Course Info

Tuesday, September 6, 2016 8:32 AM

Linear Algebra 223
Bogdan Nica
Prefers being called Bogdan
Burnside 1119

Grading scheme:
10% HW
+ 25% Midterm
+ 65% Final

Or
10% HW
+15% Midterm
+75% Final

Homework once every 2 weeks, 5 assignments
Will be put on mycourses
Given on Tuesday, give next Tuesday

Come in Tuesday morning and drop in assignment right away (on time, beginning of class)
Late assignments not accepted

Write full name, student number, staple pages

Exams: closed book, no calculator

Office Hours
Tuesday, Wednesday, Thursday
10-11

Can also talk to teaching assistant: Brahim

Tutorial not compulsory, highly recommended, for your own benefit

For emails, put MATH 223 in the subject line

Seymour Linear Algebra

30$? Plenty of exercises

Going to use it as a backbone for the course
Using more or less the material in the book

Come prepared, take good notes, stay focused, ask questions, enjoy

Compared to MATH 133 - Linear Algebra and Geometry

This course will be more abstract, talk more about structures

Talk to the teacher to let him know where you're standing, during class, ask questions, office hours

Start assignments early
Do many exercises

Introduction

Tuesday, September 6, 2016 8:58 AM

What is linear algebra?
Practical activity

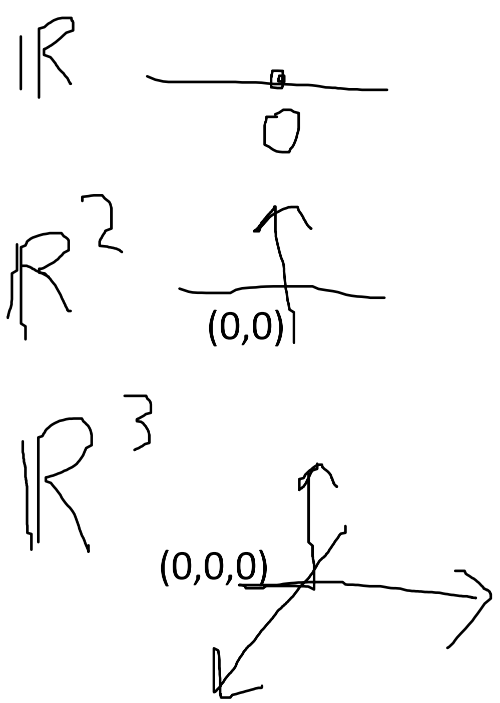
Calculus & linear algebra are 2 pillars of general math
Algebraic counterpart of Cal II

-low-level abstraction

-study of linear spaces and linear mappings

<u>The n-space</u>
What's around us?
In linear algebra, it's the 3-dimensional space $\mathbb{R}^3$

What would we like to do?



<u>Addition</u>	<u>(skalar) Multiplication</u>
a, b -> a+b	k, a -> ka
$u=(a_1,a_2)$ $v=(b_1,b_2)$ $u+v=(a_1+b_1,a_2+b_2)$	k real $u=(a_1,a_2)$ $ku=(ka_1,ka_2)$
$u=(a_1,a_2,a_3)$ $v=(b_1,b_2,b_3)$ $u+v= (a_1+b_1,a_2+b_2,a_3+b_3)$	k real $u=(a_1,a_2,a_3)$ ku=(

$\mathbb{R}^n$ = set of all n=tuples of real numbers
= $\{u = (a_1, ..., a_n) a_1, ..., a_n \in \Re\}$

$u = (a_1, ..., a_n)$ point/vector

Entries/components/coordinates

Addition

$u, v \in \Re^n$
$u = (a_1, ..., a_n)\ v = (b_1, .., b_n)$
$u + v = (a_1 + b_1, .., a_n + b_n)$

Scalar multiplication

SCALAR
$k \in \Re\ u \in \Re^n$
$u = (a_1, .., a_n)$
$k \cdot u = (k \cdot a_1, ..., k \cdot a_n)$

Multiple of u

Given $u_1,…,u_n \in \Re^n$

$k_1 \cdot u_1 + ... + k_m \cdot u_n = \Re^n$ <u>linear combination</u> of $u_1, ..., u_n$
<u>Example:</u> let $u_1 = (1, 0, 0), eu_2 = (0, 1, 0), u_3 = (0, 0, 1)0$
Then $7u_1 - u_2 + 5u_3 = (7, -1, 5)$ this is a linear combination

Basic vectors

Tuesday, September 6, 2016 9:16 AM

Basic vectors in $\Re^n$

$$
\begin{aligned}
e_1 &= (1, 0, ..., 0)\\
e_2 &= (0, 1, ..., 0)\\
&\dots\\
e_n &= (0, 0, ..., 1)
\end{aligned}
$$

Theorem: Every vector in $\mathbb{R}^n$ is a linear combination of the basic vectors $e_1,\dots,e_n$

Proof

Let $u = (a_1, \dots, a_n) \in \Re^n$

$$
u = \underbrace{(a_1, 0, \dots, 0)}_{a_1 \cdot e_1} + \dots + \underbrace{(0, \dots, 0, a_n)}_{a_n \cdot e_n}
$$

$$
= a_1 \cdot e_1 + \dots a_n \cdot e_n
$$

(coefficients are the coordinates)

$\square$

End of proof

Basic properties of vector addition & scalar multiplication

Theorem

for $u, v, w \in \Re^n$

1. $u + (v + w)) = (u + v) + w$
2. $u + v = v + u$
3. $u + 0 = 0 + u = u$
4. $u + (-u) = (-u) + u = 0$

Theorem

for $u, v \in \Re^n$ and $k, l \in \Re$

1. $k(u + v) = k \cdot u + k \cdot v$
2. $(k + l)u = k \cdot u + l \cdot u$
3. $k(lu) = (kl) \cdot u$

Norm & dot product

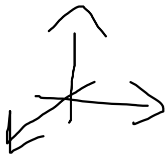
$\mathbb{R}^2$: $u = (a_1, a_2)$

$\|u\| =$ distance from u to 0

3. $k(lu) = (kl) \cdot u$

$\mathbb{R}^2$  $u = (a_1, a_2)$ $\quad ||u|| =$ distance from u to 0

$= \sqrt{a_1^2 + a_2^2}$

$\mathbb{R}^3$  $u = (a_1, a_2, a_3)$

$||u|| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

$\mathbb{R}^n$ $\quad u = (a_1, \ldots, a_n)$

Norm of u $\quad ||u|| = \sqrt{\underbrace{a_1^2 + \cdots + a_n^2}}$

Norm $\qquad$ Non-negative real

$|| \; || : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$

Norm: takes n space gives non negative real numbers

Example Let $e_1, \ldots, e_n$ in $\mathbb{R}^n$

$||e_1|| = 1$

Any other vector of norm 1 is a unit vector

$(1, 1)$ $\quad ||e_1 + e_2|| = \sqrt{2}$

Behavior of norm wrt addition/scalar multiplication

Theorem $\qquad$ Triangle inequality/Minkowski inequality

(i) $||u+v|| \leq ||u|| + ||v||$ $\qquad$ For all u,v $\in \mathbb{R}^n$

(ii) $||k \cdot u|| = |k| \, ||u||$

Proof of (ii)

Let $u = (a_1, \ldots, a_n) \in \Re^n$
$ku = (ka_1, \ldots, ka_n)$
$||ku||^2 = (ka_1^2) + \ldots + (ka_n^2)$
$= k^2 \underbrace{(a_1^2 + \ldots + a_n^2)}_{||u||^2}$
$||ku||^2 = (k||u||)^2$
$|ku| = |k| \underbrace{||u||}_{\geq 0} | = |k| \cdot ||u||$

$\square$

Dot product

$u, v \in \Re^n$ $u \cdot v$ (or (u, v) or $< u, v >$) $\in \Re$
$u = (a_1, ..., a_n), v = (b_1, ..., b_n)$
$uv = a_1 b_1 + ... + a_n b_n$
note $u \cdot u = ||u||^2$
Example $e_1, ..., e_n \in \Re^n$
$e_i \cdot e_j = 0$ if i≠j, 1 if i=j
Exercises 1.5, 1.6, 1.9, 1.11, 1.13

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Lecture 2

Last time we talked about

$\mathbb{R}^n$: the n-space

We defined addition, subtraction of vectors, multiplicati

Started talking about dot product

Orthonor

$e_i \cdot e_j = \begin{cases} 1 & \text{if } i \\ 0 & \text{if } i \end{cases}$

orthogon

$u, v \in \mathbb{R}^n$ $\quad u \cdot v \in \mathbb{R}$

$u \cdot v \to$ scalar

$u \cdot u = \|u\|^2$

Basic properties of dot product

Theorem: For $u, v, w \in \mathbb{R}^n$, $k \in \mathbb{R}$

(i) $u(v+w) = u \cdot v + u \cdot w$

(ii) $u \cdot v = v \cdot u$

(iii) $(ku) \cdot v = kuv = u(kv)$

Theorem: (Cauchy-Schwarz)

For $u, v \in \mathbb{R}^n$ we have $|u \cdot v| \leq \|u\| \cdot \|v\|$

Theorem (Minkowski / Triangle inequality)

For $u, v \in \mathbb{R}^n$, $\|u+v\| \leq \|u\| + \|v\|$

Proof: We compare $\|u+v\|^2$ and $(\|u\|+\|v\|)^2$

$$\|u+v\|^2 = (u+v)\cdot(u+v) = u \cdot u + u \cdot v + v \cdot u + v \cdot v$$
$$= \|u\|^2 + 2u \cdot v + \|v\|^2$$

$$(\|u\|+\|v\|)^2 = \|u\|^2 + 2\|u\| \cdot \|v\| + \|v\|^2$$

A-HA: Cauchy-Schwarz

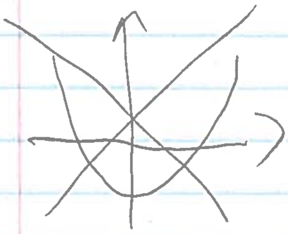
$u\cdot v \le |u\cdot v| \le \|u\| \, \|v\|$ $\square$

Proof of C.-S.: For every $t\in\mathbb{R}$ we have $(tu+v)\cdot(tu+v)\ge 0$

$(tu+v)\cdot(tu+v) = t^2\|u\|^2 + 2t\,u\cdot v + \|v\|^2$

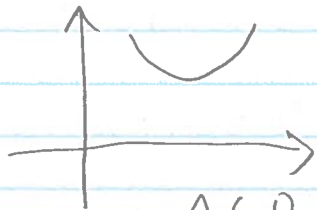
think of it as a quadratic polynomial

$= \underset{\|u\|^2}{\underset{\shortparallel}{a}}\,t^2 + \underset{2u\cdot v}{\underset{\shortparallel}{b}}\,t + \underset{\|v\|^2}{\underset{\shortparallel}{c}} \ge 0$ for every $t\in\mathbb{R}$



$\Delta > 0$

$\Delta = b^2 - 4ac$



$\Delta \le 0$ $\quad b^2 \le 4ac$

$b^2 \le 4ac$ means

$(\cancel{2}u\cdot v)^{\cancel{2}} \le \cancel{4}\|u\|^{\cancel{2}}\|v\|^{\cancel{2}}$

$|u\cdot v| \le \|u\|\cdot\|v\|$ $\square$

$\mathbb{C}$ complex numbers

$=\{z = a+bi : a,b\in\mathbb{R}\}$ $\quad \boxed{i^2=-1}$

what is i?

(a,b) $\quad i=(0,1)$

Read section in book

imaginary part ← Im z

$z = a + bi$

real part ← Re z

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$\overline{\overline{z}} = z$

$\overline{z_1+z_2} = \overline{z_1}+\overline{z_2}$

$\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$

multiplication of complex numbers, not dot product

conjugate of a complex number

$\overline{z} = a - bi$

$|z| = \sqrt{a^2+b^2}$

$z \cdot \overline{z} = |z|^2$

$\frac{1}{z} = \frac{\overline{z}}{|z|^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} \cdot i$ provided $z \neq 0$

$\mathbb{C}$, just as $\mathbb{R}$, is a field

What is a field?
System of numbers which can be added/subtracted and multiplied/divided ($\neq 0$)

There is an expansion of numbers

$$\underbrace{\mathbb{N} \leq \mathbb{Z}}_{\text{Not fields}} \leq \underbrace{\mathbb{Q} \leq \mathbb{R} \leq \mathbb{C}}_{\text{Fields}}$$

There are Finite fields (Don't need to know, extra info)

$\{0,1\}$ binary operations

$1+1=0$

This list of numbers are either primes or powers of a prime

2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 19

$\mathbb{C}^n$: the complex n-space

$= \{u = (a_1, \dots, a_n) \cdot a_1, \dots, a_n \in \mathbb{C}\}$

we can do:

addition

scalar multiplication (scalar is in $\mathbb{C}$)

dot product $u \cdot v = a_1 \cdot \overline{b_1} + \dots + a_n \cdot \overline{b_n}$ ← complex conjugate

theorem For $u, v, w \in \mathbb{C}$, $k \in \mathbb{C}$

(i) $u(v+w) = u \cdot v + u \cdot w$

(ii) $u \cdot v = \overline{v \cdot u}$

(iii) $(ku) \cdot v = k\, u \cdot v = \cancel{u(kv)}$

$u(kv) = \overline{k}\, u \cdot v$

$u, v \in \mathbb{C}^n$

$u \cdot v = a_1\overline{b_1} + \dots + a_n\overline{b_n}$

$u = (a_1, \ldots, a_n) \quad a_i \in \mathbb{C}$

$v = (b_1, \ldots, b_n) \quad b_i \in \mathbb{C}$

$(ku) \cdot v = = (ka_1) \cdot \bar{b}_1 + \cdots + (ka_n) \cdot \bar{b}_n = k u \cdot v$

$u \cdot (kv) = a_1 \cdot \overline{kb_1} + \ldots + a_n \cdot \overline{kb_n} = \bar{k} u \cdot v$

Read section 1.7 (complex n space)

Exercises
1.30-1.34 → warmup, operations
$\boxed{1.35, 1.36, 1.39, 1.40}$ more serious

Later:
vector/linear space
(informally)
- set whose elements are "vectors"
- vectors can be added (vector)+(vector) = (vector)
- vectors can be multiplied by scalars ← come from a field $\mathbb{R}$ or $\mathbb{C}$
  (scalar)(vector) = (vector)

inner product space = vector space + inner product

13/9/16

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Lecture 3

Final may have problems with complex numbers, familiarize yoursel

Matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{m1} & a_{m2} & & a_{mn} \end{pmatrix}$$

column

row.

entries denoted by $a_{ij} \in \mathbb{R}$ (or $\mathbb{C}$)

$A = (a_{ij})$ to denote size $\to$ $m \times n$ matrix

$M_{m \times n}(\mathbb{R})$: set of all $m \times n$ matrices with real entrie

$$= \{A = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}} : a_{ij} \in \mathbb{R}\}$$

case $m = n$ (square matrices)

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

diagonal (for square matrices only)

Example identity matrix

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \\ & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

Matrix addition

A, B matrices same size $\rightsquigarrow A+B$

$$\begin{cases} A=(a_{ij}) & B=(b_{ij}) \\ A+B = (a_{ij}+b_{ij}) \end{cases}$$

note if $m=1$
vector addition
in $\mathbb{R}^n$ (or $\mathbb{C}^n$)

Scalar multiplication

scalar $k \in \mathbb{R}$, A matrix $\to kA$

$$\begin{cases} A=(a_{ij}) \\ kA=(ka_{ij}) \end{cases}$$

note if $m=1$
scalar multiplication
for $\mathbb{R}^n$

Theorem: For $A, B, C \in M_{m \times n}(\mathbb{R})$, and $k, \ell \in \mathbb{R}$ the following hold

(i) $(A+B)+C = A+(B+C)$
(ii) $A+B=B+A$
(iii) $k(A+B) = kA+kB$
(iv) $k(\ell A) = (k\ell)A$
More in textbook

Later on
$M_{m\times n}(\mathbb{R})$
linear space

Proof:
(iii) let $A=(a_{ij})$, $B=(b_{ij})$
ij-entry of $k(A+B)$ is $k(a_{ij}+b_{ij})$
ij-entry of $kA+kB$ is $ka_{ij}+kb_{ij}$

$\Rightarrow kA+kB$ and $k(A+B)$ are equal
(they are equal entry by entry) $\square$

Rest.
Exercise

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223 P2

The trace (for square matrices)

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}}_{A} \longmapsto \underbrace{a_{11}+a_{22}+\dots+a_{nn}}_{tr(A)}$$

$$tr(I_n) = n$$

$tr: M_{n\times n}(\mathbb{R}) \longrightarrow \mathbb{R}$ trace map

trace is a mapping on $n\times n$ matrices that gives real numbers

T / F: The trace map is one-to-one / injective
outputs can repeat $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ $\begin{pmatrix}2&13\\0&0\end{pmatrix}$

T / F: —— " —— onto / surjective
diagonals can add to any real number

given $a \in \mathbb{R}$

$$tr\begin{pmatrix} a & 0 & & 0 \\ 0 & & \ddots & \\ & & & 0 \end{pmatrix} = a$$

signifies entries not dimension

$$A = (a_{ij})_{1\le i,j\le n} \qquad tr(A) = \sum_{i=1}^{n} a_{ii}$$

Theorem The following hold

(i) $tr(A+B) = tr(A) + tr(B)$
(ii) $tr(kA) = k\,tr(A)$

for all $A, B \in M_{n\times n}(\mathbb{R})$ and $k \in \mathbb{R}$

Proof: (i) Let $A=(a_{ij})$, $B=(b_{ij})$ the $A+B=(a_{ij}+b_{ij})$

$$tr(A+B) = \sum_{i=1}^{n} (a_{ii}+b_{ii})$$

$$tr(A)+tr(B) = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii}$$

(ii) $$tr(kA) = \sum_{i=1}^{n} ka_{ij}$$

$$k\,tr(A) = k \sum_{i=1}^{n} a_{ij}$$

Q.: Is there something like basic vectors $e_1, \dots, e_n$ For $M_{m\times n}(\mathbb{R})$?

They are basic vectors because any vector in $\mathbb{R}^n$ is a lc of basic vectors.

Yes: elem. matrices

$$\begin{pmatrix} 0 & \cdots & & 0 \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & & 0 \end{pmatrix} = e_{ij}$$

basis

proof that every $m\times n$ matrix is a linear combination of e_{ij}

$$A=(a_{ij})$$

$$A = \sum_{i,j} a_{ij} e_{ij}$$

Later on

trace map is linear (theorem from before)

Matrix multiplication

$\left.\begin{array}{lll} A & m\times n & \text{matrix} \\ B & n\times p & \text{matrix} \end{array}\right\} \rightarrow AB \quad m\times p \quad \text{matrix}$

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223 p3

$A = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$ $\quad B = (b_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le p}}$

$AB = (c_{ij})$ $\quad \boxed{c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}}$

Example

$A \in M_{n \times n}(\mathbb{R})$

$A I_n = A$

$I_n A = A$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Read sections 2.1-2.4 / 5.2

Exercises 2.14 / 5.1-5.8

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Lecture 4

Matrix multiplication

$$\left.\begin{array}{ll} A & m \times n \text{ matrix} \\ B & n \times p \text{ matrix} \end{array}\right\} \rightsquigarrow AB \quad m \times p \text{ matrix}$$

$$A = (a_{ij}) \qquad B = (b_{ij}) \qquad AB = \Big(\underbrace{\sum_{k=1}^{n} a_{ik} b_{kj}}_{ij\text{-entry}}\Big)$$

One of the most important cases

<u>$m = n = p$</u>

$$\underset{n \times n}{\boxed{}} \cdot \underset{n \times n}{\boxed{}} = \underset{n \times n}{\boxed{}}$$

$M_{n \times n}(\mathbb{R})$ <u>closed</u> under matrix multiplication
(means whenever you operate in the system, you stay in the system)

$A \cdot I_n = I_n \cdot A = A \qquad$ for all $A \in M_{n \times n}(\mathbb{R})$

↑ identity matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Kronecker delta

using $\sum$ notation $\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

$A = (a_{ij})$ $\quad I_n = (\delta_{ij})$

$A \cdot I_n$ ij-entry is $\sum_{k=1}^{n} a_{ik}\delta_{kj} = a_{ij}$

$I_n \cdot A$ ij-entry is $\sum_{k=1}^{n} \delta_{ik} a_{kj} = a_{ij}$

! matrix multiplication is not commutative: in general, $AB \neq BA$

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \boxed{a} & b \\ 2c & \boxed{2d} \end{pmatrix}$$

not =

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \boxed{a} & 2b \\ c & \boxed{2d} \end{pmatrix}$$

if A $m \times n$ $\quad AB$ $m \times m$ matrix
B $n \times m$ $\quad BA$ $n \times n$ matrix

<u>Theorem:</u> For all matrices A, B, C

(i) $A(BC) = (AB)C$

(ii) $A(B+C) = AB + AC$ and $(B+C)A = BA + CA$

(iii) $(kA)B = k(AB) = A(kB)$

(provided correct sizes)

223 p2

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Proof (i) Let $A=(a_{ij})$ $B=(b_{ij})$ $C=(c_{ij})$

ij-entry of $A(BC)$

$$\sum_k a_{ik} \underbrace{\left(\sum_\ell b_{k\ell} c_{\ell j}\right)}_{kj\text{-entry of } BC}$$

ij-entry of $(AB)C$

$$\sum_k \underbrace{\left(\sum_\ell a_{i\ell} b_{\ell k}\right)}_{ik \text{ entry of } AB} c_{kj}$$

Distribute and put both sigmas on left

Theorem. The trace satisfies $tr(AB)=tr(BA)$

Proof: Let $A=(a_{ij})$ $B=(b_{ij})$

$$tr(AB) = \sum_k \underbrace{\left(\sum_\ell a_{k\ell} b_{\ell k}\right)}_{kk\text{ entry of } AB}$$

$k \longleftrightarrow \ell$

$$tr(BA) = \sum_k \underbrace{\left(\sum_\ell b_{k\ell} a_{\ell k}\right)}_{kk \text{ entry of } BA}$$

Definition: A square matrix $A \in M_{n\times n}(\mathbb{R})$ is invertible/non-singular if there exists another $B \in M_{n \times n}(\mathbb{R})$ such that $AB=BA=I_n$

Fact: If such B exists, then it is unique

Suppose B_1 and B_2 both satisfy

$$AB_1 = B_1A = I_n, \quad AB_2 = B_2A = I_n$$

$\boxed{B_1AB_2}$

$$\left.\begin{array}{l} (B_1A)B_2 = I_n \cdot B_2 = B_2 \\ B_1(A \cdot B_2) = B_1 \cdot I_n = B_1 \end{array}\right\} \Rightarrow B_1 = B_2$$

and we call B the inverse of A
(because it's unique)

and we denote it by A^{-1}

note: A invertible, then so is A^{-1}
$(A^{-1})^{-1} = A$

Example I_n invertible, $I_n^{-1} = I_n$

Example O_n singular
zero-matrix

Example: $A = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}$ if all $a_i \neq 0$

