Category Theory in Context
Answer Key

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Contents

Chapter 1

Categories, Functors, Natural Transformations

1.1 Abstract and concrete categories

Exercise 1.1.i.

(i) Show that a morphism can have at most one inverse isomorphism.

Proof. Let $f : x \to y$ be an arbitrary morphism, and let $g : y \to x$ and $g : y \to x$ be two inverse isomorphisms of f. That is to say, $gf = hf = 1_x$ and $fg = fh = 1_y$. Consider the composition gfh. This is a valid composition, since the domain of g is equal to the codomain of f, and the domain of f is equal to the codomain of h. Composition is associative, so $(gf)h = g(fh)$.
Evaluate each of these expressions independently:

- Evaluated as $(gf)h$, we find that $gf = 1_x$, so $(gf)h = 1_x h = h$.
- Evaluated as $g(fh)$, we find that $fh = 1_y$, so $g(fh) = g1_y = g$.

Since both expressions are equal, we can conclude that $h = g$. So any two inverse isomorphisms of f must be equal. Since f was arbitrary, we can generalize to conclude that any morphism can have at most one (distinct) inverse isomorphism. □

(ii) Consider a morphism $f : x \to y$. Show that if there exist a pair of isomorphisms $g, h : y \to x$ so that $gf = 1_x$ and $fh = 1_y$, then $g = h$ and f is an isomorphism.

Proof. Let $f : x \to y$ be an arbitrary morphism, and let $g, h : y \to x$ be morphisms such that $gf = 1_X$ and $fh = 1_y$. Similarly to above, evaluate the composition gfh as $(gf)h = 1_x h = h$ and as $g(fh) = g1_y = g$. Due to associativity, $(gf)h = g(fh)$, so we can conclude that $g = h$. $fh = 1_y$ was given, and using our previous conclusion, we can substitute g for h to obtain $fg = 1_y$. Since we were also given $gf = 1_x$, we can conclude that f is an isomorphism. □

Definition 1.1.11. A **groupoid** is a category in which every morphism is an isomorphism.

Exercise 1.1.ii. Let C be a category. Show that the collection of isomorphisms in C defines a subcategory, the **maximal groupoid** inside C.

Proof. We want to show that, for a category C, restricting the class $\operatorname{mor}\mathsf{C}$ to a class of morphisms that are only the isomorphisms, call it $\operatorname{mor}\mathsf{C}_{iso}$, while preserving all of the objects of C, gives us a subcategory C_{iso} of C.

We start by showing that the identity morphisms in C_{iso} are isomorphisms. For an identity morphism $1_x \colon x \to x$, since the composition of $1_x 1_x = 1_x = 1_x 1_x$ shows that 1_x is both a right and left inverse of itself, then 1_x is an isomorphism. Thus, the identity morphisms of C_{iso} are indeed isomorphisms.

We now want to show that compositions of isomorphisms in C_{iso} yield isomorphisms. Take two morphisms $f \colon x \to y$ and $g \colon u \to x$ in C_{iso}. Since f is an isomorphism, then there is a morphism $h \in \operatorname{mor}\mathsf{C}_{iso}$ with $h \colon y \to x$, such that $fh = 1_y$ and $hf = 1_x$. Likewise, since g is an isomorphism, then there is a morphism $j \in \operatorname{mor}\mathsf{C}_{iso}$ with $j \colon x \to u$, such that $gj = 1_x$ and $jg = 1_u$. We can take the composition fg, since $\operatorname{dom}(f) = \operatorname{cod}(g)$. We also have the composition jh, since $\operatorname{dom}(j) = \operatorname{cod}(h)$. And again, respecting domains and codomains, we have the composition $(fg)(jh)$, since $\operatorname{dom}(fg) = \operatorname{cod}(jh)$. From the associativity of the parent category C, then $(fg)(jh) = f(gj)h = f(1_x)h = fh = 1_y$. Thus jh is the right inverse of the composition fg. Similarly, since $\operatorname{cod}(fg) = \operatorname{dom}(jh)$, we have the composition $(jh)(fg)$, which again from the associativity of the category C, $(jh)(fg) = j(hf)g = j1_x g = jg = 1_u$. So, jh is the left inverse of fg, and fg is an isomorphism.

