Category Theory in Context
Answer Key

Anthony Agwu Jay Chernak Bruce James
Matthew Kousoulas Mark Laczin Amarit Matharu
Emily McGovern Keith Pardue Teddy Weinberg

Autumn, 2018

Contents

Chapter 1

Categories, Functors, Natural Transformations

1.1 Abstract and concrete categories

Exercise 1.1.i.

(i) Show that a morphism can have at most one inverse isomorphism.

Proof. Let $f\colon x \to y$ be an arbitrary morphism, and let $g\colon y \to x$ and $h\colon y \to x$ be two inverse isomorphisms of f. That is to say, $gf = hf = 1_x$ and $fg = fh = 1_y$. Consider the composition gfh. This is a valid composition, since the domain of g is equal to the codomain of f, and the domain of f is equal to the codomain of h. Composition is associative, so $(gf)h = g(fh)$.
Evaluate each of these expressions independently:

- Evaluated as $(gf)h$, we find that $gf = 1_x$, so $(gf)h = 1_x h = h$.
- Evaluated as $g(fh)$, we find that $fh = 1_y$, so $g(fh) = g1_y = g$.

Since both expressions are equal, we can conclude that $h = g$. So any two inverse isomorphisms of f must be equal. Since f was arbitrary, we can generalize to conclude that any morphism can have at most one (distinct) inverse isomorphism. □

(ii) Consider a morphism $f\colon x \to y$. Show that if there exist a pair of isomorphisms $g, h\colon y \to x$ so that $gf = 1_x$ and $fh = 1_y$, then $g = h$ and f is an isomorphism.

Proof. Let $f\colon x \to y$ be an arbitrary morphism, and let $g, h\colon y \to x$ be morphisms such that $gf = 1_x$ and $fh = 1_y$. Similarly to above, evaluate the composition gfh as $(gf)h = 1_x h = h$ and as $g(fh) = g1_y = g$. Due to associativity, we have $(gf)h = g(fh)$. So we can conclude that $g = h$. Since $fh = 1_y$ was given, using our previous conclusion, we can substitute g for h to obtain $fg = 1_y$. Since we were also given $gf = 1_x$, we can conclude that f is an isomorphism. □

Definition 1.1.11. A **groupoid** is a category in which every morphism is an isomorphism.

Exercise 1.1.ii. Let C be a category. Show that the collection of isomorphisms in C defines a subcategory, the **maximal groupoid** inside C.

Proof. Let C_{iso} denote the maximal groupoid of C. We wish to show that C_{iso} is a category. C_{iso} inherits composition, associativity, and objects from C. Notice that the identity morphism for each object is in C is clearly an isomorphism as it is both right and left invertible, so the identity morphisms for each object are in C_{iso}. Because composition in C_{iso} is the same as in C, each object will have the same identity morphism as in C. To show C_{iso} is closed under composition, take two morphisms $f\colon x \to y$ and $g\colon u \to x$ in C_{iso}. Since f is an isomorphism, then there is a morphism $h \in \operatorname{mor}\mathsf{C}_{\text{iso}}$ with $h\colon y \to x$, such that $fh = 1_y$ and $hf = 1_x$. Likewise, since g is an isomorphism, then there is a morphism $j \in \operatorname{mor}\mathsf{C}_{\text{iso}}$ with $j\colon x \to u$, such that $gj = 1_x$ and $jg = 1_u$. We can take the composition fg, since $\operatorname{dom}(f) = \operatorname{cod}(g)$. We also have the composition jh, since $\operatorname{dom}(j) = \operatorname{cod}(h)$. And again, respecting domains and codomains, we have the composition $(fg)(jh)$, since $\operatorname{dom}(fg) = \operatorname{cod}(jh)$. From the associativity of the parent category C, then $(fg)(jh) = f(gj)h = f(1_x)h = fh = 1_y$. Thus jh is the right inverse of the composition fg. Similarly, since $\operatorname{cod}(fg) = \operatorname{dom}(jh)$, we have the composition $(jh)(fg)$, which again from the associativity of the category C, $(jh)(fg) = j(hf)g = j1_xg = jg = 1_u$. So, jh is the left inverse of fg, and fg is an isomorphism.

We have shown that C_{iso} is a category, with all of the objects of C and morphisms of C restricted to the isomorphisms of C. So the groupoid C_{iso} is a subcategory of C. Presented with any other subcategory D, of C, that is strictly larger than C_{iso}, there must be a morphism in D that is not in C_{iso}. Then this morphism must not be an isomorphism, and hence, D cannot be a groupoid. So, the category C_{iso} is a maximal groupoid that is a subcategory of C. □

Exercise 1.1.iii. For any category C and any object c in C show that:

1. There is a category c/C whose objects are morphisms $f\colon c \to x$ with domain c and in which a morphism from $f\colon c \to x$ to $g\colon c \to y$ is a map $h\colon x \to y$ between the codomains so that the triangle

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] & \\ x \arrow[rr, "h"'] & & y \end{tikzcd}$$

commutes, i.e., so that $g = hf$.

2. There is a category C/c whose objects are morphisms $f\colon x \to c$ with codomain c and in which a morphism from $f\colon x \to c$ to $g\colon y \to c$ between the domains so that the

triangle

$$\begin{tikzcd} x \arrow[rr, "h"] \arrow[dr, "f"'] & & y \arrow[dl, "g"] \\ & c & \end{tikzcd}$$

commutes, i.e., so that $f = gh$

The categories c/C and C/c are called **slice categories** of C **under** and **over** c, respectively.

Proof. First we must determine the form of the objects and morphisms in c/C. The objects of c/C are diagrams of the following form.

$$c \xrightarrow{f} x$$

The morphisms in c/C are diagrams of the following form.

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] & \\ x \arrow[rr, "h"'] & & y \end{tikzcd}$$

Though this is notation is by no means standard, to help distinguish between morphisms in C and morphisms in the slice categories, we will define h' as a short hand for the diagram with the morphism h as the bottom arrow (or top in C/c). Notice that both the objects are commutative diagrams in C. We could also think of the objects as functors[1] from the category $\mathbb{2}$ and the morphisms as functors from the category $\mathbb{3}$. By the way we have defined morphisms the only reasonable choices for the domain and codomain of h' are f and g respectively.

We can see how to compose two compatible morphisms $i' : e \to f$ and $h' : f \to g$ in c/C by looking at the following diagram in C.

$$\begin{tikzcd} & c \arrow[dl, "e"'] \arrow[d, "f"'] \arrow[dr, "g"] & \\ z \arrow[r, "i"'] & x \arrow[r, "h"'] & y \end{tikzcd}$$

Since $(hi)e = h(ie) = hf = g$ in C the diagram commutes, and the composition hi can be thought of as a member of c/C denoted as $(hi)'$ with domain and codomain e and g, respectively. Using the diagram notation $(hi)'$ is denoted as follows.

$$\begin{tikzcd} & c \arrow[dl, "e"'] \arrow[dr, "g"] & \\ x \arrow[rr, "hi"'] & & y \end{tikzcd}$$

[1]Functors are defined in section 1.3. Right now the diagrams here are just helpful tools to keep track of equations. Diagrams are made formal in section 1.6.

Because we have defined composition in c/C in terms of composition in C, c/C inherits the associativity of C That is for composable morphisms h', i', and j' in C we have

$$(h'i')j' = (hi)j = h(ij) = h'(i'j').$$

We need to obtain the identity morphism of each object f in c/C. To do so notice that the follow diagram commutes, because C is a category.

$$\begin{tikzcd} c \arrow[d, "f"'] \arrow[dr, "f"] & \\ x \arrow[r, "1_x"'] & x \end{tikzcd}$$

Looking at the same diagram from a different perspective we see that 1_x actually acts as the identity morphism for f in c/C. Since we were careful when defining the morphisms in c/C, this identity is well defined. If we had defined the morphisms in c/C to be anything less than a commutative diagrams, it would seem as $1'_x$ could serve as the identity for multiple objects in c/C. This issue is not restricted to identity morphisms, but this ambiguity is most obvious in the case of identity morphisms. However, since we defined morphisms appropriately, we can use the notation defined earlier to write $1'_x = 1_f : f \to f$ without any ambiguity. This notion can be used to obtain the left and right identities by considering the following commutative diagram in C

$$\begin{tikzcd} & & c \arrow[dll, "f"'] \arrow[dl, "f"] \arrow[dr, "g"'] \arrow[drr, "g"] & & \\ x \arrow[r, "1_x"'] & x \arrow[rr, "h"'] & & y \arrow[r, "1_y"'] & y \end{tikzcd}$$

Translating the above diagram into the slice category notation we have that $h'1_f = h' = 1_g h'$. We have shown that c/C satisfies all the axioms of a category.

We can use the same procedure to show that C/c is also a category. The only difference is direction of each arrow. Hence this proof will be relatively terse. The objects in c/C are the following diagram.

$$x \xrightarrow{f} c$$

A morphisms h' in C/c has the form of the following diagram.

$$\begin{tikzcd} x \arrow[rr, "h"] \arrow[dr, "f"'] & & y \arrow[dl, "g"] \\ & c & \end{tikzcd}$$

The domain of h' is f and the codomain of h is g. We define composition on C/c by taking compatible in C/c morphisms $i' : e \to f$ and $h' : f \to g$ and observing the following diagram in C.

$$
\begin{tikzcd}
z \arrow[r, "i"] \arrow[dr, "e"'] & x \arrow[r, "h"] \arrow[d, "f"] & y \arrow[dl, "g"] \\
& c &
\end{tikzcd}
$$

Again this diagram commutes since $g(hi) = (gh)i = fi = e$ in C. We see that (ih) is member of C/c with domain e and codomain g and is denoted as follows.

$$
\begin{tikzcd}
z \arrow[rr, "ih"] \arrow[dr, "e"'] & & y \arrow[dl, "g"] \\
& c &
\end{tikzcd}
$$

Just like we did in the previous case, we have defined composition in terms of the composition in C. Hence the associativity is inherited. That is given composable morphisms i', h', and j' we have

$$(j'h')i' = (jh)i = j(hi) = j'(h'i').$$

We can obtain the identity element for each object f in C/c in the exact same way as before.

$$
\begin{tikzcd}
c \arrow[d, "f"'] \arrow[dr, "f"] & \\
x \arrow[r, "1_x"'] & x
\end{tikzcd}
$$

Since the above diagram commutes we can write $1'_x = 1_f : f \to f$ To get an identity for an arbitrary element observe that the diagram below commutes and gives us that $1_f h' = h' = h' 1_g$.

$$
\begin{tikzcd}
x \arrow[r, "1_x"] & x \arrow[rr, "h"] & & y \arrow[r, "1_y"] & y \\
& & c \arrow[ull, "f"] \arrow[ul, "f"'] \arrow[ur, "g"] \arrow[urr, "g"'] & &
\end{tikzcd}
$$

Therefore both c/C and C/c are categories in their own right. $\square$

If you looked over to the next page and read the definition of opposite categories, you should notice that $((c/(\mathsf{C})^{\mathrm{op}}))^{\mathrm{op}} = (\mathsf{C}/c)$. If we knew about opposite categories beforehand we could have just proved that the c/C is a category and then cited this result and been done (since the opposite category is category, it's in the name after all), without all the extra tedium of swapping arrows.

1.2 Duality

Exercise 1.2.i. Show that $\mathsf{C}/c \cong (c/(\mathsf{C}^{\mathrm{op}}))^{\mathrm{op}}$. Defining C/c to be $(c/(\mathsf{C}^{\mathrm{op}}))^{\mathrm{op}}$, deduce Exercise 1.1.iii(ii) from Exercise 1.1.iii(i).

Proof. This exercise asks us to prove that two categories are isomorphic, which is a notion that we have not yet encountered. But, I will prove that the two categories are equal!

This exercise uses definitions from Exercise 1.1.iii. There are so many layers in the present exercise that to keep things straight, it will help to add one more piece of notation. Recall that for an object c of a category C, the slice category C/c of C over c has as objects the morphisms $f\colon x \to c$ in C. A morphism from f to g in C/c, where f has domain x and g domain y in C, is a morphism $h\colon x \to y$ in C such that $gh = f$. To distinguish between h as viewed in C and in C/c, let's write $h'\colon f \to g$ when we want to consider h as a morphism in C/c and $h\colon x \to y$ when we want to consider it in C[2]. We can use similar notation for the slice category c/C of C under c.

Since we will make systematic and careful use of opposite categories, recall that the objects and morphisms of C and of C^{op} are *precisely the same*. Only, the assignment of domains and codomains are swapped, allowing order of composition to be swapped. If f is a morphism in C, then f^{op} is precisely the same morphism, but the op reminds us that we are considering it in C^{op} rather than in C so that we have different assignments for domain and codomain.

Now, I claim that $\mathsf{C}/c = (c/(\mathsf{C}^{\mathrm{op}}))^{\mathrm{op}}$. We must first check that they have the same objects, though we notate f in the first category as f^{op} in the second. Then, for every pair of objects f and g in C/c we must see that

$$(\mathsf{C}/c)(f, g) = (c/(\mathsf{C}^{\mathrm{op}}))^{\mathrm{op}}(f^{\mathrm{op}}, g^{\mathrm{op}}).$$

(Note that we use this notation even when C is not locally small, so that each side of the equality might be a proper class.) Finally, we must see that composition of morphisms is the same in each category.

Since the objects of a category and its opposite category are the same, the objects in $(c/(\mathsf{C}^{\mathrm{op}}))^{\mathrm{op}}$ are the objects in $c/(\mathsf{C}^{\mathrm{op}})$, which are morphisms $f^{\mathrm{op}}\colon c \to x$ in C^{op}. But, these are the same as morphisms $f\colon x \to c$ in C, which is to say objects of C/c as claimed.

For the rest, let $f\colon x \to c$, $g\colon y \to c$ and $h\colon z \to c$ be three morphisms in C. A morphism

$$i^{\mathrm{op}\prime\mathrm{op}}\colon f^{\mathrm{op}} \to g^{\mathrm{op}}$$

in $(c/(\mathsf{C}^{\mathrm{op}}))^{\mathrm{op}}$ is just a morphism

$$i^{\mathrm{op}\prime}\colon g^{\mathrm{op}} \to f^{\mathrm{op}}$$

in $c/(\mathsf{C}^{\mathrm{op}})$. This in turn is a morphism $i^{\mathrm{op}}\colon y \to x$ such that $i^{\mathrm{op}} g^{\mathrm{op}} = f^{\mathrm{op}}$ in C^{op}, together with the ordered pair $(g^{\mathrm{op}}, f^{\mathrm{op}})$ giving the domain and codomain of $i^{\mathrm{op}\prime}$. Unravelling one

[2] The notation h' can still be ambiguous since there might also be other $i\colon c \to x$ and $j\colon c \to y$ such that $jh = i$ so that we also have $h'\colon i \to j$. This leads to different morphisms labeled h', but they are distinguished by their domains and codomains

more layer, this is in turn a morphism $i\colon x \to y$ such that $gi = f$ in C together with the ordered pair (f, g). This in turn corresponds to a morphism $i'\colon f \to g$ in C/c. Each of these correspondences is actually an equality of classes. So, we have argued that

$$\begin{aligned}
&(c/(\mathsf{C}^{\mathrm{op}}))^{\mathrm{op}}(f^{\mathrm{op}}, g^{\mathrm{op}}) \\
&= (c/(\mathsf{C}^{\mathrm{op}}))(g^{\mathrm{op}}, f^{\mathrm{op}}) \\
&= \{i^{\mathrm{op}} \in \mathsf{C}^{\mathrm{op}}(y, x) \mid i^{\mathrm{op}} g^{\mathrm{op}} = f^{\mathrm{op}}\} \times \{(g^{\mathrm{op}}, f^{\mathrm{op}})\} \\
&= \{i \in \mathsf{C}(x, y) \mid gi = f\} \times \{(f, g)\} \\
&= (\mathsf{C}/c)(f, g)
\end{aligned}$$

as required.

Notice also in this correspondence that when $f = g$ that the identities in each class are the same. Altogether, $1_{f^{\mathrm{op}}}^{\mathrm{op}} = 1_f$.

Now, we must see that the composition laws are the same. We have already established above that $i^{\mathrm{op}\prime\mathrm{op}} = i'$. Similarly, if $j'\colon g \to h$, then we have $j^{\mathrm{op}\prime\mathrm{op}} i^{\mathrm{op}\prime\mathrm{op}}\colon f^{\mathrm{op}} \to h^{\mathrm{op}}$. But, examining the definitions of opposite categories and of slice categories, we have equality of each of the following morphisms interpreted in the categories shown

$$\begin{array}{rll}
j^{\mathrm{op}\prime\mathrm{op}} i^{\mathrm{op}\prime\mathrm{op}}\colon f^{\mathrm{op}} \to h^{\mathrm{op}} & \text{in} & (c/(\mathsf{C}^{\mathrm{op}}))^{\mathrm{op}} \\
i^{\mathrm{op}\prime} j^{\mathrm{op}\prime}\colon h^{\mathrm{op}} \to f^{\mathrm{op}} & \text{in} & c/(\mathsf{C}^{\mathrm{op}}) \\
(i^{\mathrm{op}} j^{\mathrm{op}})'\colon h^{\mathrm{op}} \to f^{\mathrm{op}} & \text{in} & c/(\mathsf{C}^{\mathrm{op}}) \\
i^{\mathrm{op}} j^{\mathrm{op}}\colon z \to x & \text{in} & \mathsf{C}^{\mathrm{op}} \\
ji\colon x \to z & \text{in} & \mathsf{C} \\
(ji)'\colon f \to h & \text{in} & \mathsf{C}/c \\
j'i'\colon f \to h & \text{in} & \mathsf{C}/c
\end{array}$$

This proves that the two categories share the same composition law. Thus, they are one and the same.

Now, looking back at Exercise 1.1.iii, we see that we we could have defined C/c as $(c/(\mathsf{C}^{\mathrm{op}}))^{\mathrm{op}}$, so that the existence of the category C/c follows from the existence of $c/(\mathsf{C}^{\mathrm{op}})$, which we have by the first part of Exercise 1.1.iii applied to C^{op}. □

Exercise 1.2.ii.

(i) Show that a morphism $f\colon x \to y$ is a split epimorphism in a category C if and only if for all $c \in \mathsf{C}$, post-composition $f_*\colon \mathsf{C}(c, x) \to \mathsf{C}(c, y)$ defines a surjective function.

Proof. First, assume that f is a split epimorphism and that $c \in \operatorname{ob}\mathsf{C}$. That is, there exists a function $g\colon y \to x$ such that $fg = 1_y$. Now, consider the function $f_*\colon \mathsf{C}(c, x) \to \mathsf{C}(c, y)$. We know that this function corresponds to composition on the left by f, so in order for this function to be surjective, for every $k\colon c \to y$, there must exists a $j\colon c \to x$ such that $fj = k$. Now, for an arbitrary k, consider $j = gk$. It is

easy to see that $gk\colon c \to x$, and that $f_*(gk) = f(gk) = (fg)k = 1_y k = k$. Since we can construct j in this way for every $k \in \mathsf{C}(c, y)$, we see that f_* is surjective.
Now, assume that f_* is surjective, that is, for any choice of $c \in \mathsf{C}$, and any $k \in \mathsf{C}(c, y)$, $k = fg$, for some $g \in \mathsf{C}(c, x)$. Now, suppose $c = y$ and $k = 1_y$, so we have that there exists a $g \in \mathsf{C}(y, x)$ where $fg = 1_y$, and this implies that f is a spilt epimorphism. □

(ii) Argue by duality that f is a split monomoprhism if and only if for all $c \in \mathsf{C}$, pre-composition $f^*\colon \mathsf{C}(y, c) \to \mathsf{C}(x, c)$ is a surjective function.

Proof. We know that if f^{op} is a split epimorphism, that $f^{\mathrm{op}}_*\colon C^{\mathrm{op}}(c, y) \to C^{\mathrm{op}}(c, x)$ is surjective. However, if we consider the definitions of f^{op} and split monomorphisms and epimorphisms, we see that f^{op} being a split epimorphism implies that f is a split monomorphism. We also see that $f^{\mathrm{op}}_*\colon \mathsf{C}^{\mathrm{op}}(c, y) \to \mathsf{C}^{\mathrm{op}}(c, x)$ is equivalent to $f^*\colon \mathsf{C}(y, c) \to \mathsf{C}(x, c)$. So we have that f is a split monomorphism if and only if $f^*\colon \mathsf{C}(y, c) \to \mathsf{C}(x, c)$ is surjective. □

Lemma 1.2.11.
(i) If $f\colon x \rightarrowtail y$ and $g\colon y \rightarrowtail z$ are monomorphisms, then so is $gf\colon x \rightarrowtail z$.
(ii) If $f\colon x \to y$ and $g\colon y \to z$ are morphisms so that gf is monic, then f is monic.
Dually:
(i') If $f\colon x \twoheadrightarrow y$ and $g\colon y \twoheadrightarrow z$ are epimorphisms, then so is $gf\colon x \twoheadrightarrow z$.
(ii') If $f\colon x \to y$ and $g\colon y \to z$ are morphisms so that gf is epic, then g is epic.

Exercise 1.2.iii. Prove Lemma 1.2.11 by proving either (i) or (i') and either (ii) or (ii'), then arguing by duality. Conclude that the monomorphisms in any category define a subcategory of that category and dually that the epimorphisms also define a subcategory.

First we will show the above properties for monomorphisms, and then apply duality, as the problem suggests, to prove the corresponding properties for epimorphisms.

