Q1

let S be the event where we got a share picture, and D be the event where we got a diminished picture.

let G be the event where the TV is good, F the event where its fair, and B the event where it bad.

first lets calculate $P(S)$ and $P(D)$:

$$\begin{aligned} P(S) = & \ P(S|G)P(G) + P(S|F)P(F) + P(S|B)P(B) \\ = & \ 0.9 * 0.3 + 0.5 * 0.5 + 0.2 * 0.2 = 0.56 \end{aligned}$$

$$P(D) = 1 - P(S) = 1 - 0.56 = 0.44$$

now lets calculate all of the pairs of $P(TV\ type|image\ type)$ using bayes rule:

$P(G|S)$:

$$P(G|S) = \frac{P(S|G)P(G)}{P(S)} = \frac{0.9*0.3}{0.56} = 0.482$$

$P(G|D)$:

$$P(G|D) = \frac{P(D|G)P(G)}{P(D)} = \frac{0.1*0.3}{0.44} = 0.068$$

$P(F|S)$:

$$P(F|S) = \frac{P(S|F)P(F)}{P(S)} = \frac{0.5*0.5}{0.56} = 0.446$$

$P(F|D)$:

$$P(F|D) = \frac{P(D|F)P(F)}{P(D)} = \frac{0.5*0.5}{0.44} = 0.568$$

$P(B|S)$:

$$P(B|S) = \frac{P(S|B)P(B)}{P(S)} = \frac{0.2*0.2}{0.56} = 0.07$$

$P(B|D)$:

$$P(B|D) = \frac{P(D|B)P(B)}{P(D)} = \frac{0.8*0.2}{0.44} = 0.363$$

lets denote with α_1 the decision where we buy the TV and
α_2 the decision where we dont buy it.

now lets calculate all of the pairs of $R(decision|image\ type)$:

$R(\alpha_1|S)$

$$R(\alpha_1|S) = P(G|S)*\lambda(\alpha_1|G) + P(F|S)*\lambda(\alpha_1|F) + P(B|S)*\lambda(\alpha_1|B)$$
$$= 0.482*0 + 0.446*5 + 0.07*20 = 3.63$$

$R(\alpha_1|D)$

$$R(\alpha_1|D) = P(G|D)*\lambda(\alpha_1|G) + P(F|D)*\lambda(\alpha_1|F) + P(B|D)*\lambda(\alpha_1|B)$$
$$= 0.068*0 + 0.568*5 + 0.363*20 = 10.1$$

$R(\alpha_2|S)$

$$R(\alpha_2|S) = P(G|S)*\lambda(\alpha_2|G) + P(F|S)*\lambda(\alpha_2|F) + P(B|S)*\lambda(\alpha_2|B)$$
$$= 0.482*10 + 0.446*5 + 0.07*0 = 7.05$$

$R(\alpha_2|D)$

$$R(\alpha_2|D) = P(G|D)*\lambda(\alpha_2|G) + P(F|D)*\lambda(\alpha_2|F) + P(B|D)*\lambda(\alpha_2|B)$$
$$= 0.068*10 + 0.568*5 + 0.363*0 = 3.52$$

as a result of $R(\alpha_1|S) < R(\alpha_2|S)$ we get that the optimal decision to make if the image is sharp is to buy the TV.

as a result of $R(\alpha_2|D) > R(\alpha_1|D)$ we get that the optimal decision to make if the image is not sharp is to not buy the TV.

lets calculate the total risk:

$$R(\alpha) = R(\alpha_1|S)P(S) + R(\alpha_2|D)P(D) =$$
$$3.63*0.56 + 3.52*0.44 = 3.581$$

Q3

A:

lets denote with a_j the frequence of number j in dice 1,
and with b_j the frequency of the number j in dice 2.

lets now calculate $P_i(j)$. lets assume w.l.o.g that $i = 1$. we are trying to estimate the probability. so lets calculate the likelihood for some value p:

given that the probability is p, the distribution of a_j is
$Bin(40, p)$. and so we get