We have shown that C_{iso} is a category, having all of the objects of C, restricted to the isomorphisms of C. So the groupoid C_{iso} is a subcategory of C. Presented with any other subcategory D, of C, that is strictly larger than C_{iso}, there must be a morphism in D that is not in C_{iso}. Then this morphism must not be an isomorphism, and hence, D cannot be a groupoid. So, the category C_{iso} is a maximal groupoid that is a subcategory of C. ◻

Exercise 1.1.iii. For any category C and any object c in C show that:

1. There is a category c/C whose objects are morphisms $f \colon c \to x$ with domain c and in which a morphism from $f \colon c \to x$ to $g \colon c \to y$ is a map $h \colon x \to y$ between the codomains so that the triangle

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] & \\ x \arrow[rr, "h"'] & & y \end{tikzcd}$$

   commutes, i.e., so that $g = hf$.
2. There is a category C/c whose objects are morphisms $f \colon x \to c$ with codomain c and in which a morphism from $f \colon x \to c$ to $g \colon y \to c$ between the domains so that the triangle

$$\begin{tikzcd} x \arrow[rr, "h"] \arrow[dr, "f"'] & & y \arrow[dl, "g"] \\ & c & \end{tikzcd}$$

commutes, i.e., so that $f = gh$

The categories c/C and C/c are called **slice categories** of C **under** and **over** c, respectively.

Proof. First we must determine the form of the objects and morphisms in c/C. The objects of c/C are diagrams of the following form.

$$c \xrightarrow{f} x$$

The morphisms in c/C are diagrams of the following form.

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] & \\ x \arrow[rr, "h"'] & & y \end{tikzcd}$$

Though this is notation is by no means standard, to help distinguish between morphisms in C and morphisms in the slice categories, we will define h' as a short hand for the diagram with the morphism h as the bottom arrow (or top in C/c). Notice that both the objects are commutative diagrams in C. We could also think of the objects as functors from the category $\mathbb{2}$ and the morphisms as functors from the category $\mathbb{3}$. By the way we have defined morphisms the only reasonable choices for the domain and codomain of h' is f and g respectively.

We can see how to compose two compatible morphisms $i' \colon e \to f$ and $h' \colon f \to g$ in c/C by looking at the following diagram in C.

$$\begin{tikzcd} & c \arrow[dl, "e"'] \arrow[d, "f"'] \arrow[dr, "g"] & \\ z \arrow[r, "i"'] & x \arrow[r, "h"'] & y \end{tikzcd}$$

Since $(hi)e = h(ie) = hf = g$ in C the diagram commutes, and the composition hi can be thought of as a member of c/C denoted as $(hi)'$ with domain and codomain e and g. Using the diagram notation $(hi)'$ is denoted as follows.

$$\begin{tikzcd} & c \arrow[dl, "e"'] \arrow[dr, "g"] & \\ x \arrow[rr, "hi"'] & & y \end{tikzcd}$$

Because we have defined composition in c/C in terms of composition in C c/C inherits the associativity of C That is for composable morphisms h', i', and j' in C we have

$$(h'i')j' = (hi)j = h(ij) = h'(i'j').$$

We need to obtain the identity morphism of each object f in c/C. To do so notice that the follow diagram commutes, because C is a category.

$$\begin{tikzcd} c \arrow[d, "f"'] \arrow[dr, "f"] & \\ x \arrow[r, "1_x"'] & x \end{tikzcd}$$