Proof. First, we will prove that the composition of two monomorphisms is a monomorphism. Let C be a category and $f\colon x \to y$ and $g\colon y \to z$ be monomorphisms of C. Let $h, k\colon w \to x$ be two morphisms in C so that: $(gf)h = (gf)k$. Since composition of morphisms is associative, we have $g(fh) = g(fk)$. Since g is monic, we get: $fh = fk$. Since f is monic, we ultimately get: $h = k$. Thus, gf is monic. Thus, the compositions of two monomorphisms is indeed a monomorphism.

Next we will show that if the composition of two morphisms is monic, then the rightmost morphism is monic. Take morphisms $a\colon x \to y$ and $b\colon y \to z$ from category C where ba is monic. Take $h, k\colon w \to x$ so that $ah = ak$. Left composing b on both sides of the equations results in: $b(ah) = b(ak)$. By associativity we get: $(ba)h = (ba)k$. Applying the properties of monomorphisms results in: $h = k$. Thus a is monic. So we have shown that if the composition of two morphisms is monic, then the rightmost morphism is monic.

Now we will show that the monomorphisms of any category forms a category. Suppose that D is a subcategory of C where the morphisms of D are the monomorphisms of C and

D and C have the same objects. Since for any object x in C, if we had for morphisms h and k of C with codomain x the following property: $1_x h = 1_x k$, then $h = k$, since 1_x is left cancellable, thus the identity morphism for every object in D is a monomorphism. Therefore, every object in D has an identity arrow in D. Since the composition of two monomorphisms is a monomorphism, then D contains compositions of its morphisms. Obviously, the domains and codomains of morphisms of D are contained in D since D and C have the same objects. Thus D is a subcategory of C.

We have shown that:

- The composition of two monomorphisms in C is a monomorphism,
- if the composition of two morphisms in C is monic, then the rightmost morphism is monic, and
- the class of monomorphisms of any category C forms a subcategory of C.

Now we will use duality to show the corresponding properties for epimorphisms. If we have the opposite category C^{op}, where the epimorphisms of C are the monomorphisms of C^{op}, this means that the three properties proven for monomorphisms also work for C^{op}. The properties of monomorphisms in C^{op} are the dual properties of epimorphisms in $(\mathsf{C}^{\mathrm{op}})^{\mathrm{op}} = \mathsf{C}$. We will show that:

- the composition of two epimorphisms in C is an epimorphism,
- if the composition of two morphisms in C is epic, then the leftmost morphism is epic (since $f^{\mathrm{op}}g^{\mathrm{op}}$ in C^{op} corresponds to gf in C), and
- the class of epimorphisms of any category C forms a subcategory of C.

This completes the proof. □

Definition 1.2.7. A morphism $f\colon x \to y$ in a category is

(i) a *monomorphism* if for any parallel morphisms $h, k\colon w \rightrightarrows x$, $fh = fk$ implies that $h = k$; or

(ii) an *epimorphism* if for any parallel morphisms $h, k\colon y \rightrightarrows z$, $hf = kf$ implies that $h = k$.

Exercise 1.2.iv. What are the monomorphisms in the category of fields?

Proof. In the category of fields, morphisms are field homomorphisms. Let $f\colon A \to B$ be a morphism in Field. As f is a field homomorphism, its kernel is an ideal in B. Since B is a field, there are only two ideals: $\{0\}$ and B itself. The kernel of f cannot be the whole field, since this would be the zero morphism which is not a field homomorphism. So $\ker f = \{0\}$ and from this, f is injective, and in particular it is left cancellable.

Let h and k be morphisms in Field for which composition with f makes sense and say that

$$fh = fk.$$

Since f is left cancellable, this implies that

$$h = k,$$

and f is a monomorphism by Definition 1.2.7(i) above. (In fact, injections are monomorphisms in any category in which the objects have "underlying sets".)

Thus, all of the morphisms in Field are monomorphisms. □

Exercise 1.2.v. Show that the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is both a monomorphism and an epimorphism in the category Ring of rings. Conclude that a map that is both monic and epic need not be an isomorphism.

Proof. Note first that monic and epic correspond to a map being cancellable on the left and right, whereas an isomorphism is by definition invertible. It is easy to see that invertibility implies cancellability; however the converse need not be true. Looking at the monoid of natural numbers under addition, every element is cancellable; however none except zero is invertible. Because every monoid is a category this gives us an elementary example where a map that is monic and epic is not an isomorphism. However, this example might seem simplistic and it is worth asking whether there is an example of cancellability not implying invertibility in a "larger" category where the arrows represent actual maps.

Recall that $\mathbb{Q}$ is the localisation of $\mathbb{Z}$ with respect to its cancellable elements $\mathbb{Z} \setminus \{0\}$. The immediate result of this is the existence of a natural embedding $\iota\colon \mathbb{Z} \to \mathbb{Q}$ that is an injective ring homomorphism. Further, this embedding has the following universal property: given a ring R and a homomorphism $\phi\colon \mathbb{Z} \to R$ such that $\phi(q)$ has an inverse for all $q \in \mathbb{Z}$, there is a unique ring homomorphism $\psi\colon \mathbb{Q} \to R$ such that the following diagram commutes.

$$\begin{tikzcd}
\mathbb{Z} \arrow[rr, "\phi"] \arrow[dr, "\iota"'] & & R \\
& \mathbb{Q} \arrow[ur, "\psi"'] &
\end{tikzcd}$$

Note further that there is a unique ring homomorphism from $\mathbb{Z}$ to any ring R which maps $\mathbb{Z}$ onto the subring generated by the multiplicative identity of R. This implies that there can be at most one homomorphism from $\mathbb{Q}$ to any ring R. If $\psi\colon \mathbb{Q} \to R$ is a ring homomorphism, then $\psi\iota\colon \mathbb{Z} \to R$ must be the unique homomorphism from $\mathbb{Z}$ to R and thus ψ is the unique homomorphism specified by the universal property.

Supposing $h, k\colon \mathbb{Q} \rightrightarrows R$ are parallel homomorphisms, they are equal by virtue of the fact that there is at most one homomorphism from $\mathbb{Q}$ to R, and ι vacuously fulfils the condition of an epimorphism. □

Exercise 1.2.vi. Prove that a morphism that is both a monomorphism and a split epimorphism is necessarily an isomorphism. Argue by duality that a split monomorphism that is an epimorphism is also an isomorphism.

$$\begin{tikzcd} x \arrow[loop left, "gf"] \arrow[r, bend left, "f"] & y \arrow[l, bend left, "g"] \arrow[loop right, "1_y"] \end{tikzcd}$$

Proof. Let C be a category with objects x and y and a morphism $f\colon x \to y$. If f is a split epimorphism, then there exists another morphism $g\colon y \to x$ such that $fg = 1_y$. If f is also a monomorphism, then for any object w and any parallel pair of morphisms $h, k\colon w \rightrightarrows x$, $fh = fk$ implies that $h = k$. Combining these facts with some basic algebra:

$1_y f = f 1_x$	definition of identies,
$fgf = f 1_x$	g is a right inverse of f,
$gf = 1_x$	f is left cancellable,

gives that g is also a left inverse of f.

Suppose instead that f is an epimorphism and a split monomorphism with left inverse g in the category C. Then it is also a monomorphism and a split epimorphism in C^{op}, thus f is an isomorphism in C^{op} and an isomorphism in C. □

Exercise 1.2.vii. Regarding a poset $(\mathsf{P}, \leq)$ as a category, define the supremum of a subcollection of objects $A \in \mathsf{P}$ in such a way that the dual statement defines the infimum. Prove that the supremum of a subset of objects is unique, whenever it exists, in such a way that the dual proof demonstrates the uniqueness of the infimum.

Proof. Given a subcollection C of objects $A \in \mathsf{P}$, define an upper bound as follows: a object u is an upper bound of C if for all objects x in C there is a morphism $x \leq u\colon x \to u$. (Recall that morphisms in a poset category are merely elements of the $\leq$ relation.) Note that this immediately gives us a dual notion of a lower bound by considering instead P^{op}. A lower bound of C in P is an upper bound of C in P^{op}. In other words an object l^{op} such that for all objects x^{op} in C there is a morphism $(x \leq l)^{\text{op}}\colon x^{\text{op}} \to l^{\text{op}}$, or equivalently $(l \leq x)\colon l \to x$.

Letting F be the collection of all upper bounds of C, we define the supremum of C, if it exists[3], to be a lower bound of F (as defined above) which is contained in F. The condition

[3]There are many cases where suprema fail to exist. Consider the poset category:

$$\begin{tikzcd} a & b \\ c \arrow[u] \arrow[ur] & d \arrow[u] \arrow[ul] \end{tikzcd}$$

the set $\{c, d\}$ has as upper bounds $\{a, b\}$. However, $\{a, b\}$ has as lower bounds $\{c, d\}$. Because these sets are disjoint there is no supremum of $\{c, d\}$. Even in more common orderings, like the usual ordering on the rational numbers, subcollections can fail to have suprema. For example, $\{x \in \mathbb{Q} | x^2 < 2\}$.

A poset with the property that any collection of elements has a supremum and infimum is called a complete lattice.

of containment implies uniqueness. Supposing we have two lower bounds x and y of F. If both are contained in F, then there are maps $x \leq y \colon x \to y$ and $y \leq x \colon y \to x$. Since the only endomorphisms in P are identities these must compose to identities and thus be inverses, and because P is a partially ordered set (as opposed to just being a preordered set) the only isomorphisms are identities. In familiar terms, a partial order is antisymmetric. Thus x and y are the same object.

We may thus define the infimum of C to be its supremum on P^{op}. This time we consider the collection I of lower bounds C (the upper bounds of C in P^{op}). The infimum is then an upper bound of I which is contained in I (a lower bound in P^{op}). The infimum must be unique because it's a supremum in the opposite category, and suprema are unique. $\square$

1.3 Functoriality

Exercise 1.3.i. What is a functor between groups, regarded as one-object categories?

Proof. Recall that a group as a category has a single object x, and that each element of the group is a morphism in the category. All domains and codomains are that object x. There is one identity morphism 1_x, which is the identity element in the group. Composition is the same as multiplication in this context.

A functor between groups C and D with respective objects x_1 and x_2 must trivially be such that $Fx_1 = x_2$. Our primary concern is the behavior of the functor on the morphisms. We require that for a functor $F1_x = 1_{Fx}$ for all objects $x \in \mathrm{ob}\,\mathsf{C}$, which in this case just implies that 1_{x_1} is taken to 1_{x_2}. Additionally, we require $F(\mathrm{dom}(f)) = \mathrm{dom}(Ff)$ and $F(\mathrm{cod}(f)) = \mathrm{cod}(Ff)$ for all morphisms f in the first category. This is a trivial requirement, as $F(\mathrm{dom}(f)) = \mathrm{dom}(Ff) = F(\mathrm{cod}(f)) = \mathrm{cod}(Ff) = x_2$ regardless of f. Finally we require that if f and g are a composable pair of morphisms in C, then $F(fg) = FfFg$. However, all morphisms in D are composable, and this implies that $F(f * g) = Ff * Fg$ in the notation of groups with operation $*$. This property and the preservation of identities are directly the definition of a group homomorphism, so this functor is simply a group homomorphism. □

Exercise 1.3.ii. What is a functor between preorders, regarded as categories?

Proof. Recall that a preorder regarded as a category has objects that are the elements of the underlying set of the preorder, and has morphisms that are the related pairs. Identities are the unique morphisms (x, x), which exist based on the reflexivity of the relation. Note that if (a, b) and (b, c) are in the relation, the composition will be $(b, c)(a, b) = (a, c)$.

What do the properties of a functor between preorders C (with relation R) and D (with relation S) tell us? First, we know that $F1_x = 1_{Fx}$ for all $x \in \mathrm{ob}\,\mathsf{C}$. This implies that the morphism (x, x) must be brought to the morphism (Fx, Fx). This becomes redundant with the next step.

We also know that $F(\mathrm{dom}(f)) = \mathrm{dom}(Ff)$ for all $f \in \mathrm{mor}\,\mathsf{C}$. If $f = (a, b)$, then $F(\mathrm{dom}(f)) = F(a)$ and thus Ff must be a pair $(F(a), z_1)$ for some $z_1 \in \mathrm{ob}\,\mathsf{D}$. Similarly, $F(\mathrm{cod}(f)) = \mathrm{cod}(Ff)$ implies that if $f = (a, b)$, then $F(\mathrm{cod}(f)) = Fb$ and Ff must be a pair (z_2, Fb). Combining these, we get that Ff for $f = (a, b)$ must be a pair (Fa, Fb). This means that if $(a, b) \in R$ then $(Fa, Fb) \in S$.

Thus, F provides us a preorder homomorphism, as F preserves related pairs. The final property to check for a functor is composable pairs. If two morphisms f and g are composable, then $F(fg) = FfFg$. This means

$$F((b, c)(a, b)) = F(a, c) = (Fb, Fc)(Fa, Fb) = (Fa, Fc),$$

which was already confirmed by the previous property. Thus, the functor is a preorder homomoprhism. □

Exercise 1.3.iii. Find an example to show that the objects and morphisms in the image of a functor $F \colon \mathsf{C} \to \mathsf{D}$ do not necessarily define a subcategory of D.

At first, I was suspicious of this exercise since it seemed to me that the proof that the image of a group (or monoid, or ring, . . .) homomorphism is a subgroup (or submonoid, or subring, . . .) carries through without any change. Perhaps the author meant for the exercise to be something different?

So, I asked her, and she pointed out a straightforward example that also exposed the error in my reasoning. I will give below a simplification of the example that she sent me. First, here is the error in my original argument.

Let a and b be morphisms in the image of F such that $\operatorname{dom} a = \operatorname{cod} b$ so that we can form ab in D. Since a and b are in the image of F, there are morphisms f and g in C such that $a = Ff$ and $b = Fg$. Then

$$ab = FfFg = F(fg)$$

so that ab is in the image of F.

Right? Wrong!! In order to compose f and g we need that $\operatorname{dom} f = \operatorname{cod} g$. All we know for sure is that $\operatorname{dom} Ff = \operatorname{cod} Fg$. If F is injective on objects, then the argument above is valid. But perhaps F is not injective.

The Example.

Now, I provide the example requested in this exercise. Let $\mathsf{C} = \mathbb{2}$ be the ordinal category pictured as so:

$$0 \xrightarrow{f} 1.$$

Let g be an endomorphism of some object x in some category D such that gg is equal to neither 1_x nor g. For example, we could take x to be the unique object in $B\mathbb{N}$ and $g = 3$. Composition of morphisms in $B\mathbb{N}$ is addition in $\mathbb{N}$ so that $gg = g + g = 6 \neq 3, 0$.

Then we have a functor $F : \mathbb{2} \to \mathsf{D}$ given by $F0 = F1 = x$, $F1_0 = F1_1 = 1_x$ and $Ff = g$. There are only four possible compositions of the three morphisms in $\mathbb{2}$, $1_0 1_0$, $f1_0$, $1_1 f$ and $1_1 1_1$ and it is easy to see that F preserves all four of these compositions. Thus, F is a functor.

However, the image of F has only the two morphisms 1_x and g. Since gg is not in the image, the image of F is not a subcategory of D.

Lemma 1.2.3. *The following are equivalent:*

(i) $f\colon x \to y$ *is an isomorphism in* C.

(ii) For all objects $c \in \mathsf{C}$, *post-composition with* f *defines a bijection*

$$f_*\colon \mathsf{C}(c,x) \to \mathsf{C}(c,y)$$

(iii) For all objects $c \in \mathsf{C}$, *pre-composition with* f *defines a bijection*

$$f^*\colon \mathsf{C}(y,c) \to \mathsf{C}(x,c)$$

Definition 1.3.1. A *functor* F from C to D is a functor $F\colon \mathsf{C} \to \mathsf{D}$. Explicitly, this consists of the following data:

- An object $Fc \in \mathsf{D}$, for each object $c \in \mathsf{C}$.
- A morphism $Ff\colon Fc \to Fc' \in \mathsf{D}$, for each morphism $f\colon c \to c' \in \mathsf{C}$, so that the domain and codomain of Ff are, respectively, equal to F applied to the domain or codomain of f.

The assignments are required to satisfy the following two *functoriality axioms*:

- For any composable pair f, g in C, $FgFf = F(gf)$.
- For each object c in C, $F(1_c) = 1_{Fc}$.

The functors defined in 1.3.1 are called *covariant* so as to distinguish them from another variety of functor that we now introduce.

Definition 1.3.5. A *contravariant functor* F from C to D is a functor $F\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{D}$. Explicitly, this consists of the following data:

- An object $Fc \in \mathsf{D}$, for each object $c \in \mathsf{C}$.
- A morphism $Ff\colon Fc' \to Fc \in \mathsf{D}$, for each morphism $f\colon c \to c' \in \mathsf{C}$, so that the domain and codomain of Ff are, respectively, equal to F applied to the codomain or domain of f.

The assignments are required to satisfy the following two *functoriality axioms*:

- For any composable pair f, g in C, $FfFg = F(gf)$.
- For each object c in C, $F(1_c) = 1_{Fc}$.

Definition 1.3.11. If C is locally small, then for any object $c \in \mathsf{C}$ we may define a pair of covariant and contravariant *functors represented by* c:

$$\begin{tikzcd}
\mathsf{C} \arrow[rr, "\mathsf{C}(c,-)"] & & \mathsf{Set} & \mathsf{C}^{\mathrm{op}} \arrow[rr, "\mathsf{C}(-,c)"] & & Set \\
x \arrow[d, "f"'] & \mapsto & \mathsf{C}(c,x) \arrow[d, "f_*"] & x \arrow[d, "f"'] & \mapsto & \mathsf{C}(x,c) \\
y & \mapsto & \mathsf{C}(c,y) & y & \mapsto & \mathsf{C}(y,c) \arrow[u, "f^*"']
\end{tikzcd}$$

The notation suggests the action on objects: the functor $\mathsf{C}(c, -)$ carries $x \in \mathsf{C}$ to the set $\mathsf{C}(c, x)$ of arrows from c to x in C. Dually, the functor $\mathsf{C}(-, c)$ carries $x \in \mathsf{C}$ to the set $\mathsf{C}(x, c)$.

The functor $\mathsf{C}(c, -)$ carries a morphism $f : x \to y$ to the post-composition function $f_* : \mathsf{C}(c, x) \to \mathsf{C}(c, y)$ introduced in Lemma 1.2.3(ii). Dually, the functor $\mathsf{C}(-, c)$ carries f to the pre-composition function $f^* : \mathsf{C}(y, c) \to \mathsf{C}(x, c)$ introduced in 1.2.3(iii).

Exercise 1.3.iv. Verify that the constructions in Definition 1.3.11 are functorial.

Proof. We start by showing that the assignments of $\mathsf{C}(c, -)$ satisfy the functoriality axioms for (covariant) functors. The actions of dom and cod on $\mathsf{C}(c, -)$ can be seen as follows: applying $\mathsf{C}(c, -)$ to a morphism $h : i \to j$ will give a morphism $\mathsf{C}(c, -)(h) : \mathsf{C}(c, \mathrm{dom}\, h) \to \mathsf{C}(c, \mathrm{cod}\, h)$, so $\mathrm{dom}\, \mathsf{C}(c, -)(h) = \mathsf{C}(c, i)$ and $\mathrm{cod}\, \mathsf{C}(c, -)(h) = \mathsf{C}(c, j)$.

To show composition, let $f : x \to y$ and $g : w \to x$ be a composable pair of morphisms in C. Note first that $fg : w \to y$ and that $\mathsf{C}(c, -)(f) : \mathsf{C}(c, x) \to \mathsf{C}(c, y)$, and finally that $\mathsf{C}(c, -)(g) : \mathsf{C}(c, w) \to \mathsf{C}(c, x)$.
Since $\mathrm{dom}\, \mathsf{C}(c, -)(f) = \mathrm{cod}\, \mathsf{C}(c, -)(g) = \mathsf{C}(c, x)$, we can compose as follows:

$$\mathsf{C}(c, -)(f)\,(\mathsf{C}(c, -)(g)) : \mathsf{C}(c, w) \to \mathsf{C}(c, y).$$

Since $\mathsf{C}(c, -)(fg)$ and $\mathsf{C}(c, -)(f)\,(\mathsf{C}(c, -)(g))$ are given by applying fg to a morphism in $\mathsf{C}(c, w)$, we have that $\mathsf{C}(c, -)(f)\mathsf{C}(c, -)(g) = \mathsf{C}(c, -)(fg)$, satisfying functor composition.

