Category Theory in Context
Answer Key

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Chapter 1

Categories, Functors, Natural Transformations

1.1 Abstract and concrete categories

Exercise 1.1.i.

(i) Show that a morphism can have at most one inverse isomorphism.

Proof. Let $f\colon x \longrightarrow y$ be an arbitrary morphism, and let $g\colon y \longrightarrow x$ and $g\colon y \longrightarrow x$ be two inverse isomorphisms of f. That is to say, $gf = hf = 1_x$ and $fg = fh = 1_y$. Consider the composition gfh. This is a valid composition, since the domain of g is equal to the codomain of f, and the domain of f is equal to the codomain of h. Composition is associative, so $(gf)h = g(fh)$.

Evaluate each of these expressions independently:

- Evaluated as $(gf)h$, we find that $gf = 1_x$, so $(gf)h = 1_x h = h$.
- Evaluated as $g(fh)$, we find that $fh = 1_y$, so $g(fh) = g1_y = g$.

Since both expressions are equal, we can conclude that $h = g$. So any two inverse isomorphisms of f must be equal. Since f was arbitrary, we can generalize to conclude that any morphism can have at most one (distinct) inverse isomorphism. □

(ii) Consider a morphism $f\colon x \longrightarrow y$. Show that if there exist a pair of isomorphisms $g, h\colon y \longrightarrow x$ so that $gf = 1_x$ and $fh = 1_y$, then $g = h$ and f is an isomorphism.

Proof. Let $f\colon x \longrightarrow y$ be an arbitrary morphism, and let $g, h\colon y \longrightarrow x$ be morphisms such that $gf = 1_X$ and $fh = 1_y$. Similarly to above, evaluate the composition gfh as $(gf)h = 1_x h = h$ and as $g(fh) = g1_y = g$. Due to associativity, $(gf)h = g(fh)$, so we can conclude that $g = h$. $fh = 1_y$ was given, and using our previous conclusion, we can substitute g for h to obtain $fg = 1_y$. Since we were also given $gf = 1_x$, we can conclude that f is an isomorphism. □

Definition 1.1.11. A **groupoid** is a category in which every morphism is an isomorphism.

Exercise 1.1.ii. Let C be a category. Show that the collection of isomorphisms in C defines a subcategory, the **maximal groupoid** inside C.

Proof. We want to show that, for a category C, restricting the class $\operatorname{mor}\mathsf{C}$ to a class of morphisms that are only the isomorphisms, call it $\operatorname{mor}\mathsf{C}_{iso}$, while preserving all of the objects of C, gives us a subcategory C_{iso} of C.

We start by showing that the identity morphisms in C_{iso} are isomorphisms. For an identity morphism $1_x \colon x \to x$, since the composition of $1_x 1_x = 1_x = 1_x 1_x$ shows that 1_x is both a right and left inverse of itself, then 1_x is an isomorphism. Thus, the identity morphisms of C_{iso} are indeed isomorphisms.

We now want to show that compositions of isomorphisms in C_{iso} yield isomorphisms. Take two morphisms $f \colon x \to y$ and $g \colon u \to x$ in C_{iso}. Since f is an isomorphism, then there is a morphism $h \in \operatorname{mor}\mathsf{C}_{iso}$ with $h \colon y \to x$, such that $fh = 1_y$ and $hf = 1_x$. Likewise, since g is an isomorphism, then there is a morphism $j \in \operatorname{mor}\mathsf{C}_{iso}$ with $j \colon x \to u$, such that $gj = 1_x$ and $jg = 1_u$. We can take the composition fg, since $\operatorname{dom}(f) = \operatorname{cod}(g)$. We also have the composition jh, since $\operatorname{dom}(j) = \operatorname{cod}(h)$. And again, respecting domains and codomains, we have the composition $(fg)(jh)$, since $\operatorname{dom}(fg) = \operatorname{cod}(jh)$. From the associativity of the parent category C, then $(fg)(jh) = f(gj)h = f(1_x)h = fh = 1_y$. Thus jh is the right inverse of the composition fg. Similarly, since $\operatorname{cod}(fg) = \operatorname{dom}(jh)$, we have the composition $(jh)(fg)$, which again from the associativity of the category C, $(jh)(fg) = j(hf)g = j1_x g = jg = 1_u$. So, jh is the left inverse of fg, and fg is an isomorphism.

We have shown that C_{iso} is a category, having all of the objects of C, restricted to the isomorphisms of C. So the groupoid C_{iso} is a subcategory of C. Presented with any other subcategory D, of C, that is strictly larger than C_{iso}, there must be a morphism in D that is not in C_{iso}. Then this morphism must not be an isomorphism, and hence, D cannot be a groupoid. So, the category C_{iso} is a maximal groupoid that is a subcategory of C. □

Exercise 1.1.iii. For any category C and any object c in C show that:

1. There is a category c/C whose objects are morphisms $f \colon c \to x$ with domain c and in which a morphism from $f \colon c \to x$ to $g \colon c \to y$ is a map $h \colon x \to y$ between the codomains so that the triangle

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] & \\ x \arrow[rr, "h"'] & & y \end{tikzcd}$$

commutes, i.e., so that $g = hf$.

2. There is a category C/c whose objects are morphisms $f\colon x \to c$ with codomain c and in which a morphism from $f\colon x \to c$ to $g\colon y \to c$ between the domains so that the triangle

$$\begin{tikzcd} x \arrow[rr, "h"] \arrow[dr, "f"'] & & y \arrow[dl, "g"] \\ & c & \end{tikzcd}$$

commutes, i.e., so that $f = gh$
The categories c/C and C/c are called **slice categories** of C **under** and **over** c, respectively.

Proof. First we must determine the form of the objects and morphisms in c/C. The objects of c/C are diagrams of the following form.

$$c \xrightarrow{f} x$$

The morphisms in c/C are diagrams of the following form.

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] & \\ x \arrow[rr, "h"'] & & y \end{tikzcd}$$

Though this is notation is by no means standard, to help distinguish between morphisms in C and morphisms in the slice categories, we will define h' as a short hand for the diagram with the morphism h as the bottom arrow (or top in C/c). Notice that both the objects are commutative diagrams in C. We could also think of the objects as functors from the category $\mathbb{2}$ and the morphisms as functors from the category $\mathbb{3}$. By the way we have defined morphisms the only reasonable choices for the domain and codomain of h' is f and g respectively.

We can see how to compose two compatible morphisms $i'\colon e \to f$ and $h'\colon f \to g$ in c/C by looking at the following diagram in C.

$$\begin{tikzcd} & c \arrow[dl, "e"'] \arrow[d, "f"'] \arrow[dr, "g"] & \\ z \arrow[r, "i"'] & x \arrow[r, "h"'] & y \end{tikzcd}$$

Since $(hi)e = h(ie) = hf = g$ in C the diagram commutes, and the composition hi can be thought of as a member of c/C denoted as $(hi)'$ with domain and codomain e and g. Using the diagram notation $(hi)'$ is denoted as follows.

$$\begin{tikzcd} & c \arrow[dl, "e"'] \arrow[dr, "g"] & \\ x \arrow[rr, "hi"'] & & y \end{tikzcd}$$

Because we have defined composition in c/C in terms of composition in C c/C inherits the associativity of C That is for composable morphisms $h',\ i',$ and j' in C we have

$$(h'i')j' = (hi)j = h(ij) = h'(i'j').$$

We need to obtain the identity morphism of each object f in c/C. To do so notice that the follow diagram commutes, because C is a category.

$$\begin{tikzcd} c \arrow[d, "f"'] \arrow[dr, "f"] & \\ x \arrow[r, "1_x"'] & x \end{tikzcd}$$

Looking at the same diagram from a different perspective we see that 1_x actually acts as the identity morphism for f in c/C Since we were careful when defining the morphisms in c/C this identity is well defined. If we had defined the morphisms in c/C to be anything less than a commutative diagrams, it would seem as 1_x could serve as the identity for multiple objects in c/C. This issue is not restricted to identity morphisms, but it is most obvious in the case of identity morphisms. However since we defined morphisms appropriately we can use the notation defined earlier to write $1_x = 1_f : f \to f$ without any ambiguity. This notion can be used to obtain the left and right identities by considering the following commutative diagram in C

$$\begin{tikzcd} & & c \arrow[dll, "f"'] \arrow[dl, "f"] \arrow[dr, "g"'] \arrow[drr, "g"] & & \\ x \arrow[r, "1_x"'] & x \arrow[rr, "h"'] & & y \arrow[r, "1_y"'] & y \end{tikzcd}$$

Translating the above diagram into the slice category notation we have that $h'1_f = h' = 1_g h'$. We have shown that c/C satisfies all the axioms of a category.

We can use the same method to show that C/c is also a category. The only difference is direction of each arrow. Hence this proof will be relatively terse. The objects in c/C are the following diagram.

$$x \xrightarrow{f} c$$

A morphisms h' in C/c has the form of the following diagram.

$$\begin{tikzcd} x \arrow[rr, "h"] \arrow[dr, "f"'] & & y \arrow[dl, "g"] \\ & c & \end{tikzcd}$$

The domain of h' is f and the codomain of h is g. We define composition on C/c by taking compatible in C/c morphisms $i' : e \to f$ and $h' : f \to g$ and observing the following diagram in C.

$$\begin{tikzcd} z \arrow[r, "i"] \arrow[dr, "e"'] & x \arrow[r, "h"] \arrow[d, "f"] & y \arrow[dl, "g"] \\ & c & \end{tikzcd}$$

Again this diagram commutes since $g(hi) = (gh)i = fi = e$ in C. We see that (ih) is member of C/c with domain e and codomain g and is denoted as follows.

$$\begin{tikzcd} z \arrow[rr, "ih"] \arrow[dr, "e"'] & & y \arrow[dl, "g"] \\ & c & \end{tikzcd}$$

Just like we did in the previous case, we have defined composition in terms of the composition in C. Hence the associativity is inherited. That is given composable morphisms i', h', and j' we have

$$(j'h')i' = (jh)i = j(hi) = j'(h'i').$$

We can obtain the identity element for each object f in C/c in the exact same way as before.

$$\begin{tikzcd} c \arrow[d, "f"'] \arrow[dr, "f"] & \\ x \arrow[r, "1_x"'] & x \end{tikzcd}$$

Since the above diagram commutes we can write $1'_x = 1_f : f \to f$ To get an identity for an arbitrary element observe that the diagram below commutes and gives us that $1_f h' = h' = h' 1_g$.

$$\begin{tikzcd} x \arrow[r, "1_x"] & x \arrow[rr, "h"] & & y \arrow[r, "1_y"] & y \\ & & c \arrow[ull, "f"] \arrow[ul, "f"'] \arrow[ur, "g"] \arrow[urr, "g"'] & & \end{tikzcd}$$

Therefore both c/C and C/c are categories in their own right. $\square$

If you looked over to the next page and read the definition of opposite categories, you should notice that $((c/(\mathsf{C})^{\mathrm{op}}))^{\mathrm{op}} = (\mathsf{C}/c)$ If we knew about opposite categories beforehand we could have just proved that the c/C is a category and then cited this result and been done (since the opposite category is category, it's in the name after all), without all the extra tedium of swapping arrows.

