Yule-Walker Equations in matrix form

Practical Time Series Analysis

Thistleton and Sadigov

Objectives

- Rewrite Yule – Walker equations in matrix form for AR(p) processes

AR(p) process

$$X_t = \phi_0 + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots \phi_p X_{t-p} + Z_t$$

where

$$Z_t \sim Normal(0, \sigma_Z^2)$$

Note that, if we take expectation from both sides of the model

$$X_t = \phi_0 + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots \phi_p X_{t-p} + Z_t$$

We get

$$\mu = \phi_0 + \phi_1 \mu + \phi_2 \mu + \cdots \phi_p \mu$$

Subtract side by side

$$X_t - \mu = \phi_1 (X_{t-1} - \mu) + \phi_2 (X_{t-2} - \mu) + \cdots \phi_p (X_{t-p} - \mu) + Z_t$$

If $\tilde{X}_t = X_t - \mu$, then $E[\tilde{X}_t] = 0$, and

$$\tilde{X}_t = \phi_1 \tilde{X}_{t-1} + \phi_2 \tilde{X}_{t-2} + \cdots \phi_p \tilde{X}_{t-p} + Z_t$$

AR(p) process with $\mu = 0$

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots \phi_p X_{t-p} + Z_t$$

where

$$Z_t \sim Normal(0, \sigma_Z^2)$$

Yule –Walker equations

Autocorrelation function obeys

$$\rho(k) = \phi_1 \rho(k-1) + \phi_2 \rho(k-2) + \cdots + \phi_p \rho(k-p)$$

for $k \geq 1, \rho(0) = 1$ and $\rho(k) = \rho(-k)$ for $k < 0$.

Lets write them for $k = 1,2, \ldots, p$.

Yule Walker equations

$$\rho(k) = \phi_1\rho(k-1) + \phi_2\rho(k-2) + \cdots + \phi_p\rho(k-p)$$

For $k = 1,2,\dots,p$,

$$\rho(1) = \phi_1\rho(0) + \phi_2\rho(-1) + \phi_3\rho(-2) + \cdots + \phi_p\rho(1-p)$$
$$\rho(2) = \phi_1\rho(1) + \phi_2\rho(0) + \phi_3\rho(-1) + \cdots + \phi_p\rho(2-p)$$
$$\rho(3) = \phi_1\rho(2) + \phi_2\rho(1) + \phi_3\rho(0) + \cdots + \phi_p\rho(3-p)$$
$$\vdots$$
$$\rho(p-1) = \phi_1\rho(p-2) + \phi_2\rho(p-3) + \phi_3\rho(p-4) + \cdots + \phi_p\rho(1)$$
$$\rho(p) = \phi_1\rho(p-1) + \phi_2\rho(p-2) + \phi_3\rho(p-3) + \cdots + \phi_p\rho(0)$$

Recall $\rho(k) = \rho(-k)$.

i.e., $\rho(-1) = \rho(1),\ \rho(-2) = \rho(2),\ \dots, \rho(2-p) = \rho(p-2), \rho(1-p) = \rho(p-1)$

$$\rho(-k) = \rho(k)$$

$$\rho(1) = \phi_1\rho(0) + \phi_2\rho(1) + \phi_3\rho(2) + \cdots + \phi_p\rho(p-1)$$
$$\rho(2) = \phi_1\rho(1) + \phi_2\rho(0) + \phi_3\rho(1) + \cdots + \phi_p\rho(p-2)$$
$$\rho(3) = \phi_1\rho(2) + \phi_2\rho(1) + \phi_3\rho(0) + \cdots + \phi_p\rho(p-3)$$

$$\vdots$$

$$\rho(p-1)$$
$$= \phi_1\rho(p-2) + \phi_2\rho(p-3) + \phi_3\rho(p-4) + \cdots + \phi_p\rho(1)$$
$$\rho(p)$$
$$= \phi_1\rho(p-1) + \phi_2\rho(p-2) + \phi_3\rho(p-3) + \cdots + \phi_p\rho(0)$$

$$\rho(0) = 1$$

$$\rho(1) = \phi_1 + \phi_2\rho(1) + \phi_3\rho(2) + \cdots + \phi_p\rho(p-1)$$
$$\rho(2) = \phi_1\rho(1) + \phi_2 + \phi_3\rho(1) + \cdots + \phi_p\rho(p-2)$$
$$\rho(3) = \phi_1\rho(2) + \phi_2\rho(1) + \phi_3 + \cdots + \phi_p\rho(p-3)$$

$$\vdots$$

$$\rho(p-1) = \phi_1\rho(p-2) + \phi_2\rho(p-3) + \phi_3\rho(p-4) + \cdots + \phi_p\rho(1)$$
$$\rho(p) = \phi_1\rho(p-1) + \phi_2\rho(p-2) + \phi_3\rho(p-3) + \cdots + \phi_p$$

Matrix form: Yule- Walker equations

$$\underbrace{\begin{bmatrix} \rho(1) \\ \rho(2) \\ \rho(3) \\ \vdots \\ \rho(p-1) \\ \rho(p) \end{bmatrix}}_{b} = \underbrace{\begin{bmatrix} 1 & \rho(1) & \rho(2) & & \rho(p-1) \\ \rho(1) & 1 & \rho(1) & \cdots & \rho(p-2) \\ \rho(2) & \rho(1) & 1 & & \rho(p-3) \\ & \vdots & & \ddots & \vdots \\ \rho(p-2) & \rho(p-3) & \rho(p-4) & \cdots & \rho(1) \\ \rho(p-1) & \rho(p-2) & \rho(p-3) & & 1 \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_{p-1} \\ \phi_p \end{bmatrix}}_{\phi}$$