$L(p) = \binom{40}{a_j} p^{a_j}(1-p)^{40-a_j}$. to simplify, lets calculate using log-likelihood:

$$log\big(L(p)\big) = log\left(\binom{40}{a_j}p^{a_j}(1-p)^{40-a_j}\right) = log\left(\binom{40}{a_j}\right) + a_j logp + (40-a_j)log(1-p)$$

lets now derive $log\big(L(p)\big)$ and find a p that makes the derivative 0:

$$log\big(L(p)\big)' = \frac{a_j}{p} - \frac{40-a_j}{1-p} = 0$$

$$\Longrightarrow a_j(1-p) - (40-a_j)p = 0$$
$$\Longrightarrow a_j - pa_j - 40p + a_jp = 0$$
$$\Longrightarrow a_j = 40p \Longrightarrow p = \frac{a_j}{40}$$

so we found that the MLE estimate for $P_i(j)$ is $\frac{frequence\ of\ j\ in\ dice\ i}{40}$.

so now according to this rule lets calculate $P_i(j)$ for all of the values of i, j:

i \ j	1	2	3	4	5	6
1	$\frac{1}{8}$	$\frac{3}{40}$	$\frac{1}{4}$	$\frac{1}{40}$	$\frac{1}{4}$	$\frac{11}{40}$
2	$\frac{1}{4}$	$\frac{11}{40}$	$\frac{1}{10}$	$\frac{1}{4}$	$\frac{3}{40}$	$\frac{1}{20}$

B:

lets denote with c_1 the event where the dice is the first dice and c_2 the event where the dice is the second dice.
let x denote the known 40 dice rolls.

$$P(c_1|x) = \frac{P(x|c_1)p(c_1)}{P(x)}, \ P(c_2|x) = \frac{P(x|c_2)p(c_2)}{P(x)}$$

we want to find the i that maximises $P(c_i|x)$. becuase in both cases we have devision by $P(x)$, then the i that maximises $P(c_i|x)$ is also the i that maximises $P(x|c_i)P(c_i)$. $P(c_i) = \frac{1}{2}$ becuase we have no prior about which dice is more likely to be lost, and so we want the i that maximises $P(x|c_i)$, which is the MLE estimate.

lets calculate $P(x|c_i)$ for each i:

$P(x|c_1)$:

to calculate the probability of getting those exact dice rolls, we will calculate the number of possible permutations for x, and multiply that for the probability of getting that permutation. its clear to see that all the permutations have an equal probability of happening so all we need to do it calculate the probability of a permutation.

lets look at the permutation where we first get all of the 1's, and then 2's and then 3's and so on.

the probability of this permutation is

$$P_1(1)^8 * P_1(2)^{12} * P_1(2)^6 * P_1(4)^9 * P_1(5)^4 * P_1(6) =$$
$$= \left(\frac{1}{8}\right)^8 * \left(\frac{3}{40}\right)^{12} * \left(\frac{1}{4}\right)^6 * \left(\frac{1}{40}\right)^9 * \left(\frac{1}{4}\right)^4 * \left(\frac{11}{40}\right) = 1.88 * 10^{-42}$$

lets denote the number of permutations with p.

and so we get that $P(x|c_1) = p * 1.88 * 10^{-42}$.

$P(x|c_2)$:

we will calculate the probability of each permutation in the same way:

$$P_2(1)^8 * P_2(2)^{12} * P_2(2)^6 * P_2(4)^9 * P_2(5)^4 * P_2(6) =$$
$$= \left(\frac{1}{4}\right)^8 * \left(\frac{11}{40}\right)^{12} * \left(\frac{1}{10}\right)^6 * \left(\frac{1}{4}\right)^9 * \left(\frac{3}{40}\right)^4 * \left(\frac{1}{20}\right) = 1.722 * 10^{-29}$$

lets denote the number of permutations with p.

and so we get that $P(x|c_2) = p * 1.722 * 10^{-29}$.

$P(x|c_2) = p * 1.722 * 10^{-29} > p * 1.88 * 10^{-42} = P(x|c_1)$ and so the MLE estimate
is c_2. we explained earlier why the MLE estimate is the optimal bayes estimate in this case.