Looking at the same diagram from a different perspective we see that 1_x actually acts as the identity morphism for f in c/C Since we were careful when defining the morphisms in c/C this identity is well defined. If we had defined the morphisms in c/C to be anything less than a commutative diagrams, it would seem as 1_x could serve as the identity for multiple objects in c/C. This issue is not restricted to identity morphisms, but it is most obvious in the case of identity morphisms. However since we defined morphisms appropriately we can use the notation defined earlier to write $1_x = 1_f : f \to f$ without any ambiguity. This notion can be used to obtain the left and right identities by considering the following commutative diagram in C

$$\begin{tikzcd} & & c \arrow[dll, "f"'] \arrow[dl, "f"] \arrow[dr, "g"'] \arrow[drr, "g"] & & \\ x \arrow[r, "1_x"'] & x \arrow[rr, "h"'] & & y \arrow[r, "1_y"'] & y \end{tikzcd}$$

Translating the above diagram into the slice category notation we have that $h'1_f = h' = 1_g h'$. We have shown that c/C satisfies all the axioms of a category.

We can use the same method to show that C/c is also a category. The only difference is direction of each arrow. Hence this proof will be relatively terse. The objects in c/C are the following diagram.

$$x \xrightarrow{f} c$$

A morphisms h' in C/c has the form of the following diagram.

$$\begin{tikzcd} x \arrow[rr, "h"] \arrow[dr, "f"'] & & y \arrow[dl, "g"] \\ & c & \end{tikzcd}$$

The domain of h' is f and the codomain of h is g. We define composition on C/c by taking compatible in C/c morphisms $i' : e \to f$ and $h' : f \to g$ and observing the following diagram in C.

$$\begin{tikzcd} z \arrow[r, "i"] \arrow[dr, "e"'] & x \arrow[r, "h"] \arrow[d, "f"] & y \arrow[dl, "g"] \\ & c & \end{tikzcd}$$

Again this diagram commutes since $g(hi) = (gh)i = fi = e$ in C. We see that (ih) is member of C/c with domain e and codomain g and is denoted as follows.

$$\begin{tikzcd} z \arrow[rr, "ih"] \arrow[dr, "e"'] & & y \arrow[dl, "g"] \\ & c & \end{tikzcd}$$

Just like we did in the previous case, we have defined composition in terms of the composition in C. Hence the associativity is inherited. That is given composable morphisms i', h', and j' we have

$$(j'h')i' = (jh)i = j(hi) = j'(h'i').$$

We can obtain the identity element for each object f in C/c in the exact same way as before.

$$\begin{tikzcd} c \arrow[d, "f"'] \arrow[dr, "f"] & \\ x \arrow[r, "1_x"'] & x \end{tikzcd}$$

Since the above diagram commutes we can write $1'_x = 1_f : f \to f$ To get an identity for an arbitrary element observe that the diagram below commutes and gives us that $1_f h' = h' = h' 1_g$.

$$\begin{tikzcd} x \arrow[r, "1_x"] & x \arrow[rr, "h"] & & y \arrow[r, "1_y"] & y \\ & & c \arrow[ull, "f"] \arrow[ul, "f"'] \arrow[ur, "g"] \arrow[urr, "g"'] & & \end{tikzcd}$$

Therefore both c/C and C/c are categories in their own right. $\square$

If you looked over to the next page and read the definition of opposite categories, you should notice that $((c/(\mathsf{C})^{\mathrm{op}}))^{\mathrm{op}} = (\mathsf{C}/c)$ If we knew about opposite categories beforehand we could have just proved that the c/C is a category and then cited this result and been done (since the opposite category is category, it's in the name after all), without all the extra tedium of swapping arrows.

1.2 Duality

Exercise 1.2.i. Show that $\mathsf{C}/c \cong (c/(\mathsf{C}^{\mathrm{op}}))^{\mathrm{op}}$. Defining C/c to be $(c/(\mathsf{C}^{\mathrm{op}}))^{\mathrm{op}}$, deduce Exercise 1.1.iii(ii) from Exercise 1.1.iii(i).

Proof. This exercise asks us to prove that two categories are isomorphic, which is a notion that we have not yet encountered. But, I will prove that the two categories are equal!

This exercise uses definitions from Exercise 1.1.iii. There are so many layers in the present exercise that to keep things straight, it will help to add one more piece of notation.

