13/11/2025

Aula 18

Não será avaliado o Método de Gram-Schmidt (Cap. 4 slide 11) nem da FP4 os ex. 18(a) e 21.

FP4 **15** $\{(\underbrace{\sqrt{2}/2, 0, \sqrt{2}/2}_{x_1}), (\underbrace{a, \sqrt{2}/2, -b}_{x_2})\}$ é ortonormado se:

ortogonais:

$x_1 \cdot x_2 = 0 \iff \left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right) \cdot \left(a, \frac{\sqrt{2}}{2}, -b\right) = 0 \iff \frac{\sqrt{2}}{2} a + 0 \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} b = 0$

$\iff \frac{\sqrt{2}}{2} a = \frac{\sqrt{2}}{2} b \iff a = b$

Norma 1:

$X_1 \cdot X_1 = 1 \iff \left(\frac{\sqrt{2}}{2}\right)^2 + 0^2 + \left(\frac{\sqrt{2}}{2}\right)^2 = 1$ ✓

$X_2 \cdot X_2 = 1 \iff a^2 + \left(\frac{\sqrt{2}}{2}\right)^2 + (-b)^2 = 1 \overset{\text{como } a=b}{\iff} 2a^2 = 1 - \frac{1}{2} \iff a^2 = 1/4 \iff a = \pm\frac{1}{2}$

Assim, se $a = b = 1/2$ ou $a = b = -1/2$ o conjunto é o.n..

Produto externo (só em $\mathbb{R}^3$)

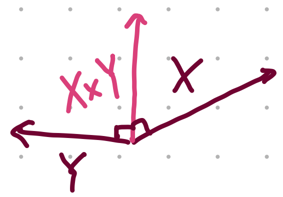
$X \times Y \rightsquigarrow \begin{vmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$ em que $i = (1,0,0)$, $j = (0,1,0)$ e $k = (0,0,1)$.

desenvolvimento sempre pela linha 1.

FP4 **5** (a) $(2,-1,1) \times (0,2,-1) = \begin{vmatrix} i & j & k \\ 2 & -1 & 1 \\ 0 & 2 & -1 \end{vmatrix} = i \begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} - j \begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix} + k \begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix}$

$= -i + 2j + 4k = -(1,0,0) + 2(0,1,0) + 4(0,0,1) = (-1,2,4)$

(b) $X \times Y$ é <u>sempre</u> ortogonal a X e a Y, isto é



$(X \times Y) \cdot X = (-1,2,4) \cdot (2,-1,1) = -2 - 2 + 4 = 0$

$(X \times Y) \cdot Y = (-1,2,4) \cdot (0,2,-1) = 0 + 4 - 4 = 0.$

Propriedades:

(1) $X \times Y = -(Y \times X)$ porque $X \times Y = \begin{vmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = -\begin{vmatrix} i & j & k \\ y_1 & y_2 & y_3 \\ x_1 & x_2 & x_3 \end{vmatrix} = -(Y \times X)$

(2) $X \times (Y + Z) = (X \times Y) + (X \times Z)$

porque

$X \times (Y+Z) = \begin{vmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1+z_1 & y_2+z_2 & y_3+z_3 \end{vmatrix} = \begin{vmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} + \begin{vmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = (X \times Y) + (X \times Z)$

(3) $\alpha(X\times Y) = (\alpha X)\times Y = X\times(\alpha Y)$ porque $(\alpha X)\times Y = \begin{vmatrix} i & j & k\\ \alpha x_1 & \alpha x_2 & \alpha x_3 \\ y_1 & y_2 & y_3\end{vmatrix} = \alpha \begin{vmatrix} i & j & k\\ x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{vmatrix}$
$= \alpha(X\times Y)$

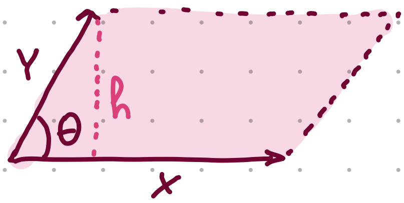
(4) $X\times X = 0$ porque $X\times X = \begin{vmatrix} i & j & k\\ x_1 & x_2 & x_3\\ x_1 & x_2 & x_3\end{vmatrix} = 0$ tem 2 linhas iguais!

(5) $X\times 0 = 0\times X = 0$ porque $X\times 0 = \begin{vmatrix} i & j & k\\ x_1 & x_2 & x_3\\ 0 & 0 & 0\end{vmatrix} = (0,0,0)$ tem uma linha nula!