$$b = R\phi$$

$$R^{-1}\, b = \phi$$

Sample ACF $r_k \approx \rho(k)$

$$\underbrace{\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_{p-1} \\ r_p \end{bmatrix}}_{\hat{b}} = \underbrace{\begin{bmatrix} 1 & r_1 & r_2 & & r_{p-1} \\ r_1 & 1 & r_1 & \cdots & r_{p-2} \\ r_2 & r_1 & 1 & & r_{p-3} \\ & \vdots & & \ddots & \vdots \\ r_{p-2} & r_{p-3} & r_{p-4} & & r_1 \\ r_{p-1} & r_{p-2} & r_{p-3} & \cdots & 1 \end{bmatrix}}_{\hat{R}} \underbrace{\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_{p-1} \\ \phi_p \end{bmatrix}}_{\hat{\phi}}$$

$$\hat{b} = \hat{R}\hat{\phi}$$

$$\hat{R}^{-1}\,\hat{b} = \hat{\phi}$$

Matrices R and $\hat{R}$

- These matrices are symmetric matrices
- They are positive semidefinite matrices
- All eigenvalues are nonnegative
- Inverses of these matrices exist

$$\begin{bmatrix} 1 & r_1 & r_2 & & r_{p-1} \\ r_1 & 1 & r_1 & \cdots & r_{p-2} \\ r_2 & r_1 & 1 & & r_{p-3} \\ & \vdots & & \ddots & \vdots \\ r_{p-2} & r_{p-3} & r_{p-4} & \cdots & r_1 \\ r_{p-1} & r_{p-2} & r_{p-3} & & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & \rho(1) & \rho(2) & & \rho(p-1) \\ \rho(1) & 1 & \rho(1) & \cdots & \rho(p-2) \\ \rho(2) & \rho(1) & 1 & & \rho(p-3) \\ & \vdots & & \ddots & \vdots \\ \rho(p-2) & \rho(p-3) & \rho(p-4) & \cdots & \rho(1) \\ \rho(p-1) & \rho(p-2) & \rho(p-3) & & 1 \end{bmatrix}$$

- $\hat{b} = \hat{R}\hat{\phi}$ has a unique solution

Example – AR(2)

We (will) estimate coefficients of the model

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + Z_t$$

by first finding r_1, r_2 using acf() routine, then solving the system of equations

$$\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} 1 & r_1 \\ r_1 & 1 \end{bmatrix} \begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{bmatrix}$$

Example – AR(3)

We (will) estimate coefficients of the model

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \phi_3 X_{t-3} + Z_t$$

by first finding r_1, r_2, r_3, using acf() routine, then solving the system of equations

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 1 & r_1 & r_2 \\ r_1 & 1 & r_1 \\ r_2 & r_1 & 1 \end{bmatrix} \begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \\ \hat{\phi}_3 \end{bmatrix}$$

What We’ve Learned

- Matrix form of Yule – Walker equations
- How to estimate the coefficients of an AR process using Yule-Walker equations

Estimating model parameters – AR(2) Simulation

Practical Time Series Analysis

Thistleton and Sadigov

Objectives

- Estimate variance of a white noise in a simulated AR(2) processes
- Estimate coefficients of a simulated AR(2) process using Yule-Walker equations in matrix form

AR(2) process (with mean zero)

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + Z_t$$

where

$$Z_t \sim Normal\ (0, \sigma_Z^2)$$

We simulate this process for

$$\phi_1 = \frac{1}{3}, \phi_2 = \frac{1}{2}, \sigma_Z = 4$$

Yule –Walker equations

We estimate coefficients of the model

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + Z_t$$

by first finding r_1, r_2 using acf() routine, then solving the system of equations

$$\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} 1 & r_1 \\ r_1 & 1 \end{bmatrix} \begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{bmatrix}$$

σ_Z Estimation

Since

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + Z_t$$

We have

$$Var(X_t)$$
$$= \phi_1^2 Var(X_{t-1}) + \phi_2^2 Var(X_{t-2}) + 2\phi_1\phi_2 Cov(X_{t-1}, X_{t-2}) + \sigma_Z^2$$

Thus

$$\sigma_Z^2 = \gamma(0)\left[1 - \phi_1^2 - \phi_2^2 - \frac{2\phi_1\phi_2\gamma(1)}{\gamma(0)}\right]$$
$$= \gamma(0)[1 - \phi_1^2 - \phi_2^2 - 2\phi_1\phi_2\rho_1]$$

Since

$$\begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} = \begin{bmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

we have

$$\rho_1 = \phi_1 + \rho_1 \phi_2$$

$$\rho_2 = \phi_1 \rho_1 + \phi_2$$

$$1 - \phi_1^2 - \phi_2^2 - 2\phi_1\phi_2\rho_1$$

$$= 1 - \phi_1^2 - \phi_1\phi_2\rho_1 - \phi_2^2 - \phi_1\phi_2\rho_1$$

$$= 1 - \phi_1(\phi_1 + \rho_1\phi_2) - \phi_2(\phi_1\rho_1 + \phi_2)$$

$$= 1 - \phi_1\rho_1 - \phi_2\rho_2$$

σ_Z Estimation cont.

Thus,

$$\sigma_Z^2 = \gamma(0)[1 - \phi_1\rho_1 - \phi_2\rho_2]$$

Yule –Walker estimator

$$\hat{\sigma}_Z^2 = c_0\,[1 - \hat{\phi}_1 r_1 - \hat{\phi}_2 r_2]$$

Simulation

- Number of data points, $n = 10000$

Routines that we use:

- arima.sim() # simulating
- plot() # plotting the series
- acf() # autocorrelation function
- matrix(,m,n) # matrix with dimensions m by n
- solve(R,b) # finds the solution to Rx=b

Code details

- sigma=4
- phi[1:2]=c(1/3,1/2)
- n=10000
- set.seed(2017)
- ar.process=arima.sim(n, model=list(ar=c(1/3,1/2)), sd=4)
- ar.process[1:5]

$$4.087685, 5.598492, 3.019295, 2.442354, 5.398302$$

Code details cont.

