Riemannian geometry 4H

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- Adapted from notes of (I think?) W. Klingenberg, Durham
- This was part of the Riemannian Geometric 4H module elective, as a follow on to the Differential geometry 3H course. Probably would help having gone through the Analysis 3H course also.

- **TODO!** diagrams

1 Differentiable manifolds

Recall that a k-manifold M is one which, locally, looked like open sets of $\mathbb{R}^k$. The manifold is build up by patching together these local sets.

Example 1. A surface of revolution M with $\gamma : [a,b] \to \mathbb{R}^3$ with $\gamma(t) = (\gamma_1(t), 0, \gamma_3(t))$ (assuming $\gamma'(t) \neq 0$ for all t and γ is simple) is given by

$$\begin{aligned} M &= \{(\gamma_1(t), \gamma_3(t)\cos\theta, \gamma_3(t)\sin\theta) \mid t \in [a,b],\ \theta \in [0, 2\pi]\} \\ &= \{(x_1, x_2, x_3) \mid \exists t \in (a,b),\ x_1 = \gamma_1(t),\ x_2^2 + x_3^2 = \gamma_3^2(t)\}. \end{aligned}$$

2. The matrix group $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ can be considered as submanifolds of the ambient space $\mathbb{R}^{n^2} = M(n, \mathbb{R}$.

3. The real projective plane/space $\mathbb{RP}^n$ is given by the set of all lines through the origin in $\mathbb{R}^{n+1}$. It can be identified with the quotient group $S^n/\sim$, where $p \sim q$ iff $p = \pm q$.

Recall that for an n-dimensional subspace $X \subset \mathbb{R}^n$, $x \in X$ is called an **inner point** of iff there exists an n-dimensional open $U \subset X$ with $x \in U$. X is **open** if every $x \in X$ is an inner point.

Let M be a set. A collection (U_α, ϕ_α) with index α is called an **atlas** of M is

1. $U_\alpha \in M$ and $\bigcup_\alpha U_\alpha = M$ (the collection of U_α covers M),
2. $\phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n$ are bijective maps with open V_α, and for each α, β, $\phi_\alpha(U_\alpha \cap U_\beta)$ is open,
3. the co-ordinate changes $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ are smooth maps between open subsets of $\mathbb{R}^n$.

Let (U_α, ϕ_α) be an atlas of M, and $X \subset M$. A point $x \in X$ is called an **inner point** of X is there exists $\phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n$ such that $\phi_\alpha(x)$ is an inner point of $\phi_\alpha(X \cap U_\alpha)$ for $x \in U_\alpha$. X is **open** if every $x \in X$ is an inner point.

A set M is a **differentiable manifold** if it carries an atlas (U_α, ϕ_α) and has the **Hausdorff property**, i.e. for any $p, q \in M$ with $p \neq q$, there exists open $A_p, A_q \subset M$ such that for $p \in A_p$ and $q \in A_q$ where $A_p \cap A_q = \emptyset$.

This would be the T_2 condition relating to topological spaces. Distinct points have disjoint neighbourhoods.

Example Consider the real line with zero removed, and we take $M = (\mathbb{R}\ \{0\}) \cup \{0_+, 0_-\}$ where $0_\pm$ are points off the real line by arbitrarily near zero. We can construct the smooth charts

$$\phi_i : (\mathbb{R}\ \{0\}) \cup \{0_i\} \to \mathbb{R}, \qquad x \mapsto \begin{cases} x, & x \neq 0_i, \\ 0, & x = 0_i, \end{cases}$$

and $\{\phi_+, \phi_-\}$ can serve as an atlas, but we do not have the Hausdorff property, and M is not a differentiable manifold.

1.1 *Manifolds and regular values*

Let $f : U \to \mathbb{R}^k$ be differentiable and U is open, and denote

$$Df(x) = \left[\frac{\partial f_i}{\partial x_j}\right]_{ij} : \mathbb{R}^n \to \mathbb{R}^k \tag{1.1}$$

be the Jacobian matrix at $x \in U$. The point $x \in U$ is a **regular point** if $Df(x)$ is surjective, i.e. of rank k. $y \in \mathbb{R}^k$ is a **regular value** of f if all $x \in f^{-1}(y)$ are regular points.