(6) $X\times Y$ é ortogonal a X e a Y.

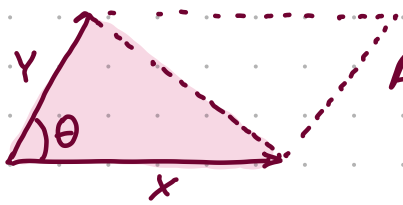
(7) $\|X\times Y\| = \|X\|\|Y\|\sin\theta$, θ é o ângulo entre X e Y.

Áreas



$\sin\theta = \dfrac{h}{\|Y\|} \Leftrightarrow h = \|Y\|\sin\theta$

$A_{\square} = b\times h$
$= \|X\|\|Y\|\sin\theta$
$= \|X\times Y\|$



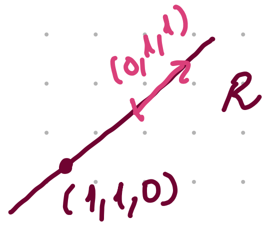
$A_{\Delta} = \dfrac{\|X\times Y\|}{2}$

Equações de retas e planos em $\mathbb{R}^3$ - ver folha extra.

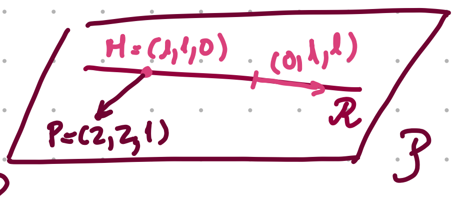
FP4 (9) $\begin{cases} x+y-z=2\\ x-y+z=0\end{cases}$ Equação cartesiana da reta R → Eq. vetorial de R

$\Leftrightarrow \begin{cases} y-z+y-z=2\\ x=y-z\end{cases} \Leftrightarrow \begin{cases} 2y-2z=2\\ x=y-z\end{cases} \Leftrightarrow \begin{cases} y=1+z\\ x=1+\cancel{z}-\cancel{z}\end{cases} \Leftrightarrow \begin{cases} y=1+z\\ x=1\end{cases}$

$(x,y,z)\in R \Leftrightarrow x=1$ e $y=1+\alpha$, $\alpha\in\mathbb{R}$
$\Leftrightarrow (x,y,z) = (1, 1+\alpha, \alpha)$
$\Leftrightarrow (x,y,z) = (1,1,0) + (0,\alpha,\alpha)$
$\Leftrightarrow (x,y,z) = (1,1,0) + \alpha(0,1,1)$, $\alpha\in\mathbb{R}$



$\overrightarrow{MP} = P-M = (2,2,1)-(1,1,0) = (1,1,1)$ vetor em $\mathcal{P}$



• Eq. vetorial de $\mathcal{P}$ é:
$(x,y,z) = (2,2,1) + \alpha(0,1,1) + \beta(1,1,1)$, $\alpha, \beta\in\mathbb{R}$

Eq. paramétrica → Eq. cartesiana de $\mathcal{P}$

$$\begin{cases} x=2+\beta \\ y=2+\alpha+\beta \\ z=1+\alpha+\beta\end{cases} \Leftrightarrow \begin{cases}\beta=x-2\\ y=\cancel{2}+\alpha+x-\cancel{2}\\ z=1+\alpha+x-2\end{cases}\Leftrightarrow \begin{cases}\beta=x-2\\ y=\alpha+x\\ z=\alpha+x-1\end{cases}\Leftrightarrow\begin{cases}\beta=x-2\\ \alpha=y-x\\ z=y-\cancel{x}+\cancel{x}-1\end{cases}\Leftrightarrow\begin{cases}—\\ —\\ z=y-1\end{cases}$$

- Eq. cartesiana de $\mathcal{P}$ é $y-z=1$ (vetor $(0,1,-1)$ é ortogonal a $\mathcal{P}$).