To show that identities are preserved, note that for any object $x \in \mathsf{C}$, $1_x : x \to x$. Then $\mathsf{C}(c, -)(1_x) : \mathsf{C}(c, x) \to \mathsf{C}(c, x)$ taking 1_x to the post composition $1_x^* : \mathsf{C}(c, x) \to \mathsf{C}(c, x)$. Since for any morphism $a \in \mathsf{C}(c, x)$, 1_x^* takes $a \mapsto 1_x a$, this is the identity $a \mapsto a$. Then consider $1_{\mathsf{C}(c,-)(x)} = 1_{\mathsf{C}(c,x)}$, which is the identity of $\mathsf{C}(c, x)$, taking each element a of the set to itself: $a \mapsto a$. Thus, $\mathsf{C}(c, -)(1_x) = 1_{\mathsf{C}(c,-)(x)}$ and $\mathsf{C}(c, -)$ preserves identities, as show by the following diagram.

$$\begin{tikzcd}
c \arrow[r, "a"] \arrow[dr, "1_x a"'] & x \arrow[d, "1_x"] \\
& y = x
\end{tikzcd}$$

To see that $\mathsf{C}(-, c)$ is a contravariant functor, we argue by duality. Since $\mathsf{C}(c, -) : \mathsf{C} \to \mathsf{Set}$ is a functor for any category C, we have that $\mathsf{C}^{\mathrm{op}}(c, -) : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ is also a functor. Additionally, it is contravariant since the definition a contravariant functor is a functor from C^{op} to Set. Given that $\mathsf{C}^{\mathrm{op}}(c, -) = \mathsf{C}(-, c)$, we know that $\mathsf{C}(-, c)$ is a contravariant functor, completing the proof. ◻

Exercise 1.3.v. What is the difference between a functor $\mathsf{C}^{\mathrm{op}} \to \mathsf{D}$ and a functor $\mathsf{C} \to \mathsf{D}^{\mathrm{op}}$? What is the difference between a functor $\mathsf{C} \to \mathsf{D}$ and a functor $\mathsf{C}^{\mathrm{op}} \to \mathsf{D}^{\mathrm{op}}$?

PROOF. We will show that if F is a functor from $\mathsf{C} \to \mathsf{D}$, then F is also a functor from C^{op} to D^{op}, and then deduce the relation ship between a functor $\mathsf{C}^{\mathrm{op}} \to \mathsf{D}$ and a functor $\mathsf{C} \to \mathsf{D}^{\mathrm{op}}$ as a special case.

Let F be a functor from $\mathsf{C} \to \mathsf{D}$. We will show that F is also a functor from C^{op} to D^{op} directly from the functoriality axioms. That is for objects x, y, and z in C and any two composable morphisms $f\colon x \to y$ and $g\colon y \to z$ in C, we must have $F(fg) = FfFg$ and $F1_x = 1_{Fx}$. Since for each object in C and D the identity maps are the same in their respective opposite categories, we only need to verify that F respects composition when take composable morphisms from C^{op} to D^{op}. Since both the objects C^{op} and D^{op} and morphisms C and D are exactly the same, but the morphisms have their domains and codomains swapped, we let $f\colon y \to x$ and $g\colon z \to y$ be morphisms in C^{op}. Now observe that since F respects composition in C we have, $Fgf = FgFf$, and F is a functor from C^{op} to D^{op}. Now since $\mathsf{C} = (\mathsf{C}^{\mathrm{op}})^{\mathrm{op}}$ and $\mathsf{D} = (\mathsf{D}^{\mathrm{op}})^{\mathrm{op}}$ we see that there is no difference from a functor $\mathsf{C}^{\mathrm{op}} \to \mathsf{D}$ and a functor $\mathsf{C} \to \mathsf{D}^{\mathrm{op}}$ as well. □

EXERCISE 1.3.vi. Given functors $F\colon \mathsf{D} \to \mathsf{C}$ and $G\colon \mathsf{E} \to \mathsf{C}$, show that there is a category, called the **comma category** $F \downarrow G$, which has

1. as objects, triples $(d \in \mathsf{D}, e \in \mathsf{E}, f\colon Fd \to Ge \in \mathsf{C})$, and
2. as morphisms $(d, e, f) \to (d', e', f')$, a pair of morphisms $(h\colon d \to d', k\colon e \to e')$ so that the square

$$\begin{tikzcd} Fd \arrow[r, "f"] \arrow[d, "Fh"'] & Ge \arrow[d, "Gk"] \\ Fd' \arrow[r, "f'"'] & Ge' \end{tikzcd}$$

commutes in C, i.e., so that $f'Fh = Gkf$.

Define a pair of projection functors $\mathrm{dom}\colon F \downarrow G \to \mathsf{D}$ and $\mathrm{cod}\colon F \downarrow G \to \mathsf{E}$

PROOF. Before we prove that the comma category $F \downarrow G$ is actually a category, we need to give a motivating example for a major issue in the proof.

Let $A\colon \mathbb{2} \to \mathsf{Set}$ and $B\colon \mathbb{2} \to \mathsf{Set}$ be functors where $A0 = \{0\}$, $A1 = \{0, 1, 2\}$, $B0 = \{0, 1\}$, $B1 = \{0, 1, 2, 3\}$ and where A and B maps the unique morphism $f\colon 0 \to 1$ to the inclusion functions $\iota\colon \{0\} \to \{0, 1, 2\}$ and $\iota\colon \{0, 1\} \to \{0, 1, 2, 3\}$ respectively. Let us take our supposed objects $(0, 0, \iota)$ and $(1, 1, \alpha)$ where ι is the inclusion function and our supposed morphism $(f\colon 0 \to 1, f\colon 0 \to 1)$ so that the diagram

$$\begin{tikzcd} A0 \arrow[r, "\iota"] \arrow[d, "Af"'] & B0 \arrow[d, "Bf"] \\ A1 \arrow[r, "\alpha"'] & B1 \end{tikzcd}$$

commutes. Now there are at least two functions for α that would allow the diagram above to commute. The first is if α was simply an inclusion function so the functions αAf and

$Bf\iota$ are inclusion functions from the singleton set $A0$ to $B1$, thus $\alpha Af = Bf\iota$. The second function which I will denote α' is defined as follows:

$$\alpha'(0) = 0, \alpha'(1) = 2, \alpha'(2) = 1.$$

Since α' still maps 0 in $A1$ to itself in $B1$, the diagram above still commutes. Thus our supposed morphism $(f : 0 \to 1, f : 0 \to 1)$ would have two codomains ($(1, 1, \alpha)$ and $(1, 1, \alpha')$) for our domain $(0, 0, \iota)$.

This example shows we need additional notation to distinguish between arrows that are represented the same but have different domains and codomains. Returning to the notation established in the first paragraph, we will append morphism pairs (f, f') to the end of some morphism $(h : d \to d', k : e \to e')$ so that we specify that the intended domain and codomain of the morphism is (d, e, f) and (d', e', f') respectively. Thus the uniqueness of the domain and codomain of some morphism $(h : d \to d', k : e \to e')(f, f')$ follows from the uniqueness of the domain and codomain of h and k, and additionally from our notation which specifies unique morphisms f and f'. Now we can complete the rest of the proof using the notation established in the first paragraph.

For an object (d, e, f),denoted as c, we can define an identity morphism for c as the following:

$$1_c = (1_d, 1_e)(f, f)$$

where 1_d and 1_e are the respective identities of d and e. Thus the diagram

$$\begin{tikzcd}
Fd \arrow[r, "f"] \arrow[d, "F1_d"'] & Ge \arrow[d, "G1_e"] \\
Fd \arrow[r, "f"'] & Ge
\end{tikzcd}$$

trivially commutes. The unique domain and codomain of 1_c, both being (d, e, f), are derived from the unique domain and codomains of the identities 1_d and 1_e, and the uniquely specified f.

Now let us define morphism composition between two morphisms.

$$(h : d \to d_1, k : e \to e_1)(f, f_1) \text{ and } (h : d_1 \to d_2, k : e_1 \to e_2)(f_1, f_2)$$

which we will denote

$$\alpha : (d, e, f) \to (d_1, e_1, f_1) \text{ and } \beta : (d_1, e_1, f_1) \to (d_2, e_2, f_2).$$

The composition of α and β shall be defined as follows:

$$\beta\alpha = (h'h : d \to d_2, k'k : e \to e_2)(f, f_2)$$

resulting in the diagram

$$\begin{tikzcd}
Fd \arrow[r, "f"] \arrow[d, "Fh"'] & Ge \arrow[d, "Gk"] \\
Fd_1 \arrow[r, "f_1"] \arrow[d, "Fh'"'] & Ge_1 \arrow[d, "Gk'"] \\
Fd_2 \arrow[r, "f_2"] & Ge_2
\end{tikzcd}$$

which commutes since functors preserve composition of morphisms and the top and bottom squares commute by construction of α and β[4]. Thus we have that the following diagram commutes:

$$\begin{tikzcd}
Fd \arrow[r, "f"] \arrow[d, "Fh'Fh"'] & Ge \arrow[d, "Gk'Gk"] \\
Fd_2 \arrow[r, "f_2"] & Ge_2
\end{tikzcd}$$

which is the commutative square for the composed morphism $\beta\alpha$. The morphism $\beta\alpha$ derives its unique domain from the unique domains of $h'h$ and $k'k$ and the specified function f which gives the domain (d, e, f) which is the domain of α, and the unique codomain is derived similarly resulting in the codomain (d_2, e_2, f_2) which is the codomain of β. So the composition of morphisms α and β gives a morphism $\beta\alpha$ with the domain of α and the codomain of β.

Now that we have defined the identity morphism and composition of morphism, we can show that the identity morphism is left and right cancellable, and that composition is associative.

Let $(h\colon d{\to}d', k\colon e{\to}e')(f, f')$, denoted as α, be a morphism with domain and codomain (d, e, f) and $(d', e', f',)$ respectively, denoted as c and c' respectively. Starting with the composition of α and 1_c, we can show the following chain of equalities:

$$\begin{aligned}
\alpha 1_c &= (h1_d, k1_e)(f, f') \\
&= (h, k)(f, f') \\
&= \alpha.
\end{aligned}$$

Composing $1_{c'}$ and α gives us a similar result:

$$\begin{aligned}
1_{c'}\alpha &= (1_{d'}h, 1_{e'}k)(f, f') \\
&= (h, k)(f, f') \\
&= \alpha.
\end{aligned}$$

Thus we have shown that $1_{c'}\alpha = \alpha = \alpha 1_c$. Therefore the identity morphism is left and right cancellable.

[4]See diagram 1.6.10.

Finally, we will show that the composition of morphisms is associative. Take morphisms $(h\colon d\to d_1, k\colon e\to e_1)(f, f_1)$, $(h\colon d_1\to d_2, k\colon e_1\to e_2)(f_1, f_2)$, and $(h\colon d_2\to d_3, k\colon e_2\to e_3)(f_2, f_3)$, denoted as α, β, and γ respectively. Now observe that:

$$\begin{aligned}(\gamma\beta)\alpha &= ((h_2h_1), (k_2k_1))(f_1, f_3)\alpha \\ &= ((h_2h_1)h, (k_2k_1)k)(f, f_3) \\ &= (h_2(h_1h), k_2(k_1k))(f, f_3) \\ &= \gamma((h_1h), (k_1k))(f, f_2) \\ &= \gamma(\beta\alpha).\end{aligned}$$

Thus the composition of morphisms is associative, and we have shown that $F \downarrow G$ is a category.

Now we will define the functors $\mathrm{dom}\colon F \downarrow G\to \mathsf{D}$ and $\mathrm{cod}\colon F \downarrow G\to \mathsf{E}$ for object (d, e, f) and morphism $(h\colon d\to d', k\colon e\to e')(f, f')$ as follows:

$$\begin{aligned}\mathrm{dom}\,(d, e, f) = d, \mathrm{dom}\,(h, k)\,(f, f') &= h \\ \mathrm{cod}\,(d, e, f) = e, \mathrm{cod}\,(h, k)\,(f, f') &= k.\end{aligned}$$

Now we will verify that both dom and cod are indeed functors.

Now let us take the morphism $(h\colon d\to d_1, k\colon e\to e_1)(f, f_1)$ denoted $\alpha\colon (d, e, f)\to(d_1, e_1, f_1)$. Applying dom to α gives us $h\colon d\to d_1$ where $\mathrm{dom}\,(d, e, f) = d$ and $\mathrm{dom}\,(d_1, e_1, f_1) = d_1$. Applying cod to α gives $k\colon e\to e_1$ where $\mathrm{cod}\,(d, e, f) = e$ and $\mathrm{cod}\,(d_1, e_1, f_1) = e_1$. Thus $\alpha\colon (d, e, f)\to(d_1, e_1, f_1)$ gets mapped to $\mathrm{dom}\,\alpha\colon \mathrm{dom}\,(d, e, f)\to\mathrm{dom}\,(d_1, e_1, f_1)$ and $\mathrm{cod}\,\alpha\colon \mathrm{cod}\,(d, e, f)\to\mathrm{cod}\,(d_1, e_1, f_1)$ by dom and cod respectively.

For object (d, e, f), the identity arrow $(1_d, 1_e)\,(f, f)$ get mapped to 1_d and 1_e by dom and cod respectively. Since $\mathrm{dom}\,(d, e, f) = d$ and $\mathrm{cod}\,(d, e, f) = e$, This shows that dom and cod preserve identities.

Finally take morphisms $(h\colon d\to d_1, k\colon e\to e_1)(f, f_1)$ denoted $\alpha\colon (d, e, f)\to(d_1, e_1, f_1)$ and $(h'\colon d_1\to d_2, k'\colon e_1\to e_2)(f_1, f_2)$ denoted $\beta\colon (d_1, e_1, f_1)\to(d_2, e_2, f_2)$. Then we have the following equalities:

$$\mathrm{dom}(\beta\alpha) = \mathrm{dom}\,(h'h, k'k)\,(f, f_2) = h'h = \mathrm{dom}\,\beta\,\mathrm{dom}\,\alpha$$

and

$$\mathrm{cod}(\beta\alpha) = \mathrm{cod}\,(h'h, k'k)\,(f, f_2) = k'k = \mathrm{cod}\,\beta\,\mathrm{cod}\,\alpha.$$

Thus dom and cod perserve morphism composition, and dom and cod are functors. □

Exercise 1.3.vii. Define functors to construct the slice categories c/C and C/c as special cases of comma categories. What are the projection functors?

Proof. We want to choose functors in a comma category, so that the comma category behaves like a slice category. Since the slice category treats morphisms like objects within a single category, and comma categories are defined generally with three categories in mind, we have some room to reduce the structure of the comma category as we construct a slice category.

Using the notation in the text's definition of the comma category, let $F \downarrow G$ be our comma category. Also, let $\mathsf{D} = \mathsf{C}$ and let the functor F be the identity functor, 1_{C}. With this choice of functor, the square in the definition of $F \downarrow G$ still commutes.

At this point, the objects of the comma category are $(d \in \mathsf{C}, e \in E, f\colon d \to Ge)$. The morphisms are $(h\colon d \to d', k\colon e \to e')$, such that $(d, e, f) \to (d', e', f')$.

The above choice of the identity functor collapses the amount of data represented. Yet there remain extra data at this point in the construction to qualify as a slice category. We desire to keep one of the morphisms h and k, while reducing the object-triple $(d \in \mathsf{C}, e \in E, f\colon d \to Ge)$ to a suitable object-morphism pair, that allows us to fix an element in C, and take morphisms as objects.

To this end, let E be the ordinal category, $\mathbb{1}$, with one object represented as $\emptyset$, and only the identity morphism. The objects in the comma category now are the triples $(d \in \mathsf{C}, \emptyset, f\colon d \to c \in \mathsf{C})$. With the functor G acting on the one object of $\mathbb{1}$, then G sends $\emptyset$ to c in C. Since, in $\mathsf{E} = \mathbb{1}$, $k = 1_{\mathbb{1}}$, then k can only send $\emptyset$ to $\emptyset$. So the objects in the comma construction are rendered as pairs, with the relevant data c and f, represented as $(d \in \mathsf{C}, f\colon d \to c)$.

Letting the morphism h take d to d', while the morphism $k = 1_{\mathbb{1}}$ sends $\emptyset$ to $\emptyset$, we have a codomain of the comma morphism (h, k) represented as $(d' \in \mathsf{C}, f'\colon d' \to c)$. This is a class of morphisms of C taken as objects, and the comma category is reduced to the C/c slice category. □

Exercise 1.3.viii. Lemma 1.3.8 shows that functors preserve isomorphisms. Find an example to demonstrate that functors need not **reflect isomorphisms**: that is, find a functor $F : \mathsf{C} \to \mathsf{D}$ and a morphism f in C so that Ff is an isomorphism in D but f is not an isomorphism in C.

Consider the functor $F\colon \mathbb{2} \to \mathbb{1}$ that maps everything in $\mathbb{2}$ to the identity in $\mathbb{1}$. So F trivially satisfies all the properties of a functor. Because $\mathbb{2}$ is not a groupoid there exists at least one morphism f in mor $\mathbb{2}$ that is not an isomorphism. In this construction Ff will also go to the best isomorphism, the identity map, even though f itself is not an isomorphism.

Exercise 1.3.ix. For any group G, we may define other groups:

- the **center** $Z(G) = \{h \in G \mid hg = gh \ \forall g \in G\}$
- the **commutator subgroup** $C(G)$, the subgroup generated by the element $ghg^{-1}h^{-1}$ for any $g, h \in G$, and
- the **automorphism group** $\mathrm{Aut}(G)$, the group of isomorphisms $\phi\colon G \to G$ in Group.

Trivially, all three constructions define a functor from the discrete category of groups (with only indentity morphsims) to Group. Are these constructions functorial in

- the isomorphisms of groups? That is, do they extend to functors $\mathsf{Group}_{\mathrm{iso}} \to \mathsf{Group}$?
- the epimorphisms of groups? That is, do they extend to functors $\mathsf{Group}_{\mathrm{epi}} \to \mathsf{Group}$?

- the homomorphisms of groups? That is, do they extend to functors $\mathsf{Group} \to \mathsf{Group}$?

Proof. First, consider the functor $F_Z\colon \mathsf{Group}_{\text{id}} \to \mathsf{Group}$, where $G \to Z(G)$. We will now show that there exists a similar functor $F_Z\colon \mathsf{Group}_{\text{epi}} \to \mathsf{Group}$. We define F_Z as the following: $F_Z(G) = Z(G)$ and if $f\colon G \to H$, then $F_Z f = f|_{Z(G)}$. We must show that this functor satisfies the properties of a functor.

1. Because each group has a unique center and each morphism f with $\mathrm{dom}(f) = G$ has a unique restriction to $Z(G)$, this functor is well defined and satisfies the first two properties (0 and 1).
2. We see that $F_Z 1_G = 1_G|_{Z(G)} = 1_{Z(G)} = 1_{F_Z G}$, so the functor preserves identities.
3. We easily see that by definition of function restriction, if $f\colon G \to H$, then $\mathrm{dom}(F_Z f) = F_Z\,\mathrm{dom}(f)$. We choose $\mathrm{cod}(F_Z f) = Z(H) = F_Z(\mathrm{cod}(f))$. To see that our morphisms are still well-defined from when the domain and codomain are restricted by this functor, we show that if $g \in Z(G)$, then $f(g) \in Z(H)$. To do this, consider, for any $k \in H$, that since f is an epimorphism and therefore surjective, that $k = f(h)$ for some $h \in G$, so
$$f(g)k = f(g)f(h) = f(gh).$$
Since $g \in Z(G)$,
$$f(gh) = f(hg) = f(h)f(g) = kf(g)$$
So $f(g) \in Z(H)$ and therefore we have a well-defined morphism from $F_Z G$ to $F_Z H$.
4. If $f\colon G \to H$ and $g\colon H \to K$, we see that $F_Z(gf) = gf|_{Z(G)}$. By the property we proved in the previous part
$$gf|_{Z(G)} = g|_{Z(H)} f|_{Z(G)} = F_Z g F_Z f.$$
So this functor also preserves morphism composition.

We have seen that F_Z satisfies all properties of a functor. We also note that F_Z will be a functor from $\mathsf{Group}_{\text{iso}} \to \mathsf{Group}$.

To show that there is no such functor between Group and Group, consider the composition of the homomorphism $\mathrm{sgn}\colon S_n \to \{\pm 1\}$ and $\iota\colon \{1, (1\ 2)\} \to S_4$. We say $g(x) = \mathrm{sgn}(\iota(x))$. We see that g is an isomorphism, and so $F_Z g$ should also be an isomorphism. However, this is not possible under any function of morphism, as S_4 has a trivial center and so any morphism from $Z(\{1, (1\ 2)\} \to Z(S_4) \to Z(\{\pm 1\})$ must be trivial. So F_Z cannot be a functor from $\mathsf{Group} \to \mathsf{Group}$.

Now consider the functor $F_C\colon \mathsf{Group}_{\text{id}} \to \mathsf{Group}$. We will show there exists a similarly constructed functor from $\mathsf{Group} \to \mathsf{Group}$ defined as the following: for $G \in \mathrm{ob}(\mathsf{Group})$, $F_C G = C(G)$, where $C(G)$ is the commutator subgroup of G. If $f\colon G \to H$, $F_C f = f|_{C(G)}$, We will show that this satisfies all the properties of a functor.

1. Because each group has a unique commutator subgroup and each morphism f with $\mathrm{dom}(f) = G$ has a unique restriction to $C(G)$, this functor is well defined and satisfies the first two properties (0 and 1).
2. We see that $F_C 1_G = 1_G|_{C(G)} = 1_{C(G)} = 1_{F_C G}$, so the functor preserves identities.
3. We easily see that by definition of function restriction, if $f\colon G \to H$, then $\mathrm{dom}(F_Z f) = F_Z\,\mathrm{dom}(f)$. We choose $\mathrm{cod}(F_C f) = C(H) = F_C(\mathrm{cod}(f))$. To see that our morphisms

are well defined when we restrict the domain and codomain, we show that if $g \in C(G)$, then $f(g) \in C(H)$. If $g \in C(G)$, $g = \prod_{i=1}^{n} a_i$, where each $a_i = hkh^{-1}k^{-1}$ for some $h, k \in G$. So

$$\begin{aligned}
f(g) &= f\left(\prod_{i=1}^{n} h_i k_i h_i^{-1} k_i^{-1}\right) \\
&= \prod_{i=1}^{n} f\left(h_i k_i h_i^{-1} k_i^{-1}\right) \\
&= \prod_{i=1}^{n} f(h_i) f(k_i) f\left(h_i^{-1}\right) f\left(k_i^{-1}\right) \\
&= \prod_{i=1}^{n} f(h_i) f(k_i) f(h_i)^{-1} f(k_i)^{-1}
\end{aligned}$$

But since $f(k_i), f(h_i) \in H$, this is an element of $C(H)$. So if $g \in C(G)$, $f(G) \in C(H)$ and therefore we have well-defined morphisms from our restricted domain to our restricted co-domain.

