Analysis 3H

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- Adapted from notes of D. Schütz, Durham
- This was part of the Analysis 3H module elective. This is a course on real analysis, touching on metric spaces, tangent spaces, vector fields, manifolds, and differential forms.

- **TODO!** diagrams

1 Metric spaces

1.1 Basic notions

The field of real numbers $\mathbb{R}$ is a totally ordered field which also satisfies the **completeness** axiom, i.e. a non-empty bounded set $A \subseteq \mathbb{R}$ has a **supremum** and/or an **infimum**. The supremum of $A \subseteq \mathbb{R}$ is a real number s where $a \leq s$ for all $a \in A$. If m is also such that $a \leq m$ for $a \in A$, then $s \leq m$, denoted $\sup A$. The infimum of A is where the inequalities signs are swapped, denoted $\inf A$.

Lemma 1.1.1 *Let $I_n = [a_n, b_n]$ be a sequence of closed intervals such that $a_n \leq a_{n+1} < b_{n+1} \leq b_n$ for all $n \geq 1$, then $\cap_{n=1}^{\infty} I_n$ is non-empty.* □

Proof Let $a = \sup\{a_n\}$. Since $a_n \leq b_1$ for all n exists by completeness axiom, $a_n \leq b_k$ for any value of n and k, and so $a \leq b_k$. Hence $a_k \leq a \leq b_k$ for all k, and that $a \in \cap_{n=1}^{\infty} I_n$.

Let M be a set. A function $d : M \times M \to [0, \infty)$ is called a **metric** on M if

1. $d(x, y) = 0$ iff $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in M$;
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in M$.

The pair (M, d) is then called a **metric space**. It is easy to see any $N \subseteq M$ is also a metric space using the same d.

Example 1. On $\mathbb{R}$, $d(x, y) = |y - x|$ gives a metric.

2. On $\mathbb{R}^2$, $d_1(\boldsymbol{x}, yb) = |y_1 - x_1| + |y_2 - x_2|$ is also a metric, but notice that, for example, $d_1((1,1),(0,0)) = 2$ as opposed to the expected distance of $\sqrt{2}$.

   The standard (Euclidean) metric in $\mathbb{R}^2$ is given by

$$d(\boldsymbol{x}, \boldsymbol{y}) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}.$$

Let V be a real vector space. An **inner product** on V is a function $(\cdot, \cdot) : V \times V \to \mathbb{R}$ that, for all $\boldsymbol{x}, \boldsymbol{y} \in V$, satisfies the following:

- linearity in the first factor;
- $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{y}, \boldsymbol{x})$;
- $(\boldsymbol{x}, \boldsymbol{x}) \geq 0$ and is zero iff $\boldsymbol{x} = 0$.

Example 1. For $V = \mathbb{R}^n$, the standard inner product is given by $(\boldsymbol{x}, \boldsymbol{y}) = x_i y_i$ (where Einstein notation is understood). If $\mathbf{A}$ is a symmetric matrix, then $(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}^T \mathbf{A} \boldsymbol{y}$ is an inner product if all eigenvalues of $\mathbf{A}$ are positive.

2. For $V = C[a,b]$, $(f,g) = \int_a^b f(x)g(x)\,\mathrm{d}x$ is an inner product since V is a vector space of continuous functions, and the only function that is everywhere zero and continuous is $f(x) = 0$ for all $x \in [a,b]$.

Theorem 1.1.2 (Cauchy–Schwartz inequality) *Let V be a real vector space, and $(\cdot,\cdot)$ an inner product on V. Then*

$$|(\boldsymbol{x}, \boldsymbol{y})| \leq \|\boldsymbol{x}\| \cdot \|\boldsymbol{y}\|,$$

where $\|\cdot\|$ is the standard Euclidean norm of the vector, and there is equality iff $\boldsymbol{x} = \lambda \boldsymbol{y}$ for some $\lambda \in \mathbb{R}$.

