Aula 6

02/10/2025

FP1 30 $\begin{bmatrix}1&0\\0&-1\end{bmatrix} X \begin{bmatrix}1&1\\3&4\end{bmatrix} = \begin{bmatrix}1&0\\0&2\end{bmatrix} \Leftrightarrow \boxed{AXB} = \boxed{C} \Leftrightarrow \underbrace{A^{-1}A}_{=I_2}XB = A^{-1}C \Leftrightarrow \boxed{XB} = \boxed{A^{-1}C} \Leftrightarrow X\underbrace{BB^{-1}}_{=I_2} = A^{-1}CB^{-1}$

$\Leftrightarrow X = A^{-1}CB^{-1}$

Calcular A^{-1}: $[A|I_2] \underset{L_2=-L_2}{\sim} \left[\begin{array}{cc|cc}1&0&1&0\\0&1&0&-1\end{array}\right] = [I_2|A^{-1}]$

Calcular B^{-1}: $[B|I_2] \underset{L_2=L_2-3L_1}{\sim} \left[\begin{array}{cc|cc}1&1&1&0\\0&1&-3&1\end{array}\right] \underset{L_1=L_1-L_2}{\sim} \left[\begin{array}{cc|cc}1&0&4&-1\\0&1&-3&1\end{array}\right] = [I_2|B^{-1}]$

Assim, $X = \begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}4&-1\\-3&1\end{bmatrix} = \begin{bmatrix}4&-1\\6&-2\end{bmatrix}$

Exame Recurso 24/25

1 (c) $\boxed{M\overset{?}{X}M^{-1}} = \boxed{MB + AM} \Leftrightarrow \underbrace{M^{-1}M}_{=I}XM^{-1} = M^{-1}(MB + AM)$

$\Leftrightarrow XM^{-1} = \underbrace{M^{-1}M}_{=I}B + M^{-1}AM \Leftrightarrow \boxed{XM^{-1}} = \boxed{B + M^{-1}AM}$

$\Leftrightarrow X\underbrace{M^{-1}M}_{=I} = (B + M^{-1}AM)M \Leftrightarrow X = BM + M^{-1}AM^2$

Aula de hoje:

(1) Matrizes elementares

(2) Dada A, como calcular $A=LU$?

(3) Como resolver $AX=B$ quando $A=LU$?

(1) Matriz elementar, é equivalente à matriz identidade por uma única operação elementar (é sempre invertível)

exp

$I_2 \underset{L_2=L_2+3L_1}{\sim} \begin{bmatrix}1&0\\3&1\end{bmatrix} = E$ é uma matriz elementar

Corresponde à operação elementar $L_2=L_2+3L_1$, isto é, seja $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ temos que:

$$A \underset{L_2=L_2+3L_1}{\sim} \begin{bmatrix}a&b\\c+3a&d+3b\end{bmatrix} = EA$$

(2) Como fatorizar $A=LU$?

$$\overset{\checkmark}{A}_{m\times n} = \underbrace{\overset{?}{L}_{m\times m}}_{\text{triang. inf. com 1's na diagonal.}} \underbrace{\overset{?}{U}_{m\times n}}_{\text{escalonada } A\sim U \text{ sem trocar linhas!}}$$

Sejam $E_1,\dots,E_p$ matrizes elementares que correspondem às operações elementares de escalonar $A \sim U$.

$$E_p\dots E_1A = U \quad \text{então} \quad A = \underbrace{(E_p\dots E_1)^{-1}}_{=L}U$$

Assim, L é a inversa de $E_p\dots E_1$. Logo, $\boxed{L \overset{\sim}{\underset{\text{fazer as mesmas op. elementares } A\sim U}{}} I}$

Exp. Fatorizar $A = \begin{bmatrix} 2 & -2 & 4 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix} = \overset{?}{L}\overset{?}{U}$

1º Escalonar $A \sim U$ $\left(\cancel{L_i \leftrightarrow L_j},\ \cancel{L_i = \alpha L_i},\ \alpha \neq 0 \right)$