$A^{-1} = \begin{pmatrix} \frac{1}{a_1} & & & \\ & \frac{1}{a_2} & & \\ & & \ddots & \\ & & & \frac{1}{a_n} \end{pmatrix}$

if some $a_i = 0$

$$\begin{array}{r} \\ i^{th} \text{ row} \\ \\ \end{array} \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots \ \boxed{0} & \\ & & & a_n \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ & & & \\ b_{n1} & \cdots & & b_{nn} \end{pmatrix} = \begin{pmatrix} & & & \\ 0 & 0 & \cdots & 0 \\ & & & \end{pmatrix} \begin{array}{l} \\ i^{th} \text{ row} \\ \\ \end{array}$$

all 0, cannot be I_n

$\neq I_n$

15/9/16

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223 P3

(same size)

Theorem: A, B invertible $\Rightarrow AB$ invertible

Proof A^{-1}, B^{-1} inverses of A, B

$$AB(B^{-1}A^{-1}) = I_n$$

$$(B^{-1}A^{-1})(AB) = I_n$$

$$(AB)^{-1} = B^{-1}A^{-1} \quad \blacksquare$$

Read: Sections 2.5, 2.7-2.9

Exercises: 2.4-2.11

2x2 matrices

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad tr(A) = a+d$

$\det A = ad - bc$

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = \underbrace{(ad-bc)}_{\det A} I_2$$

if $\det A \neq 0$ $\quad A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

if $\det A = 0$ $\quad A^2 = 0$

$A\underset{\uparrow \text{random matrix}}{B} \overset{?}{=} 0$

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Lecture 5

2x2 matrices

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\det(A) = ab - bc$
$tr(A) = a + d$

$A' = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ $AA' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \det A & 0 \\ 0 & \det A \end{pmatrix}$

$A'A = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Case 1 $\det A \neq 0$
A non-singular, with inverse
$A^{-1} = \frac{1}{\det A} \underbrace{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}_{A'}$

Case 2 $\det A = 0$ $\cancel{A^2 = 0}$ A singular

why? $AA' = 0$, assume A is non-singular
some A^{-1} exists

A' | $\underbrace{\underbrace{(A^{-1}A)}_{I_2}A'}_{A'} = \underbrace{A^{-1} \cdot 0}_{0}$ $A' = 0$

$\Rightarrow a = b = c = d = 0$
$\Rightarrow A = 0$, contradiction

if $AX = 0$, $X \neq 0$
$\Rightarrow$ A singular

Theorem (Cayley-Hamilton)

Let A 2×2 matrix. Then $A^2 - tr(A)\cdot A + det(A) I_2 = 0$

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$A^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+bc & \cancel{ab}+bd \\ \cancel{ac}+dc & cb+d^2 \end{pmatrix}$$

$$tr(A)\cdot A = \begin{pmatrix} (a+d)a & (a+d)b \\ (a+d)c & (a+d)d \end{pmatrix} = -\begin{pmatrix} a^2+ad & \cancel{ab}+ad \\ \cancel{ac}+dc & ad+d^2 \end{pmatrix}$$

$$= \begin{pmatrix} bc-ad & \\ & bc-ad \end{pmatrix} = -det(A)\cdot I_2$$

Remark: if $det A \neq 0$ then

$$det A \cdot I_2 = \underbrace{tr(A)A - A^2}_{A(tr(A)I_2 - A)}$$

$$(tr(A)I_2 - A)\cdot A$$

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \cancel{a}+d & 0 \\ 0 & a+\cancel{d} \end{pmatrix} - \begin{pmatrix} \cancel{a} & b \\ c & \cancel{d} \end{pmatrix}$$

Examples: Inverses

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$$

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}^{-1} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

2019/16

223 p2

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Example Compute $\begin{pmatrix} 1+i & 2 \\ 1 & 1-i \end{pmatrix}^{10}$

you can do diagonilisation, but we don't know yet

Cayley-Hamilton: $A=\begin{pmatrix} 1+i & 2 \\ 1 & 1-i \end{pmatrix}$, $tr(A)=2$, $det(A)=0$

$A^2-2A=0$, $A^2=2A$

(multiply each side by A) $A^3=2A^2=2(2A)=2^2A$

$A^{10}=2^9A=512A$

Types of square matrices.

(1) Diagonal

(2) Triangular

(3) Symmetric

(1) A $n\times n$ matrix $A=\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$

examples $\begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix}$, $KI_n=\begin{pmatrix} K & & \\ & \ddots & \\ & & K \end{pmatrix}$, 0_n

$A=(a_{ij})$ $a_{ij}=0$ whenever $i\neq j$

Theorem: The set of $n\times n$ diagonal matrices is closed under the following operations.

- addition
- scalar multiplication
- matrix multiplication
- inverse (if defined)

$$\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} + \begin{pmatrix} b_1 & & \\ & \ddots & \\ & & b_n \end{pmatrix} = \begin{pmatrix} a_1+b_1 & & \\ & \ddots & \\ & & a_n+b_n \end{pmatrix}$$

$$K\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} = \begin{pmatrix} Ka_1 & & \\ & \ddots & \\ & & Ka_n \end{pmatrix}$$

$$\begin{pmatrix} \boxed{a_1} & & \\ & \ddots & \\ & & a_n \end{pmatrix}\begin{pmatrix} \boxed{b_1} & & \\ & \ddots & \\ & & b_n \end{pmatrix} = \begin{pmatrix} a_1b_1 & & \\ & \ddots & \\ & & a_nb_n \end{pmatrix}$$

$$\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{a_1} & & \\ & \ddots & \\ & & \frac{1}{a_n} \end{pmatrix}$$

How do we write down if A, B diagonal $\Rightarrow AB$ diagonal

$A = (a_{ij})$ $\quad \underbrace{a_{ij} = 0 \text{ for } i \neq j}_{\text{all off diagonal entries vanish}}$

$B = (b_{ij})$ $\quad b_{ij} = 0$ for $i \neq j$

ij entry of AB
$i \neq j$

$$\sum_{k=1}^{n} a_{ik}b_{kj} \overset{\boxed{k=i}}{=} \underbrace{a_{ii}}_{\text{only non-zero}} \underbrace{b_{ij}}_{0} = 0$$

22/9/16
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Lecture 6

Types of square matrices (square matrices behave better)

{diagonal} (best behaved class) $\subseteq$ {triangular}
$\cap$ |
{symmetric}

sub?

② Triangular matrices

upper triangular matrix

lower triangular matrix

examples: diagonal matrices: upper and lower,

$$\begin{pmatrix} 1 & 2 & 3 \\ & 1 & 2 \\ & & 1 \end{pmatrix} \text{ upper triangular}$$

if upper & lower → diagonal

$$\begin{pmatrix} 0 & 2 & 3 \\ & 0 & 2 \\ & & 0 \end{pmatrix} \text{ (strictly) upper triangular}$$

upper & lower include diagonal

• elementary matrices e_{ij}

upper: $i \le j$ lower: $i \ge j$

(i,j)-entry

$$\begin{pmatrix} & & \\ & 1 & \\ & & \end{pmatrix} = e_{ij}$$

Test $\cancel{e_{11}}$, $e_{12} \to \begin{pmatrix} 0 & 1 & \dots & 0 \end{pmatrix}$

$A = (a_{ij})$ upper triangular: $a_{ij} = 0 \quad i > j$

lower triangular: $a_{ij} = 0 \quad i < j$

Theorem: Upper triangular matrices (of fixed size) closed under:
- addition
- Scalar multiplication
- matrix multiplication
- inversion (whenever defined)

Try to prove matrix multiplication (with a_{ij} notation)

illustration

some complicated thing

$$\begin{pmatrix} a_{11} & * & * \\ & a_{22} & * \\ & & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & * & * \\ & b_{22} & * \\ & & b_{33} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & * & * \\ 0 & a_{22}b_{22} & * \\ 0 & 0 & a_{33}b_{33} \end{pmatrix}$$

gets more complicated in comp.

Example:

$$3\times 3 \begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix}^{-1}$$

$$\begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$a + x = 0 \qquad x = -a$

$c + y = 0 \qquad y = -c$

$z + ay + b = 0 \qquad z = ac - b$

$$\begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -a & ac-b \\ & 1 & -c \\ & & 1 \end{pmatrix}$$

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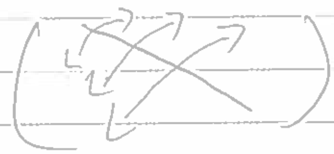
223 p2

(3) Symmetric matrices



$A = (a_{ij})$ <u>symmetric</u> if $a_{ij} = a_{ji}$

<u>example</u>: acquaintance (not real name) matrix (appears in real life)

group of people, if 2 people know each other $\to$ 1

	Andreas	Veronica	Gluida
Andreas	1	0	0
Veronica	0	1	0
Gluida	0	0	1

if 2 people know each other, symmetric entries

<u>example</u>: distance matrix (thousands of km)

	Montreal	Montevideo	Montreux
Montreal	0	9.1	5.9
Montevideo	9.1	0	11
Montreux	5.9	11	0

<u>Transpose</u>

A : $m \times n$ matrix $\quad A = (a_{ij})$

ij - entry of A^T = ji - entry of A

A^T : $n \times m$ matrix

$$A = \begin{pmatrix} \boxed{R_1} \\ \boxed{R_2} \\ \cdots \end{pmatrix} \qquad A^T = \begin{pmatrix} \boxed{R_1} & \boxed{R_2} & \cdots \end{pmatrix}$$

Examples: $\begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$

$(1 \quad 2 \quad 3)^T = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$(A^T)^T = A$
$(A+B)^T = A^T + B^T$
$(kA)^T = kA^T$
$(AB)^T = B^T A^T$

proof: $A = (a_{ij})$ $B = (b_{ij})$

ij-entry of $(AB)^T$ = ji-entry of $AB = \sum_{k=1} a_{jk} \cdot b_{ki}$

ij-entry of $B^T A^T = \sum_{k} \underbrace{b_{ki}}_{ik\text{-entry of } B^T} \underbrace{a_{jk}}_{kj\text{-entry of } A^T}$

T/F: A invertible
$\Rightarrow A^T$ invertible

$A^T \cdot (A^{-1})^T = (A^{-1}A)^T = I^T = I$

$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$

$\boxed{\therefore (A^T)^{-1} = (A^{-1})^T}$

$(A^T)^T = A$

$(AB)^T = A^T + B^T$

$(kA)^T = kA^T$

$(AB)^T = B^T A^T$

2/19/16
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223 p3

A symmetric $\iff A = A^T$

Theorem: Symmetric matrices (of fixed size)
closed: - addition
- scalar multiplication
- inversion
~~- matrix multiplication~~

Proof: A, B symmetric

$(A+B)^T = A^T + B^T = A + B$

$\Rightarrow A+B$ symmetric

$(AB)^T = B^T A^T = BA \neq AB$

Exercise: counter example

A invertible, symmetric
? A^{-1} symmetric

$(A^{-1})^T = (A^T)^{-1} = A^{-1}$

dot product or matrix multiplication

$u, v \in \mathbb{R}^n$ $\quad u = (a_1, \dots, a_n)$
$\quad v = (b_1, \dots, b_n)$

$\boxed{\text{dot product} \quad u \cdot v = a_1 b_1 + \dots + a_n b_n}$

matrix multiplication

$= (a_1 \dots a_n) \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \boxed{u v^T}$

? dot product in $\mathbb{C}^n$ -> we know
Transpose in $\mathbb{C}$? $u \cdot v = u \cdot v^H$

Conjugate transpose - (for complex matrices!)

A complex matrix example

A^H or A^* or A^+ $\begin{pmatrix} 2 & 1+i \\ 0 & 1 \end{pmatrix}^H = \begin{pmatrix} 2 & 0 \\ i & 1 \end{pmatrix}$

$A^H = \overline{(A^T)} = (\bar{A})^T$

$(A^H)^H = A$

Real square matrices		Complex square matrices	
symmetric	$A = A^T$	hermitian	$A = A^H$
orthogonal	$A^{-1} = A^T$	unitary	$A^{-1} = A^H$
normal	$AA^T = A^TA$	normal	$AA^H = A^HA$

Read sections 2.6, 2.10, 2.11

Exercises 2.12, 2.14, 2.24-2.32

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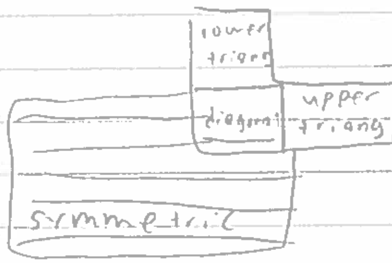
Lecture 7

Types of square matrices

- diagonal
- triangular
- symmetric



$$\{\underbrace{\text{diagonal}}_{a_{ij}=0,\ i\neq j}\} = \{\underbrace{\text{upper triangular}}_{a_{ij}=0\ i>j}\} \cap \{\underbrace{\text{lower triang}}_{a_{ij}=0\ i<j}\}$$

e & 2

$$\{\text{diagonal}\} = \{\underbrace{\text{symmetric}}_{a_{ij}=a_{ji}}\} \cap \{\underbrace{\text{lower triang}}_{a_{ij}=0\ i<j}\}$$

Real square matrices			Complex square matrices	
symmetric	$A=A^T$	$\Rightarrow$	hermitian	$A=A^H$
orthogonal	$A^{-1}=A^T$		unitary	$A^{-1}=A^H$
normal	$AA^T=A^TA$		normal	$AA^H=A^HA$

Remark symmetric (hermitian) $\Rightarrow$ normal
orthogonal (unitary) $\Rightarrow$ normal

Example: $\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{P_1}$, $\underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{P_2}$, $\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{P_3}$ Pauli matrices

All 3 are hermitian

hermitian

$$P_2^{H} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}^{T} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = P_2$$

2 × 2 hermitian matrices

$$A = \begin{pmatrix} u & v \\ w & z \end{pmatrix} \in M_2(\mathbb{C})$$

$$A^{H} = \begin{pmatrix} \bar{u} & \bar{w} \\ \bar{v} & \bar{z} \end{pmatrix}$$

A hermitian

$$\begin{pmatrix} u & v \\ w & z \end{pmatrix} = \begin{pmatrix} \bar{u} & \bar{w} \\ \bar{v} & \bar{z} \end{pmatrix}$$

$$\begin{cases} u = \bar{u}, & z = \bar{z} \iff u, z \in \mathbb{R} \\ v = \bar{w}, & w = \bar{v} \iff w = \bar{v} \end{cases}$$

hermitian $\begin{pmatrix} a & v \\ \bar{v} & b \end{pmatrix}$ $a, b \in \mathbb{R}$, $v \in \mathbb{C}$

Fact. Every 2×2 hermitian matrix is a linear combo of I_2, P_1, P_2, P_3

$$\begin{pmatrix} a & v \\ \bar{v} & b \end{pmatrix} = \begin{pmatrix} a & c+di \\ c-di & b \end{pmatrix} = cP_1 - dP_2 + \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

$$\frac{a+b}{2} I + \frac{a-b}{2} P_3$$

$v = c + di$

$c, d \in \mathbb{R}$

$$\begin{pmatrix} \frac{a+b}{2} & \\ & \frac{a+b}{2} \end{pmatrix} + \begin{pmatrix} \frac{a-b}{2} & \\ & \frac{b-a}{2} \end{pmatrix}$$

Example (orthogonal)

$$A(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \leftarrow \text{orthogonal}$$

$$A(\theta)^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = A(\theta)^{T}$$

$$B(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

orthogonal &

↓ symmetric

$$B(\theta)^{-1} = \frac{1}{-1} \begin{pmatrix} -\cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} = B(\theta)^{T}$$

$$= B(\theta)$$

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hermitian

unitary

Example $A = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}$ unitary? No : not even invertible

Example $A = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$ unitary?

$= iI$ yes

$A = \lambda \cdot I_n \quad \lambda \in \mathbb{C}$

when is A unitary? $AA^H = A^H A = I_n$

$(A^{-1} = A^H)$

$(\lambda \cdot I_n)(\bar{\lambda} \cdot I_n) = I_n$

$|\lambda|^2 \cdot I_n = I_n$

$(\lambda \cdot I_n)^H = \bar{\lambda} I_n$

$\boxed{|\lambda| = 1}$

(structure) (just like invertible matrices)

Theorem the product of two orthogonal (unitary) matrices is again, orthogonal (unitary)

The inverse of an orthogonal (unitary) matrix is orthogonal (unitary)

Proof: A, B orthogonal $A^{-1} = A^T, B^{-1} = B^T$

? AB orthogonal: ? $(AB)^{-1} = (AB)^T$

indeed! $(AB)^{-1} = \boxed{B^{-1} A^{-1}}$

$(AB)^T = \boxed{B^T A^T}$

A orthogonal ? A^{-1} orthogonal

? $(A^{-1})^{-1} = \underbrace{(A^{-1})^T}_{(A^T)^T}$

$\boxed{A = A}$

Vector space

informally: set of elements ("vectors") closed under addition, scalar multiplication

behave like diagonals, just rows instead

vs ✓ $\boxed{\mathbb{R}^n \text{ real } n\text{-space}}$ (scalar: real)

vs ✓ $\mathbb{C}^n$ complex n-space (scalar: real and complex)

vs ✓ $M_{m\times n}(\mathbb{R})$, $M_{m\times n}(\mathbb{C})$

vs ✓ {scalar $n\times n$ matrices} like $\mathbb{R}$ $(\text{scal})\cdot I_n$

vs ✓ {diagonal $n\times n$ matrices} (structure) ("algebra") also works with multiplication

vs ✓ {triangular $n\times n$ matrices}

vs ✓ {symmetric $n\times n$ matrices}

~~vs~~ {invertible $n\times n$ matrices}

~~vs~~ {orthogonal $n\times n$ matrices}

~~vs~~ {unitary $n\times n$ matrices}

scalar multiplication must include 0 as a scalar (structure) ("group")

Definition: A vector space V over a field K (of scalars) is a set endowed with 2 operations.

addition: $u, v \in V \rightsquigarrow u+v \in V$

scalar multiplication: $k \in K,\ u \in V \rightsquigarrow ku \in V$

such that the following axioms hold:

27/9/16

223 P3 Julian Lore

(A1) $(u+v)+w = u+(v+w)$ Associative

(A2) there exists a zero vector $0 \in V$ such that $u+0 = 0+u = u$

(A3) for each $u \in V$ there exists a negative $-u \in V$ such that $u+(-u) = (-u)+u = 0$

(A4) $u+v = v+u$

(M1) $k(u+v) = ku+kv$

(M2) $(k+l)u = ku + lu$ $(k, l \in K)$

(M3) $k(lu) = (kl)u$

(M4) $1 \cdot u = u$

(end)

Read sections 4.1–4.3
Exer. 2.56, 2.65–2.67, 2.71

29/9/16

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Lecture 8

Midterm: Tentative
Oct. 21
6-7:30
McMed?

Theorem: $M_{m\times n}(\mathbb{R})$ real vector space
"dim = mn"

$M_{m\times n}(\mathbb{C})$ complex vector space
(real, as well) "dim = mn"

Thm $\mathbb{R}^n$ real vector space "dim = n"
$\mathbb{C}^n$ complex vector space (real, as well)
"dim = n" "dim = 2n"

General Principle

$\left.\begin{array}{l} V \text{ vector space over a field } K \\ K' \text{ smaller field (inside } K) \end{array}\right\} \Rightarrow V$ vector space over K'

complex vector space is also a real vector space

Ex (n=1) $\mathbb{C}$ over $\mathbb{C}$ dim = 1
$\mathbb{C}$ over $\mathbb{R}$ dim = 2

$K = \mathbb{R}$ or $\mathbb{C}$

Example: The set of diagonal matrices in $M_n(K)$ is a vector subspace "dim = n"

We don't want to check all axioms, as this is a subset of something that is already a vector space (it inherits some properties). For this, we need a definition.

Def: Let V vector space (over a field of scalars K) A subset $W \subseteq V$ is a subspace if it is a vector space under the operations (vector addition / scalar multiplication) inherited from V.

(iii) 

$W = \{(a,b)\ a,b \geq 0\}$

Thm. A subset $W \subseteq V$ is a subspace if and only if.

(i) $0 \in W$

(ii) $u, v \in W \Rightarrow u+v \in W$

(iii) $u \in W$, $k \in K \Rightarrow ku \in W$

Proof: Exercise

Back to example:

Why is it a subspace?

(i) 0 diagonal

(ii) A, B diagonal $\Rightarrow A+B$ diagonal

(iii) A diagonal, $k \in K \Rightarrow kA$ diagonal

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \end{array}$$
$$\begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{pmatrix}$$

Example The set of $\left(\frac{\text{lower}}{\text{upper}}\right)$ triangular matrices in $M_n(K)$ is a vector subspace.

have to choose only 1, can't mix

"$\dim = \frac{n(n+1)}{2}$"

Example. The set of symmetric matrices in $M_n(\mathbb{R})$ is a vector subspace.

"$\dim = \frac{n(n+1)}{2}$"

Example The set of hermitian matrices in $M_n(\mathbb{C})$ is a vector subspace, when $M_n(\mathbb{C})$ real vector space

refers to type of scalars I can multiply by real

hermitian: $A^H = A$

i) 0 hermitian.

ii) A, B hermitian $\Rightarrow A+B$ hermitian. ✓ $\quad (A+B)^H = \underset{A}{A^H} + \underset{B}{B^H}$

iii) A hermitian, $\lambda \in \mathbb{C} \Rightarrow \lambda A$ hermitian

$$(\lambda A)^H = \overline{(\lambda A)}^T = \bar{\lambda} A$$

Not the same

How to fix?

Restrict λ to $\mathbb{R}$

29/9/16
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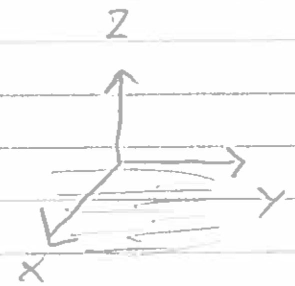
223 p2

Example. $V = \mathbb{R}^3$ real vector space



$W_1 = \{(x,y,z) \in \mathbb{R}^3 : z = 0\}$ subspace

$= \{(x,y,0) : x, y \in \mathbb{R}\}$

$W_2 = \{(x,y,z) : x = y = z\}$

$= \{(x,x,x) : x \in \mathbb{R}\}$

General Principle: given V vector space

$\{0_V\}$ subspace V subspace

whatever 0 is in V

Example: $Pol(K)$ set of all polynomials with coefficients in K

$Pol(K) = \{a_0 + a_1 t + a_2 t^2 + \dots : a_k \in K\}$

4/10/16

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Lecture 9

Vector spaces & subspaces

$Pol(K)$: set of all polynomials with coeff. in a field K.

$\{P(t) = a_0 + a_1 t + \dots + a_n t^n : a_1, \dots, a_n \in K,$ n arbitrary$\}$

Slight diff between polynomial & polynomial function

(t is just a diff thing from coeff) (can plug in a t)

$\deg P(t)$ = largest n such that $a_n \neq 0$ $\{$ DNE $\deg(0) := -\infty$

$$\deg(1 + t + 0 \cdot t^2 + t^3 + \cancel{0 \cdot t^4}) = 3$$

$\begin{cases} + & (2 - t + t^3) + (2t - t^3) = 2 + t \\ \cdot & 3(1 + t^2) = 3 + 3t^2 \end{cases}$

$\boxed{Pol(K) \text{ vector space}/K}$

When you do such a statement, have to check all axioms

$Fun(X, K)$ set of all functions from a (non-empty) set X to a field K.