Proof Note that $(\boldsymbol{x}, \boldsymbol{0}) = (\boldsymbol{x}, \boldsymbol{x} - \boldsymbol{x}) = (\boldsymbol{x}, \boldsymbol{x}) - (\boldsymbol{x}, \boldsymbol{x}) = \boldsymbol{0}$, so we may assume that $\boldsymbol{y} \neq \boldsymbol{0}$. Then, with $\lambda = -(\boldsymbol{x}, \boldsymbol{y})/\|\boldsymbol{y}\|^2$,

$$\begin{aligned} 0 \leq (\boldsymbol{x} + \lambda \boldsymbol{y}, \boldsymbol{x} + \lambda \boldsymbol{y}) &= \|\boldsymbol{x}\|^2 + 2\lambda(\boldsymbol{x}, \boldsymbol{y}) + \lambda^2 \|\boldsymbol{y}\|^2 \\ &= \|\boldsymbol{x}\|^2 - \frac{(\boldsymbol{x}, \boldsymbol{y})^2}{\|\boldsymbol{y}\|^2}. \end{aligned}$$

So $(\boldsymbol{x}, \boldsymbol{y})^2 \leq \|\boldsymbol{x}\|^2 \|\boldsymbol{y}\|^2$ and the result follows. ∎

Lemma 1.1.3 *Let V be a real vector space with inner product $(\cdot,\cdot)$. Then $d : V \times V \to [0,\infty)$ with $d(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|$ gives a metric on V.*

Proof Clearly $d(\boldsymbol{x}, \boldsymbol{x}) = 0$ and is symmetric, so we just need to check the triangle inequality. By Cauchy–Schwartz,

$$\begin{aligned} \|\boldsymbol{a} + \boldsymbol{b}\| &= \sqrt{\|\boldsymbol{a}\|^2 + 2(\boldsymbol{a}, \boldsymbol{b}) + \|\boldsymbol{b}\|^2} \\ &\leq \sqrt{\|\boldsymbol{a}\|^2 + 2\|\boldsymbol{a}\|\|\boldsymbol{b}\| + \|\boldsymbol{b}\|^2} \\ &\leq \|\boldsymbol{a}\| + \|\boldsymbol{b}\|, \end{aligned}$$

as required. ∎

Let $f : M \to N$ be a function metric metric spaces (M, d_M) and (N, d_N). For $a \in M$, f is **continuous at** a if, for all $\epsilon > 0$, there exists $\delta > 0$ such that $d_N(f(a), f(x)) < \epsilon$ for all $x \in M$ when $d_M(a, x) < \delta$.

1.2 *Sequences and Cauchy sequences*

Let M be a metric space. A **sequence** (a_n) in M consists of elements $a_n \in M$ for all $n \in \mathbb{N}$. Let $a \in M$, and (a_n) **converges to** a if, for all $\epsilon > 0$, $d(a_n, a) < \epsilon$ for some all $n \geq n_0$. We write $\lim_{n\to\infty} a_n = a$. The sequence (a_n) is called **convergent** if there exists $a \in M$ where $a_n \to a$.

Lemma 1.2.1 *Let $f : M \to N$ be a function between metric spaces and $a \in M$. The function f is continuous at $a \in M$ iff $f(a_n) \to f(a)$ for $(a_n) \in M$ with $a_n \to a$. (Note that $f(a_n)$ is a sequence in N.)*

Proof Assume that f is continuous at $a \in M$, and let (a_n) be a sequence with $a_n \to a$. By continuity, for any $\epsilon > 0$, there exists $\delta > 0$ such that, for $d(a, y) < \delta$, $d(f(a), f(y)) < \epsilon$ for arbitrary $y \in M$. Choose $n_0 \geq 0$ such that $d(a_n, a) < \delta$ for all $n \geq n_0$, then this implies $d(f(a_n), f(a)) < \epsilon$, and thus $f(a_n) \to f(a)$ as required.

On the other hand, assume $f(a_n) \to f(a)$ for all sequences such that $a_n \to a$. Given $\epsilon > 0$, assume that instead there is no $\delta > 0$ such that, for $d(a, y) < \delta$, $d(f(a), f(y)) < \epsilon$ for arbitrary $y \in M$. Then we can find $a_n \in M$ with $d(a, a_n) < 1/n$. However, this means $d(f(a), f(a_n)) \geq \epsilon$, which contradicts the assumption that $f(a_n) \to f(a)$ even though $a_n \to a$. So such δ exists and we have continuity. ■

Lemma 1.2.2 *The limit of a sequence is unique.*

Proof Assume there are two limits a and b for the sequence a_n. Then $d(a, b) \leq d(a, a_n) + d(a_n, b)$. As $n \to \infty$, the RHS tends to zero so $a = b$. ■

A **Cauchy sequence** (a_n) in the metric space M is a sequence such that, for all $\epsilon > 0$, there exists $n_0 \geq 0$ such that $d(a_p, a_q) < \epsilon$ for all $p, q \geq n_0$.

