20/10/2025

Aula 11

- Teste 1 até ao Cap. 2.1. slide 12

(fp2) (20) (e) $[(1,5)]_S = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ tal que $(1,5) = a_1\underbrace{(1,2)}_{\text{base } S} + a_2\underbrace{(0,1)}_{} \Leftrightarrow \begin{cases} a_1 = 1 \\ a_2 = 3 \end{cases}$

$[(1,5)]_S = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

(5) (d) $\langle (1,1,1),(1,0,0),(2,2,2)\rangle = \{(x,y,z) = \alpha_1(1,1,1)+\alpha_2(1,0,0)+\alpha_3(2,2,2),\ \alpha_1,\alpha_2,\alpha_3 \in \mathbb{R}\}$

$= \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix},\ \alpha_1,\alpha_2,\alpha_3 \in \mathbb{R} \right\}$

Quando é que o sistema $AX=B \Leftrightarrow \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ é possível/car$(A)=$ car$([A|B])$?

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & x \\ 1 & 0 & 2 & y \\ 1 & 0 & 2 & z \end{array}\right] \underset{\substack{L_2 = L_2 - L_1 \\ L_3 = L_3 - L_1}}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 2 & x \\ 0 & -1 & 0 & y-x \\ 0 & -1 & 0 & z-x \end{array}\right] \underset{L_3 = L_3 - L_2}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 2 & x \\ 0 & -1 & 0 & y-x \\ 0 & 0 & 0 & z-y \end{array}\right]$$

Temos que car$(A)=2$ para $AX=B$ ser possível temos de ter car$(A)=$car$([A|B])=2$ o que implica que $z-y=0$. Assim, $\langle (1,1,1),(1,0,0),(2,2,2)\rangle = \{(x,y,z)\in\mathbb{R}^3 : z-y=0\}$

(15) $K = \{(1,0,1,0),(0,1,-1,0)\}$

K é l.i.? $(0,0,0,0) = \alpha_1(1,0,1,0)+\alpha_2(0,1,-1,0) \Leftrightarrow \begin{cases} \alpha_1 = 0 \\ \alpha_2 = 0 \end{cases}$ Poss. Det.

K é l.i. mas não gera $\mathbb{R}^4$ porque só temos 4 vetores.

$\langle K\rangle = \{(x,y,z,t) = \alpha_1(1,0,1,0)+\alpha_2(0,1,-1,0),\ \alpha_1,\alpha_2 \in \mathbb{R}\}$.

$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \Leftrightarrow AX=B$ quando é possível /car$(A)=$ car$([A|B])$?

$$[A|B] = \left[\begin{array}{cc|c} 1 & 0 & x \\ 0 & 1 & y \\ 1 & -1 & z \\ 0 & 0 & t \end{array}\right] \underset{L_3 = L_3 - L_1}{\sim} \left[\begin{array}{cc|c} 1 & 0 & x \\ 0 & 1 & y \\ 0 & -1 & z-x \\ 0 & 0 & t \end{array}\right] \underset{L_3 = L_3 + L_2}{\sim} \left[\begin{array}{cc|c} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & z-x+y \\ 0 & 0 & t \end{array}\right]$$

car$(A)=2$ para que car$([A|B])=2$ temos de ter $t=0$ e $z-x+y=0$

Assim,
$\langle K\rangle = \{(x,y,z,t)\in\mathbb{R}^4 : \underbrace{z-x+y=0}_{\Leftrightarrow x = y+z} \text{ e } t=0\} \neq \mathbb{R}^4$

Por exemplo, $(0,0,0,1)$ e $(0,0,1,0) \notin \langle K\rangle$ e $\{(0,0,0,1), (0,0,1,0)\}$ é l.i. porque $(0,0,0,0) = \alpha_1(0,0,0,1)+\alpha_2(0,0,1,0) \Rightarrow \alpha_1=\alpha_2=0$.

Pelo Teorema do slide 8, o conjunto $K \cup \{(0,0,0,1),(0,0,1,0)\}$ ainda é l.i. E pelo Corolário do slide 12, como $\dim \mathbb{R}^4 = 4$ qualquer conjunto com 4 vetores de $\mathbb{R}^4$ l.i. é automaticamente uma base de $\mathbb{R}^4$.

Portanto, $((1,0,1,0),(0,1,-1,0),(0,0,0,1),(0,0,1,0))$ é uma base de $\mathbb{R}^4$.

1º teste 24/25

2 $\alpha \in \mathbb{R}$

(a) A tem inversa $\Leftrightarrow$ car(A) = 3 (nº de colunas)

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ \alpha & 0 & 3 \end{bmatrix} \underset{L_3 = L_3 - \alpha L_1}{\sim} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & \alpha & 3-2\alpha \end{bmatrix} \underset{L_3 = L_3 - \alpha L_2}{\sim} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 3-3\alpha \end{bmatrix} \text{ Escalonada}$$

car(A) = 3 se e só se $3 - 3\alpha \neq 0 \Leftrightarrow 3\alpha \neq 3 \Leftrightarrow \alpha \neq 1$.