Check all

$Fun(\{1, 2, 3\}, \mathbb{R})$ like $\mathbb{R}^3$ vector space over K

(isomorphism, see later)

Thm Let V vector space over a Field K. Then a subset $W \subseteq V$ is a subspace iff

(i) $0 \in W$

(ii) $u, v \in W \Rightarrow u+v \in W$

(iii) $k \in K, u \in W \Rightarrow ku \in W$

Example: $Pol_n(K)$ = set of all polynomials of degree $\leq n$ subspace of $Pol(K)$ $\{p(t) = a_0 + a_1 t + \dots + a_n t^n : a_i \in K\}$

fix n

(i) $0 \in Pol_n(K)$

could be less, cancel each other

(ii) $P, Q \in Pol_n(K)$ $\deg(P+Q) \leq \underbrace{\max\{\underbrace{\deg P}_{\leq n}, \underbrace{\deg Q}_{\leq n}\}}_{\leq n}$

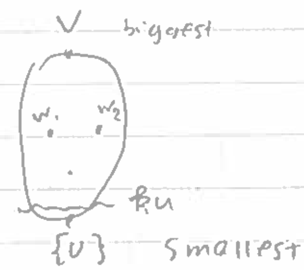
(iii) $k \in K$, P $\deg(kP) = \deg(P)$ $(k \neq 0)$

closed interval

Example: subspaces of $Fun([a,b], \mathbb{R})$

- continuous functions $[a,b] \to \mathbb{R}$
- differential functions $[a,b] \to \mathbb{R}$
  - 2^{nd} differential, 3^{rd}, ...
  - more stuff

V: vector space over K ? subspaces



Example: $\{0\}$ & V

Example: $u \in V$ vector

$\langle u \rangle = \{ku : k \in K\}$ subspace

$0 \cdot u = 0$ ✓

ku, lu $ku + lu = (k+l)u$

$k(lu) = (kl)u$

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223 P2

Example: W_1, W_2 subspaces $\Rightarrow W_1 \cap W_2$ subspace

(i) $0 \in W_1 \cap W_2$ ✓ since $0 \in W_1, 0 \in W_2$

(ii) $u, v \in W_1 \cap W_2$? $u+v \in W_1 \cap W_2$

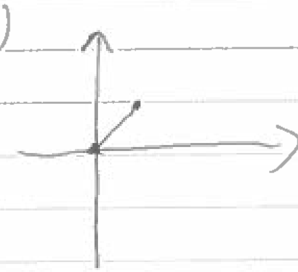
$u, v \in W_1 \Rightarrow u+v \in W_1$

$u, v \in W_2 \Rightarrow u+v \in W_2$

(iii) $k \in K, u \in W_1 \cap W_2 \Rightarrow ku \in W_1 \cap W_2$ ■

T/F: W_1, W_2 subspaces $\overset{?}{\Rightarrow} W_1 \cup W_2$ subspace

$V = \mathbb{R}^2$ (vector space over $\mathbb{R}$)



$\boxed{W_1 \cap W_2 = \{(0,0)\}}$ $W_1 = \{(x,0) : x \in \mathbb{R}\}$

$W_2 = \{(0,y) : y \in \mathbb{R}\}$

$W_1 \cup W_2$

(i) $(0,0) \in W_1 \cup W_2$

(ii) $u, v \in W_1 \cup W_2 \not\Rightarrow u+v \in W_1 \cup W_2$

$(1,0) \in W_1, (0,1) \in W_2 \quad (1,0)+(0,1)=(1,1) \notin W_1 \cup W_2$

Theorem: W_1, W_2 subspaces. Then TFAE The following are equivalent

(i) $W_1 \cup W_2$ subspace

(ii) either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

Proof $\boxed{i \Rightarrow ii}$

$\boxed{ii \Rightarrow i}$ easy. say $W_1 \subseteq W_2 \Rightarrow W_1 \cup W_2 = W_2$ is a subspace

$\boxed{i \Rightarrow ii}$ by contradiction

assume ii does not hold: $W_1 \not\subseteq W_2$ and $W_2 \not\subseteq W_1$

Pick $w_1 \in W_1$, $w_1 \notin W_2$ $\quad$ $\underline{\underline{w_1 + w_2}} \in W_1 \cup W_2$

$w_2 \in W_2$, $w_2 \notin W_1$

$$\left.\begin{array}{l} \text{say } w_1 + w_2 \in W_1 \\ \text{but } w_1 \in W_1 \end{array}\right\} \Rightarrow$$

$\Rightarrow w_2 \in W_1$ $\quad$ <u>contradiction</u>

W_1, W_2 subspaces

sum (direct sum)

define $W_1 + W_2 = \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}$

<u>Thm</u> (i) $W_1 + W_2$ subspace

(ii) $W_1 + W_2$ contains W_1 and W_2

(iii) $W_1 + W_2$ smallest subspace containing W_1 and W_2

(i) $0 \in W_1 + W_2$ $\quad 0 = 0 + 0$

(ii) $w_1 + w_2 \in W_1 + W_2$ $\quad (w_1 + w_2) + (w_1' + w_2')$

$w_1' + w_2' \in W_1 + W_2$ $\quad = \underbrace{(w_1 + w_1')}_{\in W_1} + \underbrace{(w_2 + w_2')}_{\in W_2}$

Scalar ... Exercise

(ii) given arbitrary $w_1 \in W_1$

$w_1 = w_1 + \underset{w_2}{0} \in W_1 + W_2$

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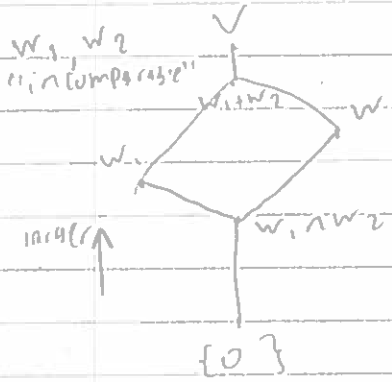
Lecture 10

W_1, W_2 subspaces of V

$W_1 + W_2 = \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}$

Thm: (i) $W_1 + W_2$ subspace containing W_1, W_2

(ii) $W_1 + W_2$ smallest as such.



Ex. $V = M_n(\mathbb{R})$

$W_1 = \{$upper triang$\}$
$W_2 = \{$lower triang$\}$

$W_1 + W_2 = V$
show by $\subseteq, \supseteq$
$\subseteq$ is given
$\supseteq$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$

Linear combinations

V vector space over K.
$u_1, \ldots, u_m$ vectors (elem. of V)

$\underbrace{v = a_1 u_1 + \ldots + a_n u_m}_{\text{linear combination of } \{v_1, \ldots, v_n\}}$ $(a_i \in K)$

- spanning set "sufficiency"
- independent set "no redundancy"
- basis (spanning & independent)

Def: $\{u_1, \dots, u_m\}$ spanning set

if each $v \in V$ is a linear combo of $\{u_1, \dots, u_m\}$

Example $V = \mathbb{R}^3$

$\{(1,0,0), (0,1,0), (0,0,1)\}$ spans

$\{(1,0,0), (0,1,0), (1,1,0)\}$

$a(1,0,0) + b(0,1,0) + c(1,1,0) = (a+c, b+c, 0)$

vectors such as $(0,0,1)$ are not linear combos.

Example $\{(1,2), (2,1), (2,2)\}$ in $\mathbb{R}^2$

$x(1,2) + y(2,1) + z(2,2) = (x+2y+2z, 2x+y+2z)$

$[(2,2)$ linear combo of $(1,2)]$

given $(a,b) \in \mathbb{R}^2$ solve for x, y, z (some solution)

$$\begin{cases} a = x + 2y + 2z \\ b = 2x + y + 2z \end{cases}$$

$a + b = 3(x+y)$

take $z = 0$

$y = a - \frac{a+b}{3} = \frac{2a-b}{3}$

$x = b - \frac{a+b}{3} = \frac{2b-a}{3}$

Remark. S spanning set.

a superset of S: spanning

a subset of S: may or may not be.

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223 P2

<u>Example</u> $\{u_1, \dots, u_m\}$ spanning

$\Rightarrow \{u_1, \dots, u_m, w\}$ spanning

$\uparrow$ a new vector

<u>Why?</u> given $v \in V$

$v = \sum a_i u_i + 0 \cdot w$

<u>Example:</u> $\{u_1, \dots, u_m, w\}$ spanning and w linear combo of $\{u_1, \dots, u_m\}$

$\Rightarrow \{u_1, \dots, u_m\}$ spanning

<u>Why?</u> $v \in V$ $v = \sum a_i u_i + aw$

$w = \sum c_i u_i$

$\Rightarrow v = \sum a_i u_i + a \sum c_i u_i = \sum (a_i + a c_i) u_i$

<u>Def</u> $\{u_1, \dots, u_m\}$ linearly dependent if

[there exists scalars $a_1, \dots, a_n$ not all 0 s.t.

$a_1 u_1 + \dots + a_m u_m = 0$

and linearly independent otherwise

if $a_1 u_1 + \dots + a_m u_m = 0$ then $a_1 = \dots a_n = 0$

$V = \mathbb{R}^3$

$\{(1,0,0), (0,1,0), (0,0,1)\}$ lin. indep

if $a(1,0,0) + b(0,1,0) + c(0,0,1) = (0,0,0)$

then $(a,b,c) = (0,0,0)$

$a = b = c = 0$

$\{(1,0,0), (0,1,0), (1,1,0)\}$ linearly dep.

$(1,0,0)+(0,1,0)-(1,1,0)=(0,0,0)$

Example $\{0, v_1, \ldots, v_k\}$
linearly dependent.

$1 \cdot 0 + 0 \cdot v_1 + \ldots + 0 \cdot v_k = 0$

Example $V = \mathbb{R}^3$

$\{(1,0,1), (1,1,0), (0,1,1)\}$

$a+b, b+c, a+c$

$a=-b \quad -b=c \quad a=-c$

$-b=-c$

$c=0$ independent

Remark. S independent

a superset of S: may or may not be.

a subset of S: indep.

Example $\{u_1, \ldots, u_m, w\}$ indep
$\Rightarrow \{u_1, \ldots, u_m\}$ indep

why? $\sum a_i u_i = 0$
$\sum a_i u_i + 0w = 0$
$\Rightarrow a_1 = a_2 = \ldots = a_m = 0 = 0$

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223 p3

Example $\{u_1, \dots, u_m\}$ indep

and w_1 not a linear combo of $\{u_1, \dots, u_m\}$

$\Rightarrow \{u_1, \dots, u_m, w\}$ indep.

Why? $\sum a_i u_i + a w = 0$

if $a \neq 0$ $w = -\sum \frac{a_i}{a} u_i$ ↯

so $a = 0$ $\sum a_i u_i = 0 \Rightarrow a_1 = \dots = a_m = 0$

Def: $\{u_1, \dots, u_m\}$ basis for V if both spanning and independent,

Thm: $\{u_1, \dots, u_m\}$ basis IFF every $v \in V$ can (spanning) be written uniquely (independence) as a linear combo of $\{u_1, \dots, u_m\}$

Proof: Exercise.

Examples:

- $\mathbb{R}^n$ $(1, 0, \dots, 0)$
  $(0, 1, \dots, 0)$
  $- - - - -$
  $(0, 0, \dots, 1)$

$e_i = (0, 0, \dots, \underset{\uparrow}{1}, 0, \dots, 0)$
i^{th} position

standard basis

$\cdot M_{m\times n}(K)$

e_{ij} = $m\times n$ matrix all whose entries are 0, except for the (i,j) entry, which is 1.

$\sum_{i,j} a_{ij} e_{ij} = 0_{m\times n}$ linear indep

$(a_{ij}) = 0_{m\times n}$

$\{e_{ij} : 1 \le i \le m, 1 \le j \le n\}$ basis (kind of the standard basis for matrices)

Ex. Upper triangular matrices.
$\{e_{ij} \quad 1 \le i \le j \le n\}$

Ex. $Pol_n(K)$
basis $\{1, x, x^2, \dots, x^n\}$

$a_0 + a_1 x + \dots + a_n x^n$
linear combo

How to get a basis

(bottom-up)

(1) Saturate an indep. set
- pick indep set. $\{v\}$ $v \ne 0$
- check if it spans; YES ✓
- if NO, pick a vector not a linear combo, add it.
// repeat

(top down)

(2) trim a spanning set
- Pick a spanning set
- check if independent; if YES ✓
- if NO, pick a vector that's a linear combo
drop it. // repeat

Sections 4.4, 4.5, 4.7, 4.8, 4.10

Exer 4.6, 4.7, 4.9-4.11, 4.15-4.24

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MATH 223 - Linear Algebra

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Lecture 11

Bases

V vector space over a field K

$S \subseteq V$ subset $S = \{s_1, \ldots, s_n\}$

Def S basis for V if S

independent & spanning

actually an iff, but only this implication is interesting

S basis $\Rightarrow$ every vector $v \in V$ can be written uniquely as a linear combo of $s_1, \ldots, s_n$.

$$v = a_1 s_1 + \ldots + a_n s_n \quad a_i \in K$$

n-tuple $(a_1, \ldots, a_n)$: coordinates of $v \in V$ with respect to basis S

! Theorem: All bases of V have the same number of elements

this allows for

Def: The dimension of V, denoted $\dim(V)$ is the number of elements in any basis

Examples: $\dim \mathbb{R}^n = n$

$\dim M_n(\mathbb{R}) = n^2$

$\dim(\{\text{upper triang}\}) = \dim(\{\text{lower triang}\}) = \frac{n(n+1)}{2}$

$\dim \mathrm{Pol}_n(K) = n+1$ (basis: all monomials $\{1, t, t^2, \dots, t^n\}$)

$\dim \{0\} = 0$ (by convention)

$\dim \mathrm{Pol}(K) = \infty$ (meaning: there is no finite basis)
a basis: all monomials $\{1, t, t^2, \dots\}$

Can an infinite dimensional vector space have a basis? Yes, above

$\mathbb{R}$ as an $\mathbb{R}$-vector space (tricky)
$\mathbb{Q}$ $\longrightarrow$ can't write a basis, but there is one

! Theorem There exists a basis for V
(There is an issue of existence of bases, need to know this so you're actually talking about something)

two algorithms
|| top-down: trim a spanning set
|| bottom-up: enrich an independent set
does it stop? Yes (tricky for infinite), by this theorem

(Proof of !)
Theorem: All bases of V have the same number of elements
using
Lemma: If $\{f_1, \dots, f_m\}$ independent
and $\{s_1, \dots, s_n\}$ spanning, then $m \leq n$

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Proof (Thm) $\{u_1, \dots, u_m\}$ basis = independent & spanning

$\{v_1, \dots, v_n\}$ basis if independent & spanning

lemma 2x $\Rightarrow m \leq n$

$n \leq m$

$m = n$

Proof (lemma)

idea: exchange f_1 for some S

exchange f_2 for some other S

- - - - - - -

exchange f_m for some other S

write $f_1 = a_1 s_1 + \dots + a_n s_n$ (can do this because s_i span)

at least one of $a_1, \dots, a_n$ is $\neq 0$

WLOG, assume $a_1 \neq 0$

$\Rightarrow$ write s_1 = linear combo of $f_1, s_2, \dots, s_n$

(move everything to the side of f_1, divide by a_1)

$\Rightarrow \{f_1, s_2, \dots s_n\}$ spanning

write $f_2 = b_1 f_1 + b_2 s_2 + \dots + b_n s_n$

at least one $b_2, \dots, b_n$ is $\neq 0$ (otherwise f_2 & f_1 wouldn't be independent)

WLOG assume $b_2 \neq 0$

$\Rightarrow$ write s_2 = linear combo of $\{f_1, f_2, s_2, \dots, s_n\}$

$\Rightarrow \{f_1, f_2, s_3, \dots, s_n\}$ spanning

and so on ... in m steps, get $\{f_1, \dots, f_m, \underbrace{\dots}_{s}\}$ spanning

Theorem: Assume $\dim V = n$. Then:

(i) any collection of more than n vectors is not independent

(ii) any collection of less than n vectors is not spanning

(condensed)

$S \subseteq V$

(i) S independent $\Rightarrow |S| \le n$

(ii) S spanning $\Rightarrow |S| \ge n$

Proof: Let β be a basis, $|B| = n$

(i) S independent $\Rightarrow |S| \le |B| = n$ (B spanning)

(ii) S spanning $\Rightarrow |S| \ge |B| = n$ (B independent)

Theorem Assume $\dim V = n$, let $S \subseteq V$ $|S| = n$. Then

S independent $\Leftrightarrow$ S spanning $\Leftrightarrow$ S basis

Example: $Pol_n(K)$ (we know $\{1, t, t^2, \dots, t^n\}$ basis)

Claim: given any $c \in K$, $\{1, t-c, (t-c)^2, \dots, (t-c)^n\}$ basis

Spanning: a bit harder ($\sim$ Taylor expansion)

independence: a bit easier

linear combo $= 0$ (Note: Not a function, t either inf range or diff nature than c)

$$\underbrace{a_n(t-c)^n + a_{n-1}(t-c)^{n-1} + \dots + a_1(t-c) + a_0}_{} = 0$$

Poly. of the form $a_n t^n$ + lower order terms (L.O.T) $= 0$

$\Rightarrow a_n = 0$

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223 p3

S independent $\Rightarrow$ S spanning
assume S not spanning; let $v \in V$ not a linear combo (from S)

$S \cup \{v\}$ independent, as well
$|S \cup \{v\}| = n+1$ $\lightning$

S spanning $\Rightarrow$ S independent
assume S dependent; let $s \in S$ expressed as linear combo
$S \setminus \{s\} \Rightarrow S \setminus \{s\}$ spanning as well.

$|S \setminus \{v\}| = n-1$ $\lightning$ ∎

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MATH 223 - Linear Algebra

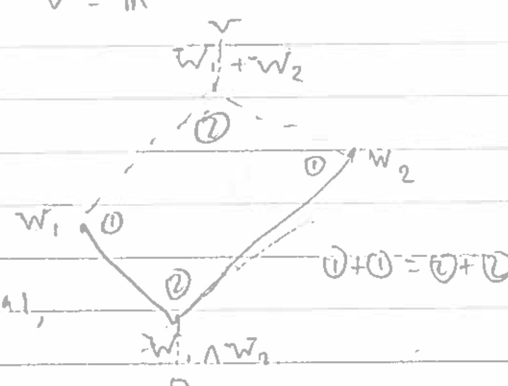
Lecture 12

Dimension & subspaces V: ambient linear space over K

Theorem: Assume $\dim V = n$. Let $W \subseteq V$ be a subspace.
Then $\dim W \leq n$. Moreover, $\dim W = n$ iff $W = V$

Proof: Let B basis of W. Then B independent in V
$\Rightarrow |B| \leq n = \dim V$
if $|B| = n \Rightarrow B$ basis for $V \Rightarrow V \subseteq W \Rightarrow V = W$ ∎

Example: $V = \mathbb{R}^3$ possible dimensions of subspaces
$0, 1, 2, 3$
only $\{0\}$ — many — only $V = \mathbb{R}^3$



Theorem Let $W_1, W_2 \subseteq V$ be subspaces that are finite-dimensional. Then $W_1 + W_2$ finite-dimensional, and

$$\boxed{\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)}$$

$\dim(W_1 + W_2) + \dim(W_1 \cap W_2) = \dim W_1 + \dim W_2$

based on **Theorem:** Assume V finite dim. Then any independent subset of V can be extended to a basis of V.

Proof (sketch)

B_0: basis of W_0
B_1: basis of W_1 extending B_0
B_2: basis of W_2 extending B_0

Claim: $B_1 \cap B_2 = B_0$

Claim: $B_1 \cup B_2$ basis for $W_1 + W_2$

$$\dim(W_1+W_2) = |B_1 \cup B_2| = \underbrace{|B_1|}_{\dim W_1} + \underbrace{|B_2|}_{\dim W_2} - \underbrace{|B_1 \cap B_2|}_{\dim(W_1 \cap W_2)}$$

Example:

M_n

{lower} {upper}

{diag}

$\dim M_m + \dim\{\text{diag}\} = \dim\{\text{upper}\} + \dim\{\text{lower}\}$

$$n^2 + n = \frac{n(n+1)}{2} + \frac{n(n+1)}{2}$$

Def: Let W_1, W_2 subspaces of V. If $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{0\}$ then we say V direct sum of W_1 and W_2
write $V = W_1 \oplus W_2$

(then $\dim V = \dim W_1 + \dim W_2$)

Linear maps

Q: what is linear algebra?
A: study of linear spaces and linear maps between them.

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Def A map $F: V \to W$, where V, W linear spaces (over same field)

is said to be linear if

$$\left(\begin{cases} F(v_1+v_2) = F(v_1)+F(v_2) & \text{for all } v_1, v_2 \in V \\ F(kv) = kF(v) & \text{for all } v \in V,\ k \in K \end{cases}\right) \Rightarrow F \text{ is structure-preserving}$$

Remark! F linear then $F(0) = 0$.

Example! the trace map $tr: M_n(\mathbb{R}) \to \mathbb{R}$ is linear.