Theorem 1.1.1 *Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}^k$ be differentiable, $k \leq n$, and $y \in \mathbb{R}^k$ be a regular value of f. Then $M = f^{-1}(y) \subset U$ is a differentiable manifold of dimension $(n-k)$.* □

Theorem 1.1.2 (Implicit function theorem) *For differentiable $f : \mathbb{R}^{n-k} \times \mathbb{R}^k \to \mathbb{R}^k$, $f(x^1, x^2) = y$, then we have*

$$df(x^1, x^2) = \left[\frac{\partial f}{\partial x^1}(x^1, x^2) \quad \frac{\partial f}{\partial x^2}(x^1, x^2)\right].$$

Assuming $\det\left[\partial f / \partial x^2(x^1, x^2)\right]$ is non-zero, then there exists neighbourhoods $V_1 \subset \mathbb{R}^{n-k}$ and $V_2 \subset \mathbb{R}^k$ (with $x^1 \in V_1$ and $x^2 \in V_2$) where there is some $\psi : V_1 \to V_2$ that is differentiable, and such that $f^{-1}(y) \cap (V_1 \times V_2) = \{(x^1, \psi(x^1)\ :\ x^1 \in V_1\}$. □

Note that for differentiable $f : \mathbb{R}^n \to \mathbb{R}^k$, by the chain rule we have

$$Df(x) \cdot z = \lim_{h \to 0} \frac{f(x + hz) - f(x)}{h}, \qquad z \in \mathbb{R}^n \tag{1.2}$$

since $Df(x) : \mathbb{R}^n \to \mathbb{R}^k$ This can be regarded as a **directional derivative in the direction** z.

Example The group $\mathrm{SO}(n)$ can be considered as a manifold. Consider $\mathrm{GL}^+(n) = \{A \in \mathrm{M}(n, \mathbb{R}) \mid |A| > 0\}$, which is an open set in $\mathrm{M}(n, \mathbb{R}) \cong \mathbb{R}^{n^2}$, as $\det : \mathrm{M}(n, \mathbb{R}) \to \mathbb{R}$ is a continuous function. Note that $AA^T = I$ means $|A| = \pm 1$, so

$$\mathrm{SO}(n) = \mathrm{O}(n) \cap \mathrm{GL}^+(n).$$

Consider $f : \mathrm{GL}^+(n) \to \mathrm{Sym}(n) = \{C \in \mathrm{M}(n, \mathbb{R}) \mid C^T = C\} \cong \mathbb{R}^{n(n+1)/2}$, with $f(A) = AA^T - I$. Then we clearly have $f^{-1}(0) = \mathrm{SO}(n)$, and observe that

$$\begin{aligned} Df|_A B &= \lim_{h\to 0} \frac{f(A+hB) - f(A)}{h} \\ &= \lim_{h\to 0} \left[\frac{(A+hB)^T(A+hB) - A^T A}{h} \right] \\ &= \lim_{h\to 0} \left[A^T B + B^T A + tB^T B \right] \\ &= A^T B + B^T A. \end{aligned}$$

We need to check if Df is surjective for all $A \in f^{-1}(0) = \mathrm{SO}(n)$. For $C \in \mathrm{Sym}(n)$, we have

$$Df|_A \left(\frac{1}{2} AC \right) = \frac{1}{2} A^T (AC) + \frac{1}{2} (AC)^T A = C$$

since $AA^T = I$ and $C = C^T$. Since C was arbitrary, Df is surjective, thus 0 is a regular value and $\mathrm{SO}(n)$ is a $n^2 - n(n+1)/2 = n(n-1)/2$ dimensional manifold.

Example Let M and N be manifolds, then the claim is that $M \times N = \{(x, y) \mid x \in M, \ y \in N\}$ is a $(m+n)$-manifold.

Let (U_α, ϕ_α) be an atlas for M and $(\tilde{U}_\beta, \tilde{\phi}_\beta)$ be an atlas for N. Then $(U_\alpha \times \tilde{U}_\beta, \psi_{\alpha\beta} = \phi_\alpha \times \tilde{\phi}_\beta)$ is an atlast for $M \times N$. The co-ordinates changes

$$\left(\psi_{\alpha\beta}^{-1} \circ \psi_{\gamma\delta} \right)(u, v) = \left(\phi_\alpha^{-1} \circ \tilde{\phi}_\gamma(u), \phi_\beta^{-1} \circ \tilde{\phi}_\delta(v) \right)$$

are clearly differentiable.