- r[1:2]=acf(ar.process, plot=F)$acf[2:3]

$$r[1] = 0.6814103$$
$$r[2] = 0.7255825$$

R=matrix(1,2,2) # matrix of dimension 2 by 2, with entries all 1's.

$$R = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

- R[1,2]=r[1] # only diagonal entries are edited
- R[2,1]=r[1] # only diagonal entries are edited

$$R = \begin{bmatrix} 1 & r_1 \\ r_1 & 1 \end{bmatrix}$$

Code details cont.

- b=matrix(r,2,1)# b- column vector entires from r

$$b = \begin{bmatrix} r[1] \\ r[2] \end{bmatrix} = \begin{bmatrix} 0.6814103 \\ 0.7255825 \end{bmatrix}$$

Code details cont.

- solve(R,b)

$$\begin{bmatrix} 0.3490720 \\ 0.4877212 \end{bmatrix}$$

- phi.hat=matrix(c(solve(R,b)[1,1], solve(R,b)[2,1]),2,1)

$$\hat{\phi}_1 = 0.3490720$$
$$\hat{\phi}_2 = 0.4877212$$

Code details cont.

- c0= acf(ar.process, type='covariance', plot=F)$acf[1]
- var.hat= c0*(1-sum(phi.hat*r))
- par(mfrow=c(3,1))
- plot(ar.process, main='Simulated AR(2)')
- acf(ar.process, main='ACF')
- pacf(ar.process, main='PACF')

Results

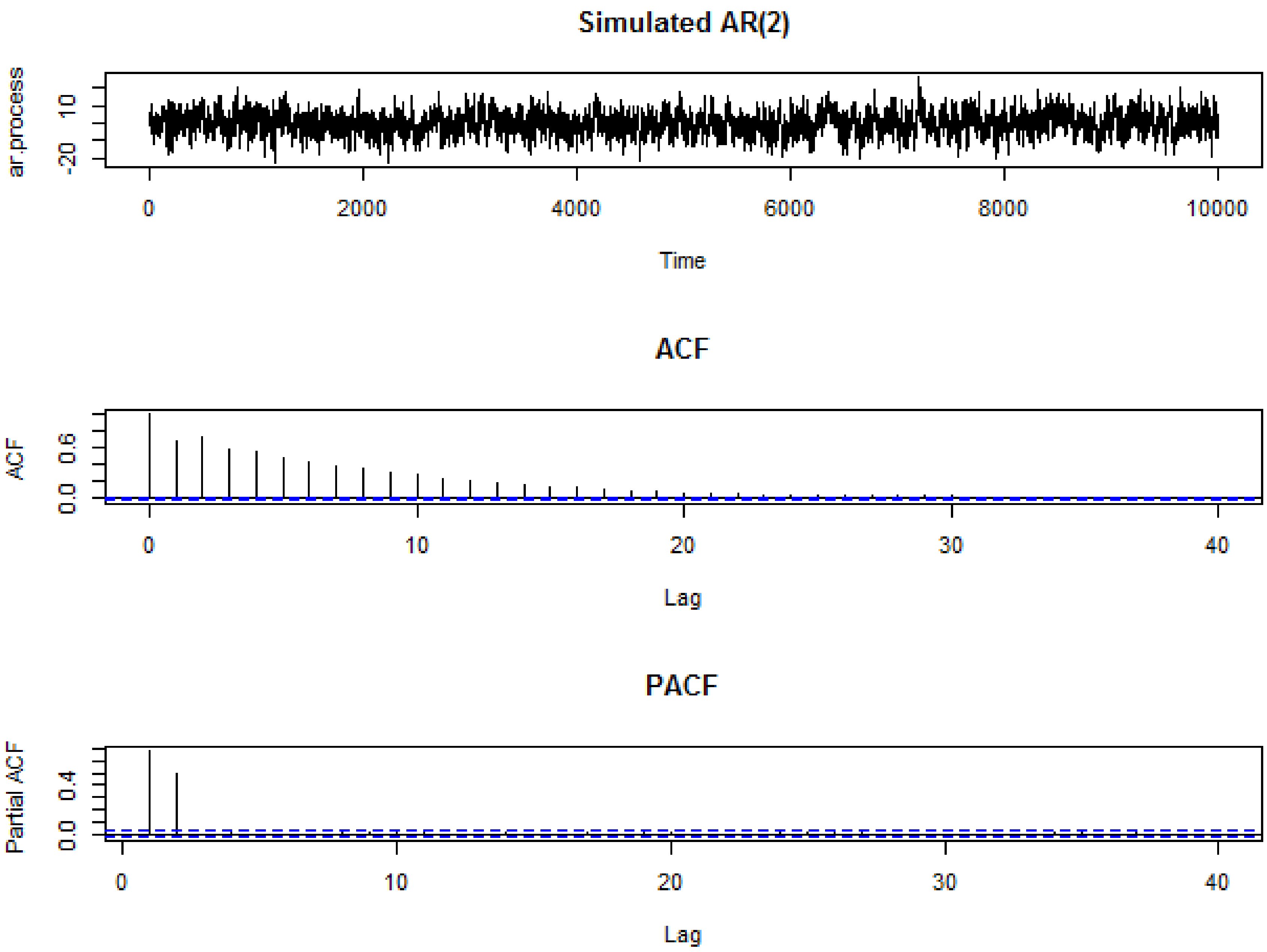
- $\phi_1 \approx \hat{\phi}_1 = 0.3490720$
- $\phi_2 \approx \hat{\phi}_2 = 0.4877212$
- $\sigma_Z^2 \approx \widehat{\sigma_Z^2} = 16.37169$