Example! $\boxed{\text{matrices} \rightleftarrows \text{linear maps}}$

given $A \in M_{m \times n}(K)$, consider the map

$F_A: K^n \to K^m \qquad F_A(v) = Av$ is linear

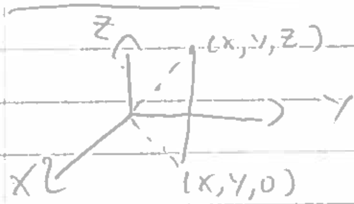
as column vectors

$M_{n\times 1}(K) \to M_{m \times 1}(K)$

Check: $F_A(v_1+v_2) = A(v_1+v_2) \underset{\text{matrix mult distributes}}{=} A\cdot v_1 + A\cdot v_2 = F_A(v_1)+F_A(v_2)$

$F_A(kv) = A(kv) = k(Av) = kF_A(v)$

Example



$F: \mathbb{R}^3 \to \mathbb{R}^3 \quad F(x,y,z) = (x,y,0)$ linear

$F\begin{pmatrix} x \\ y \\ z \end{pmatrix} \overset{?}{=} A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$

$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$G: \mathbb{R}^3 \to \mathbb{R}^3 \quad G(x,y,z) = (x, y, 1)$ Not a linear mapping

$F(0) \neq 0$ (quick way of verifying)

$$F((x_1,y_1,z_1)+(x_2,y_2,z_2)) \overset{?}{=} F(x_1,y_1,z_1) + F(x_2,y_2,z_2)$$

$$= F(x_1+x_2, y_1+y_2, z_1+z_2) \qquad (x_1, y_1, 0) + (x_2, y_2, 0)$$

$$(x_1+x_2, y_1+y_2, 0) \overset{?}{=} (x_1+x_2, y_1+y_2, 0)$$

$$F(k(x,y,z)) \overset{?}{=} k\underbrace{F(x,y,z)}_{(x,y,0)}$$

$$(kx, ky, 0) \qquad (kx, ky, 0)$$

Example $F: M_{m\times n}(K) \to M_{n\times m}(K)$

$F(A) = A^T$ linear

Example $F: M_{m\times n}(\mathbb{C}) \to M_{n\times m}(\mathbb{C})$

$F(A) = A^H$ not linear (over $\mathbb{C}$)

Def: Let $F: V \to W$ linear map

how injective $\to$ kernel of $F = \{v \in V : F(v) = 0\}$

how surjective $\to$ image of $F = \{w : w = F(v) \text{ for some } v \in V\}$

range

Example V = linear space of polynomial functions on $\mathbb{R}$

$F: V \to V \quad F(f) = f'$

? $\ker(F)$ — $\{c : c \in \mathbb{R}\}$ — dim = 1

? $\operatorname{Im}(F)$ — All polynomials

Lecture 13

October 18, 2016 8:30 AM

Linear mappings (continued)

Def. A map $F: V \to W$ between two linear spaces (over a field K) is said to be linear if

preserves addition

$F(v_1+v_2) = F(v_1)+F(v_2)$ for all $v_1, v_2 \in V$

preserves scalar mult

$F(kv) = kF(v)$ for all $v \in V$, $k \in K$

Examples (of linear mappings)

$tr: M_n(K) \to K$

* given $A \in M_{m\times n}(K)$ $F_A: K^n \to K^m$ (column vectors)

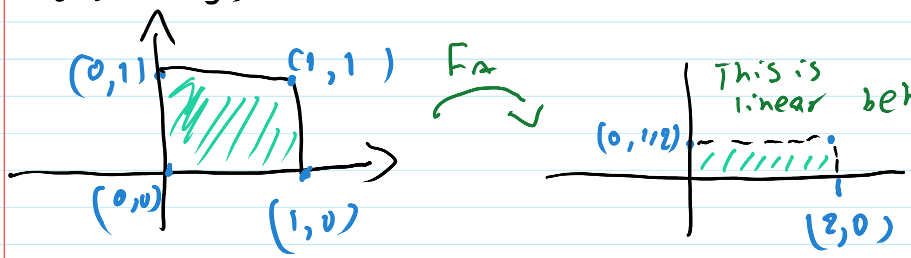
1 of the most imp

$F_A(n) = Av$

Examples want to do: $A \in M_2(\mathbb{R})$ $F_A: \mathbb{R}^2 \to \mathbb{R}^2$

① $A = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$ $F_A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ y/2 \end{pmatrix}$ $F_A\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$

(re-scaling)

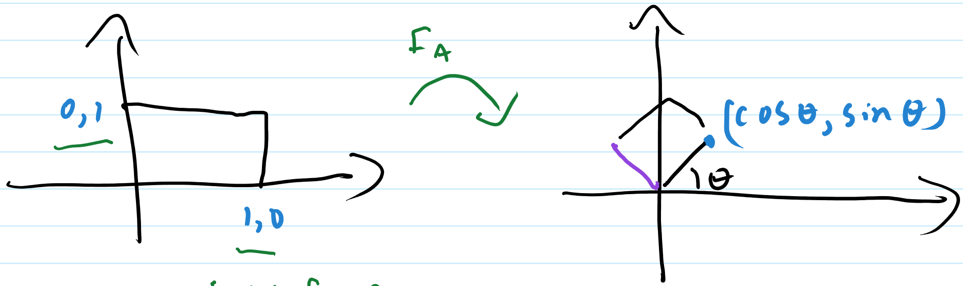


This is linear behavior, once you know what happens to the few points, can interpolate what will happen to all

② $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$F_A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$



basis for plane

computing other one

$F_A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$

Rotated by θ

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

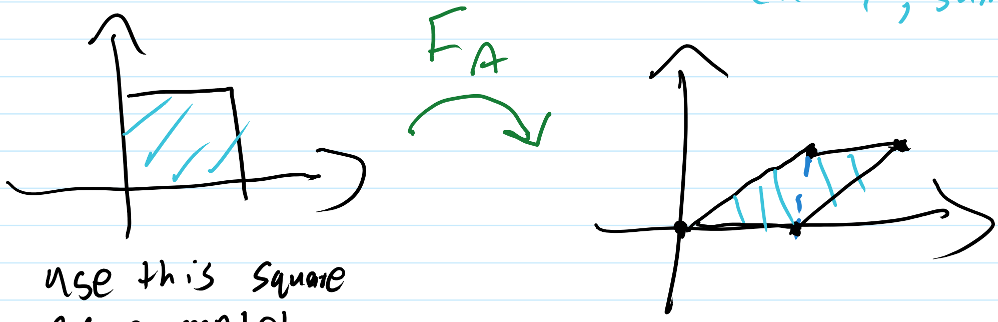
$\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$ $\quad$ $\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix}$

$\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$

for this one you can use linearity, sum $(0,1)$ & $(1,0)$

$(1,1)$ $\quad$ $(1,0)$

$(2,1)$



use this square as a model, map the 4 points (well, $(0,0)$ always 0)

All 3 of these examples have the same area.

Why?

det $= 1 \longleftrightarrow$ area-preserving

$\rightsquigarrow$ More abstract things

Theorem Let $F: V \to W$ linear map Then

all the things sent to 0

$\operatorname{Ker}(F)$ subspace of V, and

$\operatorname{Ran}(F)$ subspace of W

Proof ① $\operatorname{Ker}(F)$; need to check

$0 \in \operatorname{Ker}(F)$

clear

$v_1, v_2 \in \operatorname{Ker}(F) \Rightarrow v_1 + v_2 \in \operatorname{Ker}(F)$

then $F(v_1) = F(v_2) = 0 \Rightarrow F(v_1+v_2) = F(v_1) + F(v_2) = 0$, as desired

$k \in K, v \in \operatorname{Ker}(F) \Rightarrow kv \in \operatorname{Ker}(F)$

we know $F(v) = 0$ so $F(kv) = k\underbrace{F(v)}_{=0} = 0$

② $\operatorname{Ran}(F)$

, $w \in W$

(2) Ran(F)

$0 \in Ran(F)$ Clear: $F(0)=0$

~~$F(v)=0 \in W$ by kernel, $0 \in Ran(V)$~~

$w_1, w_2 \in Ran(F) \Rightarrow w_1+w_2 \in Ran(F)$

for some v_1, v_2, $F(v_1)$, $F(v_2)=w_2$ $\Rightarrow w_1+w_2 = F(v_1+v_2)$

$k \in K, w \in Ran(F) \Rightarrow kw \in Ran(F)$

$w = f(v)$ for some $v \in V \Rightarrow kw = kF(v) = F(kv)$

$w \in W$

$w \in Ran(F)$ if $F(v)=w$ for some $v \in V$

$\square$

Theorem (Rank-nullity)

Let $F: V \to W$ linear map, assume that V, W finite-dim (or else it gets messy) (only V needs to be finite) W doesn't play a big role

Then

$$\underbrace{\dim(Ker(F))}_{\text{Nullity of } F} + \underbrace{\dim(Ran(F))}_{\text{Rank of } F}$$

wasting your capacity

Remark Let $F: V \to W$ linear map

F surjective (or onto) means $Ran(F) = W$

F injective (or one-to-one) $\Leftrightarrow Ker(F) = \{0\}$

"$\Rightarrow$" $v \in Ker(F)$: $F(v)=0=F(0) \Rightarrow v=0$

"$\Leftarrow$" let $F(v_1)=F(v_2)$ (want get $v_1=v_2$)

$\Rightarrow F(v_1-v_2)=0$

$v_1-v_2=0 \Rightarrow v_1=v_2$

Good way to think about Range & kernel in the following way

Range(F): measures how surjective F is

Kernel(F): measures how injective F is

No mathematical notation for how injective you are, just think of it like that

Def A bijective linear map $F: V \to W$ is

said to be linear isomorphism

said to be linear isomorphism

Two spaces V, W isomorphic if there exists a linear isomorphism $F: V \to W$

$\underbrace{\text{Iso}}_{\text{sameness}}\ \underbrace{\text{morphism}}_{\text{shape}}$

Corrolary: If V, W isomorphic $\Rightarrow \dim V = \dim W$

Proof: Let $F: V \to W$ linear isomorphism.

$$\dim V = \underbrace{\dim \operatorname{Ker}(F)}_{0 \text{ (F injective)}} + \underbrace{\dim \operatorname{Ran}(F)}_{\dim W \text{ (F surjective)}}$$

$\square$

Lecture 14 Oct 20, 2016

Sketch of 2 & 3 from Assignment 3:

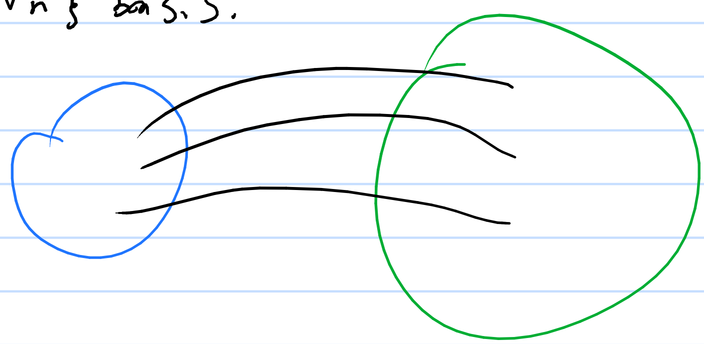
<u>Exercise 2</u>:

(via "exchange")

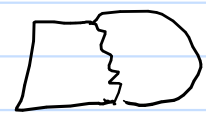
Let $S = \{s_1, \ldots, s_k\}$ independent

Pick $B = \{v_1, \ldots, v_n\}$ basis.

S



B (spanning)



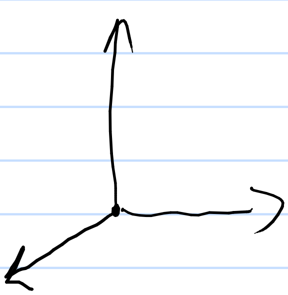
B' spanning

$|B'| = |B| = \dim V$

given V $\dim V = n$

given $k \in \{0, 1, \ldots, n\}$ Find W subspace, $\dim W = k$

Think of $\mathbb{R}^3$



Pick $\{v_1, \ldots, v_n\}$ basis

make smaller basis w/ this

consider $\mathrm{span}(\{v_1, \ldots, v_k\}) = W_k$

$= \{$linear combo of $v_1, \ldots v_k\}$

<u>$\dim W_k = k$</u>

spanning & ind & basis

-> Properties of a set of vectors

subspace -> big space/plane/w.e.

Theorem (Rank-nullity theorem).

Let $F: V \to W$ linear map, and assume V, W finite-dimensional. Then,

$$\boxed{\dim \mathrm{Ker}(F) + \dim \mathrm{Ran}(F) = \dim V}$$

Exercise (linear map)

	one-to one (redundancy)	onto (would be if not a linear map)
$\{0,1\} \to \{0,1,2\}$ $\mathbb{R}^2 \to \mathbb{R}^3$	Yes ? $(x,y) \mapsto (x,y,0)$	No ?
$\{0,1,2\} \to \{0,1\}$ $\mathbb{R}^3 \to \mathbb{R}^2$	No ?	Yes ? $(x,y,z) \mapsto (x,y)$ projection

Why is this right?

argue in terms of size, dim

relate to Rank-nullity

Say $F: \mathbb{R}^2 \to \mathbb{R}^3$ $\underline{\text{onto}}$ linear map

Then. $\underbrace{\dim \mathrm{Ker}(F)}_{\geq 0} + 3 = 2$ $\lightning$ Therefore, no such mapping

Say $F: \mathbb{R}^3 \to \mathbb{R}^2$ one-to one linear map

Then $0 + \underbrace{\dim \mathrm{Ran}(F)}_{\leq 2} = 3$ $\lightning$ Therefore, no such mapping

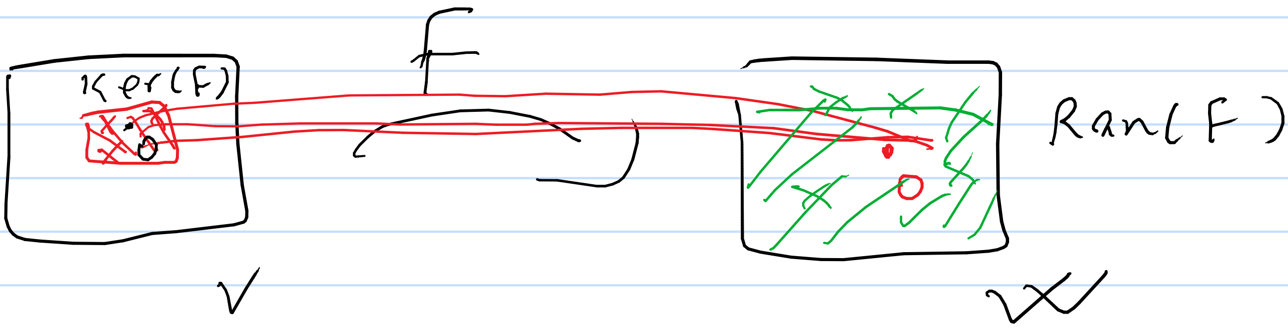
$\mathrm{Ran}\, F \subseteq \mathbb{R}^2$

Lecture 15 October 25, 2016

Thm: Let $F: V \to W$ linear map. Assume V, W finite dimensional. Then:

$$\dim V = \dim \operatorname{Ker}(F) + \dim \operatorname{Ran}(F)$$

Proof



Let $\{v_1, \dots, v_k\}$ basis for $\operatorname{Ker}(F)$

$\{w_1, \dots, w_r\}$ basis for $\operatorname{Ran}(F)$

Want to show that $k+r$ elements form a basis for V

Take the elements that map to w_i

Let $v_1', \dots, v_r' \in V$ such that $F(v_1') = w_1, \dots, F(v_r') = w_r$

This choice isn't unique

Claim: $\{v_1, \dots, v_k, v_1', \dots, v_r'\}$ basis for V.

then: $\dim V = k + r = \dim \operatorname{Ker}(F) + \dim \operatorname{Ran}(F)$.

Need to check if it's a basis

Independence

let $\underbrace{a_1 v_1 + \dots + a_k v_k}_{Ker(F)} + b_1 v_1' + \dots + b_r v_r' = 0$

Apply F: $b_1 \underbrace{F(v_1')}_{w_1} + \dots + b_r \underbrace{F(v_r')}_{w_r} = 0$

$\Rightarrow b_1 = \dots = b_r = 0$

(as $\{w_1, \dots, w_r\}$ basis)

so $a_1 v_1 + \dots + a_k v_k = 0$

$\Rightarrow a_1 = \dots = a_k = 0$

(as $\{v_1, \dots, v_k\}$ basis)

Spanning

let $v \in V$

$F(v) \in Ran(F) \Rightarrow F(v) = \sum b_i w_i$ span

$= \sum b_i F(v_i')$

$= F(\underbrace{\sum b_i v_i'}_{})$

can't say this is v, because multiple things can map to w_i

use kernel, it keeps track of this

$\boxed{F(v - \sum b_i v_i') = 0}$

$v - \sum b_i v_i' \in Ker(F)$

$v - \sum b_i v_i' = \sum a_i v_i$

so $v = \sum a_i v_i + \sum b_i v_i$

Corrolary: Let $F: V \to V$ linear map. Assume V finite-dim

TFAE:

F one-to-one $\Leftrightarrow$ F onto ($\Leftrightarrow$ F isomorphism)

Pf: "$\Rightarrow$" F one-to-one $\Rightarrow \dim Ker(F) = 0$, then, by rank nullity theorem, $\dim Ran(F) = \dim V \Rightarrow$ onto

"$\Leftarrow$" F onto $\Rightarrow \dim Ran(F) = \dim V$

$\Rightarrow \dim Ker(F) = 0 \Rightarrow$ one-to-one

Rem: Not true if V infinite-dimensional.

$V = Pol(\mathbb{R})$

$\int$ $Pol(\mathbb{R}) \to Pol(\mathbb{R})$ one-to-one not onto $(F(f(t)) = t f(t))$

D $Pol(\mathbb{R}) \to Pol(\mathbb{R})$ onto not one-to-one

differentiation

Isomorphisms

$V \simeq W$ V and W **isomorphic** (there exists $\underset{\text{isom.}}{F: V \to W}$)

Example: $Pol_n(K)$ and K^{n+1} field K to the $n+1$

$$F(a_0 + a_1 t + \ldots + a_n t^n) = (a_0, \ldots, a_n)$$

onto: take elements of K^{n+1}, change coefficients of polynomial to that

Thm: Let V, W be finite-dimensional

Then $V \simeq W \iff \dim V = \dim W$

Proof: "$\Rightarrow$" by rank-nullity theorem

Interesting direction we haven't seen yet

"$\Leftarrow$" let $\{v_1, \ldots, v_n\}$ basis for V where $n = \dim V = \dim W$

$\{w_1, \ldots, w_n\}$ basis for W

define $F: V \to W$ as follows:

because $\{v_1, \ldots, v_n\}$ basis

let $v \in V$, write $v = a_1 v_1 + \ldots + a_n v_n$ uniquely determined

(so: $a_1, \ldots, a_n$ coeff. of v in $\{v_1, \ldots, v_n\}$)

set $F(v) = a_1 w_1 + \ldots + a_n w_n$ (well-defined)

Functions must be uniquely determined

(e.g. $F(v_1) = w_1, \ldots, F(v_m) = w_m$)

- F linear
- F bijective

• F linear

$v, v' \in V$ $(a_1, \ldots, a_n)$ coord in $\{v_1, \ldots, v_n\}$

well-behaved under addition $(a_1', \ldots, a_n')$

$\Rightarrow$ coord of $v+v'$ are $a_1+a_1', \ldots, a_n+a_n'$

$F(v+v') = \sum (a_k+a_k')w_k$

$= \sum a_k w_k + \sum a_k' w_k = F(v)+F(v')$

scalar mult, similar

• F bijective

F one-to-one: $\left.\begin{array}{l} F(v)=0 \\ v=a_1v_1+\ldots+a_nv_n \end{array}\right\} \Rightarrow a_1w_1+\ldots+a_nw_n=0$

since w_i linearly ind.

$\Rightarrow a_1 = \ldots = a_n = 0$

$\Rightarrow v=0$

could finish here, since dim is same.

Try: F onto: let $w \in W$, $w = \sum b_i w_i$

since w_i span W

then $F(v)=w$ for $v = \sum b_i v_i$. ∎

Remember, this statement means that there exists an isomorphism between V & W if their dim is the same, not that any mapping is isomorphic.

i.e. $\mathbb{R}^3 \to \mathbb{R}^3$

$(x,y,z) \mapsto (x,y,0)$ not isomorphic, but $(x,y,z) \mapsto (x,y,z)$ is

Isomorphisms

Cor If $\dim V = n$ then $V \simeq \underline{\underline{K}}^n$

In any dimension, there is a single space

Why study different spaces?

$\mathbb{R}^{n+1}$ $\qquad$ $Pol_n(\mathbb{R})$ has a very natural mapping associated to it

diff

Thm: $V \simeq V$

$V \simeq W \Rightarrow W \simeq V$

$V \simeq W$ & $W \simeq U \Rightarrow V \simeq U$

Think of $\simeq$ as being very similar

Lecture 16 Oct 27, 2016

Last time (Isomorphisms)

$V \simeq W$ V, W isomorphic

had a **Thm.** $V \simeq W \iff \dim V = \dim W$ (assume V, W finite dim)

Thm Relation of isomorphism is like itself, 3 prop

$V \simeq V$ | ① identity $(v \mapsto v)$

$V \simeq W \Rightarrow W \simeq V$ | ② F .... F^{-1}

$V \underset{F}{\simeq} W, W \underset{G}{\simeq} U \Rightarrow V \simeq U$ | ③ F, G ... $G \circ F$

$G \circ F : V \to W \to U$

exercise write our details

Matrix similarity

Def: $A, B \in M_n(K)$ are similar if $B = P^{-1}AP$ for some (invertible) $P \in M_n(K)$

similarity equivalence relation on square matrices

notation: $A \sim B$ A similar to B

(informally, weak notion of equality)

Theorem:
$A \sim A$ — reflexivity
$A \sim B \Rightarrow B \sim A$ — symmetry
$A \sim B$ and $B \sim C \Rightarrow A \sim C$ — transitivity

Isomorphisms don't imply equivalence relations, but it's in the same spirit

Proof ① $A = I_n^{-1} A I_n$

② $A \sim B$, means $B = P^{-1}AP$

$P \cdot$ $\quad \cdot P^{-1}$ $\quad Q^{-1}BQ \quad Q = P^{-1}$

$A = PBP^{-1} = (P^{-1})^{-1} B P^{-1}$ so $B \sim A$

③ $A \sim B$, $B \sim C$

$B = (P^{-1}AP)$ $\quad C = Q^{-1}BQ$ $\quad A \sim C$

$C = Q^{-1}(P^{-1}AP)Q = (PQ)^{-1} A (PQ)$ ∎

These play similar roles to the 3 properties of isomorphisms

Finding good similar matrices is tricky, we'll get back to it

Theorem: $A \sim B \Rightarrow \operatorname{tr} A = \operatorname{tr} B$
$\det A = \det B$

Proof: $\boxed{\det(AB) = \det(A)\det(B)}$ (2×2 matrices)

$B = P^{-1}AP \Rightarrow \det B = \det(P^{-1}AP) = \det(P)^{-1} \det A \det P$
$= \det A$.

trace $\boxed{\operatorname{tr}(AB) = \operatorname{tr}(BA)}$

$\operatorname{tr}(B) = \operatorname{tr}(P^{-1}AP) = \operatorname{tr}(\cancel{PP^{-1}}A) = \operatorname{tr}(A)$, ∎

Example. Find similarities between

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

How to start? Need to check if they can be similar!
For isomorphisms, you check dim. For matrices, you check trace & det

$$\underbrace{\begin{matrix} tr = 3 \\ det = 2 \end{matrix} \qquad \begin{matrix} tr = 3 \\ det = 2 \end{matrix} \qquad \begin{matrix} tr = 3 \\ det = 2 \end{matrix}}_{\text{? possible}} \qquad \begin{matrix} tr = 1 \\ det = -2 \end{matrix} \qquad \begin{matrix} tr = 0 \\ det = -2 \end{matrix}$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

similar

in 2 steps?
interchange rows
then columns

$$\begin{pmatrix} x & y \\ z & t \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x & y \\ z & t \end{pmatrix}$$

Note:

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = P^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} P$$

$$\Rightarrow P \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = \cancel{P} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} P$$

$$\begin{pmatrix} x & x+2y \\ z & z+2t \end{pmatrix} = \begin{pmatrix} x & y \\ 2z & 2t \end{pmatrix} \Rightarrow \begin{matrix} x+y=0 \\ z=0 \end{matrix}$$

lots of freedom, can take:

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

Def A is $\boxed{\text{diagonalizable}}$ if A similar to diagonal matrix.

Similarity is like a mask for matrix
Diagonalization important

$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ diagonalizable in previous ex

Example: Is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ diagonalizable?

say $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ $\begin{cases} a+b=2 & \text{(traces)} \\ ab=1 & \text{(determinants)} \end{cases}$

$\Rightarrow a(2-a)=1$

$a^2-2a+1=0$

so $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$a=b=1$

Contradiction

only $I_2 \sim I_2$
also only $kI_2 \sim kI_2$

Example Is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ diagonalizable?