Lemma 1.2.3 *A convergent sequence is a Cauchy sequence (the converse is not true).*

Proof Suppose $a_n \to a$. Then, for all $\epsilon > 0$, there is some $n_0 \geq 0$ such that $d(a_n, a) < \epsilon/2$ for $n \geq n_0$. Let $p, q \geq n_0$, then $d(a_n, a_q) \leq d(a_p, a) + d(a_q, a) < \epsilon$, so the sequence is Cauchy. ■

A metric space M is **complete** if all Cauchy sequences in M converges.

Theorem 1.2.4 *The real line $\mathbb{R}$ is complete.*

Proof Let (a_n) be a Cauchy sequence in $\mathbb{R}$. Define the sequence of integers (n_k) where $n_0 = 1$, and n_{k+1} is the smallest integer bigger than n_k where $|a_p - a_q| < 2^{-(k+2)}$ for $p, q \geq n_{k+1}$. Define the intervals $I_k = [a_{n_k} - 2^{-k}, a_{n_k} + 2^{-k}]$ and let $x \in I_{k+1}$. Now, since $x \in I_{k+1}$, this implies that $|x - a_{n_{k+1}}| < 2^{-(k+1)}$. By definition of the integer sequence, $|a_{n_k} - a_{n_{k+1}}| < 2^{-(k+1)}$, so then, by triangle inequality,

$$|a_{n_k} - x| \leq |x - a_{n_{k+1}}| + |a_{n_{k+1}} - a_{n_k}| < 2 \cdot 2^{-(k+1)} = 2^{-k},$$

so $x \in I_k$. However, $x \in I_{k+1}$, so $I_{k+1} \subset I_k$. By Lemma 1.1.1, $\cap_{k=1}^{\infty} I_k \neq \emptyset$, so assume $a \in \cap_{k=1}^{\infty} I_k$. For $m \geq n_k$,

$$|a - a_m| \leq |a - a_{n_k}| + |a_{n_k} - a_m| \leq 2^{-k} + 2^{-(k+1)} \to 0$$

as $m \geq n_k \to \infty$. Thus $a_m \to a$ and this arbitrary Cauchy sequence converges in $\mathbb{R}$ and thus $\mathbb{R}$ is complete. ■

Proposition 1.2.5 *For $X \neq \emptyset$, let $\mathcal{B}(X)$ be the set of functions $f : X \to \mathbb{R}$ such that f is bounded. For $f, g, \in \mathcal{B}(X)$, let $d(f, g) = \sup_{x \in X} |f(x) - g(x)$. Then $(\mathcal{B}(X), d(f, g))$ defines a complete metric space.*

Proof d is clearly a metric. For completeness, let (f_n) be a Cauchy sequence in $\mathcal{B}(X)$. For $x \in X$, $(f_n(x))$ is a Cauchy sequence of real numbers because, by definition of $d(f, g)$, $|f_q(x) - f_p(x)| \leq d(f_p - f_q)$, and since $\mathcal{R}$ is complete, the sequence $(f_n(x))$ converges.

Defining $f : X \to \mathbb{R}$ such that $f(x) = \lim_{n\to\infty} f_n(x)$, we need to show that $f \in \mathcal{B}(X)$, and that indeed $f_n(x) \to f(x)$ regardless of $x \in X$. Be definition of a Cauchy sequence, for $\epsilon > 0$, there exists $n_0 \geq 0$ such that $d(f_p, f_q) < \epsilon/2$ for $p, q \geq n_0$. Note also that, for all $x \in X$, there exists $n_1(x) \geq n_0$ such that $|f_{n_1(x)} - f| < \epsilon/2$. Then, let $x_{\in} X$ and $n \geq n_0$, we have

$$|f_n(x) - f(x)| \leq |f_n - f_{n_1(x)}| + |f_{n_1(x)} - f| < \epsilon.$$

Additionally note, $|f(x)| \leq |f(x) - f_{n_0(x)}| + |f_{n_0(x)}| \leq \epsilon + c_{f_{n_0}}$ since $f_{n_0(x)}$ is bounded, so $f \in \mathcal{B}(X)$. Further, $d(f_n - f) = \sup |f_n - f| = \delta < \epsilon$, so f_n converges to $f \in \mathcal{B}(x)$. Thus every Cauchy sequence converges and thus the space is complete and equipped with a metric. ■