To show the Hausdorff property, let $(x, y) \neq (z, w)$ on $M \times N$. Choosing appropriate open neighbourhoods $U_{x,z} \subset M$ and $\tilde{U}_{y,w} \subset N$, we note individually they do not intersect if $x \neq z$ and $y \neq w$, since M and N are manifolds. By construction, $U_x \times \tilde{U}_y \subset M \times N$ and $U_z \times \tilde{U}_w \subset M \times N$ are open neighbourhoods of (x, y) and (w, z), and they have empty intersections, so we have the Hausdorff property.

1.2 *Differentiable maps, tangent vectors, and differentials*

Let M and N be m- and n-manifolds. A function $f : M \to N$ is **differentiable at** $x \in M$ if there are co-ordinate charts

$$\phi : U \to V \subset \mathbb{R}^m, \ x \in U \subset M, \qquad \tilde{\phi} : \tilde{U} \to \tilde{V} \subset \mathbb{R}^n, \ f(x) \in \tilde{U} \subset N,$$

such that

$$\tilde{\phi} \circ f \circ \phi^{-1} : \phi\left(U \cap f^{-1}(\tilde{U}) \right) \to \tilde{V} \tag{1.3}$$

is differentiable at $\phi(x) \in V \subset \mathbb{R}^m$. The function f is **differentiable** if it is differentiable for all $x \in M$. Note that the definition is independent of co-ordinate choice, since the co-ordinate changes are differentiable.

Differentiable maps $c : (a, b) \to M$ are called **curves**.

Let M be a manifold and $x \in M$, and $D(M, x) = \{f : M \to \mathbb{R} \mid f \text{ differentiable at } x\}$. Let $c : (a, b) \to M$ be a curve with $c(t_0) = x$. The **directional derivative** of $f \in D(M, x)$ along c at $x = c(t_0)$ is denoted by

$$c'(t_0)(f) = \lim_{t\to 0} \frac{f(c(t_0 + t)) - f(c(t_0))}{t} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} (f \circ c)(t) \tag{1.4}$$

Remark $D(M, x)$ is an algebra over $\mathbb{R}$ (a vector space over $\mathbb{R}$ and $fg \in D(M, x)$ for $f, g \in D(M, x)$). The directional derivative along c at $x = c(t_0)$ has the following properties:

1. $c'(t_0)(\lambda f + \mu g) = \lambda c'(t_0)(f) + \mu c'(t_0)g$ for $\lambda, \mu \in \mathbb{R}$,
2. $c'(t_0)(fg) = g(x)c'(t_0)(f) + f(x)c'(t_0)(g)$.

A map $D(M, x) \to \mathbb{R}$ with the above properties is called a **linear derivative** of the algebra $D(M, x)$.

Example Note that different curves $c_{1,2} : \mathbb{R} \to M$ with $c_1(0) = c_2(0) = x$ can define the same directional derivative at x. For example, let $c_{1,2} : \mathbb{R} \to \mathbb{R}^2$ with $c_1(t) = (t, 0)$ and $c_2(t) = (t, t^2)$. The directional derivatives of some f at some point corresponding to $t = 0$ are clearly the same, since $c_1'(0) = c_2'(0) = (1, 0)$.

One can generally check that if $c_{1,2}(a, b) \to M$ and $c_1(t_0) = c_2(t_0) = x$ with $c_1'(t_0) = c_2'(t_0)$, then $c_{1,2}'$ as directional derivatives are linear derivations iff there is a co-ordinate chart $\phi : U \to V \subset R^n$, $x \in U \subset M$ such that $(\phi \circ c_1)'(t_0) = (\phi \circ c_2)'(t_0)$ as ordinary vectors in $\mathbb{R}^n$. This is again independent of co-ordinate choice.

Let M be a manifold, $x \in M$. A **tangent vector** of M at x is the directional derivative $c'(t_0) : D(M, x) \to \mathbb{R}$ of a curve $c : (a, b) \to M$ with $c(t_0) = x$. The set of all tangent vectors defines the **tangent space** $T_x(M)$ at $x \in M$.