Recall that for an object c of a category C, the slice category C/c of C over c has as objects the morphisms $f\colon x \to c$ in C. A morphism from f to g in C/c, where f has domain x and g domain y in C, is a morphism $h\colon x \to y$ in C such that $gh = f$. To distinguish between h as viewed in C and in C/c, let's write $h'\colon f \to g$ when we want to consider h as a morphism in C/c and $h\colon x \to y$ when we want to consider it in C[1]. We can use similar notation for the slice category c/C of C under c.

Since we will make systematic and careful use of opposite categories, recall that the objects and morphisms of C and of C^{op} are **precisely the same**. Only, the assignment of domains and codomains are swapped, allowing order of composition to be swapped. If f is a morphism in C, then f^{op} is precisely the same morphism, but the op reminds us that we are considering it in C^{op} rather than in C so that we have different assignments for domain and codomain.

Now, I claim that $\mathsf{C}/c = (c/(\mathsf{C}^{\mathrm{op}}))^{\mathrm{op}}$. We must first check that they have the same objects, though we notate f in the first category as f^{op} in the second. Then, for every pair of objects f and g in C/c we must see that

$$(\mathsf{C}/c)(f, g) = (c/(\mathsf{C}^{\mathrm{op}}))^{\mathrm{op}}(f^{\mathrm{op}}, g^{\mathrm{op}}).$$

(Note that we use this notation even when C is not locally small, so that each side of the equality might be a proper class.) Finally, we must see that composition of morphisms is the same in each category.

Since the objects of a category and its opposite category are the same, the objects in $(c/(\mathsf{C}^{\mathrm{op}}))^{\mathrm{op}}$ are the objects in $c/(\mathsf{C}^{\mathrm{op}})$, which are morphisms $f^{\mathrm{op}}\colon c \to x$ in C^{op}. But, these are the same as morphisms $f\colon x \to c$ in C, which is to say objects of C/c as claimed.

For the rest, let $f\colon x \to c$, $g\colon y \to c$ and $h\colon z \to c$ be three morphisms in C. A morphism

$$i^{\mathrm{op}\prime\mathrm{op}}\colon f^{\mathrm{op}} \to g^{\mathrm{op}}$$

in $(c/(\mathsf{C}^{\mathrm{op}}))^{\mathrm{op}}$ is just a morphism

$$i^{\mathrm{op}\prime}\colon g^{\mathrm{op}} \to f^{\mathrm{op}}$$

in $c/(\mathsf{C}^{\mathrm{op}})$. This in turn is a morphism $i^{\mathrm{op}}\colon y \to x$ such that $i^{\mathrm{op}}g^{\mathrm{op}} = f^{\mathrm{op}}$ in C^{op}, together with the ordered pair $(g^{\mathrm{op}}, f^{\mathrm{op}})$ giving the domain and codomain of $i^{\mathrm{op}\prime}$. Unravelling one more layer, this is in turn a morphism $i\colon x \to y$ such that $gi = f$ in C together with the ordered pair (f, g). This in turn corresponds to a morphism $i'\colon f \to g$ in C/c. Each of these correspondences is actually an equality of classes. So, we have argued that

$$\begin{aligned}
&(c/(\mathsf{C}^{\mathrm{op}}))^{\mathrm{op}}(f^{\mathrm{op}}, g^{\mathrm{op}})\\
&= (c/(\mathsf{C}^{\mathrm{op}}))(g^{\mathrm{op}}, f^{\mathrm{op}})\\
&= \{i^{\mathrm{op}} \in \mathsf{C}^{\mathrm{op}}(y, x) | i^{\mathrm{op}}g^{\mathrm{op}} = f^{\mathrm{op}}\} \times \{(g^{\mathrm{op}}, f^{\mathrm{op}})\}\\
&= \{i \in \mathsf{C}(x, y) | gi = f\} \times \{(f, g)\}\\
&= (\mathsf{C}/c)(f, g)
\end{aligned}$$

[1] The notation h' can still be ambiguous since there might also be other $i\colon c \to x$ and $j\colon c \to y$ such that $jh = i$ so that we also have $h'\colon i \to j$. This leads to different morphisms labeled h', but they are distinguished by their domains and codomains

as required.