Say $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \sim \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ $\begin{cases} ab = \cos^2\theta + \sin^2\theta = 1 \\ a+b = 2\cos\theta \end{cases}$

$b = 2\cos\theta - a$

candidates are

$(2\cos\theta - a)a = 1$

$2\cos\theta a - a^2 = 1$

$a^2 - 2a\cos\theta + 1 = 0$

$\begin{pmatrix} \cos\theta + i\sin\theta & \\ & \cos\theta - i\sin\theta \end{pmatrix}$

$\sim$ (from example earlier, $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$)

$\begin{pmatrix} \cos\theta - i\sin\theta & \\ & \cos\theta + i\sin\theta \end{pmatrix}$

$$\Delta = 4\cos^2\theta - 4 = -4\sin^2\theta$$

$$\boxed{a = \frac{2\cos\theta \pm\sqrt{\Delta}}{2} = \cos\theta \pm i\sin\theta}$$

$$\underbrace{\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}}_{\text{Think about these, can guess sometimes}} \begin{pmatrix} \cos\theta + i\sin\theta & \\ & \cos\theta - i\sin\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$$

Answer: yes (over $\mathbb{C}$)
no (over $\mathbb{R}$) (except for $\pm I_2$ when $\sin\theta = 0$)

---

matrices $\rightsquigarrow$ linear maps

$A \in M_n(K)$ $\qquad F_A : K^n \to K^n$

$$F_A(v) = Av$$

want to go backwards

Prop Let $F: K^n \to K^n$ linear. Then $F = F_A$ for some unique $A \in M_n(K)$ $\quad (F(v) = Av)$

Proof: Let $\{e_1, \dots, e_n\}$ standard basis of K^n

want A such that $F(e_i) = F_A(e_i) = Ae_i$

$$\begin{pmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2i} & \cdots & a_{2n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{ni} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \quad \text{(1 = } i^{\text{th}} \text{ entry)}$$

$\to$ want matrix A

$= i^{\text{th}}$ column of A

$= \boxed{a_{1i}e_1 + a_{2i}e_2 + \dots + a_{ni}e_n}$

back to proof:

expand $F(e_1) \dots F(e_n)$

$$F(e_1) = a_{11}e_1 + a_{21}e_2 + \dots + \underbrace{a_{n1}}_{\text{Careful notation}} e_n$$

$$\dots$$

take $A = (a_{ij})$

$$F(e_n) = a_{1n}e_1 + a_{2n}e_2 + \dots + a_{nn}e_n$$

$$\Rightarrow F(e_i) = F_A(e_i) \Rightarrow F = F_A \quad \blacksquare$$

2 mappings same on a basis $\Rightarrow$ they are the same

Uniqueness

Lecture 17 Nov 1 2016

Midterm avg: 62%.

Pick up on Thursday

<u>Last time</u>

- matrix similarity.

$A \sim B \qquad B = P^{-1}AP$

A diagonalizable
(similar to diagonal matrix)

$A \sim B$
$\Downarrow$
$\left.\begin{array}{l} \operatorname{tr} A = \operatorname{tr} B \\ \det A = \det B \end{array}\right\}$ consequence of def.

converse not true

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \not\sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

<u>Thm</u> Similarity equivalence relation

reflexivity $A \sim A$
Symmetry $A \sim B$, $B \sim A$
transitivity $A \sim B, B \sim C, A \sim C$

<u>Prop.</u> $F : K^n \to K^n$ linear $\Rightarrow$ there exists a unique matrix $A \in M_n(K)$ such that $F(v) = Av$ for all $v \in V$

<u>Idea.</u> $\{e_1, \dots, e_n\}$ standard basis of K^n.
expand $F(e_1), \dots, F(e_n)$.

more generally

V finite-dim space.

$F : V \to V$ linear map

let $S = \{v_1, \dots, v_n\}$ basis for V

expand careful of indexing, this is diff from book

$F(v_1) = a_{11}v_1 + a_{21}v_2 + \dots + a_{n1}v_n$ 1st col

$F(v_2) = a_{12}v_1 + a_{22}v_2 + \dots + a_{n2}v_n$ 2nd col

$\vdots$

$F(v_n) = a_{1n}v_1 + a_{2n}v_2 + \dots + a_{nn}v_n$

completely analogous process in any finite dim space

$\rightsquigarrow$ matrix $[F]_S = (a_{ij})$

Def $[F]_S$ matrix of F in the basis S

Mapping determines matrix

But how does matrix form mapping?

How does matrix depend on basis chosen?

Example What is the matrix of

$$D : Pol_n(\mathbb{R}) \to Pol_n(\mathbb{R})$$

in the basis $\{\underset{v_1}{1}, \underset{v_2}{t}, \underset{v_3}{t^2}, \dots, t^n\}$?

$$[D] = \begin{pmatrix} 0 & 1 & & & & \\ & & 2 & & & \\ & & & 3 & & \\ & & & & \ddots & \\ & & & & & n \\ & & & & & 0 \end{pmatrix}$$

$D(1) = 0$

$D(t) = 1$

$D(v_3) = 2v_2$ $\quad \boxed{D(t^2) = 2t}$

Another Ex. D

$\{1, t, \frac{t^2}{2!}, \dots, \frac{t^n}{n!}\}$

$\{1, t-15, (t-15)^2, \dots\}$ would give you the same matrix

$D(\overset{v_3}{(t-15)^2}) = 2\overset{v_2}{(t-15)}$

Question. How does $[F]_S$ determine F?

Thm: For all $v \in V$, $[F(v)]_S = [F]_S [v]_S$

$[v]_S$; coordinates of v in the basis S

$$v = a_1 v_1 + \dots + a_n v_n$$
$$[v]_S = (a_1, \dots, a_n)$$

unique collection of n coefficients

$[F(v)]_S$: — " — $F(v) \in V$ — " —

This gives you coordinates of $F(v)$, then need to expand

$V \overset{[\cdot]_S}{\longleftrightarrow} K^n$ need to pick a basis

$v \longrightarrow [v]_S$

$a_1 v_1 + \dots + a_n v_n \longleftarrow (a_1, \dots, a_n)$

Reverse process, expand using basis

Relates vectors to coordinates vectors once you've picked a bases

---

abstract

isomorphism of "coordinisation"/ coordinates

concrete F

$$\begin{tikzcd} V \arrow[r, leftrightarrow, "{[\cdot]_S}"] \arrow[d, "F"'] & K^n \arrow[d, "\text{multiplication by } {[F]_S}"] \\ V \arrow[r, leftrightarrow, "{[\cdot]_S}"'] & K^n \end{tikzcd}$$

(commutative) diagram

start with

$$\begin{tikzcd} v \in V \arrow[r] \arrow[d] & {[v]_S} \arrow[d] \\ F(v) \arrow[r] & \boxed{[F(v)]_S = [F]_S [v]_S} \end{tikzcd}$$

2 ways to get here

Proof expand v in the basis S $\quad v = c_1 v_1 + \dots + c_n v_n$

? $[F(v)]_S$ $\quad [v]_S = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$

$$F(v) = F\left(\sum_{k=1}^{n} c_k v_k\right) \overset{F \text{ linear}}{=} \sum_{k=1}^{n} c_k F(v_k) = \sum_{k=1}^{n} c_k \left(\sum_{j=1}^{n} a_{jk} v_j\right)$$

now you know what F does to basis

$$= \sum_k \sum_j a_{jk} c_k v_j = \sum_j \underbrace{\left(\sum_k a_{jk} c_k\right)}_{\text{coeff of } v_j} v_j$$

$$[F(v)]_S = \begin{pmatrix} \sum a_{1k} c_k \\ \vdots \\ \sum a_{nk} c_k \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

$$= [F]_S [v]_S \qquad \square$$

Question: How does $[F]_S$ change upon change of basis S.

Thm. The matrix representations of F in different bases are similar

(that is, S, S' bases $\Rightarrow [F]_S \sim [F]_{S'}$)

desired P arises as change-of-basis matrix.

$S = \{v_1, \dots, v_n\}$

$S' = \{v_1', \dots, v_n'\}$

$v_1' = a_{11} v_1 + a_{21} v_2 + \dots + a_{n1} v_n$

$v_2' = a_{12} v_1 + a_{22} v_2 + \dots + a_{n2} v_n$

$\vdots$

$v_n' = a_{1n} v_1 + \dots + a_{nn} v_n$

$\rightsquigarrow$ matrix $\underbrace{P_{S \to S'}}_{\text{pass from } S \text{ to } S'} = (a_{ij})$

$Pol_n(\mathbb{R})$ $\{1, t, t^2, \ldots, t^n\}$

$\{1, t+1, (t+1)^2, \ldots, (t+1)^n\}$

Change of basis involves $(t+1)^n = \sum_{k=0}^{n} \binom{n}{k} t^k$

Thm. $P_{S \to S'} [v]_{S'} = [v]_S$

Proof. Exercise (almost the same as $F(V)$ expansion proof)

Corollary $P_{S \to S'}$ invertible, and $P_{S \to S'}^{-1} = P_{S' \to S}$

Read chapter 6, 6.1 - 6.4

Exercises 6.10, 6.17, 6.23, 6.24, 6.40, 6.45, 6.51, 6.54, 6.60

particularly nice

Lecture 18 Nov 3, 2016

Midterm pickup

A - I : 10-10:30
J - R : 10:30-11
S - Z : 11-11:30

Last time

$F: V \to V$ linear map

could be done with diff spaces, V, W, but need 2 bases

S basis

$\Downarrow$

matrix $[F]_S$

Thm $[F(v)]_S = [F]_S [v]_S$

Q: How does $[F]_S$ depend on S?

Thm: Matrices of F wrt different bases are similar

$(S, S'$ bases of $V \Rightarrow [F]_S \sim [F]_{S'})$

change-of-basis matrix

$S = \{v_1, \ldots, v_n\}$

$S' = \{v_1', \ldots, v_n'\}$

$v_1' = a_{11}v_1 + \cdots + a_{n1}v_n$

$\vdots$

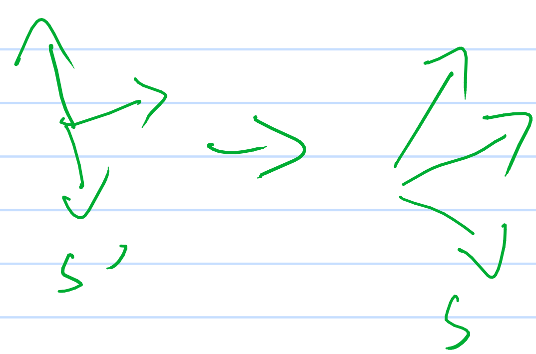
$v_n' = a_{1n}v_1 + \cdots + a_{nn}v_n$

$\rightsquigarrow P_{S \to S'} = (a_{ij})$

$$\boxed{\text{Thm. } P_{S\to S'} [v]_{S'} = [v]_S}$$

arrow dir not important

The S' and S' kind of cancel each other



Cor: $P_{S\to S'}$ invertible, and $P_{S\to S'}^{-1} = P_{S'\to S}$

Proof: want: $\begin{cases} P_{S\to S'} \cdot P_{S'\to S} = I_n \\ P_{S'\to S} \cdot P_{S\to S'} = I_n \end{cases}$ focus on this, the two are the same if you swap S & S' 6.60

$$P_{S\to S'} \underbrace{P_{S'\to S} [v]_S}_{} \underset{\text{Thm.}}{=} P_{S\to S'} [v]_{S'} \underset{\text{Thm}}{=} [v]_S \underset{\text{using remark}}{\Rightarrow} P_{S\to S'} P_{S'\to S} = I_n$$

Remark: $Av = Bv$ (A, B $n\times n$ matrices, $v \in K^n$)

$\Downarrow$

$\underline{A = B}$ specialize v, choose standard basis, show they're equal

standard basis k

$v = e_k = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$

$\underbrace{Ae_k}_{k^{th}\text{ column of } A} = \underbrace{Be_k}_{}$

Proof (of thm) $[F]_S \sim [F]_{S'}$

we show: $[F]_S = \underbrace{P_{S\to S'}}_{\text{inverse}} [F]_{S'} P_{S'\to S}$

Now do the same thing as the remark

$$\boxed{P_{S\to S'} [F]_{S'} \quad P_{S'\to S}} [v]_S = [F(v)]_S = \boxed{[F]_S} [v]_S$$

$P_{S'\to S}[v]_S = [v]_{S'}$

$[F]_{S'}[v]_{S'} = [F(v)]_{S'}$

Do same thing on a vector, thus same

Inner product spaces (Chapter 7)

Motivation: $\mathbb{R}^3$ vector space + angles (orthogonality) length

chapter about adding additional prop so it acts like $\mathbb{R}^3$

inner product space = vector space + inner product

Def. Let V real vector space. An inner product on V is a map $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{R}$ (for each pair $u, v \in V \rightsquigarrow \langle u, v\rangle \in \mathbb{R}$) such that

① $\langle a u_1 + b u_2, v \rangle = a\langle u_1, v\rangle + b\langle u_2, v\rangle$ linear in 1^{st} coord
$\langle u, a v_1 + b v_2\rangle = a\langle u, v_1\rangle + b\langle u, v_2\rangle$ linear in 2^{nd} coord

② $\langle u, v\rangle = \langle v, u\rangle$ (Symmetry)

③ $\langle u, u\rangle \geq 0$, and $\langle u, u\rangle = 0 \iff u = 0$
positive-definite

Remark: linear in 2^{nd} follows from linear in 1^{st} + symmetry
don't compute second

Remark $\langle \sum_k a_k u_k, \sum_k b_k v_k \rangle \neq \sum_k a_k b_k \langle u_k, v_k\rangle$

$$= \sum_k \sum_i a_k b_i \langle u_k, v_i\rangle$$

Example: dot product in $\mathbb{R}^n$

$\langle u, v\rangle = u \cdot v$ (alternate $u^T v$ column vectors, $u v^T$ row vectors)

③ $\langle u, u\rangle = \sum_k u_k^2 \geq 0$
$= 0 \iff u_1 = \dots = u_k = 0$ $\quad \underline{\underline{u = 0}}$

in $\mathbb{C}^n$
$\langle u, u\rangle = \sum_k \underbrace{u_k \overline{u_k}}_{|u_k|^2}$

Example: $C[a,b]$ vector space of real-valued continuous functions defined on closed interval $[a,b]$

$$\langle f,g\rangle = \int_a^b f(t)\,g(t)\,dt$$

① $\langle af_1+bf_2, g\rangle = \int_a^b (af_1(t)+bf_2(t))g(t)\,dt$

$= \int_a^b (af_1(t)g(t)) + (bf_2(t)g(t))\,dt$

$= \int_a^b a(f_1(t)g(t)) + b(f_2(t)g(t))\,dt$

$= a\langle f_1,g\rangle + b\langle f_2,g\rangle$

② $\langle f,g\rangle = \int_a^b f(t)\,g(t)\,dt = \int_a^b g(t)f(t)\,dt = \langle g,f\rangle$

③ $\langle f,f\rangle = \int_a^b (f(t))^2 \geq 0$ if $\int_a^b (f(t))^2 = 0$

$\Rightarrow f(t)=0$

or else it would gain positive mass at some point

Example: $M_n(\mathbb{R})$ $\langle A,B\rangle = tr(B^T A)$

① $\langle aA_1+bA_2, B\rangle = tr(B^T\cdot(aA_1+bA_2))$

$= tr(B^T aA_1 + B^T bA_2) = tr(B^T aA_1) + tr(B^T bA_2)$

$= a\,tr(B^T A_1) + b(B^T A_2) = a\langle A_1,B\rangle + b\langle A_2,B\rangle$

② $\langle A,B\rangle = tr(B^T A) = tr(B^T A)^T = tr(A^T B)$

$tr(A^T) = tr(A)$

$= \langle B,A\rangle$

③ $\langle A, B\rangle = \sum_k (B^T A)_{kk}$

$= \sum_k \left(\sum_j b_{jk} a_{jk} \right)$

$= \sum_k \sum_j b_{jk} a_{jk}$

$\langle A, A\rangle = \sum_{j,k} a_{jk}^2 \geq 0 \quad = 0 \text{ iff } a_{jk} = 0$

What's with the T?

$\langle A, B\rangle = tr(BA)$ will not satisfy ③

Lecture 19 Nov 8 2016

Exercise $\mathbb{R}^n$ with usual inner product.

(i) A $n\times n$ symmetric matrix $\Rightarrow \langle A\overset{\text{column vectors}}{\widehat{u,v}}\rangle = \langle u, Av\rangle$

(ii) A $n\times n$ orthogonal $\Rightarrow \langle Au, Av\rangle = \langle u, v\rangle$

(i) $\langle Au, v\rangle = (Au)^T \cdot v = u^T A^T \cdot v = u^T \cdot \overset{\text{symm}}{A^T} v = u^T \cdot Av = \langle u, Av\rangle$

(ii) $\langle Au, Av\rangle = (Au)^T \cdot (Av) = u^T A^T \cdot Av = \overset{\text{orthog}}{u^T I v} = u^T v = \langle u, v\rangle$

$$\boxed{\begin{array}{l} u, v \text{ column vectors} \\ \langle u, v\rangle = u^T v \end{array}}$$

Recall:

V real vector space

$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ inner product if

$$\begin{cases} \langle \cdot, v\rangle : V \to \mathbb{R} \text{ linear, for all } v \\ \langle u, \cdot \rangle : V \to \mathbb{R} \text{ linear, for all } u \\ \langle u, v\rangle = \langle v, u\rangle \text{ for all } u, v \\ \langle v, v\rangle \geq 0 \text{ and } \langle v, v\rangle = 0 \iff v = 0 \end{cases}$$

Exercise was done without using any of these properties

What's coming up?

- norm
- orthogonality
- Gram-Schmidt orthogonalization

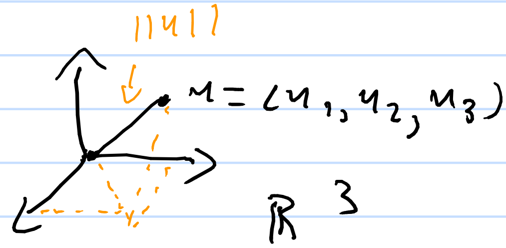
Def: V inner product space

the norm of $v \in V$ is $\|v\| = \sqrt{\langle v, v\rangle}$

seen this when we talked about usual dot product

example in $\mathbb{R}^n$, dot product

$$\|u\| = \sqrt{u_1^2 + \dots + u_n^2}$$



Thm (Cauchy-Schwarz inequality)

$$|\langle u, v\rangle| \leq \|u\| \cdot \|v\| \quad \text{for all } u, v \in V$$

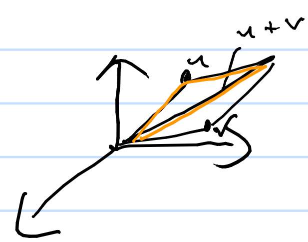
Proof: same as for $\mathbb{R}^n$ (Exercise 7.8)

Thm The following hold,

(i) $\|v\| \geq 0$ and $\|v\| = 0 \iff v = 0$,

(ii) $\|kv\| = |k|\,\|v\|$

(iii) $\|u+v\| \leq \|u\| + \|v\|$



(iii) $\|u+v\| = \sqrt{\langle u+v, u+v\rangle}$

$\|u\| + \|v\| = \sqrt{\langle u,u\rangle} + \sqrt{\langle v,v\rangle}$

$$\|u+v\|^2 = \langle u+v, u+v\rangle = \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle$$

$$= \|u\|^2 + 2\langle u,v\rangle + \|v\|^2$$

$\leq$ Cauchy-Schwarz

$$\|u\|^2 + 2\|u\|\,\|v\| + \|v\|^2 = (\|u\| + \|v\|)^2$$

Norm

Def: V real vector space
A norm of V is a map $\|\cdot\| : V \to \mathbb{R}$
satisfying (i), (ii), (iii)

what we're doing is:
inner product $\rightsquigarrow$ norm

more general

can get norm without inner product in some cases?

still have: length / distance

don't have: angle, orthogonality

Example: In $\mathbb{R}^n$

$$\|(a_1, \dots, a_n)\|_2 = \sqrt{|a_1|^2 + \dots + |a_n|^2}$$

$$\|(a_1, \dots, a_n)\|_1 = |a_1| + \dots + |a_n|$$ obviously not the same

$$\|(a_1, \dots, a_n)\|_\infty = \max\{|a_1|, \dots, |a_n|\}$$

Generally for norm, take abs val of some combination of values / entries

Different types of norms, these tell you how long vector is, but in different ways / scales

More norms than inner products

Example: $C[a,b]$

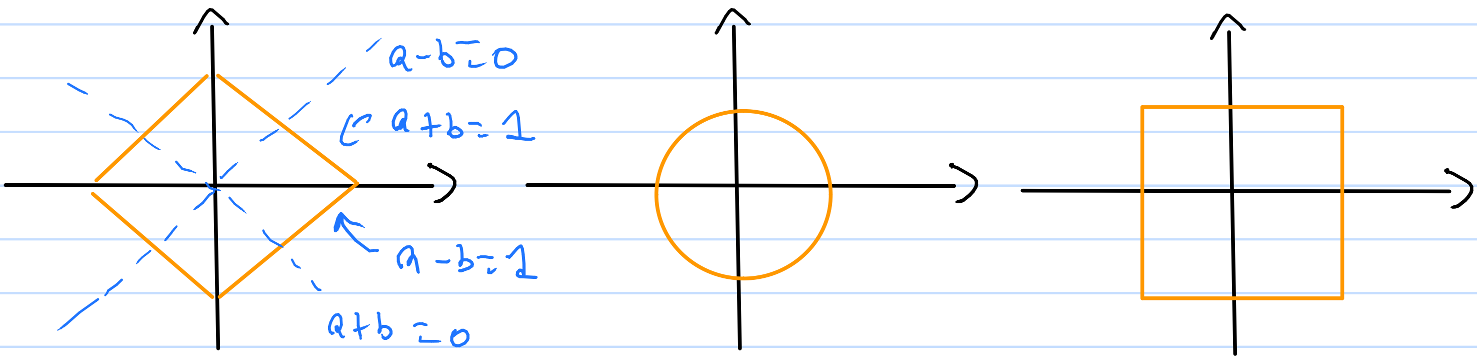
$$\|f\|_2 = \left(\int_a^b |f(t)|^2 \, dt\right)^{1/2}$$ $\leftarrow$ defined by $\langle f, g \rangle = \int_a^b f(t) g(t) \, dt$

$$\|f\|_1 = \int_a^b |f(t)| \, dt$$

$$\|f\|_\infty = \max_{t \in [a,b]} |f(t)|$$

Visual representations of norms

$\mathbb{R}^2$



$\|(a,b)\|_1 \le 1$ $\qquad$ $\|(a,b)\|_2 \le 1$ $\qquad$ $\|(a,b)\|_\infty \le 1$

$|a|+|b| \le 1$ $\qquad$ $\sqrt{|a|^2+|b|^2}$

$\max\{|a|,|b|\}$ 1

$|a| \le 1$

and $|b| \le 1$

$\|a\|_1 \ge \|a_2\| \ge \|a\|_\infty$

Changes direction for functions

Read. Section 7.10 $\qquad$ Exer 7.54 – 7.56

Def: V inner product space

$u,v \in V$ orthogonal if $\langle u,v\rangle = 0$

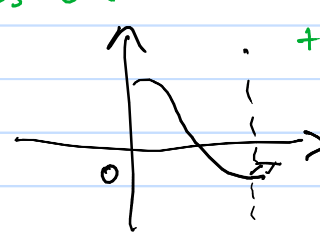
(notation: $u \perp v$)

Remark More generally, angle (notion of angle in inner space)

given $u,v \in V$ non-zero, the angle between u & v is $\theta \in [0,\pi]$ satisfying

$$\boxed{\cos\theta = \frac{\langle u,v\rangle}{\|u\|\cdot\|v\|}}$$

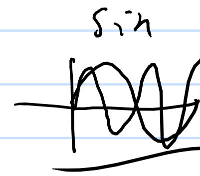
(cos one-to-one on this interval



OK: Cauchy-Swarz says $\langle u,v\rangle$ works

Def: S subset of non-zero is orthogonal if vectors in S are pairwise orthogonal.
($u, v \in S$ distinct $\Rightarrow u \perp v$)

Example: $\mathbb{R}^n$ $\{e_i\}$ standard basis



Example: $C[0, 2\pi]$
continuous func on this interval

$$\begin{cases} f_n(x) = \cos(nx) \\ g_n(x) = \sin(nx) \end{cases}$$

orthogonal set,

$\boxed{\text{Fourier series}}$

Lecture 20 Nov 10 2016

Warm-up V inner product space

Pythagorean Theorem: if $u \perp v \Rightarrow \|u+v\|^2 = \|u\|^2 + \|v\|^2$

Parallelogram Law: $\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$

1. $\|u+v\|^2 = \langle u+v, u+v\rangle = \langle u,u\rangle + \underbrace{\langle u,v\rangle}_{0} + \underbrace{\langle v,u\rangle}_{0} + \langle v,v\rangle$

$$= \|u\|^2 + \|v\|^2$$

2. $\|u+v\|^2 + \|u-v\|^2 = \|u\|^2 + \cancel{2\langle u,v\rangle} + \|v\|^2$
$+ (\|u\|^2 - \cancel{2\langle u,v\rangle} + \|v\|^2)$
$= 2\|u\|^2 + 2\|v\|^2$

Orthogonality V inner product space

$S \subseteq V$ is orthogonal if vectors in S are non-zero, and pairwise orthogonal.

