Formal preliminaries answer key

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1 Part 1

Prove that $A \cap B \subseteq A$.

By definition, this holds if everything in $A \cap B$ is also in A. Since any $x \in A$ and $x \in B$ if $x \in A \cap B$, then $x \in A$, as needed.

2 Part 2

Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Going left to right, we show that if $x \in A \cap (B \cup C)$, then $x \in (A \cap B) \cup (A \cap C)$. Since $x \in A$ and $x \in B \cup C$, it must be that $x \in B$ or $x \in C$. If the former, then $x \in A \cap B$, and hence $x \in (A \cap B \cup A \cap C)$. If the latter, then $x \in A \cap C$, and hence $x \in (A \cap B \cup A \cap C)$.

Going right to left, we show that if $x \in (A \cap B) \cup (A \cap C)$, then $x \in A \cap (B \cup C)$. There are two cases to consider: either $x \in A \cap B$ or $x \in A \cap C$. In the first case $x \in A$ and $x \in B$; thus $x \in B \cup C$ and, hence, $x \in A \cap (B \cup C)$. In the second case, $x \in A$ and $x \in C$; thus again $x \in B \cup C$ and, hence, $x \in A \cap (B \cup C)$.

3 Part 3

Prove that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

If $A \subseteq B$, then any $x \in A$ is such that $x \in B$. If $B \subseteq C$, then any $y \in B$ is such that $y \in C$. Since any $x \in A$ is such that $x \in B$, x is such a y, and so $x \in C$.

4 Part 4

Prove that if $A \subseteq B$, then $A = A \cap B$.

Going left to right, any $x \in A$ is such that $x \in B$, since $A \subseteq B$. Hence, $x \in A \cap B$ by definition.

Going right to left, any $x \in A \cap B$ is such that $x \in A$, by definition.

5 Part 5

Let R be a relation on A. Define R^* as

$$R^* = \bigcup_{i \in \mathbb{N}} R^i$$

Prove that for any relation S on A which is transitive and reflexive and such that $R \subseteq S$, $R^* \subseteq S$.

Since $\langle x, y \rangle \in R^*$ just in case $\langle x, y \rangle \in R^i$ for some $i \in \mathbb{N}$, we just need to show $R^i \subseteq S$ for arbitrary i.

First, note that $R^0 \subseteq S$, since S is reflexive.

Now, assume $R^i \subseteq S$: we want to show that $R^{i+1} \subseteq S$ as well. For any $\langle u, v \rangle \in R^{i+1} = R^i \circ R$, there must be a z such that $\langle u, z \rangle \in R^i$ and $\langle z, v \rangle \in R$. Since S contains R^i, $\langle u, z \rangle \in S$, and since S contains R, $\langle z, v \rangle \in S$. Because S is transitive, it is therefore also true that $\langle u, v \rangle \in S$, as needed.

Hence, $R^i \subseteq S$ for all i (as needed) by induction on i.

6 Part 6

Let $A = \{a, b, c, d\}$ and $R = \{\langle a, a \rangle, \langle a, b \rangle, \langle b, c \rangle, \langle c, c \rangle\}$. Then:

- $R^{-1} = \{\langle a, a \rangle, \langle b, a \rangle, \langle c, b \rangle, \langle c, c \rangle\}$
- $R^2 = \{\langle a, a \rangle, \langle a, b \rangle, \langle b, c \rangle, \langle a, c \rangle, \langle c, c \rangle\}$
- $R^3 = \{\langle a, a \rangle, \langle a, b \rangle, \langle b, c \rangle, \langle a, c \rangle, \langle c, c \rangle\}$
- $R^* = \{\langle a, a \rangle, \langle a, b \rangle, \langle b, c \rangle, \langle a, c \rangle, \langle c, c \rangle\}$
- The least equivalence relation containing R:

$$\{\langle a, a \rangle, \langle b, b \rangle, \langle a, b \rangle, \langle b, a \rangle, \langle b, c \rangle, \langle c, b \rangle, \langle a, c \rangle, \langle c, a \rangle, \langle c, c \rangle\}$$

7 Part 7

Let $R \subseteq A \times B$. Define $f_R \subseteq A \times 2^B$ such that $\langle a, X \rangle \in f_R$ iff $X = \{b \in B \mid \langle a, b \rangle \in R\}$. Show that f_R is a function from A to 2^B.

We need to show that for any given $a \in A$, X as defined above is unique. Assume both $\langle a, X \rangle \in f_R$ and $\langle a, Y \rangle \in f_R$. Then $X = \{b \in B \mid \langle a, b \rangle \in R\} = Y$, as needed.