Notice also in this correspondence that when $f = g$ that the identities in each class are the same. Altogether, $1^{\mathrm{op}}_{f^{\mathrm{op}}} = 1_f$.

Now, we must see that the composition laws are the same. We have already established above that $i^{\mathrm{op}\prime\mathrm{op}} = i'$. Similarly, if $j' \colon g \to h$, then we have $j^{\mathrm{op}\prime\mathrm{op}} i^{\mathrm{op}\prime\mathrm{op}} \colon f^{\mathrm{op}} \to h^{\mathrm{op}}$. But, examining the definitions of opposite categories and of slice categories, we have equality of each of the following morphisms interpreted in the categories shown

$$\begin{array}{rll} j^{\mathrm{op}\prime\mathrm{op}} i^{\mathrm{op}\prime\mathrm{op}} \colon f^{\mathrm{op}} \to h^{\mathrm{op}} & \text{in} & (c/(\mathsf{C}^{\mathrm{op}}))^{\mathrm{op}} \\ i^{\mathrm{op}\prime} j^{\mathrm{op}\prime} \colon h^{\mathrm{op}} \to f^{\mathrm{op}} & \text{in} & c/(\mathsf{C}^{\mathrm{op}}) \\ (i^{\mathrm{op}} j^{\mathrm{op}})' \colon h^{\mathrm{op}} \to f^{\mathrm{op}} & \text{in} & c/(\mathsf{C}^{\mathrm{op}}) \\ i^{\mathrm{op}} j^{\mathrm{op}} \colon z \to x & \text{in} & \mathsf{C}^{\mathrm{op}} \\ ji \colon x \to z & \text{in} & \mathsf{C} \\ (ji)' \colon f \to h & \text{in} & \mathsf{C}/c \\ j'i' \colon f \to h & \text{in} & \mathsf{C}/c \end{array}$$

This proves that the two categories share the same composition law. Thus, they are one and the same.

Now, looking back at Exercise 1.1.iii, we see that we we could have defined C/c as $(c/(\mathsf{C}^{\mathrm{op}}))^{\mathrm{op}}$, so that the existence of the category C/c follows from the existence of $c/(\mathsf{C}^{\mathrm{op}})$, which we have by the first part of Exercise 1.1.iii applied to C^{op}. □

Exercise 1.2.ii.

(i) Show that a morphism $f \colon x \to y$ is a split epimorphism in a category C if and only if for all $c \in \mathsf{C}$, post-composition $f_* \colon \mathsf{C}(c, x) \to \mathsf{C}(c, y)$ defines a surjective function.

Proof. First, assume that f is a split epimorphism and that $c \in \mathrm{ob}\,\mathsf{C}$. That is, there exists a function $g \colon y \to x$ such that $fg = id_y$. Now, consider the function $f_* \colon \mathsf{C}(c, x) \to \mathsf{C}(c, y)$. We know that this function corresponds to composition on the left by f, so in order for this function to be surjective, for every $k \colon c \to y$, there must exists a $j \colon c \to x$ such that $fj = k$. Now, for an arbitrary k, consider $j = gk$. It is easy to see that $gk \colon c \to x$, and that $f_*(gk) = f(gk) = (fg)k = id_y k = k$. We can construct j in this way for every $k \in \mathsf{C}(c, y)$, so we see that f_* is surjective.

Now, assume that f_* is surjective, that is, for any choice of $c \in \mathsf{C}$, and any $k \in \mathsf{C}(c, y)$, $k = fg$, for some $g \in \mathsf{C}(c, x)$. Now, suppose $c = y$ and $k = id_y$, so we have that there exists a $g \in \mathsf{C}(y, x)$ where $fg = id_y$, and this implies that f is a spilt epimorphism. □

(ii) Argue by duality that f is a split monomoprhism if and only if for all $c \in \mathsf{C}$, pre-composition $f_* \colon \mathsf{C}(y, c) \to \mathsf{C}(x, c)$ is a surjective function.