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix} \underset{\substack{L_2 = L_2 - L_1/2 \\ L_3 = L_3 - \frac{3L_1}{2}}}{\sim} \begin{bmatrix} 2 & -2 & 4 \\ 0 & -2 & -1 \\ 0 & 10 & -1 \end{bmatrix} \underset{L_3 = L_3 + 5L_2}{\sim} \begin{bmatrix} 2 & -2 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{bmatrix} = U \text{ escalonada} \checkmark$$

2º Construir L

$$\begin{array}{c|c|c} 2 & & \\ 1 & -2 & \\ 3 & 10 & -6 \end{array} \qquad \div 2 \quad \div(-2) \quad \div(-6) \longrightarrow \begin{array}{c|c|c} 1 & & \\ 1/2 & 1 & \\ 3/2 & -5 & 1 \end{array}$$

assim $L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -5 & 1 \end{bmatrix}$ trang. inf. com 1's na diagonal ✓

Podem confirmar:

- $L \underset{\substack{L_2 = L_2 - L_1/2 \\ L_3 = L_3 - \frac{3L_1}{2}}}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix} \underset{L_3 = L_3 + 5L_2}{\sim} I_3$ ✓
- $A = LU$ ✓

(3) Quando $A = LU$, resolver $AX = B \iff L\underbrace{UX}_{=Y} = B \iff LY = B$

Assim, 1º Resolver $L\overset{?}{Y} = B$

2º Resolver $UX = Y$, o X é a solução de $AX = B$.

Exp anterior: $A = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -5 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{bmatrix} = LU$, resolver $AX = \underbrace{\begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix}}_{=B}$

$AX = B \iff L\underbrace{UX}_{=Y} = B \iff LY = B$ em que $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

1º Resolver $LY = B$

$$[L \mid B] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & -5 \\ 3/2 & -5 & 1 & 7 \end{array}\right] \underset{\substack{L_2 = L_2 - \frac{L_1}{2} \\ L_3 = L_3 - \frac{3L_1}{2}}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & -5 & 1 & 7 \end{array}\right] \underset{L_3 = L_3 + 5L_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -18 \end{array}\right] = [E \mid F]$$

$LY = B \iff EY = F \iff \begin{cases} y_1 = 0 \\ y_2 = -5 \\ y_3 = -18 \end{cases} \iff Y = \begin{bmatrix} 0 \\ -5 \\ -18 \end{bmatrix}$

2º Resolver $UX = Y$

$$[U \mid Y] = \left[\begin{array}{ccc|c} 2 & -2 & 4 & 0 \\ 0 & -2 & -1 & -5 \\ 0 & 0 & -6 & -18 \end{array}\right] \underset{L_3 = \frac{-L_3}{6}}{\sim} \left[\begin{array}{ccc|c} 2 & -2 & 4 & 0 \\ 0 & -2 & -1 & -5 \\ 0 & 0 & 1 & 3 \end{array}\right] \underset{\substack{L_2 = L_2 + L_3 \\ L_1 = L_1 - 4L_3}}{\sim} \left[\begin{array}{ccc|c} 2 & -2 & 0 & -12 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right] \underset{L_1 = L_1 - L_2}{\sim}$$

$$\sim \left[\begin{array}{ccc|c} 2 & 0 & 0 & -10 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right] \underset{\substack{L_1 = L_1/2 \\ L_2 = L_2/(-2)}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right] = [G \mid H]$$

$UX = Y \iff GX = H \iff \begin{cases} x_1 = -5 \\ x_2 = 1 \\ x_3 = 3 \end{cases} \iff X = \begin{bmatrix} -5 \\ 1 \\ 3 \end{bmatrix}$ é a solução de $AX = B$.

Resumo Resolver $AX=B$ com decomposição LU:

(1) Escalonar $A \sim U$ (sem trocar linhas, nem multiplicar linhas por um escalar)

(2) Construir L

(3) Resolver $Ly=B$

(4) Resolver $UX=y$, X é a solução de $AX=B$.