0 vector is orthogonal to every other vector

Ex. $\{e_i\}$ standard basis in ($\mathbb{R}^n$, usual dot product)

$\{c_i e_i\}$ $c_i \neq 0$ and c_i is a scalar
The scalars can be different

$S \subseteq V$ is orthonormal if S orthogonal and normalized, in the sense that $\|u\| = 1$ for $u \in S$
all u

$\{e_i\}$, $\{\pm e_i\}$
with 2^n choices for $\pm$

Orthonormalization

$S = \{u_1, \dots, u_k\}$ orthogonal

$\rightsquigarrow \left\{\frac{1}{\|u_1\|} u_1, \dots, \frac{1}{\|u_k\|} u_k\right\}$ orthonormal set.

Example: A orthogonal $n \times n$ matrix $\quad AA^T = A^TA = I_n$

$\Rightarrow$ rows of A form an orthonormal set
columns of A form an orthonormal set

$$\begin{pmatrix} \boxed{\quad r_1 \quad} \\ \boxed{\quad r_2 \quad} \\ \dots \\ \boxed{\quad r_n \quad} \end{pmatrix} \begin{pmatrix} \boxed{r_1} & \boxed{r_2} & \cdots & \boxed{r_n} \end{pmatrix} = (r_i \cdot r_j)_{i,j} = I_n$$

$\boxed{r_i \cdot r_j = \delta_{ij}}$

$\{r_1, \dots, r_n\}$ orthonormal

Thm. S orthogonal $\Rightarrow$ S independent

Pf: Let $c_1 u_1 + \dots + c_k u_k = 0$, where $u_1, \dots, u_k \in S$ distinct

$\Rightarrow 0 = \langle c_1 u_1 + \dots + c_k u_k, u_j \rangle \qquad j = 1, \dots, k$ arbitrary

$= c_1 \langle u_1, u_j \rangle + \dots + c_k \langle u_k, u_j \rangle \qquad$ (S orthogonal)

$= c_j \langle u_j, u_j \rangle$

$\Rightarrow c_j = 0 \qquad$ (as $\langle u_j, u_j \rangle \neq 0$ since $u_j \neq 0$ for orthog)

∎

orthogonal => ind ↙ check spanning

Thm $S = \{u_1, \ldots, u_n\}$ orthogonal basis

Then for $v \in V$ we have

$$v = \frac{\langle v,u_1\rangle}{\langle u_1,u_1\rangle} u_1 + \frac{\langle v,u_2\rangle}{\langle u_2,u_2\rangle} u_2 + \ldots + \frac{\langle v,u_n\rangle}{\langle u_n,u_n\rangle} u_n$$

(Fourier coefficients)

Don't necessarily need to know the name

Proof: let $v = c_1 u_1 + \ldots + c_n u_n$ ($c_1, \ldots, c_n$ scalars)

$$\langle v, u_j\rangle = \langle c_1 u_1 + \ldots + c_n u_n, u_j\rangle$$

$$= c_j \langle u_j, u_j\rangle$$

$$\Rightarrow c_j = \frac{\langle v, u_j\rangle}{\langle u_j, u_j\rangle}$$

cor. $S = \{u_1, \ldots, u_n\}$ orthonormal basis

$$\Rightarrow v = \langle v,u_1\rangle u_1 + \ldots + \langle v, u_n\rangle u_n$$

as a consequence (Parseval's identity)

$$\|v\|^2 = \langle v,u_1\rangle^2 + \ldots + \langle v,u_n\rangle^2$$

pf: $\|v\|^2 = \langle v,v\rangle = \langle \sum \langle v,u_i\rangle u_i, \sum_j \langle v,u_j\rangle u_j\rangle$

$$= \sum_{i,j} \underbrace{\langle v,u_i\rangle\langle v,u_j\rangle}_{\text{Scalars jump out}} \underbrace{\langle u_i, u_j\rangle}_{\delta_{ij}} \quad (\{u_i\} \text{ orthog})$$

$$= \sum_i \langle v,u_i\rangle \langle v,u_i\rangle \langle u_i, u_i\rangle$$

$$= \sum_i \langle v, u_i\rangle^2$$

Gram-Schmidt orthogonalization

$\{v_1, \dots, v_k\} \rightsquigarrow \{w_1, \dots, w_k\}$ orthogonal

Non-orthogonal

Example. $\{v_1, v_2\} \rightsquigarrow \{w_1, w_2\}$ orthogonal

$w_1 = v_1$

$w_2 = v_2 - c v_1$

suitable c to give you orthogonality

? find c such that $\langle w_1, w_2 \rangle = 0$

$\langle v_1, v_2 \rangle - c \langle v_1, v_1 \rangle = 0$

$\Rightarrow c = \dfrac{\langle v_1, v_2 \rangle}{\langle v_1, v_1 \rangle}$

Example $\{v_1, v_2, v_3\} \rightsquigarrow \{w_1, w_2, w_3\}$ orthogonal

$w_1 = v_1$

$w_2 = v_2 - \dfrac{\langle v_1, v_2 \rangle}{\langle v_1, v_1 \rangle} v_1$

$w_3 = v_3 - c_1 w_1 - c_2 w_2$

? find c_1, c_2 such that

$\langle w_3, w_1 \rangle = 0$

$\langle w_3, w_2 \rangle = 0$

$\langle w_3, w_1 \rangle = \langle v_3 - c_1 w_1 - c_2 w_2, w_1 \rangle = 0$

$\langle v_3, w_1 \rangle - c_1 \langle w_1, w_1 \rangle = 0$

$\langle v_3, w_1 \rangle = c_1 \langle w_1, w_1 \rangle$

$\dfrac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} = c_1$

Gram-Schmidt

$\{v_1, \dots, v_k\}$ independent

$w_1 = v_1$

$w_2 = v_2 - \dfrac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1$

$\vdots$

$w_n = v_n - \dfrac{\langle v_n, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \dots - \dfrac{\langle v_n, w_{n-1} \rangle}{\langle w_{n-1}, w_{n-1} \rangle} w_{n-1}$

Lecture 21 Nov 15 2016

Last time: Gram-Schmidt

$\{v_1, \ldots, v_k\}$ independent $\rightsquigarrow \{w_1, \ldots, w_k\}$ orthogonal

recursive procedure

$$\boxed{w_i = v_i - \sum_{j<i} \text{proj}_{w_j}(v_i)}$$

$w_1 = v_1$

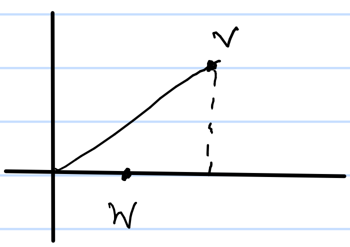
$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1$$

....

$$w_k = v_k - \frac{\langle v_k, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_k, w_2\rangle}{\langle w_2, w_2\rangle} w_2 - \ldots - \frac{\langle v_k, w_{k-1}\rangle}{\langle w_{k-1}, w_{k-1}\rangle} w_{k-1}$$

Need to make sure denominator never becomes 0

$$\boxed{\text{proj}_w(v) = \frac{\langle v, w\rangle}{\langle w, w\rangle} w}$$



Fact 1. $w_i \in \text{span}\{v_1, \ldots, v_i\}$ for $1 \le i \le k$

(exercise: by induction)
whenever you compute w_i, you only use up to v_i

Fact 2: $w_i \neq 0$ for $1 \le i \le k$

($\Rightarrow$ G-S well-defined: denominators are never zero)

want to link to independence of $v_i, \ldots, v_{i-1}$

$w_i = v_i -$ (linear combo of $w_1, \ldots, w_{i-1}$)

$\overset{\text{Fact 1}}{=} v_i -$ (linear combo of $v_1, \ldots, v_{i-1}$)

$\neq 0$ by independence

Fact 3 $\{w_1, \dots, w_k\}$ orthogonal

Proof induction on j

$\{w_1, \dots, w_j\}$ orthogonal

base case : $\{w_1, w_2\}$ $(j=2)$

Just a single vector $\{w_1\}$ would work, but so trivial

$$\langle w_1, w_2 \rangle = \quad = \left\langle w_1, v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1 \right\rangle$$

$$= \langle w_1, v_2 \rangle - \frac{\langle v_2, w_1\rangle}{\cancel{\langle w_1, w_1\rangle}} \cancel{\langle w_1, w_1\rangle}$$

$$= 0$$

induction step

Know $\{w_1, \dots, w_j\}$ orthogonal
want $\{w_1, \dots, w_{j+1}\}$ orthogonal

So we have to show

$$w_{j+1} \perp w_i \quad \text{for all } 1 \leq j \leq i$$

$$\langle w_{j+1}, w_i \rangle = \left\langle v_{j+1} - \frac{\langle v_{j+1}, w_j\rangle}{\langle w_j, w_j\rangle} w_j \quad , w_i \right\rangle$$

$$\langle v_{j+1}, w_j \rangle - \frac{\langle v_{j+1}, w_j\rangle}{\langle w_j, w_j\rangle} \underbrace{\langle w_j, w_j\rangle}_{}$$

$= 0$ by Induction hypothesis

$\blacksquare$

Example $C[0,1]$ $\langle f,g\rangle = \int_0^1 f(t)\,g(t)\,dt$

Do Gram-Schmidt on $\{1, t, t^2\}$

$f_1 = 1$

$$\langle t^a, t^b\rangle = \frac{t^{a+b+1}}{a+b+1} = \frac{1}{a+b+1}$$

$$f_2 = t - \frac{\langle t,1\rangle}{\langle 1,1\rangle} 1$$

$$= t - \frac{1/2}{1}\cdot 1 = \boxed{t - \frac{1}{2}}$$

$$f_3 = t^2 - \frac{\langle t^2, 1\rangle}{\langle 1,1\rangle}\cdot 1 - \frac{\langle t^2, t-\frac{1}{2}\rangle}{\langle t-\frac{1}{2}, t-\frac{1}{2}\rangle}\left(t - \frac{1}{2}\right)$$

$\langle t^2, 1\rangle = \frac{1}{3}$, $\langle 1,1\rangle = 1$

$$\boxed{\langle t^2, t-\tfrac{1}{2}\rangle} = \langle t^2, t\rangle - \frac{1}{2}\langle t^2, 1\rangle = \frac{1}{4} - \frac{1}{2}\cdot\frac{1}{3} = \boxed{\frac{1}{12}}$$

$$\langle t-\tfrac{1}{2}, t-\tfrac{1}{2}\rangle = \langle t,t\rangle - \langle t,1\rangle + \frac{1}{4}\langle 1,1\rangle$$

$$= \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \boxed{\frac{1}{12}}$$

So $\boxed{f_3} = t^2 - \frac{1/3}{1} - \frac{1/12}{1/12}\left(t - \frac{1}{2}\right) = t^2 - \frac{1}{3} - t + \frac{1}{2}$

$$= \boxed{t^2 - t + \frac{1}{6}}$$

Remark: Can rescale w_i's along the way

for example $f_2 = t - \frac{1}{2}$, $f_2' = 2t - 1$

$f_2'' = \frac{1}{\|f_2\|} = \sqrt{12}\left(t - \frac{1}{2}\right)$

scaling preserves orthogonality

Remark. $\{v_1, \dots, v_n\}$ basis

$\Rightarrow$ G-S yields an orthogonal basis $\{w_1, \dots, w_n\}$

why? n independent vectors $\Rightarrow$ also a basis

Cor: V finite-dim inner product space

$\Rightarrow V$ has an orthogonal (orthonormal basis)

Pf: Pick a basis for V, apply G-S $\square$

V has a basis

Cor: Every orthogonal set in V can be extended to an orthogonal basis.

Pf: Start with $\{w_1, \dots, w_k\}$ orthogonal
extend to $\{w_1, \dots, w_k, u_1, \dots, u_s\}$ basis

Apply Gram-Schmidt $\rightsquigarrow$ get $\{w_1, \dots, w_k, u_1', \dots, u_s'\}$ orthogonal basis

Gram-Schmidt preserves orthogonal vectors

$\blacksquare$

Orthogonal complements

V inner product space,
finite dimensional

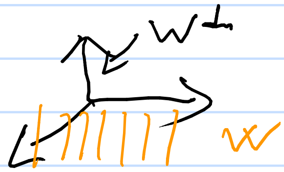
$W \subseteq V$ subspace

Def. $W^\perp = \{v \in V : \langle v, w \rangle = 0 \ \forall w \in W\}$

Example:

$\mathbb{R}_3$

1^{st} case



2^{nd} case

Ex. $\{0\}^\perp = V$

$V^\perp = \{0\}$

Thm. i) $W^\perp$ subspace of V

ii) $V = W \oplus \underline{W^\perp}$

orthogonal complement

Pf: (i) exercise

(ii) $W \cap W^\perp = \{0\}$

and $V = W + W^\perp$

$w \in W \cap W^\perp$

$\Rightarrow \langle \underbrace{w}_{W}, \underbrace{w}_{W^\perp} \rangle = 0$

$\Rightarrow w = 0$

Show $V = W + W^\perp$

take $\{w_1, \ldots, w_k\}$ orthogonal basis for W

complete to $\{w_1, \ldots, w_k, u_1, \ldots, u_s\}$ orthogonal basis for V

given $v \in V$ write $v = (\text{linear combo of } w_1, \ldots, w_k) \} W$

$+ (\text{linear combo of } u_1, \ldots, u_s) \} ? \in W^\perp$

each $u_i \in W^\perp$

since each u_i orthogonal to w_j

Also prove subset

Lecture 22 Nov 17 2016

Read Chapter 7 (including §7.9 (complex case) but not required)

Exercises: 7.7, 7.26, 7.55, 7.56, 7.71, 7.82

Determinants (Chapter 8)

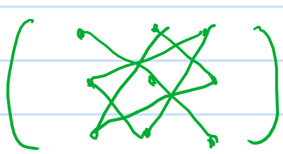
$A \in M_n(\mathbb{K}) \rightsquigarrow |A| = \det(A) \in \mathbb{K}$

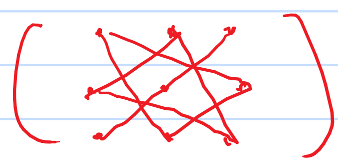
What is it? (More complicated than what it does)

What does it do?

For a 3×3:

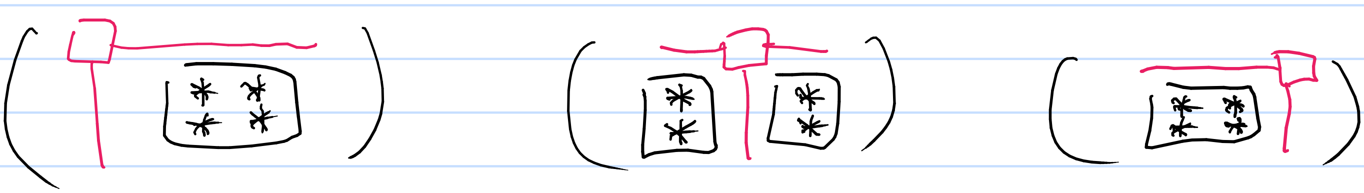
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

Positive terms 

negative terms 

Or:

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$



Basic Properties

① $\det I_n = 1$

② $\det(AB) = \det(A)\det(B)$

Cor A invertible, $\det(A^{-1}) = \frac{1}{\det A}$

Pf: $AA^{-1} = I \rightarrow \det A \cdot \det A^{-1} = 1$

Cor $AB = I_n \Rightarrow BA = I_n$

Pf: $\det(AB) = \det(I_n) = 1$
$\Rightarrow \det A \neq 0 \Rightarrow A$ invertible

$A^{-1} \cdot$ on both sides $AB = I_n \Rightarrow B = A^{-1} \Rightarrow BA = A^{-1}A = I_n$ ∎

Cor A, B similar $\Rightarrow \det(A) = \det(B)$

Pf: $A = P^{-1}BP \Rightarrow |A| = |P^{-1}|\,|B|\,|P| = |B|$ ∎

determinant of a linear map $L: V \rightarrow V$
(change of basis → similar matrix, always same det)

③ A invertible $\Leftrightarrow \det(A) \neq 0$

3.5 $\det A^T = \det A$

④ (elementary operations) starting with A ...

B: interchange two rows/columns $\Rightarrow |B| = -|A|$

B: multiply a row/column by a constant k $\Rightarrow |B| = k|A|$

B: add a multiple of a row/column to another row $\Rightarrow |B| = |A|$

Laplace expansion (first instance: 3x3 det with 3 det earlier)

for each (i,j) get a new $(n-1)\times(n-1)$ matrix

given A $n\times n$ matrix

$$A = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \qquad M_{ij} = \begin{pmatrix} & & \\ & a_{ij} & \\ & & \end{pmatrix}$$

erase ith row and j^{th} column

Thm.

plus-minus pattern

$$|A| = a_{11}|M_{11}| - a_{12}|M_{12}| + \dots = \sum_{k=1}^{n} (-1)^{k+1} a_{1k} |M_{1k}|$$

(expansion along 1^{st} row)

$$= a_{11}|M_{11}| - a_{21}|M_{21}| + \dots = \sum_{k=1}^{n} (-1)^{k+1} a_{k1} |M_{k1}|$$

(expansion along 1^{st} column)

Notation: $A_{ij} = (-1)^{ij} |M_{ij}|$ (i,j) cofactor

rewrite

$$|A| = a_{11}A_{11} + a_{12}A_{12} + \dots + a_{1n}A_{1n}$$

$$= a_{11}A_{11} + a_{21}A_{21} + \dots + a_{n1}A_{n1}$$

Applications

1. $$\begin{vmatrix} a_{11} & * & * & * \\ 0 & a_{22} & * & * \\ \vdots & & \ddots & \\ 0 & \dots & 0 & a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & & * \\ & a_{33} & \\ 0 & & \ddots \; a_{nn} \end{vmatrix}$$

(expand along 1^{st} column)

$$= \dots = \boxed{a_{11} a_{22} \cdot \dots \cdot a_{nn}}$$

For upper triang, det is product of diagonal entries in particular $\left|\begin{pmatrix} 1 & & * \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix}\right| = 1$

② Finding the inverse of a matrix

let A $n \times n$ matrix

big A in transpose fashion

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_n \\ & * & & \end{pmatrix} \begin{pmatrix} A_{11} & \\ A_{12} & \\ \vdots & ? \\ A_{1n} & \end{pmatrix}$$

Laplace expansion

You replace 1^{st} row by 2^{nd} row

det (matrix with 2 rows the same)=0

$$= \begin{pmatrix} \boxed{\det A} & 0 \\ & \end{pmatrix}$$

when you look at the product:

$a_{i1} A_{j1} + \cdots + a_{in} A_{jn} = (i,j)$ entry of $A \cdot \text{adj}(A)$

$$= \begin{cases} \det A & i = j \quad (\text{L.E. for } A \text{ along } i^{th} \text{ row}) \\ 0 & i \neq j \quad (\text{L.E. for modified } A \text{ along } j^{th} \text{ row}) \end{cases}$$

replace i^{th} row by j^{th} row

Classical adjoint

$\text{adj}(A) = (A_{ij})^T$

$A \cdot \text{adj}(A) = |A| I_n$

$\text{adj}(A) \cdot A = |A| I_n$

$a_{ij} A_{1i} + \cdots + a_{nj} A_{ni} = (i,j)$ entry of $\text{adj}(A) \cdot A$

$$= \begin{cases} \det A & i = j \quad (\text{L.E. for } A \text{ along } i^{th} \text{ column}) \\ 0 & i \neq j \end{cases}$$

→ explicit (but tedious) formula for inverse of A when $|A| \neq 0$

$$A^{-1} = \frac{1}{|A|} \text{adj}(A)$$

Example (vandermonde determinant)

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ 0 & b^3-ab^2 & c^3-ac^2 & d^3-ad^2 \end{vmatrix}$$

Start from bottom-up:
subtract r_4 by $a \cdot r_3$
r_3 by $a \cdot r_2$
r_2 by $a \cdot r_1$

$$= \begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ 0 & b^2-ab & c^2-ac & d^2-ad \\ 0 & b^3-ab^2 & c^3-ac^2 & d^3-ad^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & b-a & c-a & d-a \\ 0 & b^2-ab & c^2-ac & d^2-ad \\ 0 & b^3-ab^2 & c^3-ac^2 & d^3-ad^2 \end{vmatrix} = \begin{vmatrix} b-a & c-a & d-a \\ b^2-ab & c^2-ac & d^2-ad \\ b^3-ab^2 & c^3-ac^2 & d^3-ad^2 \end{vmatrix}$$

factor

$$= (b-a)(c-a)(d-a) \underbrace{\begin{vmatrix} 1 & 1 & 1 \\ b & c & d \\ b^2 & c^2 & d^2 \end{vmatrix}}_{(c-b)(d-b)\begin{vmatrix} 1 & 1 \\ c & d \end{vmatrix}}$$

Think about nxn case

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} = \underbrace{\prod_{1 \le j < i \le n}}_{\text{Product}} (x_i - x_j)$$

Prove it! (induction)

Lecture 23 Nov 22 2016

Determinants

usually $\mathbb{R}$ or $\mathbb{C}$

$A \in M_n(\mathbb{K}) \rightsquigarrow \det(A) \in \mathbb{K}$

Important way of getting a scalar from a square matrix

2 questions last class:

(1) What is it?

