Continuous Optimization **Spring 2023**

Exercise 4: Constrained Optimization

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Problem 1 (Constrained problem):

Consider the following 2-dimensional problem

$$\begin{aligned} \min f(x,y) &:= x(1-y^2) \\ \text{s.t. } x^2+y^2 &= 1. \end{aligned}$$

1. Write the stationary and primal feasibility conditions.
2. Derive the optimal solution (x^*, y^*).

Problem 2 (KKT problem with two constraints):

Consider the following 3-dimensional problem

$$\begin{aligned} &\min f(x,y,z) := x+y+z \\ &\text{s.t. } x^2-y^2 = 1 \text{ and } 2x+z-1=0. \end{aligned}$$

1. Write the stationary and primal feasibility conditions.
2. Derive all the optimal solutions.
3. Can you comment on the results?

Problem 3 (Projection onto hyperplane):

Consider the projection of a vector onto a hyperplane identified by the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$, i.e.

$$\mathbf{x} = \mathrm{Proj}_{\mathbf{A}\mathbf{x}=\mathbf{b}}(\mathbf{y}) = \underset{\mathbf{x}:\mathbf{A}\mathbf{x}=\mathbf{b}}{\mathrm{argmin}} \frac{1}{2}\|\mathbf{y}-\mathbf{x}\|^2. \tag{1}$$

1. Write down the Lagrangian corresponding to the constrained problem defined in Eq. (1).
2. Calculate the optimal value of $\mathbf{x}$ (using the KKT conditions). Show that

   $$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{A}^\top(\mathbf{A}\mathbf{A}^\top)^{-1}\mathbf{b},$$

   where $\mathbf{P} := (\mathbf{I} - \mathbf{A}^\top(\mathbf{A}\mathbf{A}^\top)^{-1}\mathbf{A})$ is a projection matrix.

Problem 4 (Normal cones):

Consider the following two sets:

$$\Omega_\infty := \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\|_\infty \le 1\},$$

and

$$\Omega_2 := \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\|_2 \le 1\}.$$

1. Show that Ω_∞ and Ω_2 are non-empty, convex and closed.
2. Determine the normal cones of Ω_∞ and Ω_2 for $d = 2$ at the point $\mathbf{x} = (1, 0)$.

Problem 5 (Programming):

Complete the notebooks provided as you implement the PGD algorithm on the:

1. ℓ^1 ball

2. The Positive Semidefinite Cone of Matrices $\mathbf{S}_+^n$.

For the first, we recommend reading the following paper or similar works: `https://research.google/pubs/pub36229/`

For the second, we recall that $\mathbf{S}_+^n = \left\{ A \in \mathbf{S}^n : x^T A x \geq 0, \quad \forall x \in \mathbb{R}^n \right\}$ where $\mathbf{S}^n$ denotes the vector space of $n \times n$ real symmetric matrices.