1.2 Duality

Exercise 1.2.i. Show that $\mathsf{C}/c \cong (c/(\mathsf{C}^{\mathrm{op}}))^{\mathrm{op}}$. Defining C/c to be $(c/(\mathsf{C}^{\mathrm{op}}))^{\mathrm{op}}$, deduce Exercise 1.1.iii(ii) from Exercise 1.1.iii(i).

Proof. This exercise asks us to prove that two categories are isomorphic, which is a notion that we have not yet encountered. But, I will prove that the two categories are equal!

This exercise uses definitions from Exercise 1.1.iii. There are so many layers in the present exercise that to keep things straight, it will help to add one more piece of notation. Recall that for an object c of a category C, the slice category C/c of C over c has as objects the morphisms $f\colon x \to c$ in C. A morphism from f to g in C/c, where f has domain x and g domain y in C, is a morphism $h\colon x \to y$ in C such that $gh = f$. To distinguish between h as viewed in C and in C/c, let's write $h'\colon f \to g$ when we want to consider h as a morphism in C/c and $h\colon x \to y$ when we want to consider it in C[1]. We can use similar notation for the slice category c/C of C under c.

Since we will make systematic and careful use of opposite categories, recall that the objects and morphisms of C and of C^{op} are *precisely the same*. Only, the assignment of domains and codomains are swapped, allowing order of composition to be swapped. If f is a morphism in C, then f^{op} is precisely the same morphism, but the op reminds us that we are considering it in C^{op} rather than in C so that we have different assignments for domain and codomain.

Now, I claim that $\mathsf{C}/c = (c/(\mathsf{C}^{\mathrm{op}}))^{\mathrm{op}}$. We must first check that they have the same objects, though we notate f in the first category as f^{op} in the second. Then, for every pair of objects f and g in C/c we must see that

$$(\mathsf{C}/c)(f, g) = (c/(\mathsf{C}^{\mathrm{op}}))^{\mathrm{op}}(f^{\mathrm{op}}, g^{\mathrm{op}}).$$

(Note that we use this notation even when C is not locally small, so that each side of the equality might be a proper class.) Finally, we must see that composition of morphisms is the same in each category.

Since the objects of a category and its opposite category are the same, the objects in $(c/(\mathsf{C}^{\mathrm{op}}))^{\mathrm{op}}$ are the objects in $c/(\mathsf{C}^{\mathrm{op}})$, which are morphisms $f^{\mathrm{op}}\colon c \to x$ in C^{op}. But, these are the same as morphisms $f\colon x \to c$ in C, which is to say objects of C/c as claimed.

For the rest, let $f\colon x \to c$, $g\colon y \to c$ and $h\colon z \to c$ be three morphisms in C. A morphism

$$i^{\mathrm{op}\prime\mathrm{op}}\colon f^{\mathrm{op}} \to g^{\mathrm{op}}$$

in $(c/(\mathsf{C}^{\mathrm{op}}))^{\mathrm{op}}$ is just a morphism

$$i^{\mathrm{op}\prime}\colon g^{\mathrm{op}} \to f^{\mathrm{op}}$$

in $c/(\mathsf{C}^{\mathrm{op}})$. This in turn is a morphism $i^{\mathrm{op}}\colon y \to x$ such that $i^{\mathrm{op}} g^{\mathrm{op}} = f^{\mathrm{op}}$ in C^{op}, together with the ordered pair $(g^{\mathrm{op}}, f^{\mathrm{op}})$ giving the domain and codomain of $i^{\mathrm{op}\prime}$. Unravelling one more layer, this is in turn a morphism $i\colon x \to y$ such that $gi = f$ in C together with the

[1] The notation h' can still be ambiguous since there might also be other $i\colon c \to x$ and $j\colon c \to y$ such that $jh = i$ so that we also have $h'\colon i \to j$. This leads to different morphisms labeled h', but they are distinguished by their domains and codomains

ordered pair (f, g). This in turn corresponds to a morphism $i'\colon f \to g$ in C/c. Each of these correspondences is actually an equality of classes. So, we have argued that

$$\begin{aligned}
&(c/(\mathsf{C}^{\mathrm{op}}))^{\mathrm{op}}(f^{\mathrm{op}}, g^{\mathrm{op}}) \\
&= (c/(\mathsf{C}^{\mathrm{op}}))(g^{\mathrm{op}}, f^{\mathrm{op}}) \\
&= \{i^{\mathrm{op}} \in \mathsf{C}^{\mathrm{op}}(y, x) | i^{\mathrm{op}} g^{\mathrm{op}} = f^{\mathrm{op}}\} \times \{(g^{\mathrm{op}}, f^{\mathrm{op}})\} \\
&= \{i \in \mathsf{C}(x, y) | gi = f\} \times \{(f, g)\} \\
&= (\mathsf{C}/c)(f, g)
\end{aligned}$$

as required.

Notice also in this correspondence that when $f = g$ that the identities in each class are the same. Altogether, $1_{f^{\mathrm{op}}}^{\mathrm{op}} = 1_f$.

Now, we must see that the composition laws are the same. We have already established above that $i^{\mathrm{op}\prime\mathrm{op}} = i'$. Similarly, if $j'\colon g \to h$, then we have $j^{\mathrm{op}\prime\mathrm{op}} i^{\mathrm{op}\prime\mathrm{op}}\colon f^{\mathrm{op}} \to h^{\mathrm{op}}$. But, examining the definitions of opposite categories and of slice categories, we have equality of each of the following morphisms interpreted in the categories shown

$$\begin{array}{rlll}
j^{\mathrm{op}\prime\mathrm{op}} i^{\mathrm{op}\prime\mathrm{op}}\colon & f^{\mathrm{op}} \to h^{\mathrm{op}} & \text{in} & (c/(\mathsf{C}^{\mathrm{op}}))^{\mathrm{op}} \\
i^{\mathrm{op}\prime} j^{\mathrm{op}\prime}\colon & h^{\mathrm{op}} \to f^{\mathrm{op}} & \text{in} & c/(\mathsf{C}^{\mathrm{op}}) \\
(i^{\mathrm{op}} j^{\mathrm{op}})'\colon & h^{\mathrm{op}} \to f^{\mathrm{op}} & \text{in} & c/(\mathsf{C}^{\mathrm{op}}) \\
i^{\mathrm{op}} j^{\mathrm{op}}\colon & z \to x & \text{in} & \mathsf{C}^{\mathrm{op}} \\
ji\colon & x \to z & \text{in} & \mathsf{C} \\
(ji)'\colon & f \to h & \text{in} & \mathsf{C}/c \\
j'i'\colon & f \to h & \text{in} & \mathsf{C}/c
\end{array}$$

This proves that the two categories share the same composition law. Thus, they are one and the same.

Now, looking back at Exercise 1.1.iii, we see that we we could have defined C/c as $(c/(\mathsf{C}^{\mathrm{op}}))^{\mathrm{op}}$, so that the existence of the category C/c follows from the existence of $c/(\mathsf{C}^{\mathrm{op}})$, which we have by the first part of Exercise 1.1.iii applied to C^{op}. ◻

Exercise 1.2.ii.

(i) Show that a morphism $f\colon x \to y$ is a split epimorphism in a category C if and only if for all $c \in \mathsf{C}$, post-composition $f_*\colon \mathsf{C}(c, x) \to \mathsf{C}(c, y)$ defines a surjective function.

Proof. First, assume that f is a split epimorphism and that $c \in \operatorname{ob}\mathsf{C}$. That is, there exists a function $g\colon y \to x$ such that $fg = id_y$. Now, consider the function $f_*\colon \mathsf{C}(c, x) \to \mathsf{C}(c, y)$. We know that this function corresponds to composition on the left by f, so in order for this function to be surjective, for every $k\colon c \to y$, there must exists a $j\colon c \to x$ such that $fj = k$. Now, for an arbitrary k, consider $j = gk$. It is

easy to see that $gk\colon c\to x$, and that $f_*(gk) = f(gk) = (fg)k = id_y k = k$. We can construct j in this way for every $k \in \mathsf{C}(c, y)$, so we see that f_* is surjective.
Now, assume that f_* is surjective, that is, for any choice of $c \in \mathsf{C}$, and any $k \in \mathsf{C}(c, y)$, $k = fg$, for some $g \in \mathsf{C}(c, x)$. Now, suppose $c = y$ and $k = id_y$, so we have that there exists a $g \in \mathsf{C}(y, x)$ where $fg = id_y$, and this implies that f is a split epimorphism.□

(ii) Argue by duality that f is a split monomoprhism if and only if for all $c \in \mathsf{C}$, precomposition $f_*\colon \mathsf{C}(y, c)\to\mathsf{C}(x, c)$ is a surjective function.