Actual Model

$$X_t = 0.\overline{3}\, X_{t-1} + 0.5 X_{t-2} + Z_t, \qquad Z_t \sim N(0,16)$$

Fitted model

$$X_t = 0.3490720\, X_{t-1} + 0.4877212\, X_{t-2} + Z_t,$$
$$Z_t \sim N(0,16.37169)$$


Simulated AR(2)
ar.process
10
-20
0
2000
4000
6000
8000
10000
Time
ACF
ACF
0.0
0.6
0
10
20
30
40
Lag
PACF
Partial ACF
0.0
0.4
0
10
20
30
40
Lag

What We've Learned

- Estimating model parameters of a simulated AR(2) process using Yule-Walker equations in a matrix form

Estimating model parameters – AR(3) Simulation

Practical Time Series Analysis

Thistleton and Sadigov

Objectives

- Estimate model parameters of a simulated AR(3) process using Yule-Walker equations in matrix form

AR(2) process (with mean zero)

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \phi_3 X_{t-3} + Z_t$$

where

$$Z_t \sim Normal\ (0, \sigma_Z^2)$$

We simulate this process for

$$\phi_1 = \frac{1}{3}, \phi_2 = \frac{1}{2}, \phi_3 = \frac{7}{100}, \sigma_Z = 4$$

Yule –Walker equaitons

We (will) estimate coefficients of the model

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \phi_3 X_{t-3} + Z_t$$

by first finding r_1, r_2 using acf() routine, then solving the system of equations

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 1 & r_1 & r_2 \\ r_1 & 1 & r_1 \\ r_2 & r_1 & 1 \end{bmatrix} \begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \\ \hat{\phi}_3 \end{bmatrix}$$

σ_Z Estimation

Yule – Walker estimator for σ_Z^2

$$\hat{\sigma}_Z^2 = c_0(1 - \sum_{i=1}^{p} \phi_i r_i)$$

Results (set.seed(2017))

- n$= 100000$
- $\phi_1 \approx 0.3381245$
- $\phi_2 \approx 0.4984999$
- $\phi_3 \approx 0.06849712$
- $\sigma_Z^2 \approx 15.979$

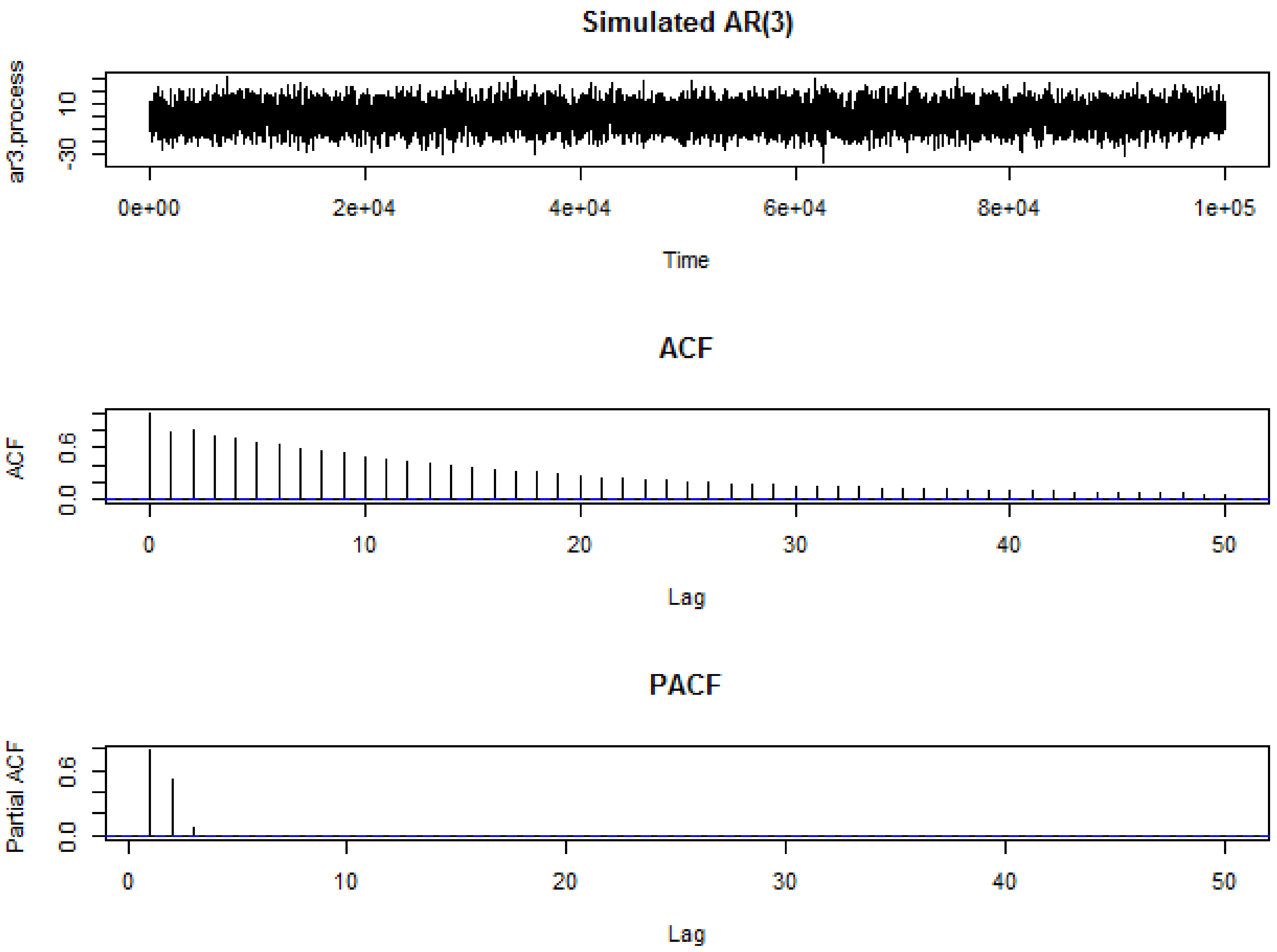
Actual Model

$$X_t = 0.\overline{3}\, X_{t-1} + 0.5X_{t-2} + 0.07X_{t-3} + Z_t, \qquad Z_t \sim N(0,16)$$