(2) What does it do? ← Answered this last class

Recap:

Properties:

- $\det(AB) = \det(A) \cdot \det(B)$
- A invertible $\iff \det A \neq 0$ — got inverse from adjoint
- effect of elem. operations — switching rows or columns, multiplying, adding
- Laplace expansion
  very useful recursive sequence to calculate determinants

Example: Vandermonde matrix (finish from last class)

$$V(x_1, \dots, x_n) = \begin{pmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ - & - & - & - \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{pmatrix}$$

$$\det V(x_1, \dots, x_n) = \prod_{n \geq i > j \geq 1} (x_i - x_j)$$

# of factors $\frac{n(n-1)}{2} = \binom{n}{2}$

Corrolary $V(x_1, \dots, x_n)$ non-singular

$\iff x_1, \dots, x_n$ distinct

or else you'll get 0 in your product

$\Rightarrow \det = 0$

Proof:

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ & & \vdots & \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} = \begin{vmatrix} 1 & 1 & & 1 \\ 0 & x_2 - x_1 & & x_n - x_1 \\ & \cdots & & \\ 0 & x_2^{n-2} - x_1 x_2^{n-3} & & x_n^{n-2} - x_1 x_n^{n-3} \\ 0 & x_2^{n-1} - x_1 x_2^{n-2} & & x_n^{n-1} - x_1 x_n^{n-2} \end{vmatrix}$$

reductions in this manner

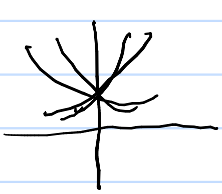
$$= (x_2 - x_1) \cdots (x_n - x_1) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_2 & x_3 & & x_n \\ & \cdots & & \\ x_2^{n-2} & x_3^{n-2} & & x_n^{n-2} \end{vmatrix}$$

$$= (x_2 - x_1) \cdots (x_n - x_1) \underbrace{\det V(x_2, \dots, x_n)}_{\text{induction}}$$

$$= \prod_{n \geq i > j \geq 2} (x_i - x_j)$$

$$= \prod_{i > j} (x_i - x_j) \qquad \blacksquare$$

Application $\{e^{ax} : a \in \mathbb{R}\}$ in $C[-1, 1]$ is independent



say $c_1 e^{a_1 x} + \dots + c_n e^{a_n x} = 0$

with $a_1, \dots, a_n$ distinct,

? $c_1 = c_2 = \dots = c_n = 0$

(1) set $x=0$: $c_1 + \dots + c_n = 0$

(2) differentiate, set $x=0$ $\quad a_1 c_1 + \dots + a_n c_n = 0$

(3) diff $\times 2$, set $x=0$ $\quad a_1^2 c_1 + \dots + a_n^2 c_n = 0$

Do it n times $\quad a^{n-1} c_1 + \dots + a_n^{n-1} c_n = 0$

$$V(a_1, \dots, a_n) \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = 0$$

non-singular

(distinct a)

$\Rightarrow c_1 = \dots = c_n = 0$

Puzzle. Is $\{e^{ax} : a \in \mathbb{R}\}$ spanning?

Definition Let A $n \times n$ matrix

The determinant of A is defined to be

contains exactly on entry from each row and column

Loads of products

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, \underbrace{a_{1\sigma(1)} \cdots a_{n\sigma(n)}}$$

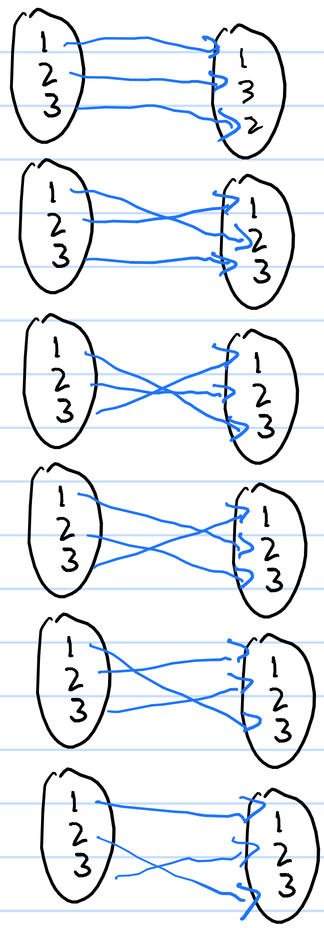
$\hookrightarrow n!$ terms

What do the terms mean?
We'll go over them now
(That's why it's easier to go over what det does, rather than what it is)

Permutations

Permutation of $\{1, \dots, n\}$ is a bijection $\{1, \dots, n\} \to \{1, \dots, n\}$

Example



S_n = set of all permutations on $\{1, \dots, n\}$

$|S_n| = n!$ why? could do this by induction

Structure of S_n

$\sigma, \tau \in S_n \Rightarrow \sigma \circ \tau \in S_n$

$\sigma \in S_n \Rightarrow \sigma^{-1} \in S_n$

(one says S_n group) (not covered in this course group theory)

Isomorphism has similar properties

Also, invertible matrices

Example $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$, where $2 = \sigma(1)$, $1 = \sigma(2)$, $3 = \sigma(3)$ $\quad \tau = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

do τ then σ

$\sigma \circ \tau = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ Non-commutative

$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ $(= \sigma)$

$\tau^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

Sign

Def: Let $\sigma \in S_n$ permutation.

$\text{inv}(\sigma) = \#$ inversions in σ

$= \#\{(i,j) : i < j \text{ but } \sigma(i) > \sigma(j)\}$

Breaking of increasing function, opposite would just be the def of increasing function

σ even, $\text{sgn}(\sigma) = 1$ if $\text{inv}(\sigma)$ even

σ odd, $\text{sgn}(\sigma) = -1$ if $\text{inv}(\sigma)$ odd

Example

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \qquad \tau = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

count disorder

$\text{inv}(\sigma) = 1$ $\qquad$ $\text{inv}(\tau) = 3$

$1 < 2$

$\sigma(1) = 3 > \sigma(2) = 2$

$1 < 3$

$\sigma(1) = 3 > \sigma(3) = 1$

$2 < 3$

$\sigma(2) = 2 > \sigma(3) = 1$

Example transpositions

$$(i\,j) = \begin{pmatrix} 1\,2 & \cdots & i & \cdots & j & \cdots & n \\ 1\,2 & & j & \cdots & i & \cdots & n \end{pmatrix}$$

$i < j$

$\text{inv}((i\,j)) = 2(j-i) - 1$

$\rightsquigarrow$ transpositions are odd, sign $= -1$ EXERCISE

Example identity even (0 permutations)

In def of det, corresponds to products of terms on diagonal (positive)

Sign rule

$\mathrm{sgn}(\sigma \circ \tau) = \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\tau)$

Cor: $\mathrm{sgn}(\sigma^{-1}) = \mathrm{sgn}(\sigma)$

$$\mathrm{sgn}(\sigma) \cdot \mathrm{sgn}(\sigma^{-1}) = \mathrm{sgn}(\sigma \circ \sigma^{-1}) = \mathrm{sgn}(\text{identity}) = 1$$

Exercise use the def of det for $n=3$

Patchy chapter, skipped over some proofs (not enough time)

Moving on to characteristic polynomial, then eigen values, eigen vectors

Read chapter 8 (8.10-8.15 optional)

Exercises 8.6, 8.16, 8.17, 8.22, 8.23, 8.25, 8.32, 8.46, 8.47, 8.67, 8.69

Lecture 24 Nov 24 2016

Last time

A: $n \times n$ matrix $A = (a_{ij})$

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

Theorem $\det A^T = \det A$
(useful for Laplace, same for columns/rows)

pf $A = (a_{ij})$ $A^T = (a_{ji})$

swap since transpose

$$\det A^T = \sum_{\sigma} \operatorname{sgn}(\sigma)\, a_{\sigma(1)1}\, a_{\sigma(2)2} \cdots a_{\sigma(n)n}$$

inverse to switch order of index

$$a_{\sigma(1)1} \cdots a_{\sigma(n)n} = a_{\sigma(1)\sigma^{-1}(\sigma(1))} \cdots a_{\sigma(n)\sigma^{-1}(\sigma(n))}$$

Think of $\{\sigma(1), \dots, \sigma(n)\} = \{1, \dots, n\}$ disordered

$$= a_{1\sigma^{-1}(1)} \cdots a_{n\sigma^{-1}(n)}$$

also: $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma^{-1})$

$$\Rightarrow \det A^T = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma^{-1})\, a_{1\sigma^{-1}(1)} \cdots a_{n\sigma^{-1}(n)}$$

$\sigma \mapsto \sigma^{-1}$ bijection on S_n

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} \cdots a_{n\sigma(n)} = \det A \quad \blacksquare$$

Characteristic polynomial

Def: Let A $n \times n$ matrix

The characteristic polynomial of A is $\Delta_A(t)$ given by

usually on variable t

$$\Delta_A(t) = \det(t \cdot I_n - A)$$

Example

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ 2×2 matrix

$$\Delta_A(t) = \det\left(t\cdot\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} t-a & -b \\ -c & t-d \end{pmatrix} = (t-a)(t-d) - bc$$

$$= t^2 - (a+d)t + ad - bc$$

Or, without looking at entries

$$\boxed{\text{or} \quad \Delta_A(t) = t^2 - (\operatorname{tr} A)t + \det A}$$

Cayley hamilton for 2×2

Example

$A = kI_n$

$$\Delta_A(t) = \det(t\cdot I_n - k\cdot I_n) = \det((t-k)\cdot I_n)$$

$$= (t-k)^n$$

Example

$$A = \begin{pmatrix} k & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & k \end{pmatrix} \quad n\times n \text{ matrix}$$

$n=2$ $\begin{pmatrix} k & 1 \\ 0 & k \end{pmatrix}$; $n=3$ $\begin{pmatrix} k & 1 & 0 \\ & k & 1 \\ 0 & & k \end{pmatrix}$

det of upper triangular is product of diagonals

$$\Delta_A(t) = \begin{vmatrix} t-k & -1 & & 0 \\ & t-k & -1 & \\ & & \ddots & -1 \\ 0 & & & t-k \end{vmatrix} \underset{\uparrow \text{ induction}}{=} \boxed{(t-k)^n}$$

Keep expanding on 1st col

Theorem. Similar matrices have the same characteristic polynomial.

$$A \sim B \Rightarrow \Delta_A(t) = \Delta_B(t)$$

Remark converse true?
NO! last 2 examples

$$P^{-1}(k I_n) P = k I_n \neq 2^{nd} \text{ matrix}$$

for $n = 2$, same trace & same determinant does not imply similarity!

Proof: $B = P^{-1} A P$ some P invertible

$\Delta_B(t) = \det(t \cdot I_n - P^{-1} A P)$

$= \det(P^{-1}(t \cdot I_n - A) P)$

can't directly rearrange Ps because of subtraction

can't seperate subtraction either, not a linear mapping

2 options, either take Ps out (since it's now multiplicative, or say $(P^{-1}(t \cdot I_n - A) P) \sim (t \cdot I_n - A)$ so det is same

$= \det(t \cdot I_n - A) = \Delta_A(t)$ ∎

Theorem Let A $n \times n$ matrix.

Then $\Delta_A(t)$ is a monic polynomial of degree n

$$\Delta_A(t) = t^n + c_{n-1} t^{n-1} + \ldots + c_1 t + c_0 \quad \text{and.}$$

easiest coefficient to get

$c_{n-1} = -\operatorname{tr} A$, $c_0 = (-1)^n \det A$

Remark: $n=2$: all coefficients

in general, intermediate coefficients more complicated.

Proof: $A=(a_{ij})$

$$tI_n - A = \begin{pmatrix} t-a_{11} & -a_{12} & \cdots & \\ & t-a_{22} & & \\ & & \ddots & \\ & & & t-a_{nn} \end{pmatrix}$$

$$(a_{ij}') = tI_n - A$$

$$\Delta_A(t) = \det(t\cdot I_n - A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \underbrace{a'_{1\sigma(1)} \cdots a'_{n\sigma(n)}}_{\text{polynomial of degree } \leq n}$$

in fact, the diagonal term $(t-a_{11})\cdots(t-a_{nn})$ (corresponding to σ = identity) is the only one having degree n

the rest is of degree $\leq n-2$

Have to be off-diagonal for at least 2 entries, cannot switch only 1

$$\Delta_A \underbrace{(t-a_{11})\cdots(t-a_{nn})}_{} + \underbrace{(\cdots)}_{\text{degree } \leq n-2}$$

This will still give some terms with deg $n-1$, etc.

$$= t^n - (a_{11}+a_{22}+\cdots+a_{nn})t^{n-1} + \underbrace{(\cdots)}_{\text{degree } \leq n-2}$$

Think of
$(t-a)(t-b)(t-c)$
$-(a+b+c)t^2$

So $\Delta_A(t)$ degree n, monic and $c_{n-1} = -\operatorname{tr} A$

$\Delta_A(0) = c_0$

$$c_0 = \Delta_A(0) = \det(0\cdot I_n - A) = \det(-A) = (-1)^n \det A$$

Example Consider $f(t) = t^n + c_{n-1}t^{n-1} + \dots + c_1 t + c_0$
a monic polynomial of degree n
Associate an $n \times n$ matrix $C(f)$
Companion matrix of $f(t)$

(somewhat reversing process)

$$C(f) = \begin{pmatrix} 0 & & & 0 & -c_0 \\ 1 & & & & -c_1 \\ & \ddots & 0 & & \vdots \\ & & \ddots & & \\ 0 & & & 1 & -c_{n-1} \end{pmatrix}$$

I_{n-1} "bordered" identity with coeffs outside

Claim: $\Delta_{C(f)}(t) = f(t)$

$$\Delta_{C(f)}(t) = \begin{vmatrix} t & & & & c_0 \\ -1 & t & & & c_1 \\ & -1 & \ddots & t & \vdots \\ & & \ddots & -1 & t + c_{n-1} \end{vmatrix}$$

expand by 1st row

$$= t \underbrace{\underbrace{\begin{vmatrix} t & & & c_1 \\ & t & & \vdots \\ & & \ddots & \\ & & -1 & t + c_{n-1} \end{vmatrix}}_{(n-1)\times(n-1)}}_{t^{n-1} + c_{n-1}t^{n-2} + \dots + c_1} + (-1)^{n+1} \underbrace{c_0 \underbrace{\begin{vmatrix} -1 & t & & \\ & \ddots & \ddots & \\ & & \ddots & t \\ & & & -1 \end{vmatrix}}_{(n-1)\times(n-1) \; = (-1)^{n-1}}}_{c_0}$$

smaller version of 1st

by induction

$$= t(t^{n-1} + c_{n-1}t^{n-2} + \dots + c_1) + c_0 = f(t)$$

Theorem (Cayley-Hamilton)

Every matrix is a root of its characteristic polynomial.

$$\boxed{\Delta_A(A)=0}$$

for $n=2$: old Cayley-Hamilton

$$A^2 - \mathrm{tr}A + \det A \cdot I_2 = 0$$

$\underline{n=3}$

$$A^3 - (\mathrm{tr}A)A^2 + (\quad)A - \det A \cdot I_3 = 0$$

$\underbrace{A_{11}+A_{22}+A_{33}}_{\text{cofactors}}$

Lecture 25 Nov 29 2016

Last time

A : $n \times n$ matrix

$\Delta_A(t) = \det(t \cdot I_n - A)$

char poly of A

- Similar matrices have the same char. polynomial (but not conversely!)
- Cayley-Hamilton. $\Delta_A(A) = 0$

$\Delta_A(t)$ monic polynomial of degree n

Example

$f(t) = t^n + c_{n-1}t^{n-1} + \dots + c_1 t + c_0$

companion matrix $C(f) = \begin{pmatrix} 0 & \cdots & 0 & -c_0 \\ 1 & & & -c_1 \\ & \ddots & & \vdots \\ & & 1 & -c_{n-1} \end{pmatrix}$

$\underbrace{}_{I_{n-1}}$

$$\Delta_{C(f)} = \begin{vmatrix} t & & & & c_0 \\ -1 & t & & & c_1 \\ & -1 & \ddots & & \vdots \\ & & \ddots & t & \\ & & & -1 & t + c_{n-1} \end{vmatrix}$$

Try: expand along last column $= f(t)$

Eigenvalues and Eigenvectors

(chapter 9 - matrix point of view)

A : $n \times n$ matrix over $\mathbb{C}$

Recall:

A diagonalizable if A similar to a diagonal matrix

$$A = P \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} P^{-1} \quad \text{some invertible } P$$

Q: when and how is A diagonalizable?

Multiply by P on the right

$$AP = P \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

$$A \underbrace{\left(u_1 \middle| u_2 \middle| \cdots \middle| u_n \right)}_{P \text{ sliced along columns}} = \left(u_1 \middle| u_2 \middle| \cdots \middle| u_n \right) \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

$$= \left(\lambda_1 u_1 \middle| \lambda_2 u_2 \middle| \cdots \middle| \lambda_n u_n \right)$$

Column-wise: $\begin{cases} A u_1 = \lambda_1 u_1 \\ \vdots \\ A u_n = \lambda_n u_n \end{cases}$

$$\boxed{A u = \lambda u}$$

Def: λ eigenvalue of A if $Av = \lambda v$ for some non-zero vector v, called an eigenvector for λ.

How do you find eigenvalues?

Theorem The eigenvalues of A are the roots of $\Delta_A(t)$.

cor. Similar matrices have the same eigenvalues

Proof (of Thm)

$$\underbrace{\lambda \text{ eigenvalue of } A}_{\substack{\exists v \neq 0 \text{ s.t } Av = \lambda v \\ \exists v \neq 0 \text{ s.t } (\lambda I - A)v = 0}} \iff \underbrace{\Delta_A(\lambda) = 0}_{\underbrace{\det(\lambda I - A) = 0}_{\lambda I - A \text{ singular}}}$$

apply to $M = \lambda I - A$ the following general fact.

Theorem: M $n \times n$ matrix TFAE

(i) M non-singular

(ii) $Mv = 0 \Rightarrow v = 0$

(iii) columns of M are linearly indep.

Exercise: tutorial

Negation of (ii) makes the 2 equal

why $\mathbb{C}$?

Fundamental Theorem of Algebra

Over $\mathbb{C}$, every polynomial of degree $\underline{n}$ has $\underline{n}$ roots, possibly repeated according to multiplicity.

Example

$t^2 - 1$ two roots $-1, 1$

$(t-1)^2$ one root, counted twice

$t^2 - 1$ no roots in $\mathbb{R}$

two roots in $\mathbb{C}$ $\pm i$

Cor. $A \in M_n(\mathbb{C})$ has $\underline{n}$ eigenvalues (possible multiplicities)

Def. λ eigenvalue of A the eigen<u>space</u> of λ is

$$E_\lambda = \{v : Av = \lambda v\}$$

· **remark**: E_λ subspace $= \ker(\lambda I - A)$

· **remark**: non-zero vectors in E_λ are eigenvectors λ

Thm. $\underbrace{\dim E_\lambda}_{\text{geometric multiplicity of } \lambda} \le \underbrace{\text{multiplicity of } \lambda \text{ as a root of } \Delta_A}_{\text{algebraic multiplicity of } \lambda}$

Example $A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ λ_i's distinct

$$\Delta_A(t) = (t-\lambda_1) \cdots (t-\lambda_n)$$

eigenvalue of A = roots of $\Delta_A = \underbrace{\lambda_1, \dots, \lambda_n}_{\text{each of multiplicity } 1}$

$$E_{\lambda_i} = \{v : Av = \lambda_i v\}$$

$$\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \lambda_i \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

$$\begin{cases} \lambda_1 v_1 = \lambda_i v_1 & \Rightarrow v_1 = 0 \\ \lambda_2 v_2 = \lambda_i v_2 & \Rightarrow v_2 = 0 \\ \quad\vdots & \boxed{v_i = ??} \\ \lambda_n v_n = \lambda_i v_n & \Rightarrow v_n = 0 \end{cases}$$

$\boxed{E_{\lambda_i} = \text{span}(e_i)}$ 0 for everything except v_i, can be whatever it wants

Thm: $\boxed{1=1}$

Example:

$$A = \begin{bmatrix} \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_1 & & & \\ & & & \ddots & & \\ & & & & \lambda_k & \\ & & & & & \ddots \\ & & & & & & \lambda_k \end{pmatrix} \begin{matrix} \}\, m_1 \\ \vdots \\ \}\, m_k \end{matrix} \end{bmatrix}$$

$m_1 + \dots + m_k = n$

$\Rightarrow n$ roots with multiplicity

Can assume they're blocked together

$$\begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 1 & \\ & & & 2 \end{pmatrix} \text{ same eigen problem as } \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 2 & \\ & & & 2 \end{pmatrix}$$

because they're similar!

$\Delta_A(t) = (t-\lambda_1)^{m_1} \cdots (t-\lambda_k)^{m_k}$

eigenvalues of A: $\underset{m_1}{\lambda_1}, \ldots, \underset{m_k}{\lambda_k}$ multiplicities

$\dim E_{\lambda_i} = m_i$ Thm: $\boxed{\underset{\text{geo}}{m_i} = \underset{\text{alg.}}{m_i}}$

for diagonal matrices geom. mult = alg. mult

Example

$$\begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$$

- eigenvalues
- dim. eigenspace

$\Delta_A(t) = (t-\lambda)^n$

Eigenvalues $= \lambda$ with multiplicity n

$$\begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

$\lambda v_1 + v_2 = \lambda v_1 \Rightarrow v_2 = 0$

$\lambda v_2 + v_3 = \lambda v_2 \Rightarrow v_3 = 0$

$\vdots \qquad\qquad \vdots$

$\lambda v_n = \lambda v_n \qquad v_n = 0$??

- dim. eigenspace span$(e_1) = E_\lambda$

$\dim(E_\lambda) = 1$ geom. mult = 1

Thm: Let $\lambda_1, \dots, \lambda_k$ <u>distinct</u> eigenvalues, and let $u_1, \dots, u_k$ corresponding eigenvectors

$\Rightarrow \{u_1, \dots, u_k\}$ independent.

Pf. $\underbrace{c_1 u_1}_{v_1} + \dots + \underbrace{c_k u_k}_{v_k} = 0$

$v_1 + \dots + v_k = 0$

A·\ multiply by A

$\lambda_1 v_1 + \dots + \lambda_k v_k = 0$

$\begin{cases} \lambda_1 v_1 = A v_1 \\ \vdots \\ \lambda_k v_k = A v_k \end{cases}$

A·\

$\lambda_1^2 v_1 + \dots + \lambda_k^2 v_k = 0$

$\dots$

$\lambda_1^{k-1} + \dots + \lambda_k^{k-1} v_k = 0$

System of linear equations, put it in a matrix

Vandermonde

$\underbrace{V(\lambda_1, \dots, \lambda_k)}_{\text{invertible}} \begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix} = 0$

$\Rightarrow v_1 = \dots = v_k = 0$

$c_1 u_1 = 0, \; u_i \neq 0 \Rightarrow c_1 = 0$

$\vdots$

$c_2 = 0$

Lecture 26 Dec 1 2016

Recall * $A_{n\times n}$ is diagonizable if it is similar to a diagonal matrix. i.e. $\exists$ a non-singular matrix P and a diagonal matrix D s.t. $A = PDP^{-1}$

* $\lambda \in \mathbb{C}$ is an eigenvalue for a matrix $A_{n\times n}$ if $Av = \lambda v$, for some non-zero vector.

Fact If λ has an algebraic multiplicity m, then λ yields at least one eigenvector and at most m eigenvectors.

If you don't get m, not diagonizable

Thm Let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues and $v_1, \ldots, v_{k'}$ are the corresponding eigenvectors.

Some λ may give more than 1 eigenvector, i.e. $k' \geq k$

eigenvectors always linearly independent

Then $\{v_1, \ldots, v_{k'}\}$ are linearly independent.

Fact An $n\times n$ matrix has at most n distinct eigenvalues

Corollary

If $A_{n\times n}$ has exactly n distinct eigenvalues, A is diagonalizable.