Proof. We know that if f^{op} is a split epimorphism, that $f_*^{\mathrm{op}}\colon C^{\mathrm{op}}(c, y)\to C^{\mathrm{op}}(c, x)$ is surjective. However, if we consider the definitions of f^{op} and split monomorphisms and epimorphisms, we see that f^{op} being a split epimorphism implies that f is a split monomorphism. We also see that $f_*^{\mathrm{op}}\colon \mathsf{C}^{\mathrm{op}}(c, y)\to\mathsf{C}^{\mathrm{op}}(c, x)$ is equivalent to $f^*\colon \mathsf{C}(y, c)\to\mathsf{C}(x, c)$. So we have that f is a split monomorphism if and only if $f^*\colon \mathsf{C}(y, c)\to\mathsf{C}(x, c)$ is surjective. □

Lemma 1.2.11.
(i) If $f\colon x\rightarrowtail y$ and $g\colon y\rightarrowtail z$ are monomorphisms, then so is $gf\colon x\rightarrowtail z$.
(ii) If $f\colon x\to y$ and $g\colon y\to z$ are morphisms so that gf is monic, then f is monic.
Dually:
(i') If $f\colon x\twoheadrightarrow y$ and $g\colon y\twoheadrightarrow z$ are epimorphisms, then so is $gf\colon x\twoheadrightarrow z$.
(ii') If $f\colon x\to y$ and $g\colon y\to z$ are morphisms so that gf is epic, then g is epic.

Exercise 1.2.iii. Prove Lemma 1.2.11 by proving either (i) or (i') and either (ii) or (ii'), then arguing by duality. Conclude that the monomorphisms in any category define a subcategory of that category and dually that the epimorphisms also define a subcategory.

First we will show the above properties for monomorphisms, and then apply duality, as the problem suggests, to prove the corresponding properties for epimorphisms.

Proof. First, we will prove that the composition of two monomorphisms is a monomorphism. Let C be a category and $f\colon x\to y$ and $g\colon y\to z$ be monomorphisms of C. Let $h, k\colon w\to x$ be two morphisms in C so that: $gf)h = (gf)k$. Since compositions of morphisms are associative, we have $g(fh) = g(fk)$. Since g is monic, we get: $(fh) = (fk)$. Since f is monic, we ultimately get: $h = k$. Thus, gf is monic. Thus, the compositions of two monomorphisms is indeed a monomorphism.

Next we will show that If the composition of two morphisms is monic, then the rightmost morphism is monic. Take morphisms $a\colon x\to y$ and $b\colon y\to z$ from category C where ba is monic. Take $h, k\colon w\to x$ so that $ah = ak$. Left composing b on both sides of the equations results in: $b(ah) = b(ak)$. By associativity we get: $(ba)h = (ba)k$. Applying the properties of monomorphisms results in: $h = k$. Thus a is monic. So we have shown that if the composition of two morphisms is monic, then the rightmost morphism is monic.

Now we will show that the monomorphisms of any category forms a category. Suppose that D is a subcategory of C where the morphisms of D are the monomorphisms of C and

D and C have the same objects. Since for any object x in C, if we had for morphisms h and k of C with codomain x the following property: $id_x h = id_x k$, then $h = k$, since id_x is left cancellable, thus the identity morphism for every object in D is a monomorphism. Therefore, every object in D has a identity arrow in D. Since the composition of two monomorphisms is a monomorphism, then D contains compositions of its morphisms. Obviously, the domains and codomains of morphisms of D are contained in D since D and C have the same objects. Thus D is a subcategory of C.

We have shown that:

1. the composition of two monomorphisms in C is a monomorphism.
2. If the composition of two morphisms in C is monic, then the rightmost morphism is monic.
3. The class of monomorphisms of any category C forms a subcategory of C.

Now we will use duality to show the corresponding properties for epimorphisms. If we have the opposite category C^{op}, where the epimorphisms of C are the monomorphisms of C^{op}, this means that the three properties proven for monomorphisms also work for C^{op}. The properties of monomorphisms in C^{op} are the dual properties of epimorphisms in $(\mathsf{C}^{\mathrm{op}})^{\mathrm{op}} = \mathsf{C}$. Namely,

1. the composition of two epimorphisms in C is an epimorphism.
2. If the composition of two morphisms in C is epic, then the leftmost morphism is epic (Since $f^{\mathrm{op}} g^{\mathrm{op}}$ in C^{op} corresponds to gf in C).
3. The class of epimorphisms of any category C forms a subcategory of C.

This completes the proof. □

Definition 1.2.7. A morphism $f\colon x \to y$ in a category is

(i) a *monomorphism* if for any parallel morphisms $h, k\colon w \rightrightarrows x$, $fh = fk$ implies that $h = k$; or

(ii) an *epimorphism* if for any parallel morphisms $h, k\colon y \rightrightarrows z$, $hf = kf$ implies that $h = k$.

Exercise 1.2.iv. What are the monomorphisms in the category of fields?

Proof. In the category of fields, morphisms are field homomorphisms. Let $f\colon A \to B$ be a morphism in Field. As f is a field homomorphism, its kernel is an ideal in B. Since B is a field, there are only two ideals: $\{0\}$ and B itself. The kernel of f cannot be the whole field, since this would be the zero morphism which is not a field homomorphism. So $\ker f = \{0\}$ and from this, f is injective, and in particular it is left cancellable.

Let h and k be morphisms in Field for which composition with f makes sense and say that

$$fh = fk$$

Since f is left cancellable, this implies that

$$h = k$$

And f is a monomorphism by Definition 1.2.7(i) above. (In fact, injections are monomorphisms in any category in which the objects have "underlying sets".)

Thus, all of the morphisms in Field are monomorphisms. $\square$

Definition 1.2.7. A morphism $f : x \to y$ in a category is

(i) a *monomorphism* if for any parallel morphisms $h, k : w \rightrightarrows x$, $fh = fk$ implies that $h = k$; or

(ii) an *epimorphism* if for any parallel morphisms $h, k : y \rightrightarrows z$, $hf = kf$ implies that $h = k$.

Exercise 1.2.v. Show that the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is both a monomorphism and an epimorphism in the category Ring of rings. Conclude that a map that is both monic and epic need not be an isomorphism.

Proof. Note first that monic and epic correspond to a map being cancellable on the left and right, whereas an isomorphism is by definition invertible. It is easy to see that invertibility implies cancellability; however the converse need not be true. Looking at the monoid of natural numbers under addition, every element is cancellable; however none except zero is invertible. Because every monoid is a category this gives us an elementary example where a map that is monic and epic is not an isomorphism. However, this example might seem simplistic and it is worth asking whether there is an example of cancellability not implying invertibility in a "larger" category where the arrows represent actual maps.

Recall that $\mathbb{Q}$ is the localisation of $\mathbb{Z}$ with respect to its cancellable elements $\mathbb{Z} \setminus \{0\}$. The immediate result of this is the existence of a natural embedding $\iota : \mathbb{Z} \to \mathbb{Q}$ that is an injective ring homomorphism. Further, this embedding has the following universal property: given a ring R and a homomorphism $\phi : \mathbb{Z} \to R$ such that $\phi(q)$ has an inverse for all $q \in \mathbb{Z}$, there is a unique ring homomorphism $\psi : \mathbb{Q} \to R$ such that the following diagram commutes.

$$\begin{tikzcd}
\mathbb{Z} \arrow[rr, "\phi"] \arrow[dr, "\iota"'] & & R \\
& \mathbb{Q} \arrow[ur, "\psi"'] &
\end{tikzcd}$$

Note further that there is a unique ring homomorphism from $\mathbb{Z}$ to any ring R which maps $\mathbb{Z}$ onto the subring generated by the multiplicative identity of R. This implies that there can be at most one homomorphism from $\mathbb{Q}$ to any ring R. If $\psi : \mathbb{Q} \to R$ is a ring homomorphism, then $\psi\iota : \mathbb{Z} \to R$ must be the unique homomorphism from $\mathbb{Z}$ to R and thus ψ is the unique homomorphism specified by the universal property.

Supposing $h, k : \mathbb{Q} \rightrightarrows R$ are parallel homomorphisms, they are equal by virtue of the fact that there is at most one homomorphism from $\mathbb{Q}$ to R, and ι vacuously fulfils the condition of an epimorphism. $\square$

Exercise 1.2.vi. Prove that a morphism that is both a monomorphism and a split epimorphism is necessarily an isomorphism. Argue by duality that a split monomorphism that is an epimorphism is also an isomorphism.

$$\begin{tikzcd} x \arrow["gf"', loop left] \arrow[r, bend left, "f"] & y \arrow[l, bend left, "g"] \arrow["1_y"', loop right] \end{tikzcd}$$

Proof. Let C be a category with objects x and y and a morphism $f\colon x \to y$. If f is a split epimorphism, then there exists another morphism $g\colon y \to x$ such that $fg = 1_y$. If f is also a monomorphism, then for any object w and any parallel pair of morphisms $h, k\colon w \rightrightarrows x$, $fh = fk$ implies that $h = k$. Combining these facts with some basic algebra:

$$\begin{aligned} 1_y f &= f 1_x && \text{definition of identies,} \\ fgf &= f1_x && g \text{ is a right inverse of } f, \\ gf &= 1_x && f \text{ is left cancellable,} \end{aligned}$$

gives that g is also a left inverse of f.

Supposing instead that f is an epimorphism and a split monomorphism with left inverse g in the category C. Then it is also a monomorphism and a split epimorphism in C^{op}, thus f is an isomorphism in C^{op}. Since duality of categories is expressed through a functor, and by Lemma 1.3.8 functors preserve isomorphisms, f is also an isomorphism in C. □

Exercise 1.2.vii. Regarding a poset $(\mathsf{P}, \leq)$ as a category, define the supremum of a subcollection of objects $A \in \mathsf{P}$ in such a way that the dual statement defines the infimum. Prove that the supremum of a subset of objects is unique, whenever it exists, in such a way that the dual proof demonstrates the uniqueness of the infimum.

Proof. Given a subcollection, C, of objects $A \in \mathsf{P}$, define an upper bound as follows: a object u is an upper bound of C if for all objects x in C there is a morphism $x \leq u\colon x \to u$. (Recall that morphisms in a poset category are merely elements of the $\leq$ relation.) Note that this immediately gives us a dual notion of a lower bound by considering instead P^{op}. A lower bound of C in P is an upper bound of C in P^{op}. In other words an object l^{op} such that for all objects x^{op} in C there is a morphism $(x \leq l)^{\mathrm{op}}\colon x^{\mathrm{op}} \to l^{\mathrm{op}}$, or equivalently $(l \leq x)\colon l \to x$.