Distância entre pontos:

16 $A=(-1,0,2)$ e $B=(1,-1,1)$ e seja $X=(x,y,z)$ tal que

$$d(X,A)=d(X,B) \Leftrightarrow \left(\sqrt{(x+1)^2+(y-0)^2+(z-2)^2}\right)^2=\left(\sqrt{(x-1)^2+(y+1)^2+(z-1)^2}\right)^2$$

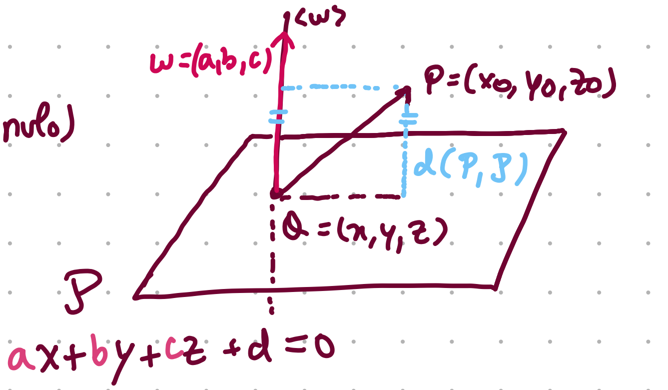
$$\Leftrightarrow \cancel{x^2}+2x+\cancel{1}+\cancel{y^2}+\cancel{z^2}-4z+4=\cancel{x^2}-2x+\cancel{1}+\cancel{y^2}+2y+1+\cancel{z^2}-2z+1$$

$$\Leftrightarrow 4x-2y-2z+2=0 \Leftrightarrow 2x-y-z+1=0$$

Eq. cartesiana de um plano. O vetor $(2,-1,-1)$ é ortogonal ao plano.

Distância ponto a um plano:

- Seja Q um ponto qualquer de $\mathcal{P}$ e $w=(a,b,c)$ um vetor ortogonal a $\mathcal{P}$ (não nulo)
- $\langle w\rangle$ é uma reta
- $\left\{\frac{w}{\|w\|}\right\}$ é uma base o.n. de $\langle w\rangle$
- $\overrightarrow{QP}=P-Q=(x_0-x,y_0-y,z_0-z)$.



Assim, a distância do ponto P ao plano $\mathcal{P}$ é:

$$d(P,\mathcal{P})=\left\|\operatorname{proj}_{\langle w\rangle}\overrightarrow{QP}\right\| \underset{\text{Teorema Slide 10}}{=} \left\|\left(\overrightarrow{QP}\cdot\frac{w}{\|w\|}\right)\frac{w}{\|w\|}\right\| \underset{\text{Slide 3}}{\overset{\text{Prop 6}}{=}}$$

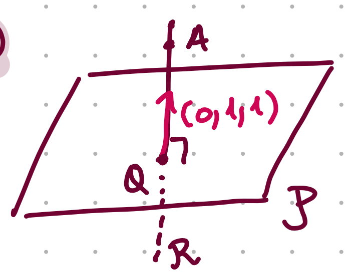
$$=\frac{|\overrightarrow{QP}\cdot w|}{\|w\|}\underbrace{\left\|\frac{w}{\|w\|}\right\|}_{=1 \text{ porque é unitário}}=\frac{|\overrightarrow{QP}\cdot w|}{\|w\|}=\frac{|(x_0-x)a+(y_0-y)b+(z_0-z)c|}{\sqrt{a^2+b^2+c^2}}$$

$$=\frac{|ax_0+by_0+cz_0-\overbrace{(ax+by+cz)}^{=d \text{ porque } Q\in\mathcal{P}}|}{\sqrt{a^2+b^2+c^2}}=\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$$

11 $A=(3,1/2,-7/2)$ e $\mathcal{P}: y+z=-1$

$\Leftrightarrow 0x+1y+1z=-1$

(a)



Eq. vetorial de $\mathcal{R}$ é:

$(x,y,z)=(3,1/2,-7/2)+\alpha(0,1,1)$

(b) Q é a interseção de R com $\mathcal{P}$. Assim, $d(A,\mathcal{P}) = d(A,Q)$

$$\begin{cases} x = 3 \\ y = 1/2 + \alpha \\ z = -7/2 + \alpha \\ y + z = -1 \end{cases} \Leftrightarrow \begin{cases} x = 3 \\ — \\ — \\ 1/2 - 7/2 + 2\alpha = -1 \end{cases} \Leftrightarrow \begin{cases} x = 3 \\ y = 3/2 \\ z = -5/2 \\ \alpha = 1 \end{cases} \therefore Q = (3, 3/2, -5/2)$$

Reta R

plano $\mathcal{P}$

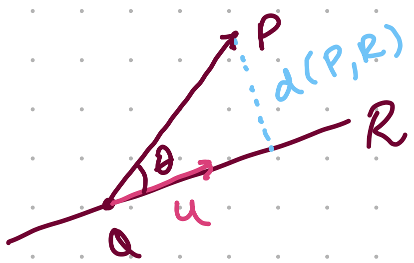
$$d(A,\mathcal{P}) = d(A,Q) = \sqrt{(3-3)^2 + (1/2 - 3/2)^2 + (-7/2 + 5/2)^2}$$
$$= \sqrt{2}$$

ou $\mathcal{P}: y + z = -1 \Leftrightarrow \underbrace{0}_{a}x + \underbrace{1}_{b}y + \underbrace{1}_{c}z + \underbrace{1}_{d} = 0$