Fitted model

$$X_t = 0.3381245\, X_{t-1} + 0.4984999\, X_{t-2} + 0.06849712\, X_{t-3} + Z_t,$$

$$Z_t \sim N(0,15.979)$$


Simulated AR(3)
ar3.process
-30
10
0e+00
2e+04
4e+04
6e+04
8e+04
1e+05
Time
ACF
ACF
0.0
0.6
0
10
20
30
40
50
Lag
PACF
Partial ACF
0.0
0.6
0
10
20
30
40
50
Lag

What We've Learned

- Estimating model parameters of a simulated AR(3) process using Yule-Walker equations in a matrix form

Parameter estimation: Recruitment

Practical Time Series Analysis

Thistleton and Sadigov

Objectives

- To fit an AR(p) model to recruitment (number of new fish) for a period of 453 months ranging over the years 1950-1987.
- Use Yule-Walker equations in matrix form to estimate parameters of the fitted model

rec {astsa}

- Recruitment (number of new fish) for a period of 453 months ranging over the years 1950-1987.
- Monthly time series
- Source: “astsa” package

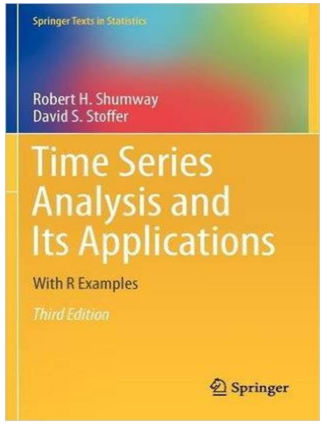
Shumway, R.H. and Stoffer, D.S. (2000)

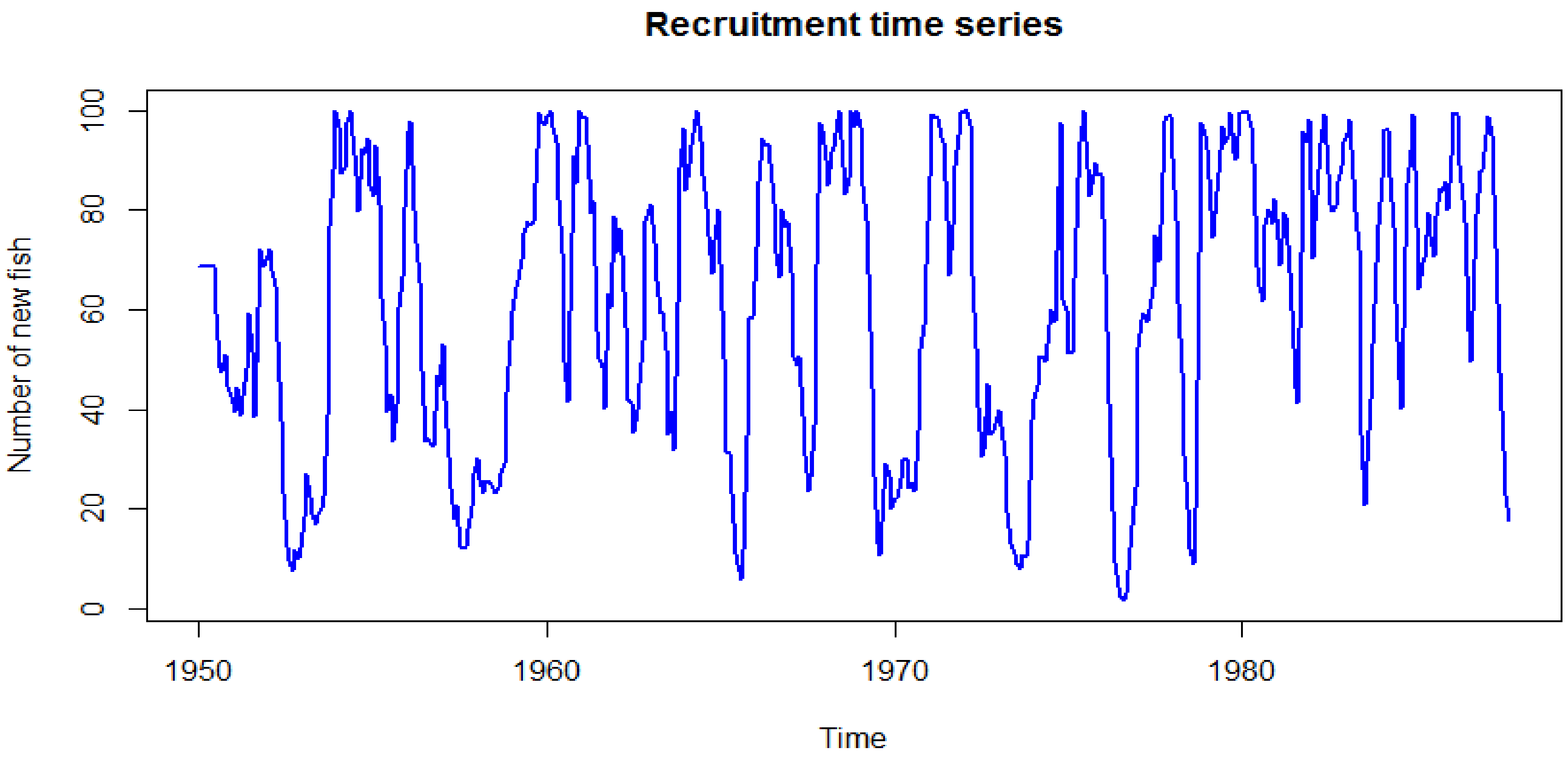
Time Series Analysis and its Applications