Letting F be the collection of all upper bounds of C, we define the supremum of C, if it exists[2], to be a lower bound of F (as defined above) which is contained in F. The condition

[2] There are many cases where suprema fail to exist. Consider the poset category:

$$\begin{tikzcd} a & b \\ c \arrow[u] \arrow[ur] & d \arrow[u] \arrow[ul] \end{tikzcd}$$

of containment implies uniqueness. Supposing we have two lower bounds x and y of F. If both are contained in F, then there are maps $x \leq y\colon x \to y$ and $y \leq x\colon y \to x$. Since the only endomorphisms in P are identities these must compose to identities and thus be inverses, and because P is a partially ordered set (as opposed to just being a preordered set) the only isomorphisms are identities. In familiar terms, a partial order is antisymmetric. Thus x and y are the same object.

We may thus define the infimum of C to be its supremum on P^{op}. This time we consider the collection I of lower bounds C (the upper bounds of C in P^{op}). The infimum is then an upper bound of I which is contained in I (a lower bound in P^{op}). The infimum must be unique because it's a supremum in the opposite category, and suprema are unique. □

1.3 Functoriality

Exercise 1.3.i. What is a functor between groups, regarded as one-object categories?

Proof. Recall that a group as a category has a single object x, and that each element of the group is a morphism in the category. All domains and codomains are that object x. There is one identity morphism 1_x, which is the identity element in the group. Composition is the same as multiplication in this context.

A functor between groups C and D with respective objects x_1 and x_2 must trivially be such that $F(x_1) = x_2$. Our primary concern is the behavior of the functor on the morphisms. We require that for a functor $F(1_x) = 1_{F(x)}$ for all objects $x \in obC$, which in this case just implies that 1_{x_1} is taken to 1_{x_2}. Additionally, we require $F(\mathrm{dom}(f)) = \mathrm{dom}(F(f))$ and $F(\mathrm{cod}(f)) = \mathrm{cod}(F(f))$ for all morphisms f in the first category. This is a trivial requirement, as $F(\mathrm{dom}(f)) = \mathrm{dom}(F(f)) = F(\mathrm{cod}(f)) = \mathrm{cod}(F(f)) = x_2$ regardless of f. Finally we require that if f and g are a composable pair of morphisms in C, then $F(fg) = F(f)F(g)$. However, all morphisms in D are composable, and this implies that $F(f * g) = F(f) * F(g)$ in the notation of groups with operation $*$. This property and the preservation of identities are directly the definition of a group homomorphism, so this functor is simply a group homomorphism. □

Exercise 1.3.ii. What is a functor between preorders, regarded as categories?

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the set $\{c, d\}$ has as upper bounds $\{a, b\}$. However, $\{a, b\}$ has as lower bounds $\{c, d\}$. Because these sets are disjoint there is no supremum of $\{c, d\}$. Even in more common orderings, like the usual ordering on the rational numbers, subcollections can fail to have suprema. For example, $\{x \in \mathbb{Q} | x^2 < 2\}$.

A poset with the property that any collection of elements has a supremum and infimum is called a complete lattice.

Proof. Recall that a preorder regarded as a category has objects that are the elements of the underlying set of the preorder, and has morphisms that are the related pairs. Identities are the unique morphisms (x, x), which exist based on the reflexivity of the relation. Note that if (a, b) and (b, c) are in the relation, the composition will be $(b, c)(a, b) = (a, c)$.

What do the properties of a functor between preorders C (with relation R) and D (with relation S) tell us? First, we know that $F(1_x) = 1_{F(x)}$ for all $x \in obC$. This implies that the morphism (x, x) must be brought to the morphism $(F(x), F(x))$. This becomes redundant with the next step.

We also know that $F(\text{dom}(f)) = \text{dom}(F(f))$ for all $f \in morC$. If $f = (a, b)$, then $F(\text{dom}(f)) = F(a)$ and thus $F(f)$ must be a pair $(F(a), z_1)$ for some $z_1 \in obD$. Similarly, $F(\text{cod}(f)) = \text{cod}(F(f))$ implies that if $f = (a, b)$, then $F(\text{cod}(f)) = F(b)$ and $F(f)$ must be a pair $(z_2, F(b))$. Combining these, we get that $F(f)$ for $f = (a, b)$ must be a pair $(F(a), F(b))$. This means that if $(a, b) \in R$ then $(F(a), F(b)) \in S$.

Thus, F provides us a preorder homomorphism, as F preserves related pairs. The final property to check for a functor is composable pairs. If two morphisms f and g are composable, then $F(fg) = F(f)F(g)$. This means $F((b, c)(a, b)) = F((a, c)) = (F(b), F(c))(F(a)(F(b)) = (F(a), F(c))$, which was already confirmed by the previous property. Thus, the functor is a preorder homomoprhism. □

Exercise 1.3.iii. Find an example to show that the objects and morphisms in the image of a functor $F\colon \mathsf{C} \to \mathsf{D}$ do not necessarily define a subcategory of D.

At first, I was suspicious of this exercise since it seemed to me that the proof that the image of a group (or monoid, or ring, . . .) homomorphism is a subgroup (or submonoid, or subring, . . .) carries through without any change. Perhaps the author meant for the exercise to be something different?

So, I asked her, and she pointed out a straightforward example that also exposed the error in my reasoning. I will give below a simplification of the example that she sent me. First, here is the error in my original argument.

Let a and b be morphisms in the image of F such that $\operatorname{dom} a = \operatorname{cod} b$ so that we can form ab in D. Since a and b are in the image of F, there are morphisms f and g in bC such that $a = Ff$ and $b = Fg$. Then

$$ab = FfFg = F(fg)$$

so that ab is in the image of F.

Right? Wrong!! In order to compose f and g we need that $\operatorname{dom} f = \operatorname{cod} g$. All we know for sure is that $\operatorname{dom} Ff = \operatorname{cod} Fg$. If F is injective on objects, then the argument above is valid. But perhaps F is not injective.

The Example.

Now, I provide the example requested in this exercise. Let $\mathsf{C} = \mathbb{2}$ be the ordinal category pictured as so:

$$0 \xrightarrow{f} 1.$$

Let g be an endomorphism of some object x in some category D such that gg is equal to neither 1_x nor g. For example, we could take x to be the unique object in $B\mathbb{N}$ and $g = 1$, so that $gg = g + g = 2 \neq 1, 0$.

Then we have a functor $F : \mathbb{2} \to \mathsf{D}$ given by $F0 = F1 = x$, $F1_0 = F1_1 = 1_x$ and $Ff = g$. There are only four possible compositions of the three morphisms in $\mathbb{2}$, $1_0 1_0$, $f1_0$, $1_1 f$ and $1_1 1_1$ and it is easy to see that F preserves all four of these compositions. Thus, F is a functor.

However, the image of F has only the two morphisms 1_x and g. Since gg is not in the image, the image of F is not a subcategory of D.

Lemma 1.2.3. *The following are equivalent:*

(i) $f : x \to y$ *is an isomorphism in* C.

(ii) For all objects $c \in \mathsf{C}$, *post-composition with* f *defines a bijection*

$$f_* : \mathsf{C}(c, x) \to \mathsf{C}(c, y)$$

(iii) For all objects $c \in \mathsf{C}$, *pre-composition with* f *defines a bijection*

$$f^* : \mathsf{C}(y, c) \to \mathsf{C}(x, c)$$

Definition 1.3.1. A *functor* F from C to D is a functor $F : \mathsf{C} \to \mathsf{D}$. Explicitly, this consists of the following data:

- An object $Fc \in \mathsf{D}$, for each object $c \in \mathsf{C}$.
- A morphism $Ff : Fc \to Fc' \in \mathsf{D}$, for each morphism $f : c \to c' \in \mathsf{C}$, so that the domain and codomain of Ff are, respectively, equal to F applied to the domain or codomain of f.

The assignments are required to satisfy the following two *functoriality axioms*:

- For any composable pair f, g in C, $Fg \cdot Ff = F(g \cdot f)$.
- For each object c in C, $F(1_c) = 1_{Fc}$.

The functors defined in 1.3.1 are called *covariant* so as to distinguish them from another variety of functor that we now introduce.

Definition 1.3.5. A *contravariant functor* F from C to D is a functor $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{D}$. Explicitly, this consists of the following data:

- An object $Fc \in \mathsf{D}$, for each object $c \in \mathsf{C}$.
- A morphism $Ff : Fc' \to Fc \in \mathsf{D}$, for each morphism $f : c \to c' \in \mathsf{C}$, so that the domain and codomain of Ff are, respectively, equal to F applied to the codomain or domain of f.

The assignments are required to satisfy the following two *functoriality axioms*:

- For any composable pair f, g in C, $Ff \cdot Fg = F(g \cdot f)$.

- For each object c in C, $F(1_c) = 1_{Fc}$.

Definition 1.3.11. If C is locally small, then for any object $c \in \mathsf{C}$ we may define a pair of covariant and contravariant *functors represented by c*:

$$\begin{tikzcd}
\mathsf{C} \arrow[rr, "\mathsf{C}(c,-)"] & & \mathsf{Set} & \mathsf{C}^{\mathrm{op}} \arrow[rr, "\mathsf{C}(-,c)"] & & Set \\
x \arrow[d, "f"'] & \mapsto & \mathsf{C}(c,x) \arrow[d, "f_*"] & x \arrow[d, "f"'] & \mapsto & \mathsf{C}(x,c) \\
y & \mapsto & \mathsf{C}(c,y) & y & \mapsto & \mathsf{C}(y,c) \arrow[u, "f^*"']
\end{tikzcd}$$

The notation suggests the action on objects: the functor $\mathsf{C}(c,-)$ carries $x \in \mathsf{C}$ to the set $\mathsf{C}(c,x)$ of arrows from c to x in C. Dually, the functor $\mathsf{C}(-,c)$ carries $x \in \mathsf{C}$ to the set $\mathsf{C}(x,c)$.

The functor $\mathsf{C}(c,-)$ carries a morphism $f\colon x \to y$ to the post-composition function $f_*\colon \mathsf{C}(c,x) \to \mathsf{C}(c,y)$ introduced in Lemma 1.2.3(ii). Dually, the functor $\mathsf{C}(-,c)$ carries f to the pre-composition function $f^*\colon \mathsf{C}(y,c) \to \mathsf{C}(x,c)$ introduced in 1.2.3(iii).