(b) $\alpha = 2$

(i)
$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \underset{L_3 = L_3 - 2L_1}{\sim} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & -2 & 0 & 1 \end{array}\right] \underset{L_3 = L_3 - 2L_1}{\sim}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -3 & -2 & -2 & 1 \end{array}\right] \underset{L_1 = L_1 + L_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -3 & -2 & -2 & 1 \end{array}\right] \underset{\substack{L_1 = L_1 + L_3 \\ L_2 = L_2 - \frac{L_3}{3}}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & -2/3 & 1/3 & 1/3 \\ 0 & 0 & -3 & -2 & -2 & 1 \end{array}\right] \sim$$

$$\underset{L_3 = L_3/3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & -2/3 & 1/3 & 1/3 \\ 0 & 0 & 1 & 2/3 & 2/3 & -1/3 \end{array}\right] = [I_3 \mid A^{-1}]$$

(ii) $X_{3\times3} = ?$

$$A^T X + 5B = DC \Leftrightarrow \boxed{A^T X} = \boxed{DC - 5B} \Leftrightarrow \underbrace{(A^T)^{-1} A^T}_{I_3} X = (A^T)^{-1}(DC - 5B)$$

$$\Leftrightarrow X = \underbrace{(A^{-1})^T}_{2(b)(i)} (DC - 5B)$$

$$= \begin{bmatrix} -1 & -2/3 & 2/3 \\ -1 & 1/3 & 2/3 \\ 1 & 1/3 & -1/3 \end{bmatrix} \left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 5 & -5 & 0 \end{bmatrix} - 5 \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} -1 & -2/3 & 2/3 \\ -1 & 1/3 & 2/3 \\ 1 & 1/3 & -1/3 \end{bmatrix} \left(\begin{bmatrix} 10 & -10 & 0 \\ 5 & -5 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 5 & -5 & 10 \\ 5 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \right)$$

$$= \begin{bmatrix} -1 & -2/3 & 2/3 \\ -1 & 1/3 & 2/3 \\ 1 & 1/3 & -1/3 \end{bmatrix} \begin{bmatrix} 5 & -5 & -10 \\ 0 & -10 & 0 \\ 0 & 0 & -5 \end{bmatrix} = \begin{bmatrix} -5 & 35/3 & 20/3 \\ -5 & 5/3 & 20/3 \\ 5 & -25/3 & -25/3 \end{bmatrix}$$

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$$[A \mid B] = \left[\begin{array}{ccc|c} 1 & \alpha-1 & \alpha & \alpha-2 \\ 0 & \alpha-1 & 0 & 1 \\ 0 & 0 & \alpha & \alpha-3 \end{array}\right] \underset{\substack{L_2 = \frac{L_2}{\alpha-1},\ \alpha \neq 1 \\ L_3 = \frac{L_3}{\alpha},\ \alpha \neq 0}}{\sim} \left[\begin{array}{ccc|c} 1 & \alpha-1 & \alpha & \alpha-2 \\ 0 & 1 & 0 & 1/(\alpha-1) \\ 0 & 0 & 1 & (\alpha-3)/\alpha \end{array}\right]$$

$AX=B$ é possível determinado $\Leftrightarrow$ $car(A) = car([A|B]) = 3$ = nº de colunas de A, isto é, $\alpha \in \mathbb{R}\setminus\{0,1\}$.

$\boxed{\text{Se } \alpha = 0 \text{ ?}}$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & -1 & 0 & -2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

como $car(A) = 2 < car([A|B]) = 3$, $AX=B$ é impossível.

$\boxed{\text{Se } \alpha = 1 \text{ ?}}$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right] \underset{L_2 \leftrightarrow L_3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

como $car(A) = 2 < car([A|B]) = 3$ $AX=B$ é impossível.

$A_{m\times n}$ com linhas $L_1, \dots, L_m$ então o espaço das linhas de A é $\mathcal{L}(A) = \langle L_1, \dots, L_m \rangle$

Lema (cap. 2.2 slide 3) exemplos

(i) $\langle (1,0,1), (0,1,-1) \rangle = \langle (0,1,-1), (1,0,1) \rangle$

(ii) $\langle (1,1) \rangle = \langle 2(1,1) \rangle = \langle (2,2) \rangle$

(iii) $\langle (1,0,1), (0,1,-1) \rangle = \langle (1,0,1), (0,1,-1) + 3(1,0,1) \rangle = \langle (1,0,1), (3,1,2) \rangle$

Teorema Se $A \sim B$ então $\mathcal{L}(A) = \mathcal{L}(B)$.

Exp $A = \begin{bmatrix} 1 & -2 & -4 & 3 \\ 2 & -4 & -7 & 5 \\ 1 & -2 & -3 & 2 \end{bmatrix} \sim \dots \underbrace{\begin{bmatrix} 1 & -2 & -4 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{= A_e \text{ escalonada}} \sim \dots \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = A_r$ reduzida

$((1,-2,-4,3), (0,0,1,-1))$ e $((1,-2,0,-1), (0,0,1,-1))$ são bases de $\mathcal{L}(A)$.

Temos que $\dim \mathcal{L}(A) = 2 = car(A)$.