1.3 *Topology of metric spaces*

Let (M, d) be a metric space with $\boldsymbol{x} \in M$ and $r > 0$. Define the **open ball** around $\boldsymbol{x}$ of radius r to be

$$B(\boldsymbol{x}; r) = \{\boldsymbol{y} \in M \mid d(\boldsymbol{x}, \boldsymbol{y}) < r\}.$$

The analogous **closed ball** $D(x; r)$ is defined with the less than or equal to sign. A set $A \subset M$ is **bounded** if it can be contained in some

$D(\boldsymbol{x};r)$ for some $\boldsymbol{x} \in M$, $r > 0$. A set $U \subset M$ is **open** if, for all $\boldsymbol{x} \in U$, there exists $r_{\boldsymbol{x}} > 0$ such that $B(\boldsymbol{x};r_{\boldsymbol{x}}) \subset U$. A set $A \subset M$ is **closed** if $M \setminus A$ is open.

Lemma 1.3.1 *Let (M,d) be a metric space, then:*

1. *M and $\emptyset$ are open;*
2. *$\bigcup_i A_i$ is open if all $A_i \subset M$ are open;*
3. *$\bigcap_i^n$ is open if all $A_i \subset M$ are open and $n < \infty$;*
4. *$B(\boldsymbol{x};r)$ is open for some $r > 0$.*

Proof The first two are obvious. For 3), suppose the open sets U_i indexed by i are open and $\boldsymbol{x} \in \bigcap_{i=1}^n U_i$. Then $xinU_i$ for all i, so there is some $B(\boldsymbol{x};r_i) \subset U_i$. Taking the minimum of such $r_i > 0$ means $B(\boldsymbol{x};r_i) \subset \bigcap_{i=1}^n U_i$, and thus the collective finite union is open.

For 4), let $\boldsymbol{y} \in B(\boldsymbol{x};r)$, $r_y = r - d(\boldsymbol{x},\boldsymbol{y}) > 0$ and $\boldsymbol{z} \in B(\boldsymbol{y};r_y)$. Then $d(\boldsymbol{x},\boldsymbol{z}) \leq d(\boldsymbol{x},\boldsymbol{y}) + d(\boldsymbol{y},\boldsymbol{z}) < d(\boldsymbol{x},\boldsymbol{y}) + r - d(\boldsymbol{x},\boldsymbol{y}) = r$, so $B(\boldsymbol{y};r_y) \subseteq B(\boldsymbol{x};r)$. ■

Corollary 1.3.2 *The following may be shown by considering the appropriate complements:*

1. *M and $\emptyset$ are closed;*
2. *$\bigcap_i A_i$ is closed if $A_i \subset M$ for all i;*
3. *$\bigcup_i A_i$ is closed if $A_i \subset M$ for all i and $n < \infty$;*
4. *$D(\boldsymbol{x};r)$ is closed.*

□

Example Open intervals are open and closed intervals are closed.

(a,∞) is open as it is a union of open bounded intervals.

$[a,\infty)$ is closed since $(-\infty,a)$ is open.

$\mathbb{Z}$ is closed as $\mathbb{R} \setminus (\cup_{n=-\infty}^{\infty}(n,n+1))$ is closed.

$\mathbb{Q}$ and $[0,1)$ are neither, while $\mathbb{R}$ is both.

Proposition 1.3.3 *Suppose M is a metric space and $A \subseteq M$. A is closed iff every sequence converges to $a \in A$.*

Proof Assume A is closed and $a_n \to a$. Assume the converse so that $a \in U = M \setminus A$ which is an open set. Then there is some $r > 0$ such that $B(a;r) \in U$, and since $a_n \to a$, there exists $n_0 \geq 0$ where $d(a_n,a) < r$ for $n \geq n_0$. This implies $a_n \in B(a;r)$ for all n, but this is a contradiction since $a_n \in A$, and thus $a \in A$.