Exercise 1.3.iv. Verify that the constructions in Definition 1.3.11 are functorial.

Proof. We start by showing that the assignments of $\mathsf{C}(c,-)$ satisfy the functoriality axioms for (covariant) functors. The actions of dom and cod on $\mathsf{C}(c,-)$ can be seen as follows: applying $\mathsf{C}(c,-)$ to a morphism $h\colon i \to j$ will give a morphism $\mathsf{C}(c,-)(h)\colon \mathsf{C}(c, \operatorname{dom} h) \to \mathsf{C}(c, \operatorname{cod} h)$, so $\operatorname{dom}\mathsf{C}(c,-)(h) = \mathsf{C}(c,i)$ and $\operatorname{cod}\mathsf{C}(c,-)(h) = \mathsf{C}(c,j)$.

To show composition, let $f\colon x \to y$ and $g\colon w \to x$ be a composable pair of morphisms in C. Note first that $f \cdot g\colon w \to y$ and that $\mathsf{C}(c,-)(f)\colon \mathsf{C}(c,x) \to \mathsf{C}(c,y)$, and finally that $\mathsf{C}(c,-)(g)\colon \mathsf{C}(c,w) \to \mathsf{C}(c,x)$.
Since $\operatorname{dom}\mathsf{C}(c,-)(f) = \operatorname{cod}\mathsf{C}(c,-)(g) = \mathsf{C}(c,x)$, we can compose as follows:

$$\mathsf{C}(c,-)(f) \cdot \mathsf{C}(c,-)(g)\colon \mathsf{C}(c,w) \to \mathsf{C}(c,y)$$

Since $\mathsf{C}(c,-)(f \cdot g)\colon \mathsf{C}(c,w) \to \mathsf{C}(c,y)$ and both this and the above are given by applying $f \cdot g$ to a morphism in $\mathsf{C}(c,w)$, we have that $\mathsf{C}(c,-)(f) \cdot \mathsf{C}(c,-)(g) = \mathsf{C}(c,-)(f \cdot g)$, satisfying functor composition.

To show that identities are preserved, note that for any object $x \in \mathsf{C}$, $1_x\colon x \to x$. Then $\mathsf{C}(c,-)(1_x)\colon \mathsf{C}(c,x) \to \mathsf{C}(c,x)$ taking 1_x to the post composition $1_x^*\colon \mathsf{C}(c,x) \to \mathsf{C}(c,x)$. Since for any morphism $a \in \mathsf{C}(c,x)$, 1_x^* takes $a \mapsto 1_x a$, this is the identity $a \mapsto a$. Then consider $1_{\mathsf{C}(c,-)(x)} = 1_{\mathsf{C}(c,x)}$, which is the identity of $\mathsf{C}(c,x)$, taking each element a of the set to itself: $a \mapsto a$. Thus, $\mathsf{C}(c,-)(1_x) = 1_{\mathsf{C}(c,-)(x)}$ and $\mathsf{C}(c,-)$ preserves identities. Diagrammatically:

$$\begin{tikzcd}
c \arrow[r, "a"] \arrow[dr, "1_x a"'] & x \arrow[d, "1_x"] \\
& y = x
\end{tikzcd}$$

To see that $\mathsf{C}(-,c)$ is a contravariant functor, we argue by duality. Since $\mathsf{C}(c,-)\colon \mathsf{C}\to\mathsf{Set}$ is a functor for any category C, we have that $\mathsf{C}^{\mathrm{op}}(c,-)\colon \mathsf{C}^{\mathrm{op}}\to\mathsf{Set}$ is also a functor. Additionally, it is contravariant since the definition a contravariant functor is a functor from C^{op} to Set. Given that $\mathsf{C}^{\mathrm{op}}(c,-) = \mathsf{C}(-,c)$, we know that $\mathsf{C}(-,c)$ is a contravariant functor, completing the proof. □

Exercise 1.3.v. What is the difference between a functor $\mathsf{C}^{\mathrm{op}} \to \mathsf{D}$ and a functor $\mathsf{C} \to \mathsf{D}^{\mathrm{op}}$? What is the difference between a functor $\mathsf{C} \to \mathsf{D}$ and a functor $\mathsf{C}^{\mathrm{op}} \to \mathsf{D}^{\mathrm{op}}$?

Proof. We will show that if F is a functor from $\mathsf{C} \to \mathsf{D}$, then F is also a functor from C^{op} to D^{op}, and then deduce the relation ship between a functor $\mathsf{C}^{\mathrm{op}} \to \mathsf{D}$ and a functor $\mathsf{C} \to \mathsf{D}^{\mathrm{op}}$ as a special case.

To show that if F is a functor from $\mathsf{C} \to \mathsf{D}$, then F is also a functor from C^{op} to D^{op}, immediately from the functoriality axioms, that is for any two composable morphisms f and g in C and for any object x in C we must have $F(fg) = FfFg$ and $F1_x = 1_{Fx}$. Since for each object in C and D the identity maps are the same in their respective opposite categories, we only need to verify that F respects composition when take composable morphisms from C^{op} to D^{op}. Since the objects C^{op} and D^{op} are exactly the same as in C and D and the morphisms are the some only with domains and codomains switched, we can apply F to the morphisms of C^{op} and get morphisms in D^{op}. This gives us that $Ffg = FfFg$, and F is a functor from C^{op} to D^{op}. Now since $\mathsf{C} = (\mathsf{C}^{\mathrm{op}})^{\mathrm{op}}$ and $\mathsf{D} = (\mathsf{D}^{\mathrm{op}})^{\mathrm{op}}$ we see that there is no difference from a functor $\mathsf{C}^{\mathrm{op}} \to \mathsf{D}$ and a functor $\mathsf{C} \to \mathsf{D}^{\mathrm{op}}$ as well. □

Exercise 1.3.vi. Given functors $F\colon \mathsf{D}\to\mathsf{C}$ and $G\colon \mathsf{E}\to\mathsf{C}$, show that there is a category, called the **comma category** $F \downarrow G$, which has

1. as objects, triples $(d \in \mathsf{D}, e \in \mathsf{E}, f\colon Fd\to Ge \in \mathsf{C})$, and
2. as morphisms $(d,e,f) \to (d',e',f')$, a pair of morphisms $(h\colon d\to d', k\colon e\to e')$ so that the square

$$\begin{tikzcd} Fd \arrow[r, "f"'] \arrow[d, "Fh"] & Ge \arrow[d, "Gk"] \\ Fd' \arrow[r, "f'"] & Ge' \end{tikzcd}$$

commutes in C, i.e., so that $f' \cdot Fh = Gk \cdot f$.

Define a pair of projection functors $dom\colon F \downarrow G\to\mathsf{D}$ and $cod\colon F \downarrow G\to\mathsf{E}$

Proof. Before we prove that the comma category $F \downarrow G$ is actually a category, we need to give a motivating example for a major issue in the proof.

Let $A\colon \mathbb{2}\to\mathsf{Set}$ and $B\colon \mathbb{2}\to\mathsf{Set}$ be functors where $A0 = \{0\}$, $A1 = \{0,1,2\}$, $B0 = \{0,1\}$, $B1 = \{0,1,2,3\}$ and where A and B maps the unique morphism $f\colon 0\to 1$ to the inclusion functions $\iota\colon \{0\}\to\{0,1,2\}$ and $\iota\colon \{0,1\}\to\{0,1,2,3\}$ respectively. Let us take our supposed objects $(0, 0, \iota)$ and $(1, 1, \alpha)$ where ι is the inclusion function and our supposed morphism $(f : 0 \to 1, f : 0 \to 1)$ so that the diagram

$$\begin{tikzcd} A0 \arrow[r, "\iota"'] \arrow[d, "A)"] & B0 \arrow[d, "Bf"] \\ A1 \arrow[r, "\alpha"] & B1 \end{tikzcd}$$

commutes. Now there are at least two functions for α that would allow the diagram above to commute. The first is if α was simply an inclusion function so the functions $\alpha \cdot Af$ and $Bf \cdot \iota$ are inclusion functions from the singleton set $A0$ to $B1$, thus $\alpha \cdot Af = Bf \cdot \iota$. The second function which I will denote α' is defined as follows:

$$\alpha'(0) = 0, \alpha'(1) = 2, \alpha'(2) = 1.$$

Since α' still maps 0 in $A1$ to itself in $B1$, the diagram above still commutes. Thus our supposed morphism $(f : 0 \to 1, f : 0 \to 1)$ would have two codomains $((1, 1, \alpha)$ and $(1, 1, \alpha'))$ for our domain $(0, 0, \iota)$.

This example shows we need additional notation to distinguish between arrows that are represented the same but have different domains and codomains. Returning to the notation established in the first paragraph, we will append morphism pairs (f, f') to the end of some morphism $(h : d \to d', k : e \to e')$ so that we specify that the intended domain and codomain of the morphism is $(d, e, f,)$ and (d', e', f') respectively. Thus the uniqueness of the domain and codomain of some morphism $(h : d \to d', k : e \to e')\,(f, f')$ follows from the uniqueness of the domain and codomain of h and k, and additionally from our notation which specifies unique morphisms f and f'. Now we can complete the rest of the proof using the notation established in the first paragraph.

For an object $(d, e, f,)$,denoted as c, we can define an identity morphism for c as the following:

$$1_c = (1_d, 1_e)\,(f, f)$$

where 1_d and 1_e are the respective identities of d and e. Thus the diagram

$$\begin{tikzcd} Fd \arrow[r, "f"'] \arrow[d, "F1_d"] & Ge \arrow[d, "G1_e"] \\ Fd \arrow[r, "f"] & Ge \end{tikzcd}$$

trivially commutes. The unique domain and codomain of 1_c, both being $(d, e, f,)$, are derived from the unique domain and codomains of the identities 1_d and 1_e, and the uniquely specified f.