With R examples

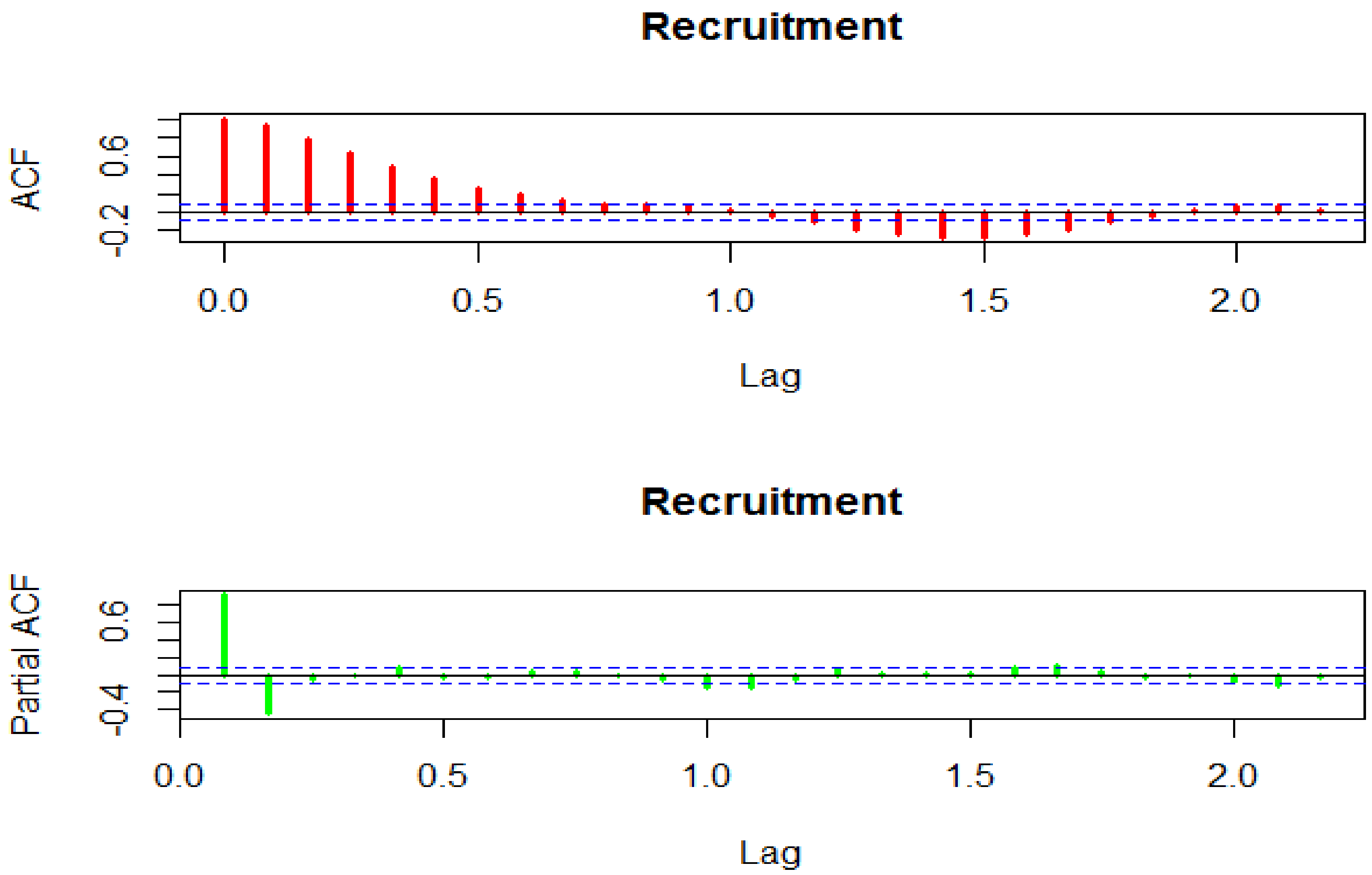
Third Edition

Springer




Recruitment time series
Number of new fish
0
20
40
60
80
100
1950
1960
1970
1980
Time

ACF and PACF


Recruitment
ACF
-0.2
0.6
0.0
0.5
1.0
1.5
2.0
Lag
Recruitment
Partial ACF
-0.4
0.6
0.0
0.5
1.0
1.5
2.0
Lag

The parsimony principle

- Choose ‘simplest explanation that fits the evidence’
- Simplest of competing theories is to be preferred
- PACF $\Longrightarrow$ AR(2)
- Yule-Walker equations in matrix form

Code

- ar.process=rec-mean(rec)

$$X_t - \mu$$

- p=2
- Yule-Walker equations: $\hat{R}\hat{\phi} = \hat{b}$
- Sample autocorrelation coefficients, vector r

```
for(i in 1:p+1){
    r[i-1]<-acf(ar.process, plot=F)$acf[i]
    }
```

Matrix $\hat{R}$

$$\begin{bmatrix} 1 & r_1 & r_2 & & r_{p-1} \\ r_1 & 1 & r_1 & \cdots & r_{p-2} \\ r_2 & r_1 & 1 & & r_{p-3} \\ & \vdots & & \ddots & \vdots \\ r_{p-2} & r_{p-3} & r_{p-4} & \cdots & r_1 \\ r_{p-1} & r_{p-2} & r_{p-3} & & 1 \end{bmatrix}$$

Realize

$$\hat{R}(i,j) = \hat{R}_{ij} = r_{|i-j|}$$

- R=matrix(1,p,p) # matrix of dimension p by p, with entries all 1's.