4. If $f\colon G \to H$ and $g\colon H \to K$, we see that $F_C(gf) = gf|_{C(G)}$. By the property we proved in the previous part

$$gf|_{C(G)} = g|_{C(H)} f|_{C(G)} = F_C g F_C f.$$

So this functor also preserves morphism composition.

So we have shown that this is a functor for Group $\to$ Group, and we note that this implies that it is also a functor for $\mathsf{Group}_{\mathrm{iso}} \to$ Group and $\mathsf{Group}_{\mathrm{epi}} \to$ Group.

Next, we show that there is a functor $F_A\colon \mathsf{Group}_{\mathrm{iso}} \to \mathsf{Group}$, defined as follows: If G is a group, then $F_A G = \mathrm{Aut}(G)$ and if ϕ is a morphism between two groups G and H, then $(F_A\phi)(f) = \phi f \phi^{-1}$. We now show that this definition satisfies the properties of a functor.

1. Each group has uniquely defined automorphism group, and each morphism ϕ conjugates elements of $\mathrm{Aut}(G)$in a unique way, so the functor is well defined.
2. $F_A(1_G)(f) = 1f1 = 1_{\mathrm{Aut}(G)}(f)$, so the functor preserves identities.
3. $F_A(\operatorname{dom}\phi) = F_A(G) = \mathrm{Aut}(G) = \operatorname{dom} F_A(\phi)$.
4. By definiton of F_A, $\operatorname{cod}(F_A\phi) = \mathrm{Aut}(H) = F_A(\operatorname{cod}(\phi))$.We see that $\phi f \phi^{-1} \in \mathrm{Aut}(H)$ for any $f \in \mathrm{Aut}(G)$ because it is a compostition of ismorphisms and therefore also an isomorphism. So when we restrict the domain and codomain of our morphisms using this functor, then they are still well-defined.
5. For two composable morphisms ϕ and τ,

$$F_A(\phi\tau)(f) = \phi\tau f(\phi\tau)^{-1} = \phi\tau f \tau^{-1}\phi^{-1} = F_A(\phi)(\tau f \tau^{-1}) = F_A(\phi)F_A(\tau)(f),$$

so this functor preserves composition.

So F_A satisfies all the properties of a functor, and there exists a functor from $\mathsf{Group}_{\mathrm{iso}} \to$ Group of the desired form.

It is unclear whether or not there is a functor from $\mathsf{Group}_{\mathrm{epi}}$ to Group and from Group to Group defined in the manner, but I believe that this is not the case, although I am having trouble finding a counterexample. □

Exercise 1.3.x. Show that the construction of the set of conjugacy classes of elements of a group is functorial, defining a functor Conj: $\mathsf{Group} \to \mathsf{Set}$. Conclude that any pair of groups whose sets of conjugacy classes of elements have differing cardinalities cannot be isomorphic.

Proof. Let the functor Conj: $\mathsf{Group} \to \mathsf{Set}$ represent the construction of the set of conjugacy classes of elements of a group, defined as follows:

- For any group s, $\mathrm{Conj}\, s = \hat{s}$.
- For any groups s and t and any group homomorphism $f\colon s \to t$, define the morphism $\mathrm{Conj}\, f\colon \hat{s} \to \hat{t}$ such that for each $[x] \in \hat{s}$, $\mathrm{Conj}\, f([x]) = [f(x)]$.[5]

First we will prove that Conj is functorial by showing that it fulfills both functoriality axioms:

- Let f and g be group homomorphisms such that gf is a valid composition, and let $[x] \in (\mathrm{dom}(f))^*$ be arbitrary.

$$\mathrm{Conj}\, g\, \mathrm{Conj}\, f([x]) = \mathrm{Conj}\, g([f(x)]) = [g(f(x))] = [gf(x)] = \mathrm{Conj}(gf([x])).$$

  Since $[x]$, f, and g were arbitrary, $\mathrm{Conj}\, g\, \mathrm{Conj}\, f = \mathrm{Conj}(gf)$. So Conj fulfills the first functoriality axiom.
- For an arbitrary group s and element $x \in s$,

$$\mathrm{Conj}\, 1_s([x]) = [1_s(x)] = [x] = 1_{\hat{s}}([x]) = 1_{\mathrm{Conj}\, s}([x]).$$

  Since s and x were arbitrary, $\mathrm{Conj}\, 1_s = 1_{\mathrm{Conj}\, s}$. So Conj fulfills the second functoriality axiom. □

Conj fulfills both axioms, so we can conclude that it is indeed functorial.

Let s and t be two isomorphic groups, and let $f\colon s \to t$ be an isomorphism. Functors preserve isomorphisms, (as per the 'first lemma in category theory') so $\mathrm{Conj}\, f\colon \hat{s} \to \hat{t}$ must also be an isomorphism. This makes $\hat{s}$ and $\hat{t}$ isomorphic, which in turn means they must have the same cardinality. So we can conclude the contrapositive: that any pair of groups whose sets of conjugacy classes have *different* cardinalities *cannot* be isomorphic.

---

[5] In order for each $\mathrm{Conj} f$ to be well-defined, it must send each $[x] \in \hat{s}$ to a single $[f(x)] \in \hat{t}$. But there may be more than one element in $[x]$, and since the definition does not mention which x should be used as a 'representative,' it might seem that there could be cases in which there were multiple possible $[f(x)]$ (and therefore, multiple possible $\mathrm{Conj} f([x])$) for a single $[x]$. So in order for $\mathrm{Conj} f$ to be well-defined, $[f(x)]$ must be the same for every possible choice of $x \in [x]$. In other words, we need to show that $[f(a)] = [f(b)]$ for any a, b in the same conjugate class.

So, let a, b be arbitrary members of the same conjugate class. Recall that this means that there is some $n \in s$ such that $b = nan^{-1}$. Furthermore, recall that group homomorphisms (like f) preserve inverses. With this in mind,

$$f(b) = f(nan^{-1}) = f(n)f(a)f(n^{-1}) = f(n)f(a)f(n)^{-1}$$

$n \in s$, so $f(n) \in t$. This means that there is some $m \in t$ such that $mf(a)m^{-1} = f(b)$. So $f(a)$ and $f(b)$ are conjugates, (and therefore $[f(a)] = [f(b)]$ by definition,) which makes $\mathrm{Conj} f$ well-defined.

1.4 Naturality

Exercise 1.4.i. Suppose $\alpha : F \Rightarrow G$ is a natural isomorphism. Show that inverses of the component morphisms define the components of a natural isomorphism $\alpha^{-1} : G \Rightarrow F$.

Proof. Suppose we have a natural isomorphism $\alpha : F \Rightarrow G$. We display the square of morphisms below for convenience.

$$\begin{tikzcd}
Fc \arrow[r, "\alpha_c"] \arrow[d, "Ff"'] & Gc \arrow[d, "Gf"] \\
Fc' \arrow[r, "\alpha_{c'}"'] & Gc'
\end{tikzcd}$$

The above diagram commutes, and $(Gf)(\alpha_c) = (\alpha_{c'})(Ff)$ takes us from Fc To Gc'. Consider the component morphisms α_c of α. Because α is a natural isomorphism, every component morphism, α_c, has an inverse, α_c^{-1}. We need to see that all the inverse component morphisms, α_c^{-1}, make a natural transformation.

To see where α_c^{-1} takes us, examine the square of morphisms below. All the components are still in D and take us from Gc to Fc'.

$$\begin{tikzcd}
Gc \arrow[r, "\alpha_c^{-1}"] \arrow[d, "Gf"'] & Fc \arrow[d, "Ff"] \\
Gc' \arrow[r, "\alpha_{c'}^{-1}"'] & Fc'
\end{tikzcd}$$

Since $(Gf)\alpha_c = \alpha_{c'}(Ff)$,

$$\alpha_{c'}^{-1}(Gf) = \alpha_{c'}^{-1}(Gf)\alpha_c\alpha_c^{-1} = \alpha_{c'}^{-1}\alpha_{c'}(Ff)\alpha_c^{-1} = (Ff)\alpha_c^{-1}.$$

So, the diagram above commutes and if we let $(\alpha^{-1})_c = \alpha_c^{-1}$ then α^{-1} is a natural transformation from G to F. $\square$

Exercise 1.4.ii. What is a natural transformation between a parallel pair of functors between groups, regarded as one-object categories?

Proof. For the abstract, one-object categories that are defined by groups, we can take any single object C_* and D_* as the specials objects of the respective groups. We then have the class of morphisms as the elements of the groups under their respective group operations. Let the operations here be group multiplication. We will start by finding what the functors are between these categories, and then find natural transformation in this context.

Given two such categories, $\mathsf{B}C$ and $\mathsf{B}D$ with their respective groups C and D, any functors F and G between $\mathsf{B}C$ and $\mathsf{B}D$ must map the object C_* to the object D_*. Since functors are functions, looking at the functor F, the role of F acting on the morphisms of $\mathsf{B}C$ is the same as a function acting on the elements of C under group multiplication. So, $F : C_* \to D_*$, for the special elements $C_* \in \mathsf{B}C$ and $D_* \in \mathsf{B}D$. Taking any two elements c, c' in C, cc' is the composition in the category $\mathsf{B}C$, and since the both the domain codomain of $\mathsf{B}C$ are equal to C, then any pair cc' is composable. As functors respect the functoriality axioms, $F(cc') = F(c)F(c')$ in D, and $F(1_C) = 1_{F_C}$ in D, then the functor F behaves as a group homomorphism between the groups C and D.

To find a natural transformation $\alpha\colon F \Rightarrow G$ between F and G, we can let $\alpha_{C_*}\colon FC_* \to GC_*$ be a class of morphisms, with $f\colon C_* \to C_*$, such that $Gf\alpha_{C_*} = \alpha_{C_*}Ff$. In our case, we have C_* as the single object of BC, and the morphism f as an element c in C. Also, we have that $FC_* = D_*$ and $GC_* = D_*$, so our morphism is now $\alpha_{C_*}\colon D_* \to D_*$. Thus, for the object C_*, the natural transformation gives the equality $\alpha_{C_*}Ff = Gf\alpha_{C_*}$.

Since α_{C_*} is a morphism from the object D_* to itself, this endomorphism (and hence automorphism) consists of the elements of the group D. Because each element has an inverse, likewise α_{C*} has an inverse, α^{-1}_{C*}. Thus $\alpha_{C_*}Ff = Gf\alpha_{C_*}$ implies $\alpha^{-1}_{C*}\alpha_{C_*}Ff = \alpha^{-1}_{C*}Gf\alpha_{C_*}$, implies $Ff = \alpha^{-1}_{C*}Gf\alpha_{C_*}$, for all f. Noting that Ff and Gf are morphisms in the category $\mathsf{B}D$, and hence is an element of the group D, then Ff and Gf are in $\mathrm{Aut}(D)$, and α_{C_*} forms a conjugacy class for these automorphisms. □

Exercise 1.4.iii. What is a natural transformation between a parallel pair of functors between preorders, regarded as categories?

Proof. Let C and D be preorder categories and $F, G\colon \mathsf{C} \rightrightarrows \mathsf{D}$ be parallel functors. Let us consider the properties of the natural transformation $\alpha\colon F \to G$. We will show that you need only find morphisms $f_c\colon Fc \to Gc$ in D for all $c \in \mathrm{ob}\,\mathsf{C}$ to produce a natural transformation from F to G. This will assist in our characterization of natural transformations.

Suppose we have morphisms $f_c\colon Fc \to Gc$ in D for $c \in \mathrm{ob}\,\mathsf{C}$. We can define $\alpha\colon F \to G$ such that $\alpha(c) = f_c$. Take morphism $g\colon c \to c'$ in D, we have that the diagram

$$\begin{tikzcd} Fc \arrow[r, "Fg"] \arrow[d, "\alpha(c)"'] & Fc' \arrow[d, "\alpha(c')"] \\ Gc \arrow[r, "Gg"'] & Gc' \end{tikzcd}$$

commutes since in a preorder category, there is at most one morphism between objects. Thus, our α defines a natural transformation from F to G. Thus it is sufficient to find morphisms $f_c\colon Fc \to Gc$ in D to define a natural transformation from F to G.

Seeing the functors F and G as monotone maps (i.e. order preserving maps) between preorders C to D, this allows us to characterize a natural transformation α as a relation over D containing only the pairs (Fc, Gc). □

Exercise 1.4.iv. In the notation of Example 1.4.7, prove that distinct parallel morphisms $f, g\colon c \rightrightarrows d$ define distinct natural transformations

$$f_*, g_*\colon \mathsf{C}(-,c) \Rightarrow \mathsf{C}(d,-)$$
$$f^*, g^*\colon \mathsf{C}(c,-) \Rightarrow \mathsf{C}(d,-)$$

Proof. These being natural transformations is shown in Example 1.4.7, so the primary concern of this problem is whether they are distinct. First, we consider f_*, g_* as natural transformations from $\mathsf{C}(-,c) \Rightarrow \mathsf{C}(-,d)$. To differentiate, consider the natural transformation defined by f_* to be α and g_* to be β. We need to show the transformations are different in at least some component. For a natural transformation, we can choose arbitrary $h\colon c_1 \to c_2$ in C to look at the functions for. In this case, take $h = 1_c$. We thus in our diagram have h^* as our Fh and Gh, which is precomposition by 1_c. This must be the case as a functor preserves identities. We then consider what this transformation does to the morphism $j = 1_c \in \mathsf{C}(c,c)$ to see what would happen to it if we put it trough the transformation. Both directions take us to $fj1_c = fj = f$. Similarly, constructing the same diagram except with g, we get $gj1_c = gj = g$. Each takes us to a different morphism in $\mathsf{C}(c,d)$ and thus the two transformations are different. Essentially the same construction works with f^* and g^*.

$$\begin{tikzcd}
\mathsf{C}(c,c) \arrow[r, "1_c^*"] \arrow[d, "f_*"'] & \mathsf{C}(c,c) \arrow[d, "f_*"] \\
\mathsf{C}(c,d) \arrow[r, "1_c^*"] & \mathsf{C}(c,d)
\end{tikzcd}$$ □

Exercise 1.4.v. Recall the construction of the comma category for any pair of functors $F\colon \mathsf{D}\to\mathsf{C}$ and $G\colon \mathsf{E}\to\mathsf{C}$ described in Exercise 1.3.vi. From this data, construct a canonical natural transformation $\alpha\colon F\,\mathrm{dom}\Rightarrow G\,\mathrm{cod}$ between the functors that form the boundary of the square

$$\begin{tikzcd}
F \downarrow G \arrow[r, "\mathrm{cod}"] \arrow[d, "\mathrm{dom}"'] & E \arrow[d, "G"] \\
D \arrow[r, "F"'] \arrow[ur, Rightarrow, "\alpha"] & C
\end{tikzcd}$$

Proof. Letting $c = (d, e, f\colon d\to e \in C)$ as above, and a morphism $m = (h\colon d\to d', k\colon e\to e')$, we can describe the actions of the functors $F\,\mathrm{dom}$ and $G\,\mathrm{cod}$ on objects:

- $F\,\mathrm{dom}\, c = Fd$,
- $G\,\mathrm{cod}\, c = Ge$,

and their actions on morphisms:

- $F\,\mathrm{dom}\, m = Fh$,

- $G \operatorname{cod} m = Gk$.

From the definition of a natural transformation, we need an $\alpha\colon \mathrm{ob}\, F \downarrow G \to \mathrm{mor}\, \mathsf{C}$, and we can get this by taking $(d, e, f) \mapsto f \in \mathsf{C}$. Then, $\alpha_c\colon Fd \to Ge$ in C. From this, we can construct the following diagram:

$$\begin{tikzcd}
Fd \arrow[r, "\alpha_c"] \arrow[d, "Fh"'] & Ge \arrow[d, "Gk"] \\
Fd' \arrow[r, "\alpha_{c'}"'] & Ge'
\end{tikzcd}$$

Which is precisely the diagram of the comma category with $f = \alpha_c$ and $f' = \alpha_{c'}$, and from the condition that $f'Fh = Gkf$, we have that this diagram commutes. Thus, α is a natural transformation. $\square$

Exercise 1.4.vi. Given a pair of functors $F\colon \mathsf{A} \times \mathsf{B} \times \mathsf{B}^{\mathrm{op}} \to \mathsf{D}$ and $G : \mathsf{A} \times \mathsf{C} \times \mathsf{C}^{\mathrm{op}} \to \mathsf{D}$ a family of morphisms

$$\alpha_{a,b,c}\colon F(a, b, b) \to G(a, c, c)$$

in D defines the components of an **extranatural transformation** $\alpha : F \Rightarrow G$ if for any $f\colon a \to a'$, $h\colon c \to c'$ the following diagrams commute in D:

$$\begin{tikzcd}
F(a,b,b) \arrow[r, "\alpha_{a,b,c}"] \arrow[d, "F(f,1_b,1_b)"'] & G(a,c,c) \arrow[d, "G(f,1_c,1_c)"] \\
F(a',b,b) \arrow[r, "\alpha_{a',b,c}"'] & G(a',c,c)
\end{tikzcd}
\quad
\begin{tikzcd}
F(a,b,b') \arrow[r, "F(1_a,1_b,g)"] \arrow[d, "F(1_a,g,1_{b'})"'] & F(a,b,b) \arrow[d, "\alpha_{a,b,c}"] \\
F(a,b',b') \arrow[r, "\alpha_{a,b',c}"'] & G(a,c,c)
\end{tikzcd}
\quad
\begin{tikzcd}
F(a,b,b) \arrow[r, "\alpha_{a,b,c'}"] \arrow[d, "G(1_a,1_{c'},1_h)"'] & G(a,c',c') \arrow[d, "G(f,1_c,1_c)"] \\
G(a,c,c) \arrow[r, "\alpha_{a',b,c}"'] & G(a,c',c)
\end{tikzcd}$$

The left-hand square asserts the at the components $\alpha_{-,b,b}\colon F(-, b, b) \Rightarrow G(-, c, c)$ define a natural transformation in a for each $b \in \mathsf{B}$ and $c \in \mathsf{C}$. The remaining squares assert that the components $\alpha_{\alpha_{a,-,-}}\colon F(a, -, -) \Rightarrow G(a, c, c)$ and $\alpha_{a,b,-}\colon F(a, b, b) \Rightarrow G(a, -, -)$ define transformations that are respectively extranatural in b and in c. Explain why functors F and G must have a common target category for this this definition to make sense.

Notice that the definition of extranatural transformation does not actually have anything to do with the question. This exercise is nothing more than a sanity check. If F and G do not have the same target category, when we try to write down any of the three above diagrams, we will see that they are simply not defined if F and G do not have the same target category.

1.5 Equivalence of categories

Lemma 1.5.1. *Fixing a parallel pair of functors $F, G : \mathsf{C} \rightrightarrows \mathsf{D}$, natural transformations $\alpha : F \Rightarrow G$ correspond bijectively to functors $H : \mathsf{C} \times \mathbb{2} \to \mathsf{D}$ such that H restricts along i_0 and i_1 to the functors F and G, i.e., so that*

$$\begin{tikzcd} \mathsf{C} \arrow[r, "i_0"] \arrow[dr, "F"'] & \mathsf{C} \times \mathbb{2} \arrow[d, "H"] & \mathsf{C} \arrow[l, "i_1"'] \arrow[dl, "G"] \\ & \mathsf{D} & \end{tikzcd}$$

Before going on, I'd like to make a set-theoretic remark about exactly what the bijection is between. Say that U is a non-empty universe and that C and D are U-categories.

Assume further that C is a U-small category and that D is a U-locally small category. Then the class of all morphisms from Fc to Gc as c varies over objects of C forms a U-set. Then the class of all functions from $\operatorname{ob}\mathsf{C}$ to this U-set is also a U-set. The natural transformations form a subclass of this U-set, and so the class of all natural transformations from F to G forms a U-set.

In this case, $\mathsf{C} \times \mathbb{2}$ is also a U-small category and so each functor $H : \mathsf{C} \times \mathbb{2} \to \mathsf{D}$ is a U-set by the Axiom of Replacement in U. We may thus form a U-class of these functors. The lemma then implies through the Axiom of Replacement again that this U-class is also a U-set.

However, if the objects of C form a proper U-class, then any natural transformation $\alpha : F \Rightarrow G$ is also a proper U-class. This is because as a function, α has domain $\operatorname{ob}\mathsf{C}$, a proper U-class. In this case, α is not an element of U and is thus not an element of any U-class.

In either case, let V be a universe such that $U \in V$. Then all of the categories mentioned in the lemma are small V-categories so that the bijection in the lemma is a bijection between V-sets.

Exercise 1.5.i. Prove the lemma above.