Now let us define morphism composition between two morphisms. $(h : d \to d_1, k : e \to e_1)\,(f, f_1)$ and $(h : d_1 \to d_2, k : e_1 \to e_2)\,(f_1, f_2)$ which we will denote $\alpha : (d, e, f) \to (d_1, e_1, f_1)$ and $\beta : (d_1, e_1, f_1) \to (d_2, e_2, f_2)$. The composition of α and β shall be defined as follows:

$$\beta \cdot \alpha = (h' \cdot h : d \to d_2, k' \cdot k : e \to e_2)\,(f, f_2)$$

resulting in the diagram

$$\begin{tikzcd}
Fd \arrow[r, "f"'] \arrow[d, "Fh"] & Ge \arrow[d, "Gk"] \\
Fd_1 \arrow[r, "f_1"] \arrow[d, "Fh'"] & Ge_1 \arrow[d, "Gk'"] \\
Fd_2 \arrow[r, "f_2"] & Ge_2
\end{tikzcd}$$

which commutes since functors preserve composition of morphisms and the top and bottom squares commute by construction of α and β. Thus we have that the following diagram commutes:

$$\begin{tikzcd}
Fd \arrow[r, "f"'] \arrow[d, "Fh' \cdot Fh"] & Ge \arrow[d, "Gk' \cdot Gk"] \\
Fd_2 \arrow[r, "f_2"] & Ge_2
\end{tikzcd}$$

which is the commutative square for the composed morphism $\beta \cdot \alpha$. The morphism $\beta \cdot \alpha$ derives its unique domain from the unique domains of $h' \cdot h$ and $k' \cdot k$ and the specified function f which gives the domain $(d, e, f,)$ which is the domain of α, and the unique codomain is derived similarly resulting in the codomain (d_2, e_2, f_2) which is the codomain of β. So the composition of morphisms α and β gives a morphism $\beta \cdot \alpha$ with the domain of α and the codomain of β

Now that we have defined the identity morphism and composition of morphism, we can show that the identity morphism is left and right cancellable, and that composition is associative.

Let $(h\colon d \to d', k\colon e \to e')(f, f')$, denoted as α, be a morphism with domain and codomain $(d, e, f,)$ and $(d', e', f',)$ respectively, denoted as c and c' respectively. Starting with the composition of α and 1_c, we can show the following chain of equalities:

$$\alpha \cdot 1_c = (h \cdot 1_d, k \cdot 1_e)(f, f')$$

$$\alpha \cdot 1_c = (h, k)(f, f')$$

$$\alpha \cdot 1_c = \alpha$$

Composing $1_{c'}$ and α gives us a similar result:

$$1_{c'} \cdot \alpha = (1_{d'} \cdot h, 1_{e'} \cdot k)(f, f')$$

$$1_{c'} \cdot \alpha = (h, k)(f, f')$$

$$1_{c'} \cdot \alpha = \alpha$$

. Thus we have shown that $1_{c'} \cdot \alpha = \alpha = \alpha \cdot 1_c$. Therefore the identity morphism is left and right cancellable.

Finally, we will show that the composition of morphisms is associative. Take morphisms $(h\colon d \to d_1, k\colon e \to e_1)(f, f_1)$, $(h\colon d_1 \to d_2, k\colon e_1 \to e_2)(f_1, f_2)$, and $(h\colon d_2 \to d_3, k\colon e_2 \to e_3)(f_2, f_3)$, denoted as α, β, and γ respectively. Starting from $(\gamma \cdot \beta) \cdot \alpha$, we will demonstrate the following equality

$$(\gamma \cdot \beta) \cdot \alpha = ((h_2 \cdot h_1), (k_2 \cdot k_1))(f_1, f_3) \cdot \alpha$$

$$(\gamma \cdot \beta) \cdot \alpha = ((h_2 \cdot h_1) \cdot h, (k_2 \cdot k_1) \cdot k)(f, f_3)$$

$$(\gamma \cdot \beta) \cdot \alpha = (h_2 \cdot (h_1 \cdot h), k_2 \cdot (k_1 \cdot k))(f, f_3)$$

$$(\gamma \cdot \beta) \cdot \alpha = \gamma \cdot ((h_1 \cdot h), (k_1 \cdot k))(f, f_2)$$

$$(\gamma \cdot \beta) \cdot \alpha = \gamma \cdot (\beta \cdot \alpha)$$

Thus the composition of morphisms is associative.

We have thus shown that $F \downarrow G$ is a category.

Now we will define the functors $dom\colon F \downarrow G \to \mathsf{D}$ and $cod\colon F \downarrow G \to \mathsf{E}$ for object (d, e, f) and morphism $(h\colon d \to d', k\colon e \to e')\,(f, f')$ as follows:

$$dom\,(d, e, f) = d, dom\,(h, k)\,(f, f') = h$$

$$cod\,(d, e, f) = e, cod\,(h, k)\,(f, f') = k$$

Now we will verify that both dom and cod are indeed functors.

Now let us take the morphism $(h\colon d \to d_1, k\colon e \to e_1)\,(f, f_1)$ denoted $\alpha\colon (d, e, f) \to (d_1, e_1, f_1)$. Applying dom to α gives us $h\colon d \to d_1$ where $dom\,(d, e, f) = d$ and $dom\,(d_1, e_1, f_1) = d_1$. Applying cod to α gives $k\colon e \to e_1$ where $cod\,(d, e, f) = e$ and $cod\,(d_1, e_1, f_1) = e_1$. Thus $\alpha\colon (d, e, f) \to (d_1, e_1, f_1)$ gets mapped to $dom\alpha\colon dom\,(d, e, f) \to dom\,(d_1, e_1, f_1)$ and $cod\alpha\colon cod\,(d, e, f) \to cod\,(d_1, e_1, f_1)$ by dom and cod respectively.

For object (d, e, f), the identity arrow $(1_d, 1_e)\,(f, f)$ get mapped to 1_d and 1_e by dom and cod respectively. Since $dom\,(d, e, f) = d$ and $cod\,(d, e, f) = e$, This shows that dom and cod preserve identities.

Finally take morphisms $(h\colon d \to d_1, k\colon e \to e_1)\,(f, f_1)$ denoted $\alpha\colon (d, e, f) \to (d_1, e_1, f_1)$ and $(h'\colon d_1 \to d_2, k'\colon e_1 \to e_2)\,(f_1, f_2)$ denoted $\beta\colon (d_1, e_1, f_1) \to (d_2, e_2, f_2)$. Then we have the following equalities:

$$dom\beta \cdot \alpha = dom\,(h' \cdot h, k' \cdot k)\,(f, f_2) = h' \cdot h = dom\beta \cdot dom\alpha$$

and

$$cod\beta \cdot \alpha = cod\,(h' \cdot h, k' \cdot k)\,(f, f_2) = k' \cdot k = cod\beta \cdot cod\alpha$$

. Thus dom and cod perserve morphism composition.

Therefore dom and cod are functors. □

Exercise 1.3.vii. Define functors to construct the slice categories c/C and C/c as special cases of comma categories. What are the projection functors?

Proof. We want to choose functors in a comma category, so that the comma category behaves like a slice category. Since the slice category treats morphisms like objects within a single category, and comma categories are defined generally with three categories in mind, we have some room to reduce the structure of the comma category as we construct a slice category.

Using the notation in the text's definition of the comma category, let F↓G be our comma category. Also, let $D = C$ and let the functor F be the identity functor, $\mathbf{1}_C$. With this choice of functor, the square in the definition of F↓G still commutes.

At this point, the objects of the comma category are $(d \in C, e \in E, f : d \to Ge)$. The morphisms are $(h : d \to d', k : e \to e')$, such that $(d, e, f) \to (d', e', f')$.

The above choice of the identity functor collapses the amount of data represented. Yet there remain extra data at this point in the construction to qualify as a slice category. We desire to keep one of the morphisms h and k, while reducing the object-triple $(d \in C, e \in E, f : d \to Ge)$ to a suitable object-morphism pair, that allows us to fix an element in C, and take morphisms as objects.

To this end, let E be the ordinal category, $\mathbf{1}$, with one object represented as $\emptyset$, and only the identity morphism. The objects in the comma category now are the triples $(d \in C, \emptyset, f : d \to c \in C)$. With the functor G acting on the one object of $\mathbf{1}$, then G sends $\emptyset$ to one object in C. Since, in $E = \mathbf{1}$, $k = id_{\mathbf{1}}$, then k can only send $\emptyset$ to $\emptyset$. So the objects in the comma construction are rendered as pairs, with the relevant data c and f, represented as $(d \in C, f : d \to c)$.

Letting the morphism h take d to d', while the morphism $k = id_{\mathbf{1}}$ sends $\emptyset$ to $\emptyset$, we have a codomain of the comma morphism (h, k) represented as $(d' \in C, f' : d' \to c)$. This is a class of morphisms of C taken as objects, and the comma category is reduced to the C/c slice category. □

Exercise 1.3.viii. Lemma 1.3.8 shows that functors preserve isomorphisms. Find an example to demonstrate that functors need not **reflect isomorphisms**: that is, find a functor $F : \mathsf{C} \to \mathsf{D}$ and a morphism f in C so that Ff is an isomorphism in D but f is not an isomorphism in C.

Consider the functor $F : \mathbb{2} \to \mathbb{1}$. This is a great functor because it maps everything in $\mathbb{2}$ to the identity in $\mathbb{1}$. So F trivially satisfies all the properties of a functor. Because $\mathbb{2}$ is not a groupoid there exists at least one morphism f in mor $\mathbb{2}$ that is not an isomorphim. In this construction Ff will also go to the best isomorphism the identity map.

Note: In this solution, I structure my work differently that the question is posed in the book, because it allows me to combine several parts of the problem.

First, consider the functor $F_Z : \mathsf{Group}_{id} \to \mathsf{Group}$, where $G \to Z(G)$. We will now show that there exists a similar functor $F_Z : \mathsf{Group}_{epi} \to \mathsf{Group}$. We define F_Z as the following: $F_Z(G) = Z(G)$ and if $f : G \to H$, then $F_Z f = f|_{Z(G)}$. We must show that this functor satisfies the properties of a functor.