Proof. We know that if f^{op} is a split epimorphism, that $f_*^{\mathrm{op}} \colon C^{\mathrm{op}}(c, y) \to C^{\mathrm{op}}(c, x)$ is surjective. However, if we consider the definitions of f^{op} and split monomorphisms and epimorphisms, we see that f^{op} being a split epimorphism implies that f is a split monomorphism. We also see that $f_*^{\mathrm{op}} \colon \mathsf{C}^{\mathrm{op}}(c, y) \to \mathsf{C}^{\mathrm{op}}(c, x)$ is equivalent to $f^* \colon \mathsf{C}(y, c) \to \mathsf{C}(x, c)$. So we have that f is a split monomorphism if and only if $f^* \colon \mathsf{C}(y, c) \to \mathsf{C}(x, c)$ is surjective. □

Lemma 1.2.1.

(i) If $f\colon x \rightarrowtail y$ and $g\colon y \rightarrowtail z$ are monomorphisms, then so is $gf\colon x \rightarrowtail z$.

(ii) If $f\colon x \to y$ and $g\colon y \to z$ are morphisms so that gf is monic, then f is monic.

Dually:

(i') If $f\colon x \twoheadrightarrow y$ and $g\colon y \twoheadrightarrow z$ are epimorphisms, then so is $gf\colon x \twoheadrightarrow z$.

(ii') If $f\colon x \to y$ and $g\colon y \to z$ are morphisms so that gf is epic, then g is epic.

Exercise 1.2.iii. Prove Lemma 1.2.11 by proving either (i) or (i') and either (ii) or (ii'), then arguing by duality. Conclude that the monomorphisms in any category define a subcategory of that category and dually that the epimorphisms also define a subcategory.

First we will show the above properties for monomorphisms, and then apply duality, as the problem suggests, to prove the corresponding properties for epimorphisms.

Proof. First, we will prove that the composition of two monomorphisms is a monomorphism. Let C be a category and $f\colon x \to y$ and $g\colon y \to z$ be monomorphisms of C. Let $h, k\colon w \to x$ be two morphisms in C so that: $gf)h = (gf)k$. Since compositions of morphisms are associative, we have $g(fh) = g(fk)$. Since g is monic, we get: $(fh) = (fk)$. Since f is monic, we ultimately get: $h = k$. Thus, gf is monic. Thus, the compositions of two monomorphisms is indeed a monomorphism.

Next we will show that If the composition of two morphisms is monic, then the rightmost morphism is monic. Take morphisms $a\colon x \to y$ and $b\colon y \to z$ from category C where ba is monic. Take $h, k\colon w \to x$ so that $ah = ak$. Left composing b on both sides of the equations results in: $b(ah) = b(ak)$. By associativity we get: $(ba)h = (ba)k$. Applying the properties of monomorphisms results in: $h = k$. Thus a is monic. So we have shown that if the composition of two morphisms is monic, then the rightmost morphism is monic.

Now we will show that the monomorphisms of any category forms a category. Suppose that D is a subcategory of C where the morphisms of D are the monomorphisms of C and D and C have the same objects. Since for any object x in C, if we had for morphisms h and k of C with codomain x the following property: $id_x h = id_x k$, then $h = k$, since id_x is left cancellable, thus the identity morphism for every object in D is a monomorphism. Therefore, every object in D has a identity arrow in D. Since the composition of two monomorphisms is a monomorphism, then D contains compositions of its morphisms. Obviously, the domains and codomains of morphisms of D are contained in D since D and C have the same objects. Thus D is a subcategory of C.

We have shown that:

1. the composition of two monomorphisms in C is a monomorphism.
2. If the composition of two morphisms in C is monic, then the rightmost morphism is monic.
3. The class of monomorphisms of any category C forms a subcategory of C.