- for(i in 1:p){

```
for(j in 1:p){
    if(i!=j)
        R[i,j]=r[abs(i-j)]
    }
}
```

- # b-column vector on the right

```
b=matrix(,p,1)# b- column vector with no entries
for(i in 1:p){
          b[i,1]=r[i]
          }
```

- # solve(R,b) solves Rx=b, and gives x=R^(-1)b vector

```
phi.hat=NULL
for(i in 1:p){
          phi.hat[i]=solve(R,b)[i,1]
          }
```

Model

$$X_t - \bar{x} = \hat{\phi}_1(X_{t-1} - \bar{x}) + \hat{\phi}_2(X_{t-2} - \bar{x}) + \cdots + \hat{\phi}_p(X_{t-p} - \bar{x}) + Z_t$$

Thus

$$X_t = \hat{\phi}_0 + \hat{\phi}_1 X_{t-1} + \hat{\phi}_2 X_{t-2} + \cdots + \hat{\phi}_p X_{t-p} + Z_t$$

where

$$\hat{\phi}_0 = \bar{x}(1 - \sum_{i=1}^{p} \hat{\phi}_i)$$

$$p = 2$$

Fitted model is

$$X_t = 7.033036 + 1.331587\, X_{t-1} - 0.4445447\, X_{t-2} + Z_t$$

$$Z_t \sim Normal\ (0{,}94.17131)$$

What We’ve Learned

- Fitting an AR(p=2) model to Recruitment (number of new fish) from ‘astsa’ package using Yule-Walker equations in matrix form

Parameter estimation: Johnson&Johnson (AR attempt)

Practical Time Series Analysis

Thistleton and Sadigov

Objectives

- To fit an AR(p) model to Quarterly earnings (dollars) per Johnson & Johnson share 1960–80.
- Use Yule-Walker equations in matrix form to estimate parameters of the fitted model

JohnsonJohnson {datasets}

- Quarterly earnings (dollars) per Johnson & Johnson share 1960–80.
- Quarterly time series
- Source: “astsa” package

Shumway, R.H. and Stoffer, D.S. (2000)
Time Series Analysis and its Applications
With R examples
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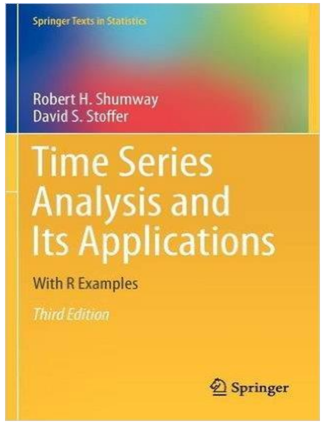
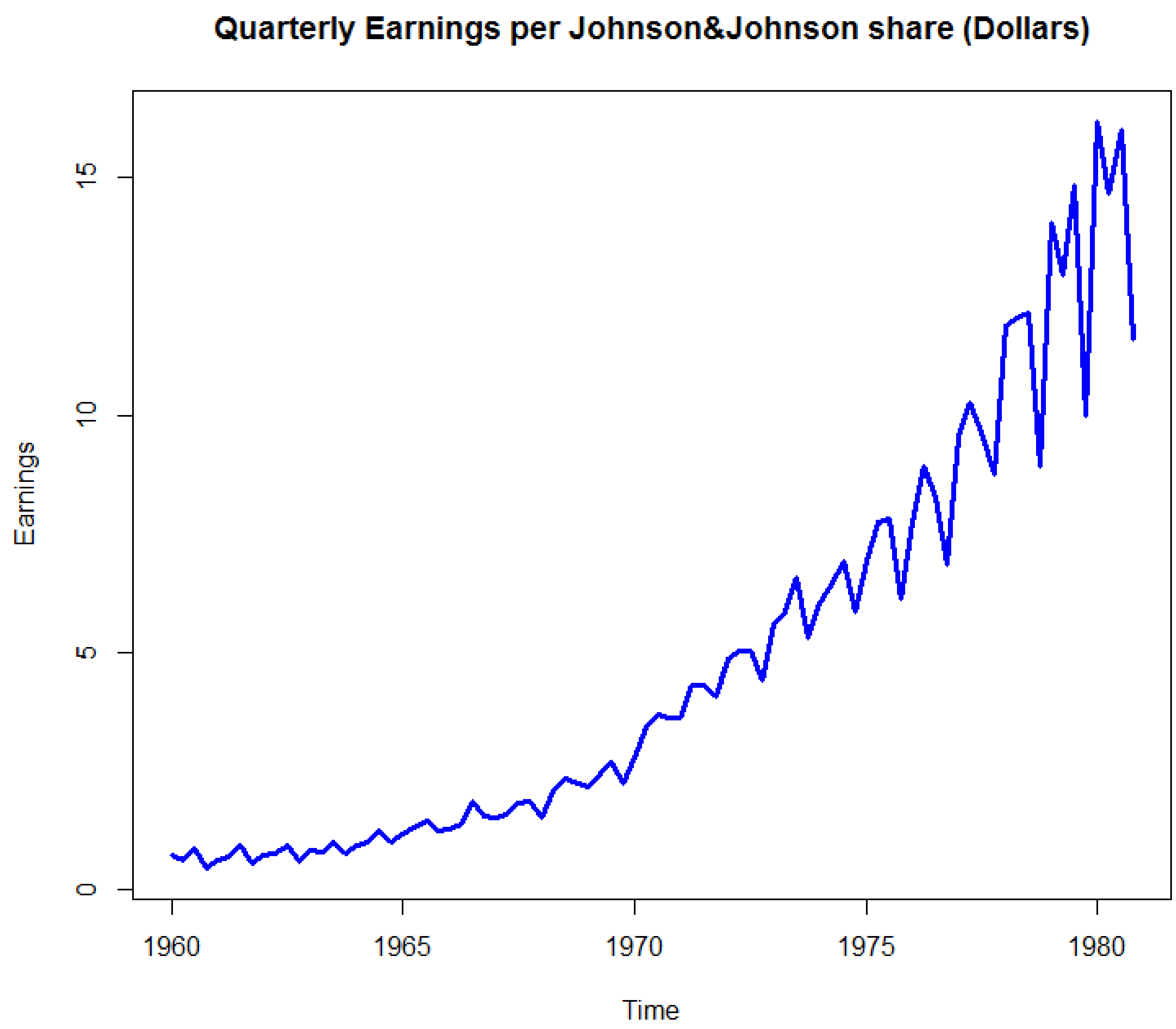