1. Because each group has a unique center and each morphism f with $dom(f) = G$ has a unique restriction to $Z(G)$, this functor is well defined and satisfies the first two properties (0 and 1).
2. We see that $F_Z 1_G = 1_G|_{Z(G)} = 1_{Z(G)} = 1_{F_Z G}$, so the functor preserves identities.
3. We easily see that by definition of function restriction, if $f : G \to H$, then $dom(F_Z f) = F_Z dom(f)$. To see that this morphism has a well-defined codomain under this functor, we show that if $g \in Z(G)$, then $f(g) \in Z(H)$. To do this, consider, for any $k \in H$, that since f is an epimorphism and therefore surjective, that $k = f(h)$ for some $h \in G$. So
$$f(g)k = f(g)f(h) = f(gh)$$
Since $g \in Z(G)$,
$$f(gh) = f(hg) = f(h)f(g) = kf(g)$$
So $f(g) \in Z(H)$ and therefore we have a well-defined codomain and
$$cod(F_Z f) = cod(f|_{Z(G)}) = Z(H) = F_Z cod(f).$$
4. If $f : G \to H$ and $G : H \to K$, we see that $F_Z(gf) = gf|_{Z(G)}$. By the property we proved in the previous part
$$gf|_{Z(G)} = g|_{Z(H)} f|_{Z(G)} = F_Z g F_Z f.$$
So this functor also preserves morphism composition.

We have seen that F_Z satisfies all properties of a functor. We also note that F_Z will be a functor from $\mathsf{Group}_{iso} \to \mathsf{Group}$.

To show that there is no such functor between $Group$ and $Group$, consider the composition of the homorphism $sgn : S_n \to \{\pm 1\}$ and $\iota : \{1, (1\ 2)\} \to S_4$. We say $g(x) = sgn(\iota(x))$. We see that g is an isomorphism, and so $F_Z g$ should also be an isomorphism. However, this is not possible under any function of morphism, as S_4 has a trivial center and so any morphism from $Z(\{1, (1\ 2)\} \to Z(S_4) \to Z(\{\pm 1\})$ must be trivial. So F_Z cannot be a functor from $\mathsf{Group} \to \mathsf{Group}$.

Now consider the functor $F_C : \mathsf{Group}_{id} \to \mathsf{Group}$. We will show there exists a similarly constructed functor from $Group \to Group$ defined as the following: for $G \in ob(Group)$, $F_C G = C(G)$, where C(G) is the commutator subgroup of G. If $f : G \to H$, $F_C f = f|_{C(G)}$, We will show that this satisfies all the properties of a functor.

1. Because each group has a unique commutator subgroup and each morphism f with $dom(f) = G$ has a unique restriction to $C(G)$, this functor is well defined and satisfies the first two properties (0 and 1).
2. We see that $F_C 1_G = 1_G|_{C(G)} = 1_{C(G)} = 1_{F_C G}$, so the functor preserves identities.
3. We easily see that by definition of function restriction, if $f : G \to H$, then $dom(F_Z f) = F_Z dom(f)$. To see that the codomain is well defined under this functor, we show that if $g \in C(G)$, then $f(g) \in C(H)$. If $g \in C(G)$, $g = \Pi_{i=1}^n a_i$, where each $a_i = hkh^{-1}k^{-1}$ for some $h, k \in G$. So
$$f(g) = f(\Pi_{i=1}^n h_i k_i h_i^{-1} k_i^{-1}) = \Pi_{i=1}^n f(h_i k_i h_i^{-1} k_i^{-1}) =$$
$$\Pi_{i=1}^n f(h_i) f(k_i) f(h_i^{-1}) f(k_i^{-1}) = \Pi_{i=1}^n f(h_i) f(k_i) f(h_i)^{-1} f(k_i)^{-1}$$

But since $f(k_i), f(h_i) \in H$, this is an element of $C(H)$. So if $g \in C(G)$, $f(G) \in C(H)$ and therefore the comdomain is well defined and

$$cod(F_C f) = cod(f|_{C(G)}) = C(H) = F_C cod(f)$$

.

4. If $f : G \to H$ and $G : H \to K$, we see that $F_C(gf) = gf|_{C(G)}$. By the property we proved in the previous part

$$gf|_{C(G)} = g|_{C(H)} f|_{C(G)} = F_C g F_C f.$$

So this functor also preserves morphism composition.

So we have shown that this is a functor for Group $\to$ Group, and we note that this implies that it is also a functor for $\text{Group}_{iso} \to$ Group and $\text{Group}_{epi} \to$ Group.

Next, we show that there is a functor F_A : $\text{Group}_{iso} \to$ Group, defined as follows: If G is a group, then $F_A G = Aut(G)$ and if ϕ is a morphism between two groups G and H, then $(F_A\phi)(f) = \phi f \phi^{-1}$. We now show that this definition satisfies the properties of a functor.

1. Each group has uniquely defined automorphism group, and each morphism ϕ conjugates elements of $Aut(G)$in a unique way, so the functor is well defined.
2. $F_A(1_G)(f) = 1f1 = 1_{Aut(G)}(f)$, so the functor preserves identities.
3. $F_A(dom\phi) = F_A(G) = Aut(G) = domF_A(\phi)$.
4. We know that $cod\phi = H$ (because $\phi : G \to H$). So $F_A(cod\phi) = Aut(H)$ $F_A\phi(f) = \phi f \phi^{-1}$, which we have seen goes from H to H, and since it is composed of isomorphisms, it is also an automorphism, so $codF_A\phi = Aut(H)$. So co-domain is well-defined under this functor and $codF_A f = F_A cod f$.
5. For two composable morphisms ϕ and τ,

$$F_A(\phi\tau)(f) = \phi\tau f(\phi\tau)^{-1} = \phi\tau f \tau^{-1}\phi^{-1} = F_A(\phi)(\tau f \tau^{-1}) = F_A(\phi)F_A(\tau)(f),$$

so this functor preserves composition.

So F_A satisfies all the properties of a functor, and there exists a functor from $\text{Group}_{iso} \to$ Group of the desired form.

It is unclear whether or not there is a functor from $\text{Group}_{epi}$to Group and from Group to Group defined in the manner, but I believe that this is not the case, although I am having trouble finding a counterexample.

Exercise 1.3.x. Show that the construction of the set of conjugacy classes of elements of a group is functorial, defining a functor Conj: Group $\to$ Set. Conclude that any pair of groups whose sets of conjugacy classes of elements have differing cardinalities cannot be isomorphic.

Proof. Let the functor F : Group $\to$ Set represent the construction of the set of conjugacy classes of elements of a group, defined as follows:

- For any group s, $Fs = s^*$.

- For any groups s and t and any group homomorphism $f: s \to t$, define the morphism $Ff: s^* \to t^*$ such that for each $[x] \in s^*$, $Ff([x]) = [f(x)]$.

  (The proof that every Ff is well-defined is non-trivial; see Subproof 1.3.x.0, below.)

First we will prove that F is functorial by showing that it fulfills both functoriality axioms:

- Let f and g be group homomorphisms such that gf is a valid composition, and let $[x] \in (\text{dom}(f))^*$ be arbitrary.

$$FgFf([x]) = Fg([f(x)]) = [g(f(x))] = [gf(x)] = F(gf([x])).$$

  Since $[x]$, f, and g were arbitrary, $FgFf = F(gf)$. So F fulfills the first functoriality axiom.

- For an arbitrary group s and element $x \in s$,

$$F1_s([x]) = [1_s(x)] = [x] = 1_{s^*}([x]) = 1_{Fs}([x]).$$

Since s and x were arbitrary, $F1_s = 1_{Fs}$. So F fulfills the second functoriality axiom. F fulfills both axioms, so we can conclude that it is indeed functorial.

Let s and t be two isomorphic groups, and let $f: s \to t$ be an isomorphism. Functors preserve isomorphisms, (as per the 'first lemma in category theory') so $Ff: s^* \to t^*$ must also be an isomorphism. This makes s^* and t^* isomorphic, which in turn means they must have the same cardinality. So we can conclude the contrapositive: that any pair of groups whose sets of conjugacy classes have *different* cardinalities *cannot* be isomorphic.

Subproof 1.3.x.0. In order for each Ff to be well-defined, it must send each $[x] \in s^*$ to a single $[f(x)] \in t^*$. But there may be more than one element in $[x]$, and since the definition does not mention which x should be used as a 'representative,' it might seem that there could be cases in which there were multiple possible $[f(x)]$ (and therefore, multiple possible $Ff([x])$) for a single $[x]$. So in order for Ff to be well-defined, $[f(x)]$ must be the same for every possible choice of $x \in [x]$. In other words, we need to show that $[f(a)] = [f(b)]$ for any a, b in the same conjugate class.

So, let a, b be arbitrary members of the same conjugate class. Recall that this means that there is some $n \in s$ such that $b = nan^{-1}$. Furthermore, recall that group homomorphisms (like f) preserve inverses. With this in mind,

$$f(b) = f(nan^{-1}) = f(n)f(a)f(n^{-1}) = f(n)f(a)f(n)^{-1}$$

$n \in s$, so $f(n) \in t$. This means that there is some $m \in t$ such that $mf(a)m^{-1} = f(b)$. So $f(a)$ and $f(b)$ are conjugates, (and therefore $[f(a)] = [f(b)]$ by definition,) which makes Ff well-defined. $\square$

1.4 Naturality

Exercise 1.4.ii. What is a natural transformation between a parallel pair of functors between groups, regarded as one-object categories?

Proof. For the abstract, one-object categories that are defined by groups, we can take any single object C_* and D_* as the specials objects of the respective groups. We then have the class of morphisms as the elements of the groups under their respective group operations. Let the operations here be group multiplication. We will start by finding what the functors are between these categories, and then find natural transformation in this context.

Given two such categories, BC and BD with their respective groups C and D, any functors F and G between BC and BD must map the object C_* to the object D_*. Since functors are functions, looking at the functor F, the role of F acting on the morphisms of BC is the same as a function acting on the elements of C under group multiplication. So, $F : C_* \to D_*$, for the special elements $C_* \in C$ and $D_* \in D$. Taking any two elements c, c' in C, cc' is the composition in the category BC, and since the domain = C = codomain of BC, then any pair cc' is composable. As functors respect the functoriality axioms, $F(cc') = F(c)F(c')$ in D, and $F(1_C) = 1_{F_C}$ in D, then the functor F behaves as a group homomorphism between the groups C and D.