NOTA: na aula enganei-me neste sinal, por favor confirmem

$$d(A,\mathcal{P}) = \frac{|a x_A + b y_A + c z_A + d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|0 \cdot 3 + 1/2 \cdot 1 + (-7/2) \cdot 1 + 1|}{\sqrt{0^2 + 1^2 + 1^2}} = \sqrt{2}$$

Distância de um ponto a uma reta:



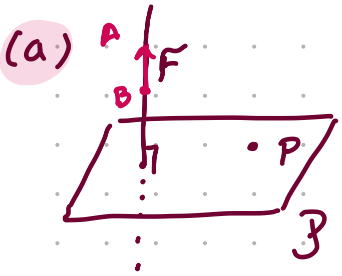
$$|\sin\theta| = \frac{d(P,R)}{\|\vec{QP}\|} \Leftrightarrow d(P,R) = \|\vec{QP}\| \, |\sin\theta|$$

Como $\|u \times \vec{QP}\| = \|u\| \|\vec{QP}\| \sin\theta \Leftrightarrow \sin\theta = \dfrac{\|u \times \vec{QP}\|}{\|u\| \|\vec{QP}\|}$

Assim, $\boxed{d(P,R) =} \ \|\vec{QP}\| \, |\sin\theta| = \dfrac{\cancel{\|\vec{QP}\|} \|u \times \vec{QP}\|}{\|u\| \cancel{\|\vec{QP}\|}} \boxed{= \dfrac{\|u \times \vec{QP}\|}{\|u\|}}$

13 $P = (-1, 1, 2)$. $\mathcal{F}$ é uma reta que contém $A = (1,0,0)$ e $B = (0,0,1)$

(a)



$\vec{BA}$ é ortogonal a $\mathcal{P}$. $\vec{BA} = A - B = (1, 0, -1)$

$1 \cdot x + 0 \cdot y - 1 \cdot z + d = 0 \Leftrightarrow x - z + d = 0$

$P = (-1, 1, 2) \in \mathcal{P}$ logo $-1 - 2 + d = 0$
$\Leftrightarrow d = 3$

Eq. cartesiana de $\mathcal{P}$: $x - z + 3 = 0$

(b) Eq. vetorial de $\mathcal{F}$: $(x,y,z) = (1,0,0) + \alpha \vec{BA} = (1,0,0) + \alpha(1,0,-1), \ \alpha \in \mathbb{R}$

Q é a interseção de $\mathcal{F}$ com $\mathcal{P}$.

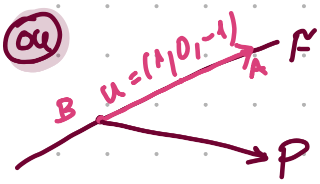
$$\begin{cases} x = 1+\alpha \\ y = 0 \\ z = -\alpha \\ x - z + 3 = 0 \end{cases} \Leftrightarrow \begin{cases} x = 1+\alpha \\ y = 0 \\ z = -\alpha \\ 1+\alpha+\alpha+3=0 \end{cases} \Leftrightarrow \begin{cases} x = -1 \\ y = 0 \\ z = 2 \\ \alpha = -2 \end{cases} \therefore Q = (-1,0,2)$$

Reta R

plano $\mathcal{F}$

$$d(P,\mathcal{F}) = d(P,Q) = \sqrt{(-1+1)^2 + (1-0)^2 + (2-2)^2} = 1$$

ou



$$\vec{BP} = P - B = (-1,1,2)-(0,0,1) = (-1,1,1)$$

$$u \times \vec{BP} = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{vmatrix} = i\underbrace{\begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix}}_{=1} - j\underbrace{\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}}_{=0} + k\underbrace{\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix}}_{=1} = (1,0,1)$$

$$d(P,\mathcal{F}) = \frac{\|u \times \vec{BP}\|}{\|u\|} = \frac{\|(1,0,1)\|}{\|(1,0,-1)\|} = \frac{\sqrt{1^2+(0)^2+1^2}}{\sqrt{1^2+0^2+(-1)^2}} = 1$$

FP4. Recomendo o 7 e 12.