Let M be a manifold and $\phi : U \to V \subset \mathbb{R}^n$, $U \subset M$ be a co-ordinate chart. For $\phi(x_1, \ldots, x_n)$ where each x_i is the co-ordinate component of ϕ, and $c_i : (-\epsilon, \epsilon) \to U$, $t \mapsto \phi^{-1}(\phi(p) + te_i)$, clearly we have $c_i(0) = p$, and so we define the **co-ordinate tangent vectors** to be

$$\left.\frac{\partial}{\partial x_i}\right|_p = c_i'(0) \in T_p(M) \,, \tag{1.5}$$

where

$$\left.\frac{\partial}{\partial x_i}\right|_p = \left.\frac{\partial}{\partial t}\right|_0 (f \circ c_i)(t) = \left.\frac{\partial}{\partial t}\right|_0 (f \circ \phi^{-1})(\phi(p) + te_i) = \frac{\partial (f \circ \phi^{-1})}{\partial x_i} (\phi(p)) . \tag{1.6}$$

Proposition 1.2.1 *Let M be a n-manifold, then $T_p(M)$ carries the structure of a n-vector space.*

Proof 1. We first aim to show that $\{\partial/\partial x_i|_p\}$ associated with a coordinate chart forms a spanning set of $T_p(M)$. Let $c : (a, b) \to M$ be a curve with $c(t_0) = p$. We show that $c'(t_0) : D(M, p) \to \mathbb{R}$ is a linear combination of $\{\partial/\partial x_i|_p\}$. Note that we have

$$c'(t_0)(f) = (f \circ c)'(t_0) = \Big((f \circ \phi^{-1}) \circ (\phi \circ c)\Big)' (t_0).$$

Here, $f \circ \phi^{-1} : \mathbb{R}^n \to \mathbb{R}$, while $\phi \circ c = (c_1, \dots c_n) = (x_1 \circ c, \dots x_n \circ c) : \mathbb{R}^n \to \mathbb{R}$. By the chain rule,

$$\begin{aligned} c'(t_0)(f) &= \Big\langle \nabla (f \circ \phi^{-1})[(\phi \circ c)(t_0)], (c_1'(t_0), \dots c_n'(t_0)) \Big\rangle \\ &= \frac{\partial (f \circ \phi^{-1})}{\partial x_i} (\phi(p)) \cdot c_i'(t_0) \\ &= \left.\frac{\partial}{\partial x_i}\right|_p (f) \cdot c_i'(t_0), \end{aligned}$$

hence our tangent vector is a linear combination of $\{\partial/\partial x_i|_p (f)\}$, and hence we have a spanning set.

2. We aim to show that the space is closed. Let $c : (-\epsilon, \epsilon) \to M$, $c(t) = \phi^{-1}(\phi(p) + (\alpha_i e_i)t)$; note that $c(0) = p$. Let

$$(c_1(t), \dots c_n(t)) = (\phi \circ c)(t) = \phi(p) + t(\alpha_1, \dots \alpha_n),$$

then $c_i'(0) = \alpha_i$, and so we have

$$c'(0)(f) = c_i'(0) \left.\frac{\partial}{\partial x_i}\right|_p (f) = \alpha_i \left.\frac{\partial}{\partial x_i}\right|_p (f),$$

so a linear combinations of the spanning set are still tangent vectors.

3. We aim to show that $\{\partial/\partial x_i|_p\}$ are linearly independent, and thus we have a basis. Suppose we have some

$$\alpha_i \left.\frac{\partial}{\partial x_i}\right|_p = 0 : D(M, p) \to \mathbb{R}; \quad f \mapsto 0.$$

We aim to show that all $\alpha_i = 0$. Choose $\phi \in C^\infty(M)$ with $\psi \equiv 1$ near $p \in M$ and $\psi \equiv 0$ outside U. Let $f_i = \psi \cdot x_i : M \to \mathbb{R}$, and so

$$f_i(q) = \begin{cases} x_i(q), & q \in U, \\ 0, & q \notin U, \end{cases}$$

and $f_i \in D(M, q)$. But then

$$\alpha_i \left.\frac{\partial}{\partial x_i}\right|_p (f_j) = \alpha_i \frac{(f_j \circ \phi^{-1})}{\partial x_i}(\phi(p)),$$

and $f_j \circ \phi^{-1}$ is the projection to the j^{th} co-ordinate near $\phi(p) \in \mathbb{R}^n$, with $(a_1, \ldots a_n) \mapsto a_j$. So

$$\alpha_i \left.\frac{\partial}{\partial x_i}\right|_p (f_j) = \alpha_i \delta_{ij} = \alpha_j = 0$$

for all j by assumption, and we thus have linear independence and therefore a basis for the n-vector space. ■