To find a natural transformation $\alpha : F \implies G$ between F and G, we can let $\alpha_{C_*} : FC_* \to GC_*$ be a class of morphisms, with $f : C_* \to C_*$, such that $Gf\alpha_{C_*} = \alpha_{C_*}Ff$. In our case, we have C_* as the single object of BC, and the morphism f as an element c in C. Also, we have that $FC_* = D_*$ and $GC_* = D_*$, so our class of morphisms now is $\alpha_{C_*} : D_* \to D_*$. Thus, for the object C_*, the natural transformation gives the equality $\alpha_{C_*}Ff = Gf\alpha_{C_*}$.

Since α_{C_*} is a morphism from the object D_* to itself, this endomorphism (and hence automorphism) consists of the elements of the group D. Because each element has an inverse, likewise α_{C*} has an inverse, α^{-1}_{C*}. Thus $\alpha_{C_*}Ff = Gf\alpha_{C_*}$ implies $\alpha^{-1}_{C*}\alpha_{C_*}Ff = \alpha^{-1}_{C*}Gf\alpha_{C_*}$, implies $Ff = \alpha^{-1}_{C*}Gf\alpha_{C_*}$, for all f. Noting that Ff and Gf are morphisms in the category BD, and hence is an element of the group D, then Ff and Gf are in $Aut(D)$, and α_{C_*} forms a conjugacy class for these automorphisms. □

Exercise 1.4.iii. What is a natural transformation between a parallel pair of functors between preorders, regarded as categories?

Proof. Let C and D be preorder categories and $F, G \colon \mathsf{C} \rightrightarrows \mathsf{D}$ be parallel functors. Let us consider the properties of the natural transformation $\alpha \colon F \to G$. We will show that you need only find morphisms $f_c \colon Fc \to Gc$ in D for all $c \in \mathrm{ob}\,\mathsf{C}$ to produce a natural transformation from F to G. This will assist in our characterization of natural transformations.

Suppose we have morphisms $f_c \colon Fc \to Gc$ in D for $c \in \mathrm{ob}\,\mathsf{C}$. We can define $\alpha \colon F \to G$ such that $\alpha(c) = f_c$. Take morphism $g \colon c \to c'$ in D, we have that the diagram

$$\begin{tikzcd}
Fc \arrow[r, "Fg"] \arrow[d, "\alpha(c)"'] & Fc' \arrow[d, "\alpha(c')"] \\
Gc \arrow[r, "Gg"'] & Gc'
\end{tikzcd}$$

commutes since in a preorder category, there is at most one morphism between objects. Thus, our α defines a natural transformation from F to G. Thus it is sufficient to find morphisms $f_c\colon Fc \to Gc$ in D to define a natural transformation from F to G.

Seeing the functors F and G as monotone maps (i.e. order preserving maps) between preorders C to D, this allows us to characterize a natural transformation α as a relation over D containing only the pairs (Fc, Gc). □

Exercise 1.4.iv. In the notation of Example 1.4.7, prove that distinct parallel morphisms $f, g : c \rightrightarrows d$ define distinct natural transformations

$$\begin{aligned} f_*, g_* &: C(-, c) \Rightarrow C(d, -) \\ f^*, g^* &: C(c, -) \Rightarrow C(d, -) \end{aligned}$$

Proof. These being natural transformations is shown in Example 1.4.7, so the primary concern of this problem is whether they are distinct. First, we consider f_*, g_* as natural transformations from $C(-, c) \Rightarrow C(-, d)$. To differentiate, consider the natural transformation defined by f_* to be α and g_* to be β. We need to show the transformations are different in at least some component. For a natural transformation, we can choose arbitrary $h : c_1 \to c_2$ in C to look at the functions for. In this case, take $h = 1_c$. We thus in our diagram have h^* as our Fh and Gh, which is precomposition by 1_c. This must be the case as a functor preserves identities. We then consider what this transformation does to the morphism $j = 1_c \in C(c, c)$ to see what would happen to it if we put it trough the transformation. Both directions take us to $fj1_c = fj = f$. Similarly, constructing the same diagram except with g, we get $gj1_c = gj = g$. Each takes us to a different morphism in $C(c, d)$ and thus the two transformations are different. Essentially the same construction works with f^* and g^*.

$$\begin{tikzcd} C(c,c) \arrow[r, "1_c^*"] \arrow[d, "f_*"] & C(c,c) \arrow[d, "f_*"'] \\ C(c,d) \arrow[r, "1_c^*"] & C(c,d) \end{tikzcd}$$

□

Exercise 1.3.vi. (For Reference.) Given functors $F\colon \mathsf{D} \to \mathsf{C}$ and $G\colon \mathsf{E} \to \mathsf{C}$, show that there is a category called the **comma category** $F \downarrow G$, which has

- as objects, triples $(d \in \mathsf{D}, e \in \mathsf{E}, f\colon Fd \to Ge \in \mathsf{C})$, and
- as morphisms $(d, e, f) \to (d', e', f')$, a pair of morphisms $(h\colon d \to d', k\colon e \to e')$ so

that the following square commutes in C, i.e., so that $f' \cdot Fh = Gk \cdot f$.

$$\begin{tikzcd} Fd \arrow[r, "f"] \arrow[d, "Fh"'] & Ge \arrow[d, "Gk"] \\ Fd' \arrow[r, "f'"'] & Ge' \end{tikzcd}$$

Define a pair of projection functors $\mathrm{dom}\colon F \downarrow G \to D$ and $\mathrm{cod}\colon F \downarrow G \to E$.

Exercise 1.4.v. Recall the construction of the comma category for any pair of functors $F\colon \mathsf{D} \to \mathsf{C}$ and $G\colon \mathsf{E} \to \mathsf{C}$ described in Exercise 1.3.vi. From this data, construct a canonical natural transformation $\alpha\colon F\,\mathrm{dom} \Rightarrow G\,\mathrm{cod}$ between the functors that form the boundary of the square

$$\begin{tikzcd} F \downarrow G \arrow[r, "\mathrm{cod}"] \arrow[d, "\mathrm{dom}"'] & E \arrow[d, "G"] \\ D \arrow[r, "F"'] \arrow[ur, Rightarrow, "\alpha"] & C \end{tikzcd}$$

Proof. Letting $c = (d, e, f\colon d \to e \in C)$ as above, and a morphism $m = (h\colon d \to d', k\colon e \to e')$, we can describe the actions of the functors $F\,\mathrm{dom}$ and $G\,\mathrm{cod}$ on objects:

- $F\,\mathrm{dom}\,c = Fd$
- $G\,\mathrm{cod}\,c = Ge$

And their actions on morphisms:

- $F\,\mathrm{dom}\,m = Fh$
- $G\,\mathrm{cod}\,m = Gk$

From the definition of a natural transformation, we need an $\alpha\colon \mathrm{ob}\,F \downarrow G \to \mathrm{mor}\,\mathsf{C}$, and we can get this by taking $(d, e, f) \mapsto f \in \mathsf{C}$. Then, $\alpha_c\colon Fd \to Ge$ in C. From this, we can construct the following diagram:

$$\begin{tikzcd} Fd \arrow[r, "\alpha_c"] \arrow[d, "Fh"'] & Ge \arrow[d, "Gk"] \\ Fd' \arrow[r, "\alpha_{c'}"'] & Ge' \end{tikzcd}$$

Which is precisely the diagram of the comma category with $f = \alpha_c$ and $f' = \alpha_{c'}$, and from the condition that $f' \cdot Fh = Gk \cdot f$, we have that this diagram commutes. Thus, α is a natural transformation. $\square$

Exercise 1.4.vi. Given a pair of functors $F\colon \mathsf{A} \times \mathsf{B} \times \mathsf{B}^{\mathrm{op}} \to \mathsf{D}$ and $G : \mathsf{A} \times \mathsf{C} \times \mathsf{C}^{\mathrm{op}} \to \mathsf{D}$ a family of morphisms

$$\alpha_{a,b,c}\colon F(a, b, b) \to G(a, c, c)$$

in D defines the components of an **extranatural transformation** $\alpha : F \implies G$ if for any $f\colon a \to a'$, $h\colon c \to c'$ the following diagrams commute in D:

$$
\begin{tikzcd}
F(a,b,b) \arrow[r, "\alpha_{a,b,c}"] \arrow[d, "F(f,1_b,1_b)"'] & G(a,c,c) \arrow[d, "G(f,1_c,1_c)"] \\
F(a',b,b) \arrow[r, "\alpha_{a',b,c}"'] & G(a',c,c)
\end{tikzcd}
\quad
\begin{tikzcd}
F(a,b,b') \arrow[r, "F(1_a,1_b,g)"] \arrow[d, "F(1_a,g,1_{b'})"'] & F(a,b,b) \arrow[d, "\alpha_{a,b,c}"] \\
F(a,b',b') \arrow[r, "\alpha_{a,b',c}"'] & G(a,c,c)
\end{tikzcd}
\quad
\begin{tikzcd}
F(a,b,b) \arrow[r, "\alpha_{a,b,c'}"] \arrow[d, "G(1_a,1_{c'},1_h)"'] & G(a,c',c') \arrow[d, "G(f,1_c,1_c)"] \\
G(a,c,c) \arrow[r, "\alpha_{a',b,c}"'] & G(a,c',c)
\end{tikzcd}
$$

The left-hand square asserts the at the components $\alpha_{-,b,b}\colon F(-,b,b) \implies G(-,c,c)$ define a natural transformation in a for each $b \in \mathsf{B}$ and $c \in \mathsf{C}$. The remaining squares assert that the components $\alpha_{\alpha_{a,-,-}}\colon F(a,-,-) \implies G(a,c,c)$ and $\alpha_{a,b,-}\colon F(a,b,b) \implies G(a,-,-)$ define transformations that are respectively extranatural in b and in c. Explain why functors F and G must have a common target category for this this definition to make sense.

Notice that the definition of natural transformation does not actually have anything to do with the question. This exercise is nothing more than a sanity check. If F and G do not have the same target category, and if we try and write down any of the three above diagrams, we will see that they are simply not defined if F and G do not have the same target category.