Proof. In the category $\mathbb{2}$, there are precisely three morphisms:

$$\begin{aligned} \phi_{00} &: 0 \to 0 \\ \phi_{01} &: 0 \to 1 \\ \phi_{11} &: 1 \to 1 \end{aligned}$$

(ϕ_{00} and ϕ_{11} are identity morphisms.) Any morphism in $\mathsf{C} \times \mathbb{2}$ is of the form

$$(f, \phi_{mn}) : (x, m) \to (y, n)$$

where $f : x \to y$ is a morphism in C and $m, n \in \{0, 1\}$ with $m \leq n$.

Let N be the collection (or more precisely U-set or V-set as above) of natural transformations from F to G, and let X be the collection of functors H as described in the statement of the lemma. We first make a function from N to X taking $\alpha \in N$ to $H_\alpha \in X$.

Define the functor $H_\alpha : \mathsf{C} \times \mathbb{2} \to \mathsf{D}$ as follows:

1. For every $c \in \operatorname{ob} \mathsf{C}$,
   (a) $H_\alpha(c, 0) = Fc$ and
   (b) $H_\alpha(c, 1) = Gc$.
2. For every morphism $f : x \longrightarrow y$ in C,
   (a) $H_\alpha(f, \phi_{00}) = Ff$,
   (b) $H_\alpha(f, \phi_{11}) = Gf$, and
   (c) $H_\alpha(f, \phi_{01}) = Gf\alpha_x = \alpha_y Ff$.

There is no ambiguity in the final case, since $\alpha : F \Rightarrow G$ being a natural transformation tells us that the following diagram commutes.

$$\begin{tikzcd} Fx \arrow[r, "Ff"] \arrow[d, "\alpha_x"'] & Fy \arrow[d, "\alpha_y"] \\ Gx \arrow[r, "Gf"'] & Gy \end{tikzcd}$$

Let's check that H really is a functor. It takes objects to objects and morphisms to morphisms. Notice that in each of the three formulas for $H_\alpha(f, \phi_{mn})$, the domain of $H_\alpha(f, \phi_{mn})$ is equal to $H_\alpha(x, m)$, either Fx or Gx as need be. Likewise, in each case the codomain of $H_\alpha(f, \phi_{mn})$ is equal to $H_\alpha(y, n)$, either Fy or Gy as need be. So, we see that $H_\alpha \circ \operatorname{dom} = \operatorname{dom} \circ H_\alpha$ and $H_\alpha \circ \operatorname{cod} = \operatorname{cod} \circ H_\alpha$ as required.

(f, ϕ_{mn}) is an identity if and only if $f = 1_x$ and $m = n$. We have that $H_\alpha(1_x, \phi_{00}) = F1_x = 1_{Fx}$, since F is a functor, and $H_\alpha(1_x, \phi_{11}) = G1_x = 1_{Gx}$, since G is a functor. So, H_α takes identities to identities as required.

Finally, if we also have $(g, \phi_{np}) : (y, n) \longrightarrow (z, p)$ then $m \leq n \leq p$ and $(g, \phi_{np})(f, \phi_{mn}) = (gf, \phi_{mp})$. There are four cases to consider:

1. $(m, n, p) = (0, 0, 0)$:

$$H_\alpha(gf, \phi_{00}) = F(gf) = FgFf = H_\alpha(g, \phi_{00})H_\alpha(f, \phi_{00}).$$

2. $(m, n, p) = (0, 0, 1)$:

$$H_\alpha(gf, \phi_{01}) = G(gf)\alpha_x = Gg(Gf\alpha_x) = Gg(\alpha_y Ff)$$

$$= (Gg\alpha_y)Ff = H_\alpha(g, \phi_{01})H_\alpha(f, \phi_{00}).$$

3. $(m, n, p) = (0, 1, 1)$:

$$H_\alpha(gf, \phi_{01}) = G(gf)\alpha_x = Gg(Gf\alpha_x) = H_\alpha(g, \phi_{11})H_\alpha(f, \phi_{01}).$$

4. $(m, n, p) = (1, 1, 1)$:

$$H_\alpha(gf, \phi_{11}) = G(gf) = GgGf = H_\alpha(g, \phi_{11})H_\alpha(f, \phi_{11}).$$

Now, the functors $i_n : \mathsf{C} \longrightarrow \mathsf{C} \times \mathbb{2}$ for $n = 0, 1$ are the following. On objects, $i_n c = (c, n)$. On morphisms, $i_n f = (f, \phi_{nn})$. So, on objects the compositions are $H_\alpha i_0 c = H_\alpha(c, 0) = Fc$ and $H_\alpha i_1 c = H_\alpha(c, 1) = Gc$. On morphisms, the compositions are $H_\alpha i_0 f = H_\alpha(f, \phi_{00}) =$

Ff and $H_\alpha i_1 f = H_\alpha(f, \phi_{11}) = Gf$. So, $H_\alpha i_0 = F$ and $H_\alpha i_1 = G$ as required. So, we have constructed a function $\alpha \mapsto H_\alpha$ from N to X.

Now, we construct a function from X to N. Given a functor $H : \mathsf{C} \times \mathbb{2} \to \mathsf{D}$ such that $Hi_0 = F$ and $Hi_1 = G$, we must construct a natural transformation $\alpha^H : F \Rightarrow G$. For an object c in C, let $\alpha_c^H = H(1_c, \phi_{01})$. We must see that this gives a natural transformation.

Using that H is a functor, we have that

$$\operatorname{dom} \alpha_c^H = \operatorname{dom} H(1_c, \phi_{01}) = H \operatorname{dom}(1_c, \phi_{01}) = H(c, 0) = Fc.$$

Similarly,

$$\operatorname{cod} \alpha_c^H = \operatorname{cod} H(1_c, \phi_{01}) = H \operatorname{cod}(1_c, \phi_{01}) = H(c, 1) = Gc.$$

So, $\alpha_c^H : Fc \to Gc$ as required.

Now, if $f : x \to y$ in C, then

$$Gf\alpha_x^H = H(i_1 f)H(1_x, \phi_{01}) = H(f, \phi_{11})H(1_x, \phi_{01}) = H(f, \phi_{01})$$

$$= H(1_y, \phi_{01})H(f, \phi_{00}) = \alpha_y^H H(i_0 f) = \alpha_y^H Ff.$$

This verifies that α^H is a natural transformation from F to G, so that we have constructed a function from X to N taking H to α^H.

Now, we must see that our two functions are inverses of each other. Starting with a natural transformation α in N, going to X and back to N gives the natural transformation α^{H_α}. For each object c in C, we must verify that $\alpha_c^{H_\alpha} = \alpha_c$. Combining the definitions of our two functions, we see that

$$\alpha_c^{H_\alpha} = H_\alpha(1_c, \phi_{01}) = G1_c\alpha_c = 1_{Gc}\alpha_c = \alpha_c$$

as required.

In the other direction, we must verify that for any $H \in X$, $H_{\alpha^H} = H$. On objects, $H_{\alpha^H}(x, 0) = Fx = Hi_0 x = H(x, 0)$ and $H_{\alpha^H}(x, 1) = Gx = Hi_1 x = H(x, 1)$. So, these two functors agree on objects.

On morphisms, we have three cases for a given $f : x \to y$ in C.

1.
$$H_{\alpha^H}(f, \phi_{00}) = Ff = Hi_0 f = H(f, \phi_{00}),$$

2.
$$H_{\alpha^H}(f, \phi_{11}) = Gf = Hi_1 f = H(f, \phi_{11},)$$

3.
$$H_{\alpha^H}(f, \phi_{01}) = Gf\alpha_x^H = Hi_1 f\alpha_x^H = H(f, \phi_{11})H(1_x, \phi_{01}) = H(f, \phi_{01}).$$

So, H and H_{α^H} agree on morphisms as well as objects, so that $H = H_{\alpha^H}$ as required. $\square$

Exercise 1.5.ii. Segal defined a category Γ as follows:

> Γ is the category whose objects are all finite sets, and whose morphisms from S to T are the maps $\theta\colon S \to P(T)$ such that $\theta(\alpha)$ and $\theta(\beta)$ are disjoint when $\alpha \neq \beta$. The composite of $\theta\colon S \to P(T)$ and $\phi\colon T \to P(U)$ is $\psi\colon S \to P(U)$, where $\psi(\alpha) = \bigcup_{\beta \in \theta(\alpha)} \phi(\beta)$.

Prove that Γ is equivalent to the opposite of the category Fin_* of finite pointed sets. In particular, the functors introduced in Example 1.3.2(xi) define presheaves on Γ.

Proof. As a preliminary matter, it is worth checking that Γ is indeed a category. The definition above makes clear that an arrow $\theta\colon S \to P(T)$ has domain S and codomain T, as well as telling us how to compose two morphisms. What is less obvious is what morphisms are the identities, and that composition is associative. For identities, let us first note that the identity on S in Γ is a function $1_S\colon S \to P(S)$ meaning the usual identity on the set S is not a possibility. However, there is a natural embedding of S in $P(S)$ which takes every element α to the singleton $\{\alpha\}$. Checking composition of 1_S and 1_T thus defined with an arbitrary map $\theta\colon S \to P(T)$ gives:

$$\theta 1_S(\alpha) = \bigcup_{\beta \in 1_S(\alpha)} \theta(\beta) = \bigcup_{\beta \in \{\alpha\}} \theta(\beta) = \theta(\alpha)$$
$$1_T\theta(\alpha) = \bigcup_{\beta \in \theta(\alpha)} 1_T(\beta) = \bigcup_{\beta \in \theta(\alpha)} \{\beta\} = \theta(\alpha)$$

verifying that we have defined the identities properly. Finally, we have to check that the composition law is associative. Let θ be as above along with $\phi\colon T \to P(U)$ and $\psi\colon U \to P(V)$ be valid morphisms in Γ. Then

$$((\psi\phi)\theta)(\alpha) = \bigcup_{\beta \in \theta(\alpha)} \phi\psi(\beta) = \bigcup_{\beta \in \theta(\alpha)} \left(\bigcup_{\gamma \in \phi(\beta)} \psi(\gamma) \right),$$

and

$$(\psi(\phi\theta))(\alpha) = \bigcup_{\gamma \in \phi\theta(\alpha)} \psi(\gamma) = \bigcup_{\gamma \in \left(\bigcup_{\beta \in \theta(\gamma)} \phi(\beta) \right)} \psi(\gamma).$$

At first appearance it is not at all clear that these ought to be the same set, but unpacking makes it clear they are in fact the same. Both assert that for any $\delta \in (\psi(\phi\theta))$ there exists some $\beta \in \theta(\alpha)$ and $\gamma \in \phi(\beta)$ such that $\delta \in \psi(\gamma)$. Thus our composition law is associative and we have indeed defined a category.

An equivalence of categories consists of two functors along with two natural isomorphisms between their compositions and the identity functor on each category. We will thus begin by defining two functors $+\colon \Gamma \leftrightarrows \mathsf{Fin}_*^{\mathrm{op}} : (-)^{-1}$. We define $+$ by the following mappings:

$$S \mapsto (S \cup \{ S \}, S)$$
$$\theta\colon S \to T \mapsto +\theta\colon +T \to +S$$

where

$$+\theta(\beta) = \begin{cases} S & \text{if } \beta = T \\ S & \text{if } \beta \notin \bigcup_{\alpha \in S} \theta(\alpha) \\ \alpha \in S & \text{if } \beta \in \theta(\alpha) \end{cases}.$$

Note first that this is a valid map from $+T$ to $+S$ in Fin_* (and thus a map from $+S$ to $+T$ in $\mathsf{Fin}_*^{\mathrm{op}}$): it takes the base point of $+T$ to the base point of $+S$, if $\alpha \in \theta(\gamma)$ for some $\gamma \in S$, then this γ must be unique since the image of distinct elements of S under θ must be disjoint. We must further check the functoriality axioms. Recall that the identity map on an object S of Γ is $1_S\colon \alpha \mapsto \{\alpha\}$. Applying this to the definition above gives:

$$+1_S(\alpha) = \begin{cases} S & \text{if } \alpha = S \\ S & \text{if } \alpha \notin \bigcup_{\alpha' \in S} 1_S(\alpha') = S \\ \alpha' \in S & \text{if } \alpha \in 1_S(\alpha') = \{ \alpha' \} \end{cases}, \quad \text{which is clearly the identity.}$$

So $+1_S = 1_{+S}$.

Further, given morphisms $\theta\colon S \to T$ and $\phi\colon T \to U$ between objects of Γ, we have that

$$+\phi(\gamma) = \begin{cases} T & \text{if } \gamma = U \\ T & \text{if } \gamma \notin \bigcup_{\beta \in T} \phi(\beta) \\ \beta \in T & \text{if } \gamma \in \phi(\beta) \end{cases}, \qquad +\theta(\beta) = \begin{cases} S & \text{if } \beta = T \\ S & \text{if } \beta \notin \bigcup_{\alpha \in S} \theta(\alpha) \\ \alpha \in S & \text{if } \beta \in \theta(\alpha) \end{cases},$$

$$\text{and} \quad +\phi\theta(\gamma) = \begin{cases} S & \text{if } \gamma = U \\ S & \text{if } \gamma \notin \bigcup_{\alpha \in S} \phi\theta(\alpha) = \bigcup_{\alpha \in S} \bigcup_{\beta \in \theta(\gamma)} \phi(\beta) \\ \alpha \in S & \text{if } \gamma \in \phi\theta(\alpha) = \bigcup_{\beta \in \theta(\alpha)} \phi(\beta) \end{cases}.$$

Letting $\gamma \in +U$ be arbitrary there are several cases. First, if $\gamma = U$, then $+\phi\theta(\gamma) = S$ and $+\phi(\gamma) = T$ so $+\theta + \phi(\gamma) = S$. Alternately, if $\gamma \neq U$ so $\gamma \in U$ we again have two cases. If there exists $\alpha \in S$ such that $\gamma \in \phi\theta(\alpha)$, then $+\phi\theta(\gamma) = \alpha$ and there exists a $\beta \in \theta(\alpha)$ such that $\gamma \in \phi(\beta)$ implying $+\phi(\gamma) = \beta$ and $+\theta(\beta) = \alpha$ so $+\theta + \phi(\gamma) = \alpha$. Finally, if there exists no such $\alpha \in S$, then $+\phi\theta(\gamma) = S$. If there also exists no $\beta \in T$ such that $\gamma \in \phi(\beta)$ then $+\phi(\gamma) = T$ so $+\theta + \phi(\gamma) = S$. However, is such $\beta \in T$ does exist so that $+\phi(\gamma) = \beta$ then it must be the case that

Next we define the functor $(-)^{-1}$ by:

$$(S, s) \mapsto S \setminus \{ s \}$$
$$f\colon (T, t) \to (S, s) \mapsto f^{-1}\colon S \setminus \{ s \} \to T \setminus \{ t \}$$

where

$$f^{-1}(\alpha) = \{\, \beta \in T \mid f(\beta) = \alpha \,\} .$$

Note that if $f^{-1}(\alpha) \cap f^{-1}(\alpha')$ is inhabited, then there is some $\beta \in T$ such that $f(\beta) = \alpha$ and $f(\beta) = \alpha'$ implying that $\alpha = \alpha'$. Thus the map we have defined satisfies the disjointness condition sufficient to be an morphism in Γ. Again we must check functoriality axioms. The identity morphism on (S, s) is merely the identity function on S, so

$$\begin{aligned} {1_S}^{-1}(\alpha) &= \{\, \alpha' \in S \mid 1_S(\alpha') = \alpha \,\} \\ &= \{\, \alpha' \in S \mid \alpha' = \alpha \,\} = \{\, \alpha \,\} = 1_{(-)^{-1}S}(\alpha). \end{aligned}$$

Given morphisms $f\colon (T, t) \to (S, s)$ and $g\colon (U, u) \to (T, t)$ we have that

$$\begin{aligned} (fg)^{-1}(\alpha) &= \{\, \gamma \in U \mid fg(\gamma) = \alpha \,\} \\ &= \bigcup_{f(\beta)=\alpha} \{\, \gamma \in U \mid g(\gamma) = \beta \,\} \\ &= \bigcup_{\beta \in f^{-1}(\alpha)} g^{-1}(\beta) = g^{-1} f^{-1}(\alpha). \end{aligned}$$

We thus have two bona fide functors connecting Γ and $\mathsf{Fin}_*^{\mathrm{op}}$. What remains is to construct natural isomorphisms $\eta\colon 1_\Gamma \cong (-)^{-1}+$ and $\epsilon\colon +(-)^{-1} \cong 1_{\mathsf{Fin}_*^{\mathrm{op}}}$. Looking first at η, given an object S of Γ we have that

$$(-)^{-1} + S = (-)^{-1}(S \cup \{\, S \,\}, S) = (S \cup \{\, S \,\}) \setminus \{\, S \,\} = S$$

meaning that the collection of maps which make up η will be automorphisms. Further, given $\theta\colon S \to T$ and $\alpha \in (-)^{-1} + S = S$, we have

$$(-)^{-1} + \theta(\alpha) = \{\, \beta \in T \mid +\theta(\beta) = \alpha \,\} .$$

Now, recall that $+\theta(\beta) = \alpha$ precisely when $\beta \in \theta(\alpha)$ and thus

$$\{\, \beta \in T \mid +\theta(\beta) = \alpha \,\} = \theta(\alpha)$$

so that $(-)^{-1}+$ is the identity functor on Γ and we may take our natural isomorphism to be the collection of identity maps 1_S for each object S of Γ.

Now for ϵ, given an object (S, s) of $\mathsf{Fin}_*^{\mathrm{op}}$ we have that

$$+(-)^{-1}(S, s) = +S \setminus \{\, s \,\} = ((S \setminus \{\, s \,\}) \cup \{\, S \setminus \{\, s \,\} \,\}, S \setminus \{\, s \,\})$$

For sanity's sake we will use the notation $(S_* \cup \{\, S_* \,\}, S_*)$ for the above set. This grotesque looking object (which amounts to replacing s with something more generic) is not S. However, there is a based isomorphism ϵ_s from S to it which takes s to $S \setminus \{\, s \,\}$ and leaves the other elements of S be. The collection of such isomorphisms will form ϵ. Similarly, given a map $f\colon (T, t) \to (S, s)$ we have that

$$+(f^{-1})(\beta) = \begin{cases} S_* & \text{if } \beta = T_* \\ S_* & \text{if } \beta \notin \bigcup_{\alpha \in S_*} f^{-1}(\alpha) = \bigcup_{\alpha \in S_*} \{\, \beta \in T \mid f(\beta) = \alpha \,\} \\ \alpha \in S_* & \text{if } \beta \in f^{-1}(\alpha) = \{\, \beta \in T \mid f(\beta) = \alpha \,\} \end{cases} .$$

To verify that these do constitute a natural isomorphism we must check that the following diagram commutes:

$$\begin{tikzcd}
(S, s) \arrow[d, "\epsilon_S"'] & (T, t) \arrow[l, "f"'] \arrow[d, "\epsilon_T"] \\
S_* & T_* \arrow[l, "+(f^{-1})"]
\end{tikzcd}$$

Let β be an element of T, t, there are two cases to consider. If $\beta = t$, then

$$\epsilon_S f(t) = \epsilon_S(s) = S_* \quad \text{and} \quad +\left(f^{-1}\right)\epsilon_T(t) = +\left(f^{-1}\right)T_* = S_*.$$

If $\beta \neq t$, then either $f(\beta) = s$ or $f(\beta) \neq s$. In the first case, there is no element $\alpha \in S \setminus \{ s \}$ such that $f(\beta) = \alpha$. Since β is not in any of the fibers from $S \setminus \{ s \}$, by definition it is taken to S_* by $+\left(f^{-1}\right)$, thus

$$\epsilon_S f(\beta) = \epsilon_S(s) = S_* \quad \text{and} \quad +\left(f^{-1}\right)\epsilon_T(\beta) = +\left(f^{-1}\right)(\beta) = S_*.$$

Finally, in the interesting case where $f(\beta) \neq s$, we have $f(\beta) = \alpha$ for some $\alpha \in S \setminus \{ s \}$, so $\beta \in f^{-1}(\alpha)$, and thus

$$\epsilon_S f(\beta) = \epsilon_S \alpha = \alpha \quad \text{and} \quad +\left(f^{-1}\right)\epsilon_T(\beta) = +\left(f^{-1}\right)(\beta) = \alpha.$$

Thus we have defined a natural isomorphism and we may conclude that Γ is equivalent to $\mathsf{Fin}_*^{\mathrm{op}}$. $\square$

Exercise 1.5.iii. Finish the following proof of Lemma 1.5.10:

Lemma 1.5.10. *Any morphism $f\colon a \to b$ and fixed isomorphisms $a \cong a'$ and $b \cong b'$ determine a unique morphism $f'\colon a' \to b'$ so that any of—or, equivalently, all of—the following four diagrams commute:*

$$\begin{tikzcd}
a \arrow[d, "f"'] & a' \arrow[l, "\cong"'] \arrow[d, "f'"] \\
b \arrow[r, "\cong"'] & b'
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, "\cong"] \arrow[d, "f"'] & a' \arrow[d, "f'"] \\
b \arrow[r, "\cong"'] & b'
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[d, "f"'] & a' \arrow[l, "\cong"'] \arrow[d, "f'"] \\
b & b' \arrow[l, "\cong"]
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, "\cong"] \arrow[d, "f"'] & a' \arrow[d, "f'"] \\
b & b' \arrow[l, "\cong"]
\end{tikzcd}$$

For legibility, I use α to denote the isomorphism $\cong\colon a' \to a$ (with α^{-1} denoting its inverse) and β to denote the isomorphism $\cong\colon b' \to b$ (with β^{-1} denoting its inverse.) Furthermore, I refer to the first, second, third, and fourth diagrams, counting from the left.