Now we will use duality to show the corresponding properties for epimorphisms. If we have the opposite category C^{op}, where the epimorphisms of C are the monomorphisms of C^{op}, this means that the three properties proven for monomorphisms also work for C^{op}. The properties of monomorphisms in C^{op} are the dual properties of epimorphisms in $(\mathsf{C}^{\mathrm{op}})^{\mathrm{op}} = \mathsf{C}$. Namely,

1. the composition of two epimorphisms in C is an epimorphism.

2. If the composition of two morphisms in C is epic, then the leftmost morphism is epic (Since $f^{\mathrm{op}}g^{\mathrm{op}}$ in C^{op} corresponds to gf in C).
3. The class of epimorphisms of any category C forms a subcategory of C.

This completes the proof. $\square$

Definition 1.2.2. A morphism $f: x \to y$ in a category is

(i) a **monomorphism** if for any parallel morphisms $h, k: w \rightrightarrows x$, $fh = fk$ implies that $h = k$; or

(ii) an **epimorphism** if for any parallel morphisms $h, k: y \rightrightarrows z$, $hf = kf$ implies that $h = k$.

Exercise 1.2.iv. What are the monomorphisms in the category of fields?

Proof. In the category of fields, morphisms are field homomorphisms. Let $f: A \to B$ be a morphism in Field. As f is a field homomorphism, its kernel is an ideal in B. Since B is a field, there are only two ideals: $\{0\}$ and B itself. The kernel of f cannot be the whole field, since this would be the zero morphism which is not a field homomorphism. So $\ker f = \{0\}$ and from this, f is injective, and in particular it is left cancellable.

Let h and k be morphisms in Field for which composition with f makes sense and say that

$$fh = fk$$

Since f is left cancellable, this implies that

$$h = k$$

And f is a monomorphism by Definition 1.2.7(i) above. (In fact, injections are monomorphisms in any category in which the objects have "underlying sets".)

Thus, all of the morphisms in Field are monomorphisms. $\square$

Definition 1.2.7. A morphism $f: x \to y$ in a category is

(i) a **monomorphism** if for any parallel morphisms $h, k: w \rightrightarrows x$, $fh = fk$ implies that $h = k$; or

(ii) an **epimorphism** if for any parallel morphisms $h, k: y \rightrightarrows z$, $hf = kf$ implies that $h = k$.

Exercise 1.2.v. Show that the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is both a monomorphism and an epimorphism in the category Ring of rings. Conclude that a map that is both monic and epic need not be an isomorphism.

Proof. Note first that monic and epic correspond to a map being cancellable on the left and right, whereas an isomorphism is by definition invertible. It is easy to see that invertibility implies cancellability; however the converse need not be true. Looking at the monoid of natural numbers under addition, every element is cancellable; however none except zero is invertible. Because every monoid is a category this gives us an elementary example where a map that is monic and epic is not an isomorphism. However, this example might seem simplistic and it is worth asking whether there is an example of cancellability not implying invertibility in a "larger" category where the arrows represent actual maps.

Recall that $\mathbb{Q}$ is the localisation of $\mathbb{Z}$ with respect to its cancellable elements $\mathbb{Z} \setminus \{0\}$. The immediate result of this is the existence of a natural embedding $\iota\colon \mathbb{Z} \to \mathbb{Q}$ that is an injective ring homomorphism. Further, this embedding has the following universal property: given a ring R and a homomorphism $\phi\colon \mathbb{Z} \to R$ such that $\phi(q)$ has an inverse for all $q \in \mathbb{Z}$, there is a unique ring homomorphism $\psi\colon \mathbb{Q} \to R$ such that the following diagram commutes.

$$\begin{tikzcd} \mathbb{Z} \arrow[rr, "\phi"] \arrow[dr, "\iota"'] & & R \\ & \mathbb{Q} \arrow[ur, "\psi"'] & \end{tikzcd}$$

Note further that there is a unique ring homomorphism from $\mathbb{Z}$ to any ring R which maps $\mathbb{Z}$ onto the subring generated by the multiplicative identity of R. This implies that there can be at most one homomorphism from $\mathbb{Q}$ to any ring R. If $\psi\colon \mathbb{Q} \to R$ is a ring homomorphism, then $\psi\iota\colon \mathbb{Z} \to R$ must be the unique homomorphism from $\mathbb{Z}$ to R and thus ψ is the unique homomorphism specified by the universal property.