If a manifold $M \subseteq \mathbb{R}^N$, then we can identify the abstract tangent vectors $c'(0) \in T_p(M)$, $c'(0) : D(M, p) \to \mathbb{R}$ with classical tangent vectors $\tau'(0) \in \mathbb{R}^N$ via

$$\tau'(0) = (c'(0)(y_1), \ldots c'(0)(y_N)),$$

with y_i the restriction of the i^{th} co-ordinate function (i.e. $(a_1, \ldots a_n) \mapsto a_i$) to M.

Lemma 1.2.2 *Let $A : (-\epsilon, \epsilon) \to GL(n, \mathbb{R})$ be a curve. Then $det A : (-\epsilon, \epsilon) \to \mathbb{R}$, $t \mapsto det(A(t))$ is differentiable and*

$$(det A)'(t) = (det A(t)) \cdot tr\left(A^{-1}(t) A'(t)\right).$$

Proof Let $A(t) = [a_1(t)| \ldots |a_n(t)]$, $a_i(t) = [a_{1i}(t), \ldots a_{ni}(t)]^T$. Then

$$\det A(t) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{\sigma(1),1} \ldots a_{\sigma(n)n},$$

so

$$\begin{aligned}(\det A)'(t) &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \left(a'_{\sigma(1),1} \ldots a_{\sigma(n)n} + \ldots + a_{\sigma(1),1} \ldots a'_{\sigma(n)n}\right) \\ &= \det[a'_1(t)| \ldots |a_n(t)] + \ldots + \det[a_1(t)| \ldots |a'_n(t)].\end{aligned}$$

Since $\det A \neq 0$ by the fact that $A \in \mathrm{GL}(n, \mathbb{R})$, $\{a_i(t)\}$ forms a basis of $\mathbb{R}^n$ and therefore there exists coefficients α_{ij} such that $a'_j(t) = \alpha_{ij}(t) a_i(t)$, or $A' = A\alpha$ where $\alpha = (\alpha_{ij})$. Then

$$\begin{aligned}(\det A)'(t) &= \det[\alpha_{11} a_1(t)| \ldots |a_n(t)] + \ldots + \det[a_1(t)| \ldots |\alpha_{nn} a_n(t)] \\ &= (\alpha_{11} + \ldots + \alpha_{nn}) \det A \\ &= \operatorname{tr}\alpha \cdot \det A(t) \\ &= \operatorname{tr}\left(A^{-1}(t) A'(t)\right) \cdot \det A(t)\end{aligned}$$

since $\alpha = A^{-1} A'$. ■

Example We show that the tangent space of $\mathrm{SL}(n,\mathbb{R}) \subset M(n,\mathbb{R}) \cong \mathbb{R}^{n^2}$ at the identity I is a (n^2-1)-manifold. Let $A : (-\epsilon,\epsilon) \to \mathbb{SL}(n,\mathbb{R})$ be a curve with $A(0) = I$. Then from above lemma,

$$(\det A)'(0) = 0 = \det A(0) \cdot \operatorname{tr}\left(A^{-1}(0)A'(0)\right) = \operatorname{tr} A'(0),$$

so the tangent space $T_I(\mathrm{SL}(n,\mathbb{R}))$ is contained in the set of zero trace matrices of dimension n^2-1. But the tangent space is a manifold and also of dimension n^2-1, so the tangent space is

This is the **Lie algebra** of $\mathrm{SL}(n,\mathbb{R})$, denoted $\mathfrak{sl}(n,\mathbb{R})$.

$$\mathfrak{sl}(n,\mathbb{R}) \equiv T_I(\mathrm{SL}(n,\mathbb{R})) = \{B \in M(n,\mathbb{R}) \mid \operatorname{tr} B = 0\}.$$

A group G that happens to have a smooth manifold structure is called a **Lie group**; the composition and inverse maps are differentiable.