Proof. The prompt implies that the first diagram, at least, is commutative; so we know that $f' = \beta f \alpha$.

At this point, it might be tempting to immediately right-compose both sides of that expression with α^{-1}, to obtain $f'(\alpha^{-1}) = \beta f \alpha(\alpha^{-1})$. Then we could simplify to obtain $f'\alpha^{-1} = \beta f$, and *voilá!* The second diagram commutes! ......Right?

Well, no, not quite. The logic above is missing a crucial step—it first assumes that we actually *can* right-compose both sides of $f' = \beta f \alpha$ with α^{-1}. It is true that since an isomorphism can always be composed with its inverse, the validity of the composition $\beta f \alpha(\alpha^{-1})$ is trivial. But the validity of $f'(\alpha^{-1})$ is decidedly non-trivial, and this must be proven before the logic above can be applied.

First, it will be useful to explicitly identify the domains and codomains of the morphisms included in each diagram, to make further references more concise. These can easily be inferred from the diagrams: $f\colon a \to b$; $f'\colon a' \to b'$; $\alpha\colon a' \to a$; and $\beta\colon b \to b'$; while the inverses of each morphism go between the same objects, but with the domain and codomain reversed. Now the remainder of the proof becomes almost trivial:

As stated above, $\mathrm{cod}(\alpha^{-1}) = \mathrm{dom}(\alpha) = a' = \mathrm{dom}(f')$. So $\mathrm{dom}(f') = \mathrm{cod}(\alpha^{-1})$, which means $f'\alpha^{-1}$ is a valid composition. With that in mind, we are now able to apply the logic quoted in the note above to show that $f'\alpha^{-1} = \beta f$; and this is sufficient to show that the second diagram commutes.

To show that the remaining diagrams commute, we must prove that $\beta^{-1} f' = f\alpha$ and that $\beta^{-1} f' \alpha^{-1} = f$, for the third and fourth diagrams respectively. Again, it is easy to obtain these expressions by composing β^{-1} with the expressions we have already determined – specifically, by taking the compositions $(\beta^{-1})f' = (\beta^{-1})\beta f \alpha = f\alpha$ for the third diagram and $(\beta^{-1})f'\alpha^{-1} = (\beta^{-1})\beta f = f$ for the fourth. The 'difficult' part is to show that these compositions are valid.

Fortunately, this is still fairly easy: the validity of $\beta^{-1}\beta$ is trivial, and $\mathrm{dom}(\beta^{-1}) = \mathrm{cod}(\beta) = b' = \mathrm{cod}(f')$, so $\beta^{-1} f'$ is valid. This means that the compositions mentioned in the previous paragraph are valid, and therefore that $\beta^{-1} f' = f\alpha$ and $\beta^{-1} f' \alpha^{-1} = f$. So the third and fourth diagrams commute. □

Exercise 1.5.iv. Show that a full and faithful functor $F\colon \mathsf{C} \to \mathsf{D}$ both **reflects** and **creates isomorphisms**. That is, show:

1. If f is a morphism in C so that Ff is an isomorphism in D, then f is an isomorphism.
2. If x and y are objects in C so that Fx and Fy are isomorphic in D, then x and y are isomorphic in C.

Proof. Consider categories C and D, and a functor $F\colon \mathsf{C} \to \mathsf{D}$ that is full and faithful. That is, for all $x, y \in \mathsf{C}$, the function $F\colon \mathsf{C}(x, y) \to \mathsf{D}(Fx, Fy)$ that takes f to Ff is bijective. Now, for a morphism $f\colon x \to y$, suppose that $Ff\colon Fx \to Fy$ is an isomorphism. This means that there exists $G\colon Fy \to Fx$ where $G(Ff) = 1_{Fx}$ and $Ff(G) = 1_{Fy}$. Now, we can apply the definition of full and faithful functor to see that $\mathsf{C}(y, x)$ is in bijection with $\mathsf{D}(Fy, Fx)$ and so there exists a unique $g \in \mathsf{C}(y, x)$ where $Fg = G$. We claim that $g = f^{-1}$.

To show this, we must show that $fg = 1_y$. We consider $F(fg)$, the image of fg under our full and faithful functor. We see that

$$F(fg) = FfFg = FfG = 1_{Fy} = F(1_y)$$

by properties of functors and our previous definitions of g and G. Since we know that $F\colon \mathsf{C}(y,y) \to \mathsf{D}(Fy, F, y)$ is bijective, $F(fg) = F(1_y)$ implies that $fg = 1_y$. We must also show that $gf = 1_x$. We use a similar method and show that

$$F(gf) = FgFf = G(Ff) = 1_{Fx} = F(1_x).$$

Since we again know that $F\colon \mathsf{C}(x,x) \to \mathsf{D}(Fx, Fx)$ is bijective, this implies that $gf = 1_x$. So we have that $fg = 1_y$ and that $gf = 1_x$. Therefore, f is an isomorphism with inverse g.

If Fx and Fy are isomorphic, we know that there exists some isomorphism $G\colon Fx \to Fy$. But since F is a full and faithful functor and therefore $\mathsf{C}(x,y)$ is in bijection with $\mathsf{D}(Fx, Fy)$, $G = Ff$ for some $f\colon x \to y$. By part i, we know that if Ff is an isomorphism, then f is also an isomorphism, so we see here that we have an isomorphism $f\colon x \to y$, and therefore x and y are isomorphic. □

Exercise 1.5.v. Find an example to show that a faithful functor need not reflect isomorphisms.

Proof. Let $F\colon \mathbb{2} \to \mathbb{1}$ be the unique morphism from $\mathbb{2}$ to $\mathbb{1}$. For $x, y \in \mathrm{ob}\,\mathbb{2}$, $\mathbb{2}(x,y)$ either contains one morphism or is empty, thus the function from $\mathbb{2}(x,y)$ to $\mathbb{1}(0,0)$ induced by F is injective. Thus, F is faithful. Let $!\colon 0 \to 1$ be the unique arrow from 0 to 1. Since $F! = 1_0$ and ! is not an isomorphism, then F does not reflect isomorphisms. Thus faithful functors need not reflect isomorphisms. □

Lemma 1.3.8. *Functors preserve isomorphisms.*

Theorem 1.5.9. (characterizing equivalences of categories). *A functor defining an equivalence of categories is full, faithful, and essentially surjective on objects. Assuming the axiom of choice, any functor with these properties defines an equivalence on categories.*

Exercise 1.5.vi.

- i Prove that the composite of a pair of full, faithful, or essentially surjective functors again has the same properties.
- ii Prove that if $\mathsf{C} \simeq \mathsf{D}$ and $\mathsf{D} \simeq \mathsf{E}$, then $\mathsf{C} \simeq \mathsf{E}$. Conclude that the equivalence of categories is an equivalence relation.

Proof.

- Let $F\colon C \to D$ and $G\colon D \to E$ be functors. If F and G are full, but GF is not full, then there is some $c \in \mathsf{C}(x, y)$, for some x, y which make sense, such that $GFc \notin \mathsf{E}(GFx, GFy)$. However, $GFc = G(Fc) = Gd \in \mathsf{E}(GFx, GFy)$ by the fullness of G, so GF is full.
  To show that GF is faithful if F and G are faithful, by a similar argument, if GF were not faithful, then there would be a morphism in $\mathsf{E}(GFx, GFy)$ mapped to by two morphisms of $\mathsf{C}(x, y)$. However, $GFc = G(Fc) = Gd$ for a unique $d \in \mathsf{D}(Fx, Fy)$, which is again injective by the faithfulness of G.
  Finally, to show that GF is essentially surjective if F and G are essentially surjective, we see from Lemma 1.3.8 that functors take isomorphisms to isomorphisms. Since G is essentially surjective, for each $d \in \mathsf{D}$, there is some $e \in \mathsf{E}$ such that $Gd \cong e$. Then, since F has the same property, d must be isomorphic to some Fc, that is, $Fc \cong d$. So we have that $GFc \cong Gd \cong e$, which is the requirement for GF to be essentially surjective.
- If $\mathsf{C} \simeq \mathsf{D}$ and $\mathsf{D} \simeq \mathsf{E}$, then the functors F and G are fully faithful and essentially surjective, and by Theorem 1.5.9, $\mathsf{C} \simeq \mathsf{E}$. We also have that $\mathsf{C} \simeq \mathsf{C}$ (reflexivity), if $\mathsf{C} \simeq \mathsf{D}$ then $\mathsf{D} \simeq \mathsf{C}$ (symmetry), and from what we just showed we get transitivity. Thus, equivalence of categories defines an equivalence relation. □

Exercise 1.5.vii. Let G be a connected groupoid and let G be the group of autmorphisms at any of its objects. The inclusion $\mathsf{B}G \hookrightarrow \mathsf{G}$ defines an equiavlence of categories. Construct an inverse equiavlence $\mathsf{G} \to \mathsf{B}G$.

Proof. To construct the inverse equivalence between G and $\mathsf{B}G$, we must find a fully faithful and essentially surjective functor between these categories. First, we see that since $\mathsf{B}G$ has just one object and any two singleton sets are isomorphic, that any functor $F\colon \mathsf{G} \to \mathsf{B}G$ will be essentially surjective. We define our functor F on the objects of G by sending any object of G to the one object of $\mathsf{B}G$. Before defining our functor on morphisms, we first show that For every trio of objects $x, y, z \in \mathsf{G}$, there exists a bijection from $G = \mathsf{G}(x, x)$ to $\mathsf{G}(y, z)$.

First, we define a class of reference morphisms for our groupoid in the following manner: Because G is connected, we have at least one morphism in $\mathsf{G}(x, y)$ for any $y \in \operatorname{ob}\mathsf{G}$. By the Axiom of Choice, for each y, we can choose a morphism f_y. Note that if G is not small, we must apply the Axiom of Choice in a larger universe. Since G is a groupoid, we also have a morphism f_y^{-1} for each y. We also note that if $x = y$, we choose $f_y = 1_y$.

Now, we use this subclass of morphisms to first determine a bijection ρ between $\mathsf{G}(x, x) \to \mathsf{G}(x, y)$. We define this bijection by sending $\gamma \in \mathsf{G}(x, x)$ to $f_y\gamma \in \mathsf{G}(x, y)$. It is easy to see that this function is injective, because f_y is invertible, and surjective, as for any $g \in \mathsf{G}(x, y)$, $g = \rho(f_y^{-1}g)$, where $f_y^{-1}g \in \mathsf{G}(x, x)$. So we have the desired bijection.

Next, we see that we can define bijections σ from $\mathsf{G}(x, x)$ to $\mathsf{G}(y, x)$, where $\sigma(\gamma) = \gamma f_y^{-1}$ which is bijective by a similar argument as above. We can then compose these bijections to define a bijection $\phi = \sigma\rho$ from $\mathsf{G}(x, x)$ to $\mathsf{G}(y, z)$, where $\phi(\gamma) = f_z\gamma f_y^{-1}$.

Clearly, we also have an inverse bijection $\phi^{-1} \colon \mathsf{G}(x, y) \to G$, where for a $g \in \mathsf{G}(y, z)$ such that $g = f_z \gamma f_y^{-1}$, $\phi^{-1}(g) = \gamma$. We will use this to define our functor on morphisms, so that for a morphism $g \colon y \to z$, $Fg = \phi^{-1}(g)$. First, we note that in this setting $f_x = 1_x$ and that for any y, $f_y = f_y \gamma f_x = f_y \gamma$, so that $F f_y = 1_x$. Also, for any y, $1_y = f_y 1_x f_y^{-1}$, so our functor preserves identities.

Now, we show that our functor preserves composition of morphisms. To do this, consider $g \colon y \to z$ and $h \colon z \to w$, where $g = f_z \gamma_g f_y^{-1}$ and $h = f_w \gamma_h f_z^{-1}$. Now, consider Fhg. We know that $hg = f_w \gamma_h f_z^{-1} f_z \gamma_g f_y^{-1} = f_w \gamma_h \gamma_g f_y^{-1}$. So $Fhg = \gamma_h \gamma_g = FhFg$. So we see that our functor preserves composition of morphisms, and therefore we have a well defined functor.

To see that our functor is fully faithful, we remember that for any $y, z \in \mathrm{ob}\,\mathsf{G}$, $\mathsf{G}(y, z)$ is in bijection with G. Since $Fy = Fz = \emptyset$, the only object of $\mathsf{B}G$, and the set $\mathsf{B}G(\emptyset, \emptyset) = G$, we have a bijection between $\mathsf{G}(y, z)$ and $\mathsf{B}G(Fy, Fz)$ and therefore a fully faithful functor. Therefore, we have a fully faithful functor that is essentially surjective on objects from G to $\mathsf{B}G$.

Now, call the functor defined in 1.5.12 $\iota \colon \mathsf{B}G \to \mathsf{G}$. We must now define natural transformations $\tau \colon F\iota \Rightarrow 1_{\mathsf{B}G}$ and $\eta \colon 1_{\mathsf{G}} \Rightarrow \iota F$. For all $\gamma \in mor\mathsf{B}G$, we have that $F\iota(\gamma) = F(\gamma) = \gamma$ and that $F\iota(\emptyset) = Fx = \emptyset$ so $F\iota = 1_{\mathsf{B}G}$, and the natural transformation is the identity transformation. Now, for each $y \in ob\mathsf{G}$, we must find η_y, so that the following diagram commutes for every $f \colon y \to z$.

$$\begin{tikzcd}
\iota F y = x \arrow[r, "\iota F f"] \arrow[d, "\eta_y"'] & \iota F z = x \arrow[d, "\eta_z"] \\
y \arrow[r, "f"'] & z
\end{tikzcd}$$

We claim that if $\eta_y = f_y$, the reference morphism picked earlier, than we will have formed a natural transformation. First, note that since f can be represented by $f_z \gamma f_y^{-1}$ for some automorphism $\gamma \colon x \to x$ and that $\gamma = f_z^{-1} f f_y$. Also, note that $\iota F f = \iota(\gamma) = \gamma$. So we must show that $f_z \gamma = f f_y$. We know that $f_z \gamma = f_z f_z^{-1} f f_y = f f_y$, so we have the desired equality. So the following diagram commutes, and η is a natural transformation.

$$\begin{tikzcd}
x \arrow[r, "\iota F f"] \arrow[d, "f_y"'] & x \arrow[d, "f_z"] \\
y \arrow[r, "f"'] & z
\end{tikzcd} \tag{1.1}$$

We also see that η_y is an isomorphism for every f_y, because G is a groupoid. So we have shown the existence of the desired natural isomorphisms. Therefore, F and ι define a equivalence between the categories $\mathsf{B}G$ and G. □

The exercise below concerns affine and projective planes as incidence geometries. For background, see Section 2.6 of Hartshorne's *Geometry: Euclid and Beyond*. I adapted the following two definitions from that source[6]. The two examples that I give are standard examples with coordinates in a field k, but note that the definitions make no use of coordinates and the constructions that we will use do not either.

However, the constructions do have natural linear algebra interpretations in the examples with coordinates. I will use some basic facts from linear algebra concerning two and three dimensional vector spaces, such as properties of cross-product. If you do not recall these, then you may ignore the examples and concentrate on the rest.

Definition. An *affine plane* is a triple of sets $\mathbb{A} = (A, L, I)$ with $A \cap L = \emptyset$ where the elements of A are called *points* and the elements of L are called *lines* satisfying the following additional requirements. $I \subseteq A \times L$ is a relation where $sI\ell$ is read as "s lies on ℓ". $\mathbb{A}$ satisfies the following axioms.

1. For any two distinct $s, t \in A$, there is a unique $\ell \in L$ such that s and t lie on ℓ.
2. For every $\ell \in L$ there are at least two distinct $s, t \in A$ that lie on ℓ.
3. There are at least three distinct $s, t, u \in A$ such that there is no $\ell \in L$ such that s, t and u all lie on ℓ.
4. Two lines ℓ and m are said to be *parallel* if either $\ell = m$ or there is no $s \in A$ that lies on both ℓ and m. Write $\ell \parallel m$ if ℓ and m are parallel. For every $\ell \in L$ and $s \in A$, there is a unique $m \in L$ such that $s \in m$ and $\ell \parallel m$.

Definition. A *projective plane* is a triple of sets $\mathbb{P} = (P, L, I)$ with $P \cap L = \emptyset$ where the elements of P are called *points* and the elements of L are called *lines* satisfying the following additional requirements. $I \subseteq P \times L$ is a relation where $sI\ell$ is read as "s lies on ℓ". $\mathbb{P}$ satisfies the following axioms.

1. For any two distinct $s, t \in P$, there is a unique $\ell \in L$ such that s and t lie on ℓ.
2. For every $\ell \in L$ there are at least three distinct $s, t, u \in P$ that lie on ℓ.
3. There are at least three distinct $s, t, u \in P$ such that there is no $\ell \in L$ such that s, t and u all lie on ℓ.
4. For every $\ell, m \in L$ there is an $s \in P$ that lies on both ℓ and m.

Here are two easy lemmas and then two standard examples.

Lemma. *Let ℓ and m be two lines in an affine or projective plane and let s, t be distinct points that lie on both ℓ and m. Then $\ell = m$.*

Proof. The first property in each definition is that there is a unique line containing s and t, so $\ell = m$. □

Lemma. *In an affine plane (A, L, I), $\parallel$ is an equivalence relation.*

Proof. $\parallel$ is clearly reflexive and symmetric. To see that it is transitive, say that $\ell \parallel m$ and $m \parallel n$. We must see that $\ell \parallel n$. If there is a point $s \in A$ such that s lies on both ℓ and n, then since there is a unique line parallel to m on which s lies, both ℓ and n are this line and $\ell = n$. Otherwise, there is no point lying on both ℓ and n. Either way, $\ell \parallel n$. □

[6] I added in each case that the set of points and the set of lines do not intersect so as to avoid annoying set-theoretic problems in the constructions

Example. Let k be a field. Then $\mathbb{A}^2(k) = (k^2, L, \in)$ where the lines are solution sets to equations $ax + by + c = 0$ where $a, b, c \in k$ and at least one of a and b is not zero. Note that if $\lambda \in k^*$, then $ax + by + c = 0$ has the same solutions as $\lambda ax + \lambda by + \lambda c = 0$. This is the only way that two different equations yield the same line. Linear algebra tells us that $\mathbb{A}^2(k)$ satisfies the first property of an affine plane. It is not hard to prove via a parameterization that every line has the same number of elements as k, which is at least 2, so that the second property holds as well. An easy computation shows that for $(0, 0)$, $(1, 0)$ and $(0, 1)$ to all be solutions to $ax + by + c = 0$ then $a = b = c = 0$. So, we have three non-collinear points. Finally, if $s = (s_1, s_2)$ is not a solution to $ax + by + c = 0$, then there is a unique $d \in k$ such that s is a solution to $ax + by + d = 0$ and $d \neq c$. The lines defined by these two equations have empty intersection. So, $\mathbb{A}^2(k)$ satisfies all requirements of an affine plane.

Note that this example shows a way to construct finite affine planes. If $k = \mathbb{F}_q$, then $\mathbb{A}^2(\mathbb{F}_q)$ has q^2 points and $\frac{q(q^2-1)}{q-1} = q(q + 1)$ lines. In fact, $\mathbb{A}^2(\mathbb{F}_2)$ with 4 points and 6 lines is the smallest possible affine plane.

Example. Let k be a field. We will construct a projective plane for which the "points" are lines through the origin in k^3, while the "lines" are planes through the origin in k^3. More specifically, consider the equivalence relation $\equiv$ on $k^3\backslash\{(0, 0, 0)\}$ given by $(\alpha, \beta, \gamma) \equiv (\delta, \epsilon, \phi)$ if there is a $\lambda \in k^*$ such that $(\alpha, \beta, \gamma) = \lambda(\delta, \epsilon, \phi)$. That is, two nonzero vectors are equivalent if they are linearly dependent. (That is, determine the same line through the origin.) Denote the equivalence class of (α, β, γ) by $(\alpha : \beta : \gamma)$. For a linear polynomial $ax + by + cz$ with at least one of a, b, c not zero, if the equation $ax + by + cz = 0$ is satisfied by (α, β, γ) then it is also satisfied by everything in its equivalence class. Thus, it makes sense to say whether or not $(\alpha : \beta : \gamma)$ is a solution to $ax + by + cz = 0$.

$\mathbb{P}^2(k) = ((k^3\backslash\{(0, 0, 0)\})/\equiv, L, \in)$ where the lines are solution sets to equations $ax + by + cz = 0$ with at least one of a, b, c nonzero. As in the previous example, for $\lambda \in k^*$ the solution sets of $ax + by + cz = 0$ and $\lambda ax + \lambda by + \lambda cz = 0$ are the same and this is the only way that two different equations yield the same line.

Starting with the second property, let $ax + by + cz = 0$ be the equation of a line ℓ. If at least two of a, b, c are not zero then $(b : -a : 0), (c : 0 : -a)$ and $(0 : c : -b)$ are three distinct points on ℓ. If $a = b = 0$ so that $c \neq 0$, then $(1 : 0 : 0), (0 : 1 : 0)$ and $(1 : 1 : 0)$ are three distinct points on ℓ. A similar construction applies if $a = c = 0$ or if $b = c = 0$.

Moving to the last property, if two distinct lines have equations $ax + by + cz = 0$ and $dx + ey + fz = 0$, then there is exactly one common solution given by the equivalence class of the cross-product $(a, b, c) \times (d, e, f)$. Thus, given lines ℓ and m, either $\ell = m$ or $\ell \cap m$ has just one point. In either case, $\ell \cap m \neq \emptyset$.