Assume $a_n \to a \in A$. Let $x \in M \setminus A$, $r > 0$, and assume there is no such $B(x;r) \subset M \setminus A$. Thus there is an intersection, i.e., $B(x;1/n) \cap A \neq \emptyset$. This implies that there is some i where $a_i \in B(x;1/n) \cap A$. However, (a_n) is a sequence in A and $d(a_m, x) < 1/n$ for $m \geq n+1$, so $a_m \to a$, but this implies $x = a$ which is not possible since $x \in M \setminus A$. So $M \setminus A$ is open which means A is closed. ■

Theorem 1.3.4 *Let M be a complete metric space and $A \subseteq M$ is closed. Then A is complete with the induced metric.*

Proof Let (a_n) be a Cauchy sequence in A. Since M is complete, (a_n) converges in M, but A is closed, so (a_n) converges in A by previous proposition, which implies A is complete. ■

Let M be a metric space. M is **compact** if every sequence $(a_n) \in M$ has a convergent subsequence (a_{n_k}).

Example • $(a_n) = (-1)^n$ is non-convergent but has a convergent sequence.

- $M = (0,1)$ is not compact since $a_n = 1/n$ and its subsequences do not converge in M.
- $\mathbb{R}$ is not compact as a_n has no subsequence converging in $\mathbb{R}$.
- $M = [0,1]$ is compact. Let (a_n) be a subsequence in M. Let I_1 be either $[0,1/2]$ or $[1/2,1]$, and let (a_{n_k}) be the subsequences in I_1. Continuing this we have a sequence of intervals $I_{m+1} \subset I_m$ with I_m of length 2^{-m}. Denote the subsequences $(a^m_{m_k})$ to be those in I_m. Taking $b_m = a^m_{n_m} \in I_m$, we see that $b_{m+1} \in I_M$ since $I_{m+1} \subset I_m$, so that $d(b_m, b_q) \leq 2^{-m}$ for $q \geq m$. Thus (b_m) is a Cauchy sequence, which is a subsequence of (a_n). Since $M \subseteq \mathbb{R}$, M is complete, so $b_m \to b \in M$, and thus M is compact.

Proposition 1.3.5 *By extension, closed n-gons in $\mathbb{R}^n$ are compact.* □

Proposition 1.3.6 *Let $f : M \to N$ be a continuous map between metric spaces. If M is compact, then $f(M) \subset N$ is compact.*

Proof Let (a_n) be a sequence in $f(M)$. Then $a_n = f(b_n)$ for some $b_n \in M$. The sequence (b_{n_k}) converges in M since M is compact, thus

$$\lim_{k\to\infty} a_{n_k} = \lim_{k\to\infty} f(b_{n_k}) = f\left(\lim_{k\to\infty} b_{n_k}\right) = f(b)$$

since f is continuous. So (a_{n_k}) is convergent, thus $f(M)$ is compact. □

Proposition 1.3.7 *A closed subset of a compact space is a compact set.*

Proof Let (a_n) be a sequence in $A \subset M$ where M is compact. Since $(a_n) \in M$, (a_{n_k}) is convergent, but A closed so $(a_{n_k}) \to a \in A$, thus A is compact. □

1.3.1 Heine–Borel theorem

Theorem 1.3.8 *A subset $A \subseteq \mathbb{R}^n$ is compact iff A is closed and bounded.*

Proof Suppose A is compact, so clearly A is closed. If A is unbounded, then there exists $(a_n) \in A$ where $d(a_n, 0) \geq n$, so (a_{n_k}) does not converge in $\mathbb{R}^n$. However A is compact, which is a contradiction, so A is bounded.

Suppose A is bounded, then $A \subseteq [a, b]^n$. If A is closed, then it is a closed subset of a compact set, so A is compact by previous proposition. ■

For example, if $f : M \to N$ with f is a scalar continuous function, then $f(M) \subset \mathbb{R}$ is closed and bounded since M is compact, and thus $f(M)$ compact implies $f(M)$ is closed and bounded.

1.4 Banach and Hilbert spaces

Let V be a real vector space. The **norm** on V is a function $\|\cdot\| : V \to [0, \infty)$ where:

1. $\|\boldsymbol{x}\| = 0$ iff $\boldsymbol{x} = \boldsymbol{0}$;
2. $\|\lambda \boldsymbol{x}\| = |\lambda| \cdot \|\boldsymbol{x}\|$ for all $\boldsymbol{x} \in V$ and $\lambda \in \mathbb{R}$;
3. $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$.

The pair $(V, \|\cdot\|)$ gives a **normed vector space**.