Examples of Lie groups include the usual matrix groups such as

- $\mathrm{GL}(n,\mathbb{R})$ (with $\dim = n^2$)
- $\mathrm{SL}(n,\mathbb{R})$ (with $\dim = n^2-1$)
- $\mathrm{SO}(n,\mathbb{R})$ (with $\dim = n(n-1)/2$)
- $\mathrm{O}(n,\mathbb{R})$ (with $\dim = n(n-1)/2$)

Let M and N be differentiable manifolds, and $f : M \to N$ be differentiable. For $p \in M$,

$$\boxed{Df(p) : T_p(M) \to T_{f(p)}(N), \quad c'(t) \mapsto (f \circ c)'(t)} \tag{1.7}$$

is called the **differential** of f at p, where c is a curve with $c(t) = p$.

Proposition 1.2.3 *Note that $v \in T_p(M) : D(M,p) \to \mathbb{R}$ is a linear derivation. Then we have*

$$\boxed{Df(p)(v) : D(N,p) \to \mathbb{R}, \quad (Df(p)(v))\,(g) = v(g \circ f)}\,. \tag{1.8}$$

Proof Let $c : (-\epsilon,\epsilon) \to M$ be a curve with $c(0) = p$, $c'(0) = v \in T_p(M)$. Then

$$\begin{aligned}
(Df(p)(c'(0)))\,(g) &= (f \circ c)'(0)(g) \\
&= (g \circ (f \circ c))'(0) \\
&= ((g \circ f) \circ c)'(0) \\
&= c'(0)(g \circ f) \\
&= v(g \circ f).
\end{aligned}$$

■

Example Suppose we have the unit 2-sphere $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$ and the cylinder $Z = \{(x, y, z) \mid x^2 + y^2 = 1, \ -1 < z < 1\}$. Let

$$f : Z \to S^2, \quad f(x, y, z) = \left(x\sqrt{1 - z^2}x, y\sqrt{1 - z^2}, z \right),$$

and suppose $p = (1, 0, z_0) \in Z$, $v_1 = (0, 1, 0)$, $v_2 = (0, 0, 1)$. Define two curves on the cylinder Z to include $v_{1,2}$ via

$$c_1(t) = (\cos t, \sin t, z_0), \qquad c_2(t) = (1, 0, z_0 + t),$$

and clearly $c_1(0) = c_2(0) = p$ and $c'_{1,2}(0) = v_{1,2}$. Then

$$\begin{aligned} Df(p)(v_1) &= (f \circ c_1)'(0) \\ &= \left(\cos t\sqrt{1 - z_0^2}, \sin t\sqrt{1 - z_0^2}, z_0 \right)' \Big|_{t=0} \\ &= \left(0, \sqrt{1 - z_0^2}, 0 \right). \end{aligned}$$

Then $f(p) = (\sqrt{1 - z_0^2}, 0, z_0)$ we can check that we have orthogonality $\langle f(p), Df(p)(v_1) \rangle = 0$, so $Df(p)(v_1) \in T_{f(p)}(S^2)$. Similarly, we have

$$\begin{aligned} Df(p)(v_2) &= (f \circ c_2)'(0) \\ &= \left(\sqrt{1 - (z_0 + t)^2}, \sin t\sqrt{1 - (z_0 + t)}, z_0 \right)' \Big|_{t=0} \\ &= \left(-\frac{z_0}{\sqrt{1 - z_0^2}}, 0, 1 \right), \end{aligned}$$

and we have $\langle f(p), Df(p)(v_2) \rangle = 0$, so $Df(p)(v_2) \in T_{f(p)}(S^2)$ as well.

1.3 *Tangent bundles, vector fields and Lie brackets*

Let M be a manifold. The tangent spaces $T_p(M)$ for points $p \in M$ are all pairwise disjoint (since their elements are maps on different spaces $D(M, p)$). Their disjoint union is called the **tangent bundle** of M, denoted

That's why one needs to be careful since we can't arbitrary add things on different tangent spaces together, even if they are all 'vectors'.

$$\boxed{\dot{\bigcup}_{p \in M} T_p(M) = T(M)}\,. \tag{1.9}$$

There is a canonical **footpoint projection** $\pi : T(M) \to M$ with $\pi(v) = p$ if $p \in T_p(M)$.

This is mapping the vector at the touching points of the tangent spaces with the manifold onto the manifold.