Quarterly Earnings per Johnson&Johnson share (Dollars)
Earnings
0
5
10
15
1960
1965
1970
1975
1980
Time

Transformation

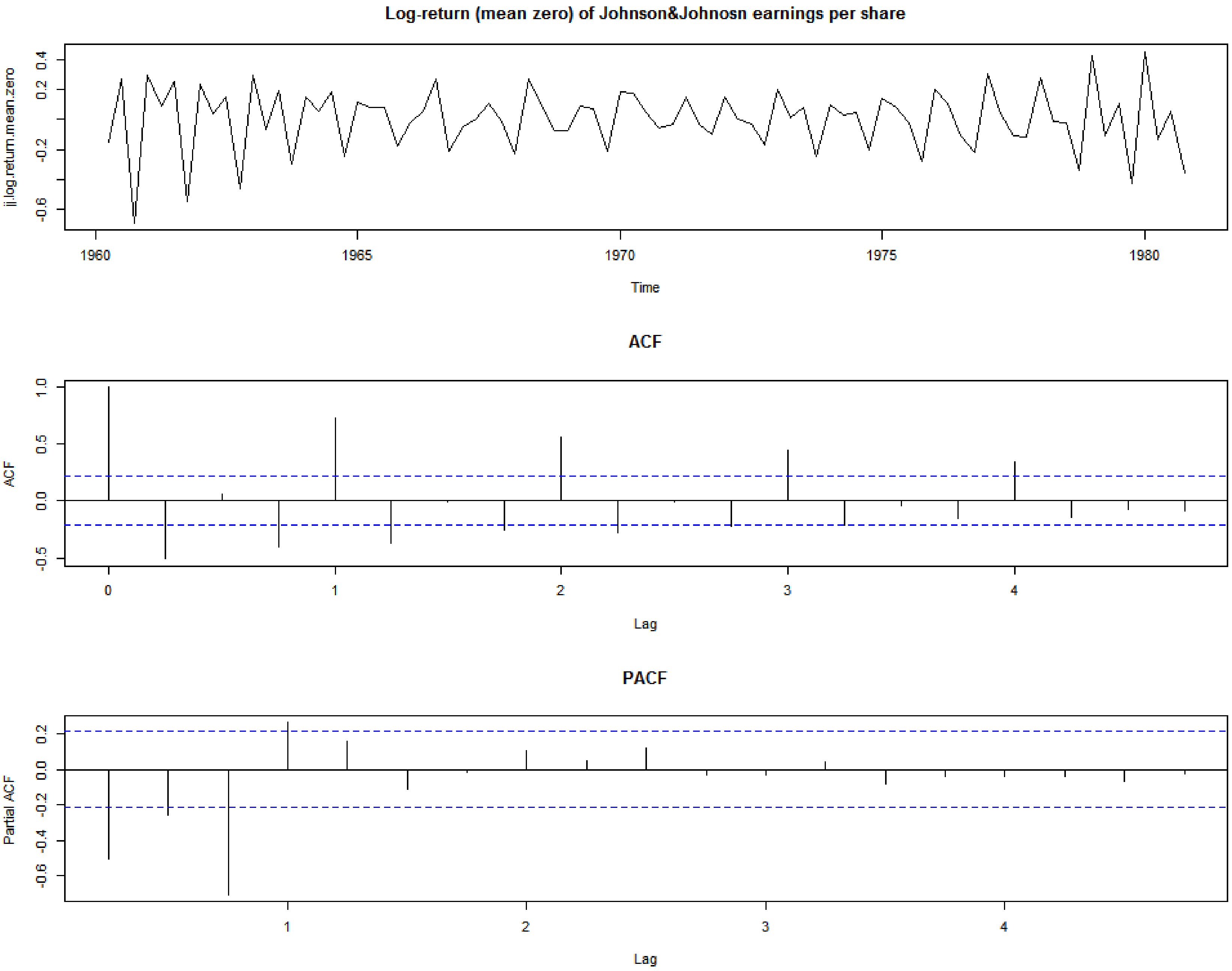
Log-return a time series $\{X_t\}$

is defined as

$$r_t = \log\left(\frac{X_t}{X_{t-1}}\right) = \log(X_t) - \log(X_{t-1})$$

In R,

$$diff(\log(\quad))$$



Log-return (mean zero) of Johnson&Johnosn earnings per share
jj.log.return.mean.zero
-0.6
-0.2
0.2
0.4
1960
1965
1970
1975
1980
Time
ACF
ACF
-0.5
0.0
0.5
1.0
0
1
2
3
4
Lag
PACF
Partial ACF
-0.6
-0.4
-0.2
0.0
0.2
1
2
3
4
Lag

The parsimony principle

- Choose ‘simplest explanation that fits the evidence’
- Simplest of competing theories is to be preferred
- PACF $\Longrightarrow$ AR(4)
- Yule-Walker equations in matrix form

$$p = 4$$

Fitted model is

$$r_t = 0.079781 - 0.6293492\, r_{t-1} - 0.5171526\, r_{t-2} - 0.4883374\, r_{t-3} + 0.2651266\, r_{t-4} + Z_t$$

$$Z_t \sim Normal\ (0, 0.01419242)$$

where

$$r_t = \log\left(\frac{X_t}{X_{t-1}}\right)$$

What We've Learned

- Fitting an AR(p=4) model to log-return of Johnson & Johnson quarterly earnings from 'astsa' package using Yule-Walker equations in matrix form