Lemma 1.4.1 *Let V be a normed vector space, then $d(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|$ defines a metric on V.*

Proof Two of the properties follow from definition. To show the reflexive property, note that

$$d(\boldsymbol{y}, \boldsymbol{x}) = \|\boldsymbol{y} - \boldsymbol{x}\| = \|(-1)(\boldsymbol{x} - \boldsymbol{y})\| = \|\boldsymbol{x} - \boldsymbol{y}\| = d(\boldsymbol{x}, \boldsymbol{y}).$$

■

Example 1. It may be shown that the metrics

$$\sum_i |x_i|, \qquad \sum_i \sqrt{|x_i|^2}, \qquad \max\{|x_i| \in \mathbb{R}\}$$

define norms on $\mathbb{R}^n$ (the ℓ^1, ℓ^2 and ℓ^∞ norms).

2. The **supremum norm** on $B(X)$ is defined by

$$\|f\|_\infty = \sup\{|f(x)| \in \mathbb{R}\ ;\ x \in X\}.$$

3. For X a metric space, $C_b(X) = \{f : x \to \mathbb{R} \mid f \text{ continuous and bounded}\}$ is also a normed vector space with the supremum norm.

If $C(X) = \{f : x \to \mathbb{R} \mid f \text{ continuous}\}$ then f does not have a supremum, however, we have the following:

Proposition 1.4.2 *If X is compact, then $C(X) = C_b(X)$, so $C(X)$ is a normed vector space.*

Proof $C_b(X) \subseteq C(X)$ regardless of X. For the converse, assume $f \in C(X)$, so that $f(X)$ is compact. This implies $f(X)$ is bounded and closed by the Heine–Borel theorem, so $C(X) \subseteq C_b(X)$. ■

Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be two normed vector spaces. A function $f : V \to W$ is continuous at $\boldsymbol{x} \in V$ if, for all $\epsilon > 0$, there exists $\delta > 0$ such that $\|\boldsymbol{x} - \boldsymbol{y}\|_V < \delta$ implies that $\|f(\boldsymbol{x}) - f(\boldsymbol{y})\|_W < \epsilon$.

Let V be a normed vector space. V is a **Banach space** if V with the metric induced by the norm is complete.

Theorem 1.4.3 *Let X be a metric space, then $C_b(X)$ with the supremum norm is a Banach space.*

Proof Since $C_b(X) \subseteq B(X)$, if C_b is closed, then C_b is complete since $B(X)$ is complete. To show this, let $(f_n) \in C_b(X)$, and let $f_n \to f \in B(X)$. The convergene of f_n implies that there exists $n_0 \geq 0$ such that $\|f_n - f\| < \epsilon/3$ for any $\epsilon > 0$ with $n \geq n_0$. Also, $\|f_{n_0}(y) - f(y)\| < \epsilon/3$ for all $y \in X$. The functions are continuous, so there exists $\delta > 0$ where, if $d(x, y) < \delta$, $\|f_{n_0}(x) - f_{n_0}(y)\| < \epsilon/3$ for $x \in X$. Thus, for $d(x, y) < \delta$,

$$|f(x) - f(y)| \leq |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(y)| + |f_{n_0}(y) - f(y)| < \epsilon,$$

so f is continuous, and $C_b(X)$ is closed and thus complete. ■

Corollary 1.4.4 *For $a < b$, $C[a, b]$ with the supremum norm is a Banach space.* □

Note that $C[a, b]$ is not a complete space with, for example, the L_2 norm

$$\|f\|_2 = \sqrt{\int_a^b (f(x))^2 \, \mathrm{d}x}.$$

For example, with $f_n = x^n$, $f_n \to 0$ but clearly $f_n(1) = 1$ for all n. The underlying reason is the sequence is not a Cauchy sequence with respect to the norm.

Convergence with respect to the supremum norm is called **uniform convergence** (cf. Complex Analysis 2H).

Let $(V, \|\cdot\|)$ be a Banach space. If there is an inner product from V which induces this norm, then V is called a **Hilbert space**.

Theorem 1.4.5 *Let (M, d) be a metric space. Then there exists $(\overline{M}, \overline{d})$ where $\overline{M}$ is complete, and there is an embedding $\iota : M \to \overline{M}$ with $d(x, y) = d(\iota(x), \iota(y))$ for all $x, y \in M$. Also, for all $\overline{x} \in \overline{M}$, there is a sequence $(x_n) \in M$ with $x_n \to \overline{x}$ as $n \to \infty$.* $\square$

Here, $\overline{M}$ is called the **completion** of M, and it is unique up to some isomorphism.