Proposition 1.3.1 *$T(M)$ a n-manifold M is a $2n$-manifold.*

Proof Let $(U_\alpha, \phi_\alpha)_{\alpha \in A}$ be an atlas for M. Let $\phi_\alpha(x_1^\alpha, \dots x_n^\alpha) : U_\alpha \to V_\alpha \subset \mathbb{R}^n$. We construct an atlas of $T(M)$ by choosing, for every $\alpha \in A$, the subset

$$\tilde{U}_\alpha = \bigcup_{p \in U_\alpha} T_p(M) \subset T(M)$$

and bijective maps

$$\psi_\alpha : \tilde{U}_\alpha \to V_\alpha \times \mathbb{R}^n,$$

$$\psi_\alpha \left(\beta_i \left. \frac{\partial}{\partial x_i^\alpha} \right|_p \right) = (\phi_\alpha(p), \beta_1, \dots \beta_n) = (\phi_\alpha(p), \beta),$$

where $\beta \in \mathbb{R}^n$ and $\beta_i \partial/\partial x_i^\alpha|_p \in T_p(M)$. The inverse map ψ_α^{-1} is

$$\psi^{-1}(x, \beta) = \beta_i \left. \frac{\partial}{\partial x_i^\alpha} \right|_{\phi_\alpha^{-1}(x)}, \quad x = \phi_x(p).$$

Clearly $\bigcup_{\alpha \in A} \tilde{U}_\alpha = T(M)$ as $\bigcup_{\alpha \in A} U_\alpha = T(M)$. For the co-ordinate changes,

$$\psi_\gamma \circ \psi_\alpha^{-1}(x, \beta) = \psi_\gamma \left(\left. \frac{\partial}{\partial x_i^\alpha} \right|_p \right) = \psi_\gamma \left(\beta_i \left(\frac{\partial (x_j^\gamma \circ \phi_\alpha^{-1})}{\partial x_i}(x) \left. \frac{\partial}{\partial x_j^\gamma} \right|_p \right) \right).$$

By swapping the order of summation, we have

$$\begin{aligned} \psi_\gamma \circ \psi_\alpha^{-1}(x, \beta) &= \psi_\alpha \left(\beta_i \left(\frac{\partial (x_j^\gamma \circ \phi_\alpha^{-1})}{\partial x_i}(x) \left. \frac{\partial}{\partial x_j^\gamma} \right|_p \right) \right) \\ &= \left(\left(\phi_\gamma \circ \phi_\alpha^{-1} \right)(x), \beta \left(\frac{\partial (x_j^\gamma \circ \phi_\alpha^{-1})}{\partial x_i}(x) \right) \right)_{1 \le i,j \le n}. \end{aligned}$$

Thus co-ordinate changes are differentiable, and so we have an atlas. We assume the Hausdorff property, and so $T(M)$ is a $2n$-manifold. ∎

A **vector field** X is a differentiable map $X : M \to T(M)$ such that $X(p) \in T_p(M)$. The space of vector fields on M is denoted $\mathcal{X}(M)$, and carries the structure of an infinite dimensional real vector space.

Note that if $X \in \mathcal{X}(M)$, then $(\pi \circ X)(p) = p$ for all $p \in M$. Locally, every vector field X can be written with respect to a co-ordinate chart $\phi = (x_1, \dots x_n) : U \to V \subset \mathbb{R}^n$ as

$$X(p) = f_i(p) \left. \frac{\partial}{\partial x_1} \right|_p \tag{1.10}$$

for all $p \in U$. Here the $f_i : U \to \mathbb{R}$ are differential functions, and are called **component functions** of X (since $\partial/\partial x_i$ is a basis for the tangent space associated with the co-ordinate choice).

Example Let S^2 be the unit 2-sphere and $X(u) = (2u_3 - u_2, u_1, -2u_2)$ be a vector field. First, notice that the (outward) normal vector on S^2 would be $n = (u_1, u_2, u_3)$, and with the standard inner product we have

$$\langle X(u), n\rangle = 2u_3u_1 - u_1u_2 + u_1u_2 - 2u_2u_3 = 0,$$

so $X(u) \in T_u(S^2)$ and $X \in \mathcal{X}(S^2)$ is a well-defined vector field on S^2. A co-ordinate chart (U, ϕ) of S^2 would be the spherical co-ordinates (but using latitude instead of co-latitude)

$$\begin{aligned} \phi^{-1} &: (-\pi/2, \pi/2) \times (0, 2\pi) \to S^2, \\ (x_1, x_2) &\mapsto (\cos x_1 \cos x_2, \cos x_1 \sin x_2, \sin x_1). \end{aligned}$$