For the first property, given two distinct points s and t with representatives (s_1, s_2, s_3) and $t = (t_1, t_2, t_3)$ if we let $s \times t = (a, b, c)$ then at least one of a, b, c is not 0, since s and t are linearly independent, and s and t are both solutions of to the equation $ax + by + cz = 0$. This line is the unique line with this property, since as we have just seen any two lines that share more than one point are the same line.

Finally, $(1 : 0 : 0), (0 : 1 : 0)$ and $(0 : 0 : 1)$ are not collinear since they are all solutions to $ax + by + cz = 0$ then $a = b = c = 0$. So, $\mathbb{P}^2(k)$ is a projective plane.

As in the previous example, this gives us a way to construct to finite projective planes as $\mathbb{P}^2(\mathbb{F}_q)$ for finite fields $\mathbb{F}_q$, having $\frac{q^3-1}{q-1} = q^2+q+1$ points and also q^2+q+1 lines. The smallest possible projective plane is $\mathbb{P}^2(\mathbb{F}_2)$, which has 7 points and 7 lines. This projective plane is also known as the *Fano plane*.

Proposition. *Let $\mathbb{P} = (P, L, I)$ be a projective plane and let $\ell_\infty \in L$. Let*

$$A = P\backslash\{s \in P | sI\ell_\infty\}, \quad L' = L\backslash\{\ell_\infty\}, \text{ and } I' = I \cap (A \times L').$$

Then (A, L', I') is an affine plane.

Proof. For any two distinct $s, t \in A$ we also have that $s, t \in P$, so that there is a unique $\ell \in L$ such that s and t lie on ℓ. Since $s, t \in A$, $\ell \neq \ell_\infty$ so that $\ell \in L'$. This proves that (A, L', I') satisfies the first axiom of an affine plane.

For $\ell \in L'$, $\ell \neq \ell_\infty$ so that there is exactly one point that lies on both ℓ and ℓ_∞. So, exactly one fewer points in A lie on ℓ than points in P lie on ℓ. Since at least three points in P lie on ℓ, at least two points in A lie on ℓ, satisfying the second axiom of an affine plane.

Since (P, L, I) is a projective plane, there are distinct $s, t, u \in P$ that are not collinear. In particular, at least one of them, say s, does not lie on ℓ_∞. Thus, $s \in A$. Let $\ell \in L$ be the line determined by s and t and let $m \in L$ be the line determined by s and u. Note that $\ell \neq m$ since s, t, u are not collinear. Since there are at least two points of A lying on ℓ and at least two points of A lying on m, there is a $v \in A$ lying on ℓ with $v \neq s$ and a $w \in A$ lying on m with $w \neq s$. If s, v, w all lay on a common line, then that line would share two points each with ℓ and m, so that it would be equal to both ℓ and m. Since $\ell \neq m$, s, v, w do not lie on a common line, satisfying the third axiom of a projective plane.

Finally, let $s \in A$ and $\ell \in L'$. Let t be the unique point that lies on both ℓ and ℓ_∞ and let m be the line in $\mathbb{P}$ determined by s and t. Since s lies on m, $m \neq \ell_\infty$ so that $m \in L'$. If s lies on ℓ then $\ell = m$. If not, then $\ell \neq m$ and since ℓ and m share the common point t in $\mathbb{P}$, they cannot share any points in A. In either case, $\ell \parallel m$ in $\mathbb{A}$. $\square$

Exercise 1.5.viii. Klein's Erlangen program studies groupoids of geometric spaces of various kinds. Prove that the groupoid **Affine** of affine planes is equivalent to the groupoid $\mathbf{Proj}^{\mathbf{l}}$ of projective planes with a distinguished line, called the "line at infinity." The morphisms in each groupoid are bijections of both points and lines (preserving the distinguished line in the case of projective planes) that preserve and reflect the incidence relation. The functor $\mathbf{Proj}^\ell \to \mathbf{Affine}$ removes the line at infinity and the points it contains. Explicitly describe an inverse equivalence.

That the functor $F : \mathbf{Proj}^\ell \to \mathbf{Affine}$ is well-defined on objects is shown by the proposition above. For a morphism $f : (P, L, I, \ell_\infty) \to (Q, M, J, m_\infty)$, the induced morphism $Ff : (A, L', I') \to (B, M', J')$ is given by restricting the bijections $P \to Q$ and $L \to M$ to A and to L'. These restrictions give bijections to B and to M' exactly because the bijection from L to M takes ℓ_∞ to m_∞. Then it is easy to also see that Ff is a morphism, that F takes identities to identities and that F respects composition of morphisms. So, F is a functor.

We will now make a functor $G : \mathbf{Affine} \to \mathbf{Proj}^\ell$. First, we describe G on objects. Let (A, L, I) be an affine plane. Let Π be the set of equivalence classes of L under the relation $\|$. Let $P = A \cup \Pi$ and let $\bar{L} = L \cup \{\ell_\infty^*\}$ for some ℓ_∞^* that is not an element of $A \cup \Pi \cup L$. For the sake of definiteness, take $\ell_\infty^* = \{A \cup \Pi \cup L\}$. Let $\bar{I} \subseteq P \times \bar{L}$ be the relation defined by $s\bar{I}\ell$ if either

1. $s \in A$, $\ell \in L$ and $sI\ell$, or
2. $s \in \Pi$ and $\ell = \ell_\infty^*$, or
3. $s \in \Pi$ and $\ell \in s$.

We must see that $(P, \bar{L}, \bar{I})$ is a projective plane. First, we mention an annoying set theoretic issue. For this to be true, we need among other things that $P \cap \bar{L} = \emptyset$, which is true so long as $\Pi \cap L = \emptyset$. But, it is possible that one of the equivalence classes s of "lines" is already another "line" in L. We can avoid this by replacing each s by the Kuratowski product $s' = s \times L = \{\{s\}, \{s, L\}\}$. This cannot be equal to any $\ell \in L$ for if it were then $\ell \in L \in \{s, L\} \in s' = \ell$, and such loops are impossible under the ZFC axioms. We will move forward as if $\Pi \cap L = \emptyset$ so as to avoid obscuring the ideas under a weight of additional notation. But, everything below can be adjusted to use s' in place of s.

We will start by showing that any two distinct lines have exactly one common point. If $\ell, m \in \bar{L}$ then at least one of them, say ℓ, is in L. If $m = \ell_\infty^*$, then from the second and third cases of the definition of $\bar{I}$, we see that the only point that lies on both ℓ and ℓ_∞^* is the equivalence class s of ℓ under $\|$. If $m \in L$ as well then since $\ell \neq m$, either they are parallel in (A, L, I), in which case the only $s \in P$ that lies on both is their common equivalence class, or they are not parallel, in which case they have one point in common via case (1), but have no point in common in Π.

Now, we will see that any two points determine a unique line. Let $s, t \in P$ be distinct points. They cannot both be on two different lines, since we have just seen that two lines have exactly one point in common. So, it suffices to show that they are on some line. If $s, t \in A$, then we already know that there is an $\ell \in L$ on which both s and t lie since (A, L, I) is an affine plane and I is preserved in $\bar{I}$ via the first case of its definition. If $s, t \in \Pi$, then s and t both lie on ℓ_∞^* by case (2). We are left with the case in which one of them, say s, is in A and the other, t, is in Π. Then by the last axiom of an affine plane, there is a unique line $\ell \in L$ in the equivalence class t on which s lies. But, t also lies on ℓ by case (3).

Now we see that at least three distinct points lie on any line $\ell \in \bar{L}$. If $\ell \in L$, then there are at least 2 points in A that lie on ℓ. But, the equivalence class of ℓ is an element of Π that also lies on ℓ, giving ℓ at least 3 distinct points. The remaining case is $\ell = \ell_\infty^*$, whose points are the elements of Π. So, we must show that there are at least 3 equivalence classes of lines in A. To see that, recall that we are guaranteed three distinct points $s, t, u \in A$ such that there is no line on which they all lie. Let ℓ, m and n be the lines in (A, L, I) determined respectively by s and t, by s and u and by t and u. These three lines are distinct by the choice of s, t, u. But, each pair of ℓ, m, n has a point in common, so no pair is parallel. Therefore, we have at least three equivalence classes of parallel lines, giving at least three points lying on ℓ_∞^*.

Finally, we must guarantee that we have at least three $s, t, u \in P$ that do not lie on any common line in $(P, \bar{L}, \bar{I})$. We may just take $s, t, u \in A$ that do not lie on a common line in (A, L, I). Then they maintain this property in $(P, \bar{L}, \bar{I})$.

So, we take $G(A, L, I) = (P, L, I, \ell^*_\infty)$, the projective plane just constructed with a distinguished line at infinity, also newly constructed.

Say that $f : (A, L, I) \to (B, M, J)$ is a morphism in **Affine**. We must describe

$$Gf : (P, \bar{L}, \bar{I}, \ell^*_\infty) \to (Q, \bar{M}, \bar{J}, m^*_\infty).$$

First note that since f is a bijection on both points and lines and also $sI\ell$ if and only if $f(s)Jf(\ell)$, it follows that $\ell \parallel m$ in (A, L, I) if and only if $f(\ell) \parallel f(m)$ in (B, M, J). So, the bijection from L to M induces a bijection between parallel equivalence classes in L and in M. Thus, f extends to a bijection from P to Q. We also extend f to a bijection from $\bar{L}$ to $\bar{M}$ by taking ℓ^*_∞ to m^*_∞. This gives a description of Gf as a function.

To see that Gf is a morphism in $\mathbf{Proj}^\ell$, it remains to be seen that $s\bar{I}\ell$ in $(P, \bar{L}, \bar{I})$ if and only if $Gf(s)\bar{J}Gf(\ell)$ in $(Q, \bar{M}, \bar{J})$. Examining the three ways in which we could have $s\bar{I}\ell$ in the definition of $\bar{I}$, we see that in each case $Gf(s)$ and $Gf(\ell)$ are in the very same case, and conversely. So, Gf is a morphism in $\mathbf{Proj}^\ell$.

Now, it is easy to that G takes identities to identities and preserves composition of morphisms. So, G is a functor.

Consider the composite functor FG : **Affine** $\to$ **Affine**. G adds new points to A and a new line to L on which the new points lie, but F takes away that new line and also all of those new points returning us to A and to L. Also $(\bar{I})' = \bar{I} \cap (A \times L) = I$. So, on objects FG is the identity. But it is on morphisms as well, since as a function Gf is an extension of f to P and $\bar{L}$, but F just restricts Gf to the original sets, giving that $FGf = f$. So, $FG = 1_{\mathbf{Affine}}$.

However, $GF \neq 1_{\mathbf{Proj}^\ell}$. Indeed, if we write $A = P \backslash \{s | sI\ell_\infty\}$ and $L' = L \backslash \{\ell_\infty\}$ and $I' = I \cap (A \times L')$ as above, then

$$GF(P, L, I, \ell_\infty) = (A \cup \Pi, L' \cup \{\ell^*_\infty\}, \overline{I'}, \ell^*_\infty).$$

But, there is a correspondence between what was taken away by F and what was added by G. Namely, if $s \in P$ lies on ℓ_∞ then the other lines on which s lies form an equivalence class $s^* \subset L'$ for $\parallel$ in (A, L', I'). Indeed, any two distinct elements of s^* share s in P, and so cannot also share any points in A. Any two lines that are parallel in A must share a point in P, necessarily one that lies on ℓ_∞. This gives a bijection

$$\{s \in P | sI\ell_\infty\} \leftrightarrow \{s^* \in \Pi\} = \{t \in A \cup \Pi | t\overline{I'}\ell^*_\infty\}.$$

We can fit these together into a natural transformation $\eta : 1_{\mathbf{Proj}^\ell} \Rightarrow GF$. Namely, if $\mathbb{P} = (P, L, I, \ell_\infty)$ then

$$\eta_{\mathbb{P}} : (P, L, I, \ell_\infty) \to (A \cup \Pi, L' \cup \{\ell^*_\infty\}, \overline{I'}, \ell^*_\infty)$$

is given by

$$s \mapsto \begin{cases} s^* \text{ if } sI\ell_\infty \\ s \text{ otherwise, and} \end{cases}$$
$$\ell \mapsto \begin{cases} \ell_\infty^* \text{ if } \ell = \ell_\infty \\ \ell \text{ otherwise.} \end{cases}$$

It is easy to see that $\eta_{\mathbb{P}}$ is a morphism, and in particular an isomorphism as all morphisms in $\mathbf{Proj}^\ell$ are.

To check that η is a natural transformation, and thus a natural isomorphism, let[7] $\mathbb{Q} = (Q, M, J, m_\infty)$ and $f : \mathbb{P} \to \mathbb{Q}$ be a morphism. Then we need to check that

$$\begin{tikzcd} \mathbb{P} \arrow[r, "f"] \arrow[d, "\eta_{\mathbb{P}}"'] & \mathbb{Q} \arrow[d, "\eta_{\mathbb{Q}}"] \\ GF\mathbb{P} \arrow[r, "GFf"] & GF\mathbb{Q} \end{tikzcd}$$

commutes. Checking along both paths from $\mathbb{P}$ to $GF\mathbb{Q}$, we find that both give

$$s \mapsto \begin{cases} f(s)^* \text{ if } sI\ell_\infty \\ f(s) \text{ otherwise, and} \end{cases}$$
$$\ell \mapsto \begin{cases} m_\infty^* \text{ if } \ell = \ell_\infty \\ f(\ell) \text{ otherwise.} \end{cases}$$

Since η is a natural isomorphism, we have that F and G define an equivalence of categories as claimed.

Exercise 1.5.ix. Show that any category equivalent to a locally small category is locally small.

Proof. First, we will prove a much more general statement; that if there exists a faithful functor $F \colon \mathsf{C} \to \mathsf{D}$, where D is a locally small category, then C must be locally small. The proof follows:

> $F \colon \mathsf{C} \to \mathsf{D}$ is faithful, so for any $x, y \in \mathsf{C}$, the map $F_{x,y} : \mathsf{C}(x, y) \to \mathsf{D}(Fx, Fy)$ is injective. And since there is an injective function $F_{x,y}$ from $\mathsf{C}(x, y) \to \mathsf{D}(Fx, Fy)$, there must be a surjective function $g \colon \mathsf{D}(Fx, Fy) \to \mathsf{C}(x, y)$[1].
>
> Per the Axiom of Replacement, the image of a function whose domain is a set must be a set, so the image of g is a set. But we just said that g is surjective over $\mathsf{C}(x, y)$, so its image is simply $\mathsf{C}(x, y)$; which means that $\mathsf{C}(x, y)$ must be a set! And since this holds for all $x, y \in \mathsf{C}$, C must be locally small. □

[7]Note that $\mathbb{Q}$ does not represent the rational numbers for the moment.

In the particular case specified by the exercise, wherein $\mathsf{C} \simeq \mathsf{D}$, there is by definition a faithful functor $F\colon \mathsf{C} \to \mathsf{D}$. So this is a special case of the general statement proven above; meaning we can immediately conclude that C is locally small.

Exercise 1.5.x. Characterize the categories that are equivalent to discrete categories. A category that is connected and essentially discrete is called **chaotic**.

Proof. We will show that a category C is equivalent to a discrete category D if and only C is preorder and a groupoid. Consider any category C that is equivalent to a discrete category D. By theorem 1.5.9 we have a full, faithful, and essentially surjective functor F from C to D. Since F is both full and faithful, we have for each x, y in $\mathrm{ob}\,\mathsf{C}$ the map $\mathsf{C}(x, y) \to \mathsf{D}(Fx, Fy)$ is a bijection. Because D is a discrete category we know that $\mathsf{D}(Fx, Fy)$ is empty if Fx and Fy are distinct or, if they are the same, consists of only the identity map. Since $\mathsf{C}(x, x) \to \mathsf{D}(Fx, Fx)$ is a bijection and $\mathsf{D}(Fx, Fx)$ only contains the identity map we can conclude that $\mathsf{C}(x, x)$ also only consists of the identity map. Now let $\mathsf{C}(x, y)$ be inhabited by f, so $\mathsf{D}(Fx, Fy)$ is inhabited as well and we must have $Fx = Fy$. Hence there must be exactly one morphism between x and y in C. Using similar reasoning we can conclude $\mathsf{C}(y, x)$ has at most one element. Since both $\mathsf{C}(x, y)$ and $\mathsf{C}(y, x)$ are mapped to the identity of Fx we must have $fg = gf = 1_x$. Hence between any objects x and y in C $\mathsf{C}(x, y)$ has exactly one member or is empty. Since every morphism in C is invertible for any two objects x and y in C there is at most one element in the $\mathsf{C}(x, y)$, C is a groupoid and preorder as desired.

Conversely consider any category C that is both a groupoid and a preorder. For insurance reasons we will work in a universe V, where C is a small category. Now look at the skeleton category of C, the category whose objects are exactly one element from each isomorphism classes (which we obtain by using the axiom of choice, and hence the reason for the insurance policy) of C, denoted as ☠. By definition we know that ☠ $\simeq \mathsf{C}$. Because C is a groupoid, if $\mathsf{C}(x, y)$ is nonempty for any two objects x and y in C, we must have that x is isomorphic to y and therefore in the same isomorphism class. Also since C is a preorder there is at most one element in $\mathsf{C}(x, y)$. So all the morphisms of C are collapsed into the identity morphisms of the appropriate isomorphism classes. Hence the only morphisms of ☠ are the identities of each isomorphism class, so ☠ is discrete category. There for C is equivalent to a discrete category and this completes the proof. □

Exercise 1.5.xi. Consider the functors $\mathsf{Ab} \to \mathsf{Group}$ (inclusion), $\mathsf{Ring} \to \mathsf{Ab}$ (forgetting multiplication), $(-)^{\times}\colon \mathsf{Ring} {\to} \mathsf{Group}$ (taking the group of units), $\mathsf{Ring} \to \mathsf{Rng}$ (inclusion), $\mathsf{Field} \to \mathsf{Ring}$ (inclusion), and $\mathsf{Mod}_R \to \mathsf{Ab}$ (forgetful). Determine which functors are full, which are faithful, and which are essentially surjective. Do any define equivalence of

[1]The principle that "For two sets A and B, if there is a surjection from A to B, then there is an injection from B to A, and vice versa" is called the Partition Principle. While the principle has been known to be a consequence of the Axiom of Choice for quite awhile, it's an open question whether or not it *implies* the Axiom of Choice – in other words, whether it is equivalent to the axiom. Bertrand Russell claimed it did, but he never provided a proof; and while set theorists as a whole have come incredibly close to one since then, they've never quite gotten there.

categories? (Warning: A few of these questions conceal research-level problems, but they can be fun to think about even if full solutions are hard to come by.)

$\mathsf{Ab} \longrightarrow \mathsf{Group}$

Proof. This is a fully faithful functor. For any two groups x and y, the functor maps the group homomorphisms $f\colon x \longrightarrow y$ to themselves. In other words $Ff = f$. This immediately gives us that the functor is faithful, as if $Ff = Fg$, then $f = g$. For it to be full, we must confirm that for any groups $x, y \in \mathsf{Ab}$ that for any $g\colon Fx \longrightarrow Fy$ we can find $f\colon x \longrightarrow y$ such that $Ff = g$. But then we note that g is of domain x and codomain y as F is just inclusion, and that any homomorphism g between two Abelian groups in the category of groups also exists in the category of Abelian groups. Thus, we have the desired surjectivity.

However, our functor is not essentially surjective on objects. We know that for $c \in \mathsf{Ab}$, $Fc = c \in \mathsf{Group}$. For it to be essentially surjective, for any $d \in \mathsf{Group}$, we would need to be able to find an Fc isomorphic to it. But this just means we need to find an Abelian group isomorphic to d. That is impossible for any group d that is not Abelian, as Abelian groups are only isomorphic to Abelian groups. It thus does not define an equivalence of categories, however the Abelian groups are a subcategory. □

$\mathsf{Ring} \longrightarrow \mathsf{Ab}$

Proof. Let F be the functor described above. The additive group of any ring is already abelian, so F takes the additive group of each $r \in \mathsf{Ring}$ (in other words, every $(r, +) \in \mathsf{Ring}$) to the same $(r, +) \in \mathsf{Ab}$; and takes every ring homomorphism $f\colon r \longrightarrow s \in \mathsf{Ring}$ to $f_+\colon (r, +) \longrightarrow (s, +) \in \mathsf{Ab}$, where we define f_+ as exactly the function f, but applied only to the additive group of r (instead of to the whole ring.)

So, in a sense, F is an 'inclusion functor,' taking the additive groups of elements of Ring to those same groups in Ab, and the ring homomorphisms between those additive group in Ring to the same homomorphisms in Ab.

When we say that F takes every morphism to 'itself', what we are really saying is the following: For any $f\colon r \longrightarrow s \in \mathsf{Ring}$, if we define $f_+\colon (r, +) \longrightarrow (s, +)$ as f applied to the additive group of r, we can say that $Ff = f_+ \in \mathsf{Ring}$. So it is trivial that for any $f, g\colon r \longrightarrow s$ such that $Ff \neq Fg$, $f_+ \neq g_+$.

But the ring homomorphisms $f, g\colon r \longrightarrow s$ can be equal only if $f(a + b) = g(a + b)$. In other words, $f = g$ only if $Ff = f_+ = g_+ = Fg$; so if $Ff \neq Fg$, then of course $f \neq g$! So $F(r, s)$ is injective by definition, and since this is the case for all $r, s \in \mathsf{Ring}$, we can conclude that F is faithful.