FP1 36 (b)

(1) Escalonar $A \sim U$ ($L_i \leftrightarrow L_j$, ~~$L_i = \alpha L_i$~~ $_{\alpha \neq 0}$)

$$A = \begin{bmatrix} 2 & -6 & 4 \\ -4 & 8 & 0 \\ 0 & -4 & 6 \end{bmatrix} \underset{L_2 = L_2 + 2L_1}{\sim} \begin{bmatrix} 2 & -6 & 4 \\ 0 & -4 & 8 \\ 0 & -4 & 6 \end{bmatrix} \underset{L_3 = L_3 - L_2}{\sim} \begin{bmatrix} 2 & -6 & 4 \\ 0 & -4 & 8 \\ 0 & 0 & -2 \end{bmatrix} = U \text{ escalonada} \checkmark$$

(2) Construir L

$$\begin{bmatrix} 2 & & \\ -4 & -4 & \\ 0 & -4 & -2 \end{bmatrix}_{\div 2 \;\; \div(-4) \;\; \div(-2)} \rightarrow \begin{bmatrix} 1 & & \\ -2 & 1 & \\ 0 & 1 & 1 \end{bmatrix} \text{ assim } L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$ triang. inf. com 1's na diagonal

(ver que $A = LU$). Como $A = LU$ temos $AX = B \Leftrightarrow L\underbrace{UX}_{=Y} = B \Leftrightarrow Ly = B$

(3) Resolver $Ly = B$ em que $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$[L|B] \underset{L_2 = L_2 + 2L_1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 6 \end{array}\right] \underset{L_3 = L_3 - L_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 6 \end{array}\right] = [E|F]$$

$$Ly = B \Leftrightarrow Ey = F \Leftrightarrow \begin{cases} y_1 = 2 \\ y_2 = 0 \\ y_3 = 6 \end{cases} \Leftrightarrow y = \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix}$$

(4) Resolver $UX = Y$

$$[U|Y] \underset{\substack{L_2 = L_2 + 4L_3 \\ L_1 = L_1 + 2L_3}}{\sim} \left[\begin{array}{ccc|c} 2 & -6 & 0 & 14 \\ 0 & -4 & 0 & 24 \\ 0 & 0 & -2 & 6 \end{array}\right] \underset{\substack{L_1 = L_1/2 \\ L_2 = L_2/(-4) \\ L_3 = L_3/(-2)}}{\sim} \left[\begin{array}{ccc|c} 1 & -3 & 0 & 7 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & -3 \end{array}\right] \underset{L_1 = L_1 + 3L_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -11 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & -3 \end{array}\right] = [G|H]$$

$$UX = Y \Leftrightarrow GX = H \Leftrightarrow \begin{cases} x_1 = -11 \\ x_2 = -6 \\ x_3 = -3 \end{cases} \Leftrightarrow X = \begin{bmatrix} -11 \\ -6 \\ -3 \end{bmatrix}$$ é a solução de $AX=B$ (confirmar que $AX = B$)

Decomposição LDU:

$$A = LU \Leftrightarrow \begin{bmatrix} 2 & -6 & 4 \\ -4 & 8 & 0 \\ 0 & -4 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -6 & 4 \\ 0 & -4 & 8 \\ 0 & 0 & -2 \end{bmatrix} \underset{\substack{L_1 = L_1/2 \\ L_2 = L_2/(-4) \\ L_3 = L_3/(-2)}}{\sim}$$

$$A = \underset{\substack{\text{triang. inf.} \\ \text{com 1's} \\ \text{na diag.}}}{L} \; \underset{\text{diagonal}}{D} \; \underset{\substack{\text{escalonada} \\ \text{c/ 1's na} \\ \text{diagonal}}}{U} \Leftrightarrow \begin{bmatrix} 2 & -6 & 4 \\ -4 & 8 & 0 \\ 0 & -4 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

FP1 toda!

Recomendo 16(c), 18, 32, 36(a).