How to Fuzz up a Type Theory

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Background

Goal

We want to figure out how a Fuzzy Type Theory should be defined and what properties it should have, particularly from a categorical point of view.

Specifically, we seek to determine:

- What exactly is fuzzy in a Fuzzy Type Theory? Terms? Types?
- What do the categorical semantics of Fuzzy Type Theory look like, especially when compared to Martin-Löf Type Theory?
- How can we generalize a Fuzzy Type Theory to make the most of our categorical approach?

Result

Fuzzy Type Theory

We define a Fuzzy Type Theory using an Encode-Decode argument, as shown below:

$$\begin{tikzcd} \textbf{Martin-Löf Type Theory} \arrow[r, "\text{Choose Model}"] & \textbf{Display-Map Categories} \arrow[d, "\text{Enrich}"] \\ \textbf{Fuzzy Type Theory} & \textbf{Fuzzy Display-Map Categories} \arrow[l, "\text{Derive Theory}"] \end{tikzcd}$$

We define enriched display-maps to not be fuzzy. Thus, in this Fuzzy Type Theory, terms are fuzzy, but types aren't. When x is a term of type A in a context Γ with fuzziness at least α, we write

$$\Gamma \vdash x :_\alpha A$$

Fuzzy Logic and Fuzzy Sets

Fuzzy logic is a logic in which the possible truth values lie in the interval $[0,1]$, rather than in the Booleans $\{0,1\}$. It has applications across fields such as control theory and artificial intelligence.

Using fuzzy logic, one can define fuzzy sets, which are sets with a fuzzified membership relation. Formally, the type of fuzzy sets can be viewed as the type

$$\sum_{S:\mathrm{Set}} (S \to [0,1])$$

Categorifying

Framework

Categorical Semantics

Display-map categories are one kind of categorical model of Martin-Löf Type Theory. These consist of a category and a distinguished collection of morphisms in this category called display-maps.

Category	Type theory
Objects	Contexts
Morphisms	Context substitution
Domains of display-maps	Types in a context
Sections of display-maps	Terms in a type
Pullbacks of display-maps	Substitution

Enriching

Enriching in Fuzzy Sets

As Natural Deduction also has a categorical model in display-map categories enriched over $\{0,1\}$, this suggests that by enriching display-map categories over both $[0,1]$ and fuzzy sets, we can obtain categorical models of fuzzy analogues of these theories.

	Logic	Type Theory
Classical	$\{0,1\}$	Set
Fuzzy	$[0,1]$	$\sum_{S:\mathrm{Set}} (S \to [0,1])$

Future Work

- Investigate how different type formers, such as dependent products and dependent sums, work in Fuzzy Type Theory
- Examine ways Fuzzy Type Theory can be applied to real world problems, such as in modeling opinions/confidences and in modeling neural networks
- Use the fact that this construction works for any quantale, not just the interval, to begin exploring the general theory of Enriched Type Theories