On the other hand, consider that there are no morphisms from the zero ring, $\{0\}$, to any nonzero $r \in \mathsf{Ring}$. $F\{0\}$ is simply $(\{0\}, +)$; that is, the trivial group. But there is automatically a group homomorphism from the trivial group to any group, which means that the set of morphisms from $F\{0\}$ to Fr is *not* empty for any $r \in \mathsf{Ring}$. So F cannot be surjective over $\{f : F\{0\} \longrightarrow Fr\}$, which means F is not full.

Suppose to the contrary that F is essentially surjective. This means that, for example, there must be some $r \in \mathsf{Ring}$ such that $Fr = (r, +) \simeq \mathbb{Q}/\mathbb{Z} \in \mathsf{Ab}$. Consider that elements of $\mathbb{Q}/\mathbb{Z}$ take the form $\left\{\frac{a}{b} + \mathbb{Z} \mid a, b \in \mathbb{Z}\right\}$; which means that for every $n \in \mathbb{Q}/\mathbb{Z}$, there is some positive integer b such that if you 'multiply' n by b (using the definition we mentioned earlier of "adding n to itself b times") you obtain that $b * n = a + \mathbb{Z} = \mathbb{Z} = 0_{\mathbb{Q}/\mathbb{Z}}$.

But consider the case where $n = 1_r$. We have just determined that there must be some positive integer b such that $b * 1 = 0$. This means that $(r, +)$ must have characteristic b, which itself means that every element of $(r, +)$ must have order $\leq b$. So since $(r, +) \simeq \mathbb{Q}/\mathbb{Z}$, every element of $\mathbb{Q}/\mathbb{Z}$ must also have order $\leq b$. But we know that there is some element of $\mathbb{Q}/\mathbb{Z}$ with order n for *any* positive n; there cannot be a finite b such that every element of $\mathbb{Q}/\mathbb{Z}$ has order $\leq b$. This brings us to a contradiction, which means that our assumption must be false – F must not be essentially surjective. □

Ring → Group

Proof. An isomorphism of groups must preserve cardinality, among other things. Since there are groups of any finite order (consider the cyclic groups), to disprove that the functor is essential surjective it suffices to show that no ring can have a multiplicative group of a specific order. In particular we will consider five.

First, note that in any ring we may consider the multiplicative order of -1, the additive inverse of 1. If 1 is distinct from -1, then -1 has order 2,[8] implying that the multiplicative group of our ring must contain a subgroup of order two, and thus must have even order by Lagrange's theorem. We thus need only consider rings where 1 does not have a distinct additive inverse, i.e. rings of characteristic two.

Suppose that we have a ring R of characteristic two. If the multiplicative group of R has order five, then it must be isomorphic to $\mathbb{Z}/5\mathbb{Z}$ and thus have some element ζ with multiplicative order five, i.e. $\zeta^5 - 1 = 0$.

Now consider the polynomial ring $\mathbb{F}_2[x]$, and the evaluation map $ev_\zeta \colon \mathbb{F}_2[x] \to R$ which takes x to z. The polynomial $f(x) = x^5 - 1$ must be in the kernel of this map, and $x^5 - 1$ factors into $x + 1$ and $x^4 + x^3 + x^2 + x + 1$, both irreducible in $\mathbb{F}_2[x]$.[9]

$\mathbb{F}_2[x]$ is a principle ideal domain, so $x^5 + 1$ must be contained in the ideal generated by itself or one of its factors.

...

So we may factor the evaluation map through the quotient of $\mathbb{F}_2[x]$ by $\ker ev_\zeta$, which must then embed $\mathbb{F}_{16}$ in R meaning that it's multiplicative group has far more than just five elements. Thus the group $\mathbb{Z}/5\mathbb{Z}$ is completely missed by our functor which fails to be essentially surjective.

Now let us consider whether this functor is full or faithful. First, consider the ring of real numbers $\mathbb{R}$ with their usual operations, this has no non-identity homomorphisms.

[8] We have $0 = -1 \cdot 0 = -1(-1 + 1) = -1 \cdot -1 + -1 \cdot 1 = -1 \cdot -1 + -1$ implying that $-1 \cdot -1$ is the additive inverse of -1, so $-1 \cdot -1 = 1$.

[9] To see this note first that it is neither divisible by x nor $x + 1$. So if it were divisible it would be so by two degree two polynomial. However, $x^2 + x + 1$ is the only irreducible degree two polynomial in $\mathbb{F}_2[x]$, and $(x^2 + x + 1)^2 = x^4 + x^2 + 1$.

However, if we consider only $\mathbb{R}^\times$, then there are many group homomorphisms. A typical homomorphism is raising an element to some power. Thus our functor cannot be full.

Finally, let R be a nonzero ring. For any polynomial p with coefficients in R, the is a ring endomorphism of $R[x]$ which takes x to p and thus any other polynomial q to $q(p)$. Note that most of these are note the identity homomorphism. However, if q is a constant polynomial, i.e. just an element of R, then $q(p) = q$. So these endomorphism are all the identity when restricted to the inclusion of R in $R[x]$.

Further, the units of $R[x]$ are precisely the units of R.[10] This means that any of the endomorphisms above will also be the identity on the units of $R[x]$ which means that our functor is not injective on hom-sets. □

Ring → Rng

Proof. Before defining the functor Ring → Rng, first the category Rng needs some description. Since a rng, denoted here by R^-, is a non-unital ring, we note that $(R^-, +)$ is a commutative group and $(R^-, *)$ is a magma with associativity (or a monoid without the condition of identity). Thus every ring, R, is a rng by the fact that R is a monoid under multiplication, and this monoid is certainly a magma with associativity. So the objects in the category Ring are included in the class of objects of Rng.

Let the functor in question be the inclusion functor $\iota\colon \mathsf{Ring} \to \mathsf{Rng}$ that maps rings in Ring to rings in Rng, and ring homomorphispms in Ring to rng homomorphisms between rings in Rng. The rng homomorphisms have all of the properties of ring homomorphisms, except the condition of mapping units in one rng to units in another.

To test whether ι is a full functor, take $\mathbf{0}$ and $\mathbb{Q}$ in Ring. Then $\iota\mathbf{0}, \iota\mathbb{Q}$ are rings included in Rng, and hence are the same $\mathbf{0}$ and $\mathbb{Q}$. Since there exists a rng homomorphism ϕ between $\mathbf{0}$ and $\mathbb{Q}$, i.e., the homomorphism that maps 0 to 0,while there is no ring homomorphisms between the same two objects in Ring, then ϕ is not mapped to by ι acting on any ring homomorphism in Ring. Thus, ι is not surjective between the morphisms fixed on any two objects in Ring. Hence ι is not full.

To test whether ι is faithful, for the objects $c \neq c'$ in Ring, take two homomorphisms ϕ_1 and ϕ_2 in Ring between c, c'. Applying ι to ϕ_1 and ϕ_2 yields $\iota\phi_1$ and $\iota\phi_2$, which remain unequal when applied to the objects $\iota c = c$ and $\iota c' = c'$. Thus ι is injective on morphisms between fixed objects in Ring. Thus ι is faithful.

To test whether ι is essentially surjective, we can take the object in Rng, E of even integers. Since $1 \notin E$, then E is not isomorphic to any ring that is also in Rng. Thus ι is not essentially surjective.

From the theorem characterizing the equivalence of categories, since ι fails to be a full functor from Ring to Rng, then there is not an equivalence of categories between Ring and Rng.

[10]Given an invertible element of R, its inverse is retained on inclusion in $R[x]$ making it a unit of $R[x]$. Conversely, multiplying non-constant polynomials increases their degree meaning they cannot multiply to 1, so these are all the units of $R[x]$.

Field $\longrightarrow$ Ring

Proof. The inclusion functor $\iota\colon$ Field$\to$Ring is faithful: it must take each morphism in the domain to itself in the codomain and is thus injective. To show it is full, let $f\colon x\to y$ be a field homomorphism in Field. It must take $f(0_x) = 0_y$ and $f(1_x) = 1_y$. Since we are only concerned with morphisms between objects we know are fields in the domain (Field) of the functor, we know that in the codomain (Ring) every morphism between ιx and ιy must also have that $\iota f(0_x) = 0_{\iota x}$ and $\iota f(1_x) = 1_{\iota y}$ (and respect the rest of the ring homomorphism requirements). But then ιf must be a field homomorphism, and is thus included in Field.

The inclusion functor is not essentially surjective on objects however. There is no object in Field that this functor takes to the zero ring (and the zero ring is unique up to isomorphism). So, the inclusion map from Field $\longrightarrow$ Ring does not define an equivalence of categories. □

Mod$_R$ $\longrightarrow$ Ab

Proof. This functor is not always essentially surjective. If we let $R = \{0\}$, then the only object Mod$_0$ is the zero module. Thus $F0$ only goes to the trivial group. So if U were to be fully faithful in this case, every Abelian group must be isomorphic to the trivial group. This is clearly not the case since $\mathbb{Z}/2\mathbb{Z}$ (or any finite Abelian group really) cannot be isomorphic to the trivial group. However U is both full and faithful as Mod$_0(0,0)$ and Ab$(F0, F0)$ = Ab$(0,0)$ both only have one element in them. So the only map between Mod$_0(0,0) \longrightarrow$ Ab$(0,0)$ takes the identity map in Mod$_0$ to the identity map in the trivial group, which is clearly bijective.

If we let $R = \mathbb{Z}$, recall that every abelian group can be uniquely expressed as a $\mathbb{Z}$-module. In this case since Mod$_\mathbb{Z}$ is exactly Ab, the forgetful functor U becomes the identity functor, so U is clearly full, faithful, and essentially surjective.

The previous two examples were both full, however, this need not always be the case. If we let $R = \mathbb{Z}_2[x]/x_2 + x + 1$, and look at R as a dimension one vector space over itself we have the following addition and multiplication tables.

$+$	0	1	α	$\alpha+1$
0	0	1	α	$\alpha+1$
1	1	0	$\alpha+1$	α
α	α	$\alpha+1$	0	1
$\alpha+1$	$\alpha+1$	α	1	0

$*$	0	1	α	$\alpha+1$
0	0	0	0	0
1	0	1	α	$\alpha+1$
α	0	α	$\alpha+1$	1
$\alpha+1$	0	$\alpha+1$	1	α

Since R is rank one free module, every endomorphism on R is defined by scalar multiplication. Now forget the scalar multiplication on R, and treat R like an Abelian group. Since R is an Abelian group and in particular a field of characteristic two, the Frobenius endomorphism $a \mapsto a^p$, where p (in this case $p = 2$) is the characteristic of R (this is a field endomorphism so it preserves the additive structure as well even though it is defined in terms of the multiplicative operation which we have technically forgotten) is a member of Ab(UR, UR). There is no module homomorphism that corresponds to the Frobenius endomorphism, since the Frobenius endomorphism fixes 0 and 1 and swaps α and $\alpha + 1$, but every module homomorphism in this case can only fix one element at a time. (We know this because every endomorphism is multiplication by a scalar, so they are all completely described in the multiplication table.) Notice that this counterexample will work for any finite field with characteristic p.

Even though U is not always full, U is always faithful. If we say f and g are distinct morphisms in $\mathsf{Mod}_R(x, y)$, then they must disagree on at least one element, say, z in x. Because Uf and Ug are exactly the same functions in $\mathsf{Ab}(Ux, Uy)$ they disagree on the same element z, so they are distinct in $\mathsf{Ab}(Ux, Uy)$ as well. Hence U is always faithful. $\square$

1.6 The art of the diagram chase

Exercise 1.6.i. Show that any map from a terminal object in a category to an initial one is an isomorphism. An object that is both initial and terminal is called a zero object.

Proof. Let $f : t \to i$, where t is terminal and i is initial. As i is initial, there exists exactly one morphism $g : i \to t$. We must show this g is the inverse of f such that $fg = 1_i$ and $gf = 1_t$. We know that f and g are composable, and we know $fg : i \to i$ and $gf : t \to t$ based on the composition law. As i is initial, there exists exactly one morphism $i \to c$ for any object c, and thus there exists only one morphism $i \to i$, the identity morphism 1_i. Thus, fg must be 1_i. Similarly, as t is terminal there exists exactly one morphism $c \to t$, and thus there exists only one morphism $t \to t$, the identity morphism 1_t. Thus gf must be 1_t. Thus, f is an isomorphism. $\square$

Exercise 1.6.ii. Show that any two terminal objects in a category are connected by a unique isomorphism.

Proof. Let t_0 and t_1 be two terminal objects in a category $f : X \to Y$. That means there is a unique homomorphism $f_0 : x \to t_0$ and $f_1 : x \to t_1$ for all x in X. We want to show that there is a unique isomorphism between t_0 and t_1. Since t_0 is a terminal object, we have a morphism $f_0 : t_1 \to t_0$. And since t_1 is a terminal object, we have a morphism $f_1 : t_0 \to t_1$. Consider the composition $f_0 f_1$. This gives us $f_0 f_1 : t_0 \to t_1 \to t_0$ or more simply $f_0 f_1 : t_0 \to t_0$. So we must have that $f_0 f_1 = 1_{t_0}$. Similarly, it must be that $f_1 f_0 = 1_{t_1}$. This is exactly what it means to be isomorphic. Therefore any two terminal objects in a category are indeed connected by a unique isomorphism. $\square$

Exercise 1.6.iii. Show that any faithful functor reflects monomorphisms. That is, if $F : \mathsf{C} \to \mathsf{D}$ is faithful, prove that if Ff is a monomorphism in D, then f is a monomorphism in C. Argue by duality that faithful functors also reflect epimorphisms. Conclude that in any concrete category, any morphism that defines an injection of underlying sets is a monomorphism and any morphism that defines a surjection of underlying sets is an epimorphism.

Proof. Suppose that $Ff : Fx \to Fy$ is a monomorphism in D, i.e. for any object w in C and parallel morphisms $h, k : w \rightrightarrows x$ then $FfFh = FfFk$ implies that $Fh = Fk$. (Because the property holds over all D, it holds in particular over the image of F in D.)

$$\begin{tikzcd} w \arrow[r, shift left, "h"] \arrow[r, shift right, "k"'] \arrow[d, Rightarrow, "F"'] & x \arrow[r, "f"] \arrow[d, Rightarrow, "F"'] & y \arrow[d, Rightarrow, "F"'] \\ Fw \arrow[r, shift left, "Fh"] \arrow[r, shift right, "Fk"'] & Fx \arrow[r, "Ff"'] & Fy \end{tikzcd}$$

Now, supposing that $fh = fk$ in C, this implies that $F(fh) = F(fk)$ by elementary properties of equality. Then by the functoriality axioms we have $FfFh = FfFk$, and by the fact that Ff is a monomorphism $Fh = Fk$. Finally since F is faithful and thus injective on $\mathsf{C}(w, x)$, $h = k$. This argument amounts to pushing equality clockwise around the above diagram.

Note that this also proves that a functor reflects epimorphisms, since an epimorphisms is just a monomorphism in the opposite category. The argument above will compose neatly with applying the op functor at the beginning and end to transport us to the right category.

Further, given a concrete category C we have a faithful functor from C to Set. Since monomorphisms in Set are completely characterised by injectivity, this becomes sufficient condition for a map to be a monomorphism in C. Similarly, surjectivity in Set will force epic in C. Frequently, the faithful functor in question is just the forgetful functor and the maps in C are just functions defined on sets with peculiar features. Thus it is sensible to talk about injectivity and surjectivity in C itself, and we can say that injectivity and surjectivity are sufficient—but not necessary—conditions for a map to monic or epic respectively. □

Exercise 1.6.iv. Find a example to show that a faithful functor need not preserve epimorphisms. Argue by duality, or by another counterexample, that a faithful functor need not preserve monomorphisms.

Proof. Consider the category Ring and the unique morphism $\phi\colon \mathbb{Z} \to \mathbb{Q}$. Exercise 1.2.v shows that ϕ is an epimorphism, however, it is easy to see that ϕ is not surjective. Now, consider a functor $F\colon \mathsf{Ring} \to \mathsf{Group}$ that takes a ring R to it's additive group and a morphism $f\colon R \to S$ to the corresponding group homomorphism on the additive group. We see that this functor is faithful, because for fixed rings R and S and morphisms $f, g\colon R \to S$, $Ff = Fg$ implies that $Ff(x) = Fg(x)$ for all $x \in G_R$, the additive group of R, which has the same elements as R. But since by our defintion $Ff(x) = f(x)$ and $Fg(x) = g(x)$, this implies that $f = g(x)$ for all $x \in R$ and so $f = g$. Therefore, for all $x, y \in \mathsf{Ring}$, there is a injection from $\mathsf{Ring}(x, y) \to \mathsf{Group}(Fx, Fy)$ and therefore F is faithful. Now, note that in Group, epimorphisms correspond exactly to surjective homomorphisms. But it is clear that $F\phi$ is not surjective, as its behavior on the elements of $\mathbb{Z}$ and $\mathbb{Q}$ is identical to that of ϕ. So $F\phi$ is not an epimorphism. Therefore, F does not preserve epimorphisms.

Now, we consider $F\colon \mathsf{Ring}^{\mathrm{op}} \to \mathsf{Group}^{\mathrm{op}}$, where F acts on objects and morphisms as before. By 1.3.v, there is no difference between a functor from Ring to Group and a functor from $\mathsf{Ring}^{\mathrm{op}}$ to $\mathsf{Group}^{\mathrm{op}}$, so F is still faithful is this setting. Now, we note that the epimorphisms in Ring and Group are precisely the monomorphisms in $\mathsf{Ring}^{\mathrm{op}}$ and $\mathsf{Group}^{\mathrm{op}}$. So ϕ is a monomorphism in $\mathsf{Ring}^{\mathrm{op}}$, but not in $\mathsf{Group}^{\mathrm{op}}$. Therefore, F does not preserve monomorphisms. So a faithful functor need not preserve monomorphisms. □

Exercise 1.6.v. Find a concrete category that contains a monomorphism whose underlying function is not injective. Find a concrete category that contains an epimorphism whose underlying function is not surjective.

Proof. For the first example, take a subcategory of Set, C, consisting of two objects, A being the set with elements 0, 1, and 2, and B being the set with elements 0, and 1. Take as morphisms between A and B the identity morphisms, 1_A and 1_B, along with the morphism, $f\colon A \to B$, defined by $f(0) = 0$, $f(1) = 0$, and $f(2) = 0$. Since $fg = fh$ implies $g = h = 1_A$, then f is a monomorphism. Yet since f sends 0 to 0 and 1 to 0 in the underlying sets, then f is not injective.

For the second example, take the objects $\mathbb{Z}$ and $\mathbb{Q}$ in the category Ring. There is a unique ring homomorphism between $\mathbb{Z}$ and $\mathbb{Q}$. This homomorphism is not surjective on the underlying sets, yet from the previous exercise 1.2.v, this homomorphism is an epimorphism. □

Exercise 1.6.vi. A **coalgebra** for an endofunctor $T\colon \mathsf{C}\to\mathsf{C}$ is an object $C \in \mathsf{C}$ equipped with a map $\gamma\colon C\to TC$. A morphism $f\colon (C,\gamma)\to(C',\gamma')$ of coalgebras is a map $f\colon C\to C'$ so that the following square commutes.

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\gamma"'] & C' \arrow[d, "\gamma'"] \\ TC \arrow[r, "Tf"'] & TC' \end{tikzcd}$$

Prove that if (C,γ) is a **terminal coalgebra**, that is a terminal object in the category of coalgebras, then the map $\gamma\colon C\to TC$ is an isomorphism.

Proof. Suppose that $T\colon \mathsf{C}\to\mathsf{C}$ is an endofunctor and (C,γ) a terminal coalgebra. Let $\chi\colon TC\to C$ be the unique morphism so that the following diagram commutes.

$$\begin{tikzcd} TC \arrow[r, "\chi"] \arrow[d, "T\gamma"'] & C \arrow[d, "\gamma"] \\ TTC \arrow[r, "T\chi"'] & TC \end{tikzcd}$$

Then we have that:

$$\gamma\chi = T(\chi\gamma)$$

Now we will to show that the diagram

$$\begin{tikzcd} C \arrow[r, "\chi\gamma"] \arrow[d, "\gamma"'] & C \arrow[d, "\gamma"] \\ TC \arrow[r, "T(\chi\gamma)"'] & TC \end{tikzcd}$$

commutes. Since $\gamma\chi = T(\chi\gamma)$, then composing γ on the right will give

$$\gamma\chi\gamma = T(\chi\gamma)\gamma,$$

which shows that the diagram above commutes. This means that the morphism $\chi\gamma$ uniquely allows the diagram above to commute. The identity morphism 1_C gives an endomorphism of the terminal coalgebra (C, γ) since $\gamma\ 1_C = T1_C\gamma = 1_{TC}\gamma$ by the properties of identity morphisms, thus $\chi\gamma = 1_C$. This will give us

$$\gamma\chi = T(\chi\gamma) = T1_C = 1_{TC}.$$

Thus χ and γ are inverses, therefore γ is an isomorphism. $\square$

An example of a co-algebra is one defined by the endofunctor $P_{\text{fin}} : \mathsf{Set} \to \mathsf{Set}$ where P_{fin} is the functor mapping a set X to the set of finite subsets of X, and maps a morphism $f : X \to Y$ to $P_{\text{fin}} f : P_{\text{fin}} X \to P_{\text{fin}} Y$ where for finite subset S of X, $P_{\text{fin}} f(S) = f(S)$.