Example let $A = \begin{pmatrix} 5 & 6 \\ 3 & -2 \end{pmatrix}$

① Find $\det(tI - A) = \det\begin{pmatrix} t-5 & -6 \\ -3 & t+2 \end{pmatrix} = t^2 + 2t - 5t - 10 - 18$
$= (t-7)(t+4)$

$\lambda_1 = 7, \quad \lambda_2 = -4$

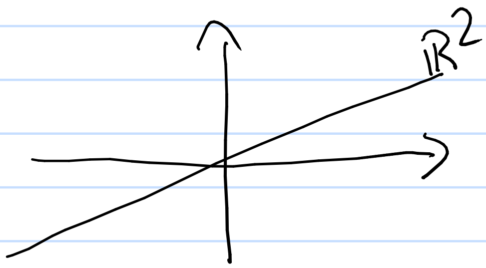
② To find the eigenvector for $\lambda_1 = 7$, we need to find a vector $v = \begin{pmatrix} a \\ b \end{pmatrix}$ s.t. $Av = \lambda_1 v$

i.e. solving $(\lambda_1 I - A)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 2 & -6 \\ -3 & 9 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 2a - 6b = 0 \\ -3a + 9b = 0 \end{cases} \Rightarrow \boxed{a = 3b}$$

could do $b = \frac{a}{3}$ get $v_1 = \begin{pmatrix} 1 \\ 1/3 \end{pmatrix}$ same thing since it's parallel

$$v = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3b \\ b \end{pmatrix} = b\begin{pmatrix} 3 \\ 1 \end{pmatrix} \qquad v_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$



If you multiply by vector on the (3,1) line, matrix acts as if it is just multiplying it (as a scalar)

$\lambda_2 = -4$

$$\begin{pmatrix} -9 & -6 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -9a - 6b = 0 \Rightarrow$$

$$v_2 = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -2/3\, b \\ b \end{pmatrix} \to v_2 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$\lambda_1 = 7, \quad \lambda_2 = -4$$

$$\downarrow \qquad \downarrow$$

$$v_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \Rightarrow D = \begin{pmatrix} 7 & 0 \\ 0 & -4 \end{pmatrix}$$

$$P = \begin{pmatrix} 3 & -2 \\ 1 & 3 \end{pmatrix}$$

can switch columns of D, but have to do same to P

$A = PDP^{-1}$

Recall A is symmetric if $A = A^T$

Def A square matrix $A_{n \times n}$ with complex entries is said to be hermitian if $A = \overline{A^T}$

A^H

Ex: $A = \begin{pmatrix} 2 & 2-i \\ 2+i & 3 \end{pmatrix}$

Fact Let $A = A^T$

$\langle Av, w \rangle = \langle v, A^T w \rangle = \langle v, Aw \rangle$

Since:

$(Av)^T \cdot w = (v^T A^T) \cdot w = v^T A^T w$

Also, $\langle Av, w \rangle = \langle v, A^H w \rangle = \langle v, Aw \rangle$ if A is Hermitian

Thm Let A be a real symmetric matrix. Then the eigenvalues of A are real.

Ex. $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow \Delta_A(t) = t^2 + 1 \quad \pm i$

Pf Given $A = A^T$ and λ an eigenvalue of A, and v the corresponding eigenvector

$\langle \underbrace{Av}_{\lambda v}, v \rangle = \langle v, A^T v \rangle = \langle v, \underbrace{Av}_{\lambda v} \rangle$ since v is eigenvector

linear antilinear in 2nd coord

$\Rightarrow \langle \lambda v, v \rangle = \langle v, \lambda v \rangle$

Don't know yet if λ is real

$\lambda \underbrace{\langle v, v \rangle}_{\neq 0} = \bar{\lambda} \langle v, v \rangle$

$(\lambda - \bar{\lambda}) \underbrace{\langle v, v \rangle}_{\neq 0} = 0$

$\boxed{\lambda - \bar{\lambda} = 0} \Rightarrow \lambda \in \mathbb{R}$

$\lambda = a + ib$

$\bar{\lambda} = a - ib$

$\lambda - \bar{\lambda} = (a+ib) - (a-ib)$
$= 2ib$
$\Rightarrow b = 0$

Thm Let A be a symmetric matrix and suppose that $\lambda_1 \neq \lambda_2$ are two distinct eigenvalues. Then the corresponding eigenvectors v_1, v_2 will be orthogonal.

Pf Let $Av_1 = \lambda_1 v_1$ $\quad Av_2 = \lambda_2 v_2$

$$\langle Av_1, v_2\rangle = \langle v_1, Av_2\rangle$$

$$\Rightarrow \langle \lambda_1 v_1, v_2\rangle = \langle v_1, \lambda_2 v_2\rangle$$

$$\Rightarrow \lambda_1 \langle v_1, v_2\rangle = \lambda_2 \langle v_1, v_2\rangle$$

$$\Rightarrow \underbrace{(\lambda_1 - \lambda_2)}_{\neq 0}\langle v_1, v_2\rangle = 0 \quad \Rightarrow \langle v_1, v_2\rangle = 0$$

$$\Rightarrow v_1 \perp v_2$$

$\blacksquare$

Remark $\lambda_1 \to v_1, \; \lambda_1 \to v_1'$, $\quad \lambda_2 \to v_2, \; \lambda_2 \to v_2', \; \lambda_2 \to v_2''$

Not necessarily orthogonal, just linearly ind.

$v_1 \perp v_2$, $v_1 \perp v_2'$, v_1, v_2''
$v_1' \perp v_2$, ...

Thm Let A be a symmetric matrix, then there exists an orthogonal matrix P s.t. $A = PDP^{-1}$, where D diagonal

Use gram-Schmidt for v_1 and v_1'
v_2, v_2', v_2''
i.e. for λ that give multiple eigenvalues

Note To find an orthogonal matrix P, we do the following:

① Find the eigenvectors corresponding to each eigenvalue λ_i i.e.: $\lambda_i \to \{v_{i_1}, v_{i_2}, \dots, v_{i_m}\} \to$ then use Gram-Schmidt to find an orthonormal collection $\{v_{i_1}', v_{i_2}', \dots, v_{i_m}'\}$

Thm If A is hermitian, then there exists a unitary matrix V s.t. $A = VDV^{-1}$ for some diagonal matrix D.

Ex $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad \Delta_A(t) = \det \begin{pmatrix} t & -1 & 0 \\ -1 & t & 0 \\ 0 & 0 & t-1 \end{pmatrix}$

$= (t-1)^2(t+1)$

multiplicity

$\lambda_1 = 1, \; m_1 = 2$

$\lambda_2 = -1, \; m_2 = 1$

$\lambda_1 = 1$

$(I - A)\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$a - b = 0, \quad c = c$

$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ a \\ c \end{pmatrix} = a\underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}_{v_1} + c\underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{v_1'}$

$\lambda_2 = -1$

$$\begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -a-b=0 \Rightarrow \boxed{a=-b} \\ -2c=0 \Rightarrow \boxed{c=0} \end{cases}$$

$$v_2 = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -b \\ b \\ 0 \end{pmatrix} = b\boxed{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}$$

$$\|(1,1,0)\| = \sqrt{1^2+1^2+0^2} = \sqrt{2}$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$A = PDP^{-1}$

columns are orthogonal, but not normalized

$$\left\|\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)\right\| = \sqrt{\frac{1}{2} + \frac{1}{2} + 0}$$

14/9/16
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MATH 223 - Linear Algebra

Tutorial 1

Brahim

Complex numbers

$z = a + ib$, $i^2 = -1$

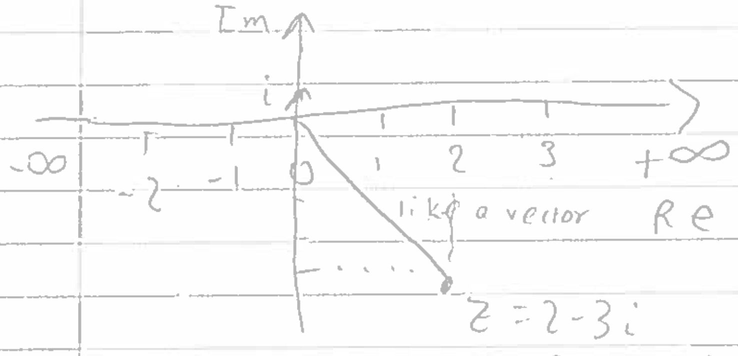
a: Real part, $Re(z)$

b: imaginary, $Im(z)$

$z = 3 + i5$

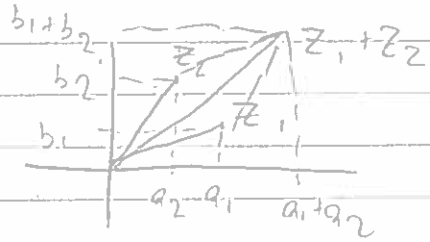
$z = 2 - i3 = 2 + i(-3)$

2: $Re(z)$ $\quad$ -3: $Im(z)$



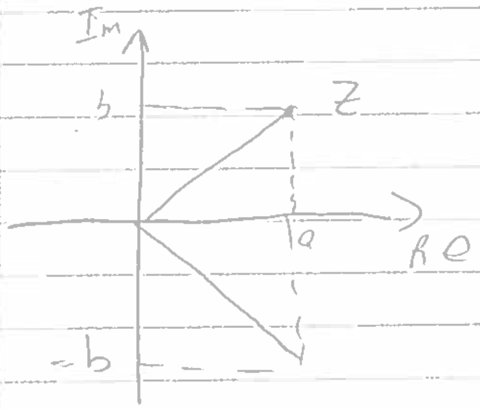
Exp $z_1 = a_1 + ib_1$,
$z_2 = a_2 + ib_2$, then

$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$
same as vector addition



The complex plane

Conjugate of $z = (a) + ib$ is $\bar{z} = (a) - ib$



Properties let $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$

(1) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ (4) $z_1 \bar{z}_1 = a_1^2 + b_1^2$

(2) $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$

(3) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

$$\frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2} = (\text{Real number}) + i\,(\text{Real number})$$

$$\frac{z_1}{z_2} = \frac{z_1}{z_2} \cdot \frac{\bar{z}_2}{\bar{z}_2} = \frac{z_1 \bar{z}_2}{a_2^2 + b_2^2}$$

absolute value of z

Def: If $z = a + ib$, then $|z| = \sqrt{a^2 + b^2}$

(5) $|z_1 z_2| = |z_1| \cdot |z_2|$

Q Does there exist z st $|z| < |\bar{z}|$

A Does not exist

$z_1 = a + ib$, then we can represent z as a vector in the plane; (a, b)

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223 p2

Exercise 1.37

complex numbers

$z, w \in \mathbb{C}$ $\quad z = a+ib, w = c+id$

$|z+w| \leq |z| + |w|$

Let $u = (a, b)$, $v = (c, d)$

Then by Minkowski's

$||u+v|| \leq ||u|| + ||v||$

$|z+w| = |\underbrace{(a+ib)}_{z} + \underbrace{(c+id)}_{w}|$

$= |(a+c) + i(b+d)|$

$= \sqrt{(a+c)^2 + (b+d)^2}$

$= ||(a+c, b+d)||$

$= ||v+u|| \leq ||v|| + ||u|| = |z| + |w|$

Dot product

$\mathbb{C}^n$ is the space of n-tuples $(z_1, z_2, z_3, \ldots, z_n)$, where $z_i \in \mathbb{C}$, $1 \leq i \leq n$

Given two vectors $u = (z_1, \ldots, z_n)$ and $u = (w_1, w_2, \ldots, w_n)$ in $\mathbb{C}^n$

We can define an inner product

Not commutative

$u \cdot v = (z_1, z_2, \dots, z_n) \cdot (w_1, w_2, \dots, w_n)$

$= z_1\overline{w_1} + z_2\overline{w_2} + \dots + z_n\overline{w_n}$

__Exp__ $u = (2+i, 5, -i)$, $v = (i, 3, i)$

$u \cdot v = (2+i)(\underset{-i}{\overline{i}}) + 5 \cdot 3 + (-i) \cdot (\underset{-i}{\overline{i}})$

$= -2i + 1 + 15 + (-1)$

$= 15 - 2i$

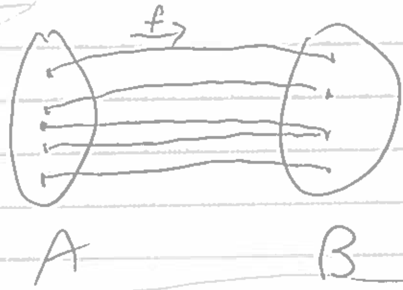
In $\mathbb{C}^n$ $u \cdot v = \overline{v \cdot u}$

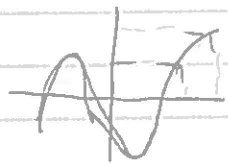
In $\mathbb{R}^n$ $u \cdot v = v \cdot u$

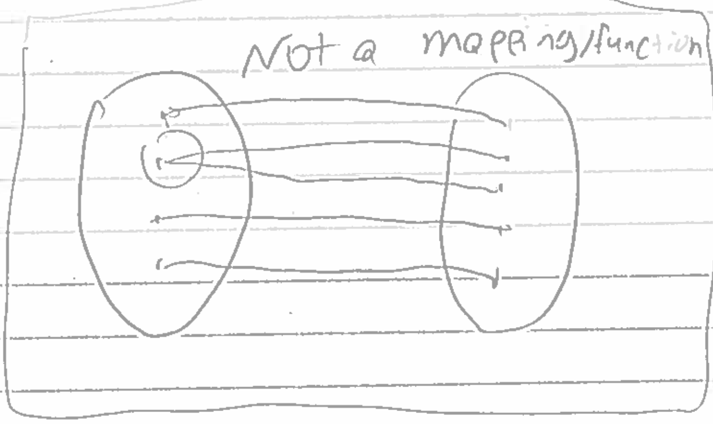
Mappings

input $\longrightarrow \boxed{f} \longrightarrow$ output

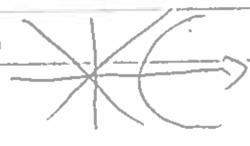
__Exp__ let $f: A \to B$







not a function



14/9/16
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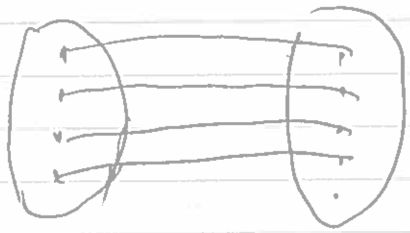
223 p3

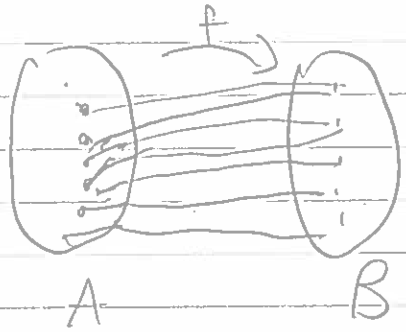
$f: A \to B$ is a one-to-one mapping (injective, 1-1) if
$f(a_1) = f(a_2)$, then $a_1 = a_2$



in other words:
if $a_1 \neq a_2$, then
$f(a_1) \neq f(a_2)$

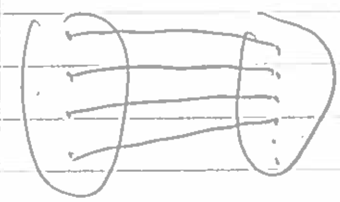
Onto: $f: A \to B$ is onto (surjective) if

for any element b in B, there exists an element a in A s.t. $f(a) = b$



Def: If $f: A \to B$ is 1-1 and onto, then we say f is bijective

1-1 but not onto



Exp Define the mapping $\|\cdot\|$

$$\|\cdot\| : \mathbb{R}^3 \to \mathbb{R}$$
$$v \mapsto \|v\|$$

$v = (3, 1, 0)$
$\|v\| = \sqrt{3^2 + 1^2 + 0^2} = \sqrt{10}$

Remark that the norm function always takes positive values therefore any negative number in $\mathbb{R}$ has no pre-image (i.e. input) so f is not onto

Note, to make $\|\cdot\|$ onto

we need to restrict the image to only $\mathbb{R}_{\geq 0}$

$\|\cdot\|: \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$
is onto

21/9/16
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MATH 223-Linear Algebra

Tutorial 2

We'll prove this:

$tr(AB) = tr(BA)$

Let A, B be nxn matrices

$A = (a_{ik})$ $B = (b_{kj})$

$AB = C = (c_{ij})$, where

$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

$tr(AB) = \sum_{i=1}^{n} c_{ii} = \sum_{i=1}^{n} \left(\sum_{k=1}^{n} a_{ik} b_{ki} \right) = \sum_{k=1}^{n} \left(\sum_{i=1}^{n} b_{ki} a_{ik} \right)$

$= tr(BA)$

$B = (b_{jk})$; $A = (a_{ki})$ $BA = D = (d_{ij})$

because $tr(BA) = \sum_{i=1}^{n} D_{ii} = \sum_{i=1}^{n} \left(\sum_{k=1}^{n} b_{ik} a_{ki} \right) = \sum_{k=1}^{n} \left(\sum_{i=1}^{n} b_{ik} a_{ki} \right)$

Now, rename i by k and k by i

$= \sum_{k=1}^{n} \left(\sum_{i=1}^{n} b_{ki} a_{ik} \right)$

Recall

1. Matrix multiplication is associative then $A \cdot B \cdot C = (AB) \cdot C$
$= A(BC)$

Hilroy

(2) If A has an inverse A^{-1}, then
$AA^{-1} = I \quad A^{-1}A = I$

Recall

$tr(AB) = tr(BA)$

Ex Prove that

$tr(B^{-1}AB) = tr(A)$

$tr(B^{-1}AB) = tr(A)$

$tr((B^{-1}A)B) = tr(B(B^{-1}A))$

$= tr(\underbrace{(BB^{-1})}_{I}A)$

$= tr(I \cdot A)$

$= tr(A)$ ∎

proof by induction

Ex Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Find A^n

Sol First let's compute A^n for $n = 1, 2, 3, 4$

$A^1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

$A^2 = AA = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$

$A^3 = A^2 A = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$

$A^n = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}$

base case, works for 1, 2, 3 shown above

Now, we suppose that $A^k = \begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix}$ is true for all $k = 1, 2, \dots, n$

we need to show that it is also true for $k = n+1$

$A^{n+1} = A^n A = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2n+2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2(n+1) \\ 0 & 1 \end{pmatrix}$ ∎

28/9/16
Julian Lore

MATH 223 - Linear Algebra

Tutorial 3

Recall

Def A set of vectors $\{u_1, u_2, \ldots, u_n\}$ is orthogonal if each vector has norm $=1$ and the dot product $u_i \cdot u_j = 0$ when $i \neq j$
in other words $u_i \cdot u_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

Exp $\{(0,1), (1,0)\}$, $\{(\frac{1}{2}, \frac{\sqrt{3}}{2}), (-\frac{\sqrt{3}}{2}, \frac{1}{2})\}$

Def A matrix A is orthogonal if $AA^T = A^TA = I$

Note this definition implies A must be a square matrix and must be invertible (why?)

Exp $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = A$

$A^T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $AA^T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Exp $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

equivalent def

Properties (both equal, check 1)

equiv

(1) A is Orthogonal $\Leftrightarrow$ the set of its row-vectors is orthonormal

(2) A is orthogonal $\Leftrightarrow$ the set of its column vectors is orthonormal

exp $\left(\begin{array}{cc} q_1 & q_2 \\ \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} & \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} \end{array} \right)$ $\quad \|q_1\| = \sqrt{\frac{1}{4}+\frac{3}{4}} = \sqrt{\frac{4}{4}} = 1$

$\|q_2\| = 1$

and $q_1 \cdot q_2 = (\frac{1}{2})(\frac{\sqrt{3}}{2}) + (-\frac{\sqrt{3}}{2})(\frac{1}{2}) = 0$

Theorem Let A be a 2×2 orthogonal matrix then for number θ, A can be written in the following form:

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Pf Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be orthogonal then norm of row (a, b) is $=1$ i.e;

$\sqrt{a^2+b^2} = 1$, after squaring both sides

$\boxed{a^2+b^2=1}$

Similarly, $c^2+d^2=1$, and for the columns we get

$a^2+c^2=1$

$b^2+d^2=1$

So from $\begin{cases} a^2+b^2=1 \quad \dots\dots (\text{I}) \\ d^2+b^2=1 \quad \dots\dots (\text{II}) \end{cases}$

28/9/16
Julian Lore

223 P2

$(\mathrm{I})-(\mathrm{II}) = (a^2+b^2)-(d^2+b^2) = 1-1 = 0$
$= a^2+b^2-d^2-b^2 = 0$
$= a^2-d^2 = 0 \Rightarrow$

$\boxed{a^2=d^2} \Rightarrow \boxed{a=\pm d}$

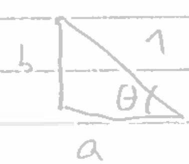
By the same method we get
$b = \pm c$

so $\boxed{A = \begin{pmatrix} a & b \\ \pm b & \pm a \end{pmatrix}}$

$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$, $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$, $\begin{pmatrix} a & b \\ -b & -a \end{pmatrix}$

the dot prod: $\underbrace{ab+ab}_{2ab} \neq 0$ $\quad \boxed{-ab+ab=0 \quad ab-ba=0} \quad -ab-ba=-2ab \neq 0$

We have $a^2+b^2=1$
By "Pythagoras" theorem



$\cos\theta = \frac{a}{1} = a$

$\sin\theta = \frac{b}{1} = b$

Def Given a complex matrix A, the conjugate transpose is

$$A^H = (\bar{A})^T = \overline{(A^T)}$$

Exp $A = \begin{pmatrix} i & 3+i \\ 2 & 1+i \end{pmatrix}$, $A^H = \overline{\begin{pmatrix} i & 2 \\ 3+i & 1+i \end{pmatrix}} = \begin{pmatrix} -i & 2 \\ 3-i & 1-i \end{pmatrix}$

Def A matrix A is unitary if $AA^H = A^H A = I$

Note A must be square and invertible

A unitary $\Leftrightarrow$ It's row vector form an orthonormal set

A unitary $\Leftrightarrow$ It's column vector form an orthonormal set

Def * A is Hermitian if $A = A^H$
* A is skew-Hermitian if $A^H = -A$

Prop (1) $(A^H)^H$ (2) $(A+B)^H = A^H + B^H$ (3) for $k \in \mathbb{C}$ $(kA)^H = \bar{k}A^H$
(4) $(AB)^H = B^H A^H$

Exercise $A = \begin{pmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{pmatrix}$ is it unitary?

Find the norm of $r_1 = \left(\frac{1+i}{2}, \frac{1-i}{2}\right)$

in $\mathbb{R}^2$; $v = (3,4)$
$\|v\| = \sqrt{3^2+4^2} = \sqrt{25} = 5$

for $\mathbb{C}$: $\left|\frac{1+i}{2}\right| = \left|\frac{1}{2} + \frac{i}{2}\right| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{2}}{2}$

have to do norms of each component first

$\left|\frac{1-i}{2}\right| = \left|\frac{1}{2} - \frac{i}{2}\right| = \sqrt{\frac{1}{4} + \frac{1}{4}} = \frac{\sqrt{2}}{2}$

Now,

$\|r_1\| = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2}$

$= \sqrt{\frac{2}{4} + \frac{2}{4}} = 1$