Example The completion of $\mathbb{Q}$ is $\mathbb{R}$ with respect to the Euclidean metric.

The completeness of $C[a, b]$ with respect to the inner product metric is denoted $L^2[a, b]$..

Note that elements of L^2 are not exactly functions, but rather *equivalence classes* (cf. $11 \equiv 1$ modulo 10)

1.4.1 *The contraction mapping theorem*

Theorem 1.4.6 *Let (M, d) be a complete metric space, $0 \leq \lambda \leq 1$ and a $f : M \to M$ with $d(f(x), f(y)) \leq \lambda d(x, y)$ for all $x, y \in M$. Then f has one unique fixed point where $f(x_0) = x_0$.*

Or, if you throw a map of the world on the floor, there is exactly one point on the map that exactly corresponds to one point on the floor.

Proof Note that f is a contraction, and continuity is automatically satisfied from the condition that $d(f(x), f(y)) \leq \lambda d(x, y)$.

Let $x \in M$, and $a_n = f^n(x)$. So we have

$$\begin{aligned} d(x, a_n) &\leq d(x, f(x)) + d(f(x), f^2(x)) + \ldots d(f^{n-1}(x), f^n(x)) \\ &= \sum_{i=0}^{n-1} d(f^i(x), f^{i+1}(x)) \\ &\leq \sum_{i=0}^{n-1} \lambda d(x, f(x)) \\ &= d(x, f(x)) \frac{1-\lambda^n}{1-\lambda} \\ &\leq \frac{d(x, f(x))}{1-\lambda}, \end{aligned}$$

by Cauchy–Schwartz and the arithmetic progression with $0 \leq \lambda < 1$. Now,

$$d(a_n, a_m) = d(f^n(x), f^m(x)) \leq \lambda^m d(f^{n-m}, x) \leq \lambda^m \frac{d(x, f(x))}{1-\lambda}$$

assuming $n > m$. For $n, m \geq n_0$, we have

$$d(a_n, a_m) \leq \lambda^{n_0} \frac{d(x, f(x))}{1-\lambda}.$$

Clearly (a_n) is a Cauchy sequence, and thus we have completeness and $a_n \to a \in M$. Now,

$$f(a) = f\left(\lim_{n\to\infty} a_n\right) = \lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} a_{n+1} = a,$$

so there is some $a \in M$ that is a fixed point.

To show uniqueness, suppose b is another fixed point. Then

$$d(a,b) = d(f(a), f(b)) \leq \lambda d(a,b),$$

and for $\lambda \neq 0$, $d(a,b) = 0$, so $a = b$. ■

1.5 A norm for matrix spaces

We want a norm reflecting the fact that matrices can be identified with linear maps. Let $\mathbf{A} = (\mathsf{A}_{ij}) \in M_{n,k}(\mathbb{R})$. We define

$$\|\mathbf{A}\| = \sup\{\|\mathbf{A}x\|_2 \ : \ x \in \mathbb{R}^k, \ \|x\|_2 \leq 1\}, \tag{1.1}$$

where $\|\cdot\|$ is the Euclidean norm. Here, $\mathbf{A}x \in \mathbb{R}^n$, and $x \mapsto \|\mathbf{A}x\|_2$ is clearly a continuous map. By the Heine–Borel theorem, $\{\|\mathbf{A}x\|_2 :$ $\|x\|_2 \leq 1\}$ is bounded and closed, so the supremum exists, and there is x with $\|x\|_2 \leq 1$ such that $\|\mathbf{A}\| = \|\mathbf{A}x\|_2$ exists.

Lemma 1.5.1 *We have*

- $\|\mathbf{A}x\|_2 \leq \|\mathbf{A}\|\|x\|_2$ *for all* $\mathbf{A}$ *and* x
- $\|\mathbf{AB}\| \leq \|\mathbf{A}\|\|\mathbf{B}\|$
- $\|\mathbf{A}\|_\infty \leq \|\mathbf{A}\| \leq k\sqrt{n}\|\mathbf{A}\|_\infty$,

where $\|\mathbf{A}\|_\infty = \max\{|\mathsf{A}_{ij}| \ : \ \mathbf{A} \in M_{n,k}(\mathbb{R})\}$. □

2 Ordinary differential equations

3 Tangent spaces and vector fields

4 Differential forms on $\mathbb{R}^n$

5 Differential forms on oriented manifolds