Supposing $h, k\colon \mathbb{Q} \rightrightarrows R$ are parallel homomorphisms, they are equal by virtue of the fact that there is at most one homomorphism from $\mathbb{Q}$ to R, and ι vacuously fulfils the condition of an epimorphism. □

1.3 Functoriality

Exercise 1.3.i. What is a functor between groups, regarded as one-object categories?

Proof. Recall that a group as a category has a single object x, and that each element of the group is a morphism in the category. All domains and codomains are that object x. There is one identity morphism 1_x, which is the identity element in the group. Composition is the same as multiplication in this context.

A functor between groups C and D with respective objects x_1 and x_2 must trivially be such that $F(x_1) = x_2$. Our primary concern is the behavior of the functor on the morphisms. We require that for a functor $F(1_x) = 1_{F(x)}$ for all objects $x \in obC$, which in this case just implies that 1_{x_1} is taken to 1_{x_2}. Additionally, we require $F(\mathrm{dom}(f)) = \mathrm{dom}(F(f))$ and $F(\mathrm{cod}(f)) = \mathrm{cod}(F(f))$ for all morphisms f in the first category. This is a trivial requirement, as $F(\mathrm{dom}(f)) = \mathrm{dom}(F(f)) = F(\mathrm{cod}(f)) = \mathrm{cod}(F(f)) = x_2$ regardless of f. Finally we require that if f and g are a composable pair of morphisms in C, then $F(fg) = F(f)F(g)$. However, all morphisms in D are composable, and this implies that $F(f * g) = F(f) * F(g)$ in the notation of groups with operation $*$. This property and the preservation of identities are directly the definition of a group homomorphism, so this functor is simply a group homomorphism. □

Exercise 1.3.iii. Find an example to show that the objects and morphisms in the image of a functor $F\colon \mathsf{C} \to \mathsf{D}$ do not necessarily define a subcategory of D.

At first, I was suspicious of this exercise since it seemed to me that the proof that the image of a group (or monoid, or ring, . . .) homomorphism is a subgroup (or submonoid, or subring, . . .) carries through without any change. Perhaps the author meant for the exercise to be something different?

So, I asked her, and she pointed out a straightforward example that also exposed the error in my reasoning. I will give below a simplification of the example that she sent me. First, here is the error in my original argument.

Let a and b be morphisms in the image of F such that $\operatorname{dom} a = \operatorname{cod} b$ so that we can form ab in D. Since a and b are in the image of F, there are morphisms f and g in bC such that $a = Ff$ and $b = Fg$. Then

$$ab = FfFg = F(fg)$$

so that ab is in the image of F.

Right? Wrong!! In order to compose f and g we need that $\operatorname{dom} f = \operatorname{cod} g$. All we know for sure is that $\operatorname{dom} Ff = \operatorname{cod} Fg$. If F is injective on objects, then the argument above is valid. But perhaps F is not injective.

The Example.

Now, I provide the example requested in this exercise. Let $\mathsf{C} = \mathbb{2}$ be the ordinal category pictured as so:

$$0 \xrightarrow{f} 1.$$

Let g be an endomorphism of some object x in some category D such that gg is equal to neither 1_x nor g. For example, we could take x to be the unique object in $B\mathbb{N}$ and $g = 1$, so that $gg = g + g = 2 \neq 1, 0$.

Then we have a functor $F : \mathbb{2} \to \mathsf{D}$ given by $F0 = F1 = x$, $F1_0 = F1_1 = 1_x$ and $Ff = g$. There are only four possible compositions of the three morphisms in $\mathbb{2}$, $1_0 1_0$, $f 1_0$, $1_1 f$ and $1_1 1_1$ and it is easy to see that F preserves all four of these compositions. Thus, F is a functor.

However, the image of F has only the two morphisms 1_x and g. Since gg is not in the image, the image of F is not a subcategory of D.