}_0 & \dot{w}_0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 - \mathbf{u}_0 \\ w_1 - w_0 \end{pmatrix} - \Delta s = 0 \tag{19}$$

Define the augmented variable $\mathbf{x} = \begin{pmatrix} \mathbf{u} \\ w \end{pmatrix}$, The arclength continuation problem becomes solving $\mathbf{f}(\mathbf{x}_1) = 0$. It can be solved with Newton-Raphson method.

Given the Jacobian matrix (gradient matrix) $\mathbf{f}_\mathbf{x}$ at $\mathbf{x}_1$, the direction vector can be solved by considering that the direction vector is orthogonal to the gradient direction.

$$\mathbf{f}_{\mathbf{x}_1} \dot{\mathbf{x}}_1 = \mathbf{0} \tag{20}$$

The solution is not unique cause $\mathbf{f}_\mathbf{x} : \mathbb{R}^{N+1} \to \mathbb{R}^N$. The graphic interpretation is that the length of the direction vector is undefined. We can get a unique solution by adding a normalization constraint

$$\dot{\mathbf{x}}_1^\top \dot{\mathbf{x}}_1 = 1 \tag{21}$$

However, the augmented solver for the direction is implicit

$$\begin{pmatrix} \mathbf{f}_\mathbf{x}(\mathbf{x}_1) \\ \dot{\mathbf{x}}_1^\top \end{pmatrix} \dot{\mathbf{x}}_1 = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix} \tag{22}$$

For $\mathbf{f}$ smooth enough and Δs small enough, we assume $\dot{\mathbf{x}}_1 \approx \dot{\mathbf{x}}_0$, and solve it like this:

$$\dot{\mathbf{x}}_1 = \begin{pmatrix} \mathbf{f}_\mathbf{x}(\mathbf{x}_1) \\ \dot{\mathbf{x}}_0^\top \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix} \tag{23}$$

Before the extra normalization process.

$$\dot{\mathbf{x}}_1 \leftarrow \frac{\dot{\mathbf{x}}_1}{\|\dot{\mathbf{x}}_1\|} \tag{24}$$