Let $p = \phi^{-1}(x_1, x_2)$, then

$$\begin{aligned} \left.\frac{\partial}{\partial x_1}\right|_p &= (-\sin x_1 \cos x_2, -\sin x_1 \sin x_2, \cos x_1), \\ \left.\frac{\partial}{\partial x_2}\right|_p &= (-\cos x_1 \sin x_2, \cos x_1 \cos x_2, 0), \end{aligned}$$

while

$$X(p) = (2\sin x_1 - \cos x_2 \sin x_2, \cos x_1 \cos x_2, 2\cos x_1 \cos x_2).$$

For $X(p) = \beta_i \partial/\partial x_i|p$, we should have

$$\begin{aligned} \beta_1(-\sin x_1 \cos x_2) + \beta_2(-\cos x_1 \sin x_2) &= 2\sin x_1 - \cos x_2 \sin x_2, \\ \beta_1(-\sin x_1 \sin x_2) + \beta_2(\cos x_1 \cos x_2) &= \cos x_1 \cos x_2, \\ \beta_1 \cos x_1 &= -2\cos x_1 \cos x_2, \end{aligned}$$

so by inspection, $\beta_1 = -2\cos x_2$ and $\beta_2 = 1 - 2\tan x_1 \sin x_2$.

Recall that a tangent vector $v \in T_p(M)$ differentiates a function $f \in D(M, p)$ in the direction v through $v(t)$. Similar, given a vector field $X \in \mathcal{X}(M)$, $f \in C^\infty(M)$, $X(f) \in C^\infty$ is defined as

$$(X(f))(p) = X(p)(f) \in \mathbb{R}. \tag{1.11}$$

Locally, if $X = g_i\partial/\partial x_i|p(f)$ with respect to $U, \phi)$, we can write

$$(X(f))(p) = g_i(p) \left.\frac{\partial}{\partial x_i}\right| p(f) = g_i(p) \left.\frac{\partial f}{\partial x_i}\right| p = g_i(p) \frac{\partial(f \circ \phi^{-1})}{\partial x_i}(\phi(p)). \tag{1.12}$$

Let $X, Y \in \mathcal{X}(M)$, then there is a $Z \in \mathcal{X}(M)$ such that, for all $f \in C^\infty(M)$,

$$Z(f) = X(Y(f)) - Y(X(f)) = [X, Y](f). \tag{1.13}$$

Here $Z = [X, Y]$ is the **Lie bracket** of X and Y and is a vector field. If

Note the similarities of this to the commutator, and similarities but subtle differences with the Poisson bracket.

we have a co-ordinate system, then we have

$$X = a_i \frac{\partial}{\partial x_i}, \quad Y = b_i \frac{\partial}{\partial x_i}, \quad Z = \left(a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i} \right) \frac{\partial}{\partial x_j}.$$

(Act this on a f and use the fact that f is differentiable and derivative operations can be swapped.)

Proposition 1.3.2 *The Lie bracket satisfies the following properties:*

1. *anti-symmetry,* $[X, Y] = -[Y, X]$
2. *distributive, for real scalars* a, b, $[aX + bY, Z] = a[X, Z] + b[Y, Z]$
3. *Jacobi identity,*

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

4. *for* $f, g \in C^\infty(M)$,

$$[fX + gY] = fg[X, Y] + f(X(g)) \cdot Y - g(Y(f)) \cdot X.$$

Note the cyclic permutations. The Lie bracket can be thought of as a derivative where $[X, Y] = \mathcal{L}_X Y$ (the **Lie derivative** of Y along X), and then the Jacobi identity is basically the equivalent product rule, since $\mathcal{L}_X[Y, Z] = [X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]] = [\mathcal{L}_X Y, Z] + [Y, \mathcal{L}_X Z]$.

2 Riemannian manifolds

Let M be a differentiable manifold. A **Riemannian metric** $g = \{g_p\}_{p\in M}$ is a family of inner products

$$g_p : T_p(M) \times T_p(M) \to \mathbb{R}, \quad p \mapsto g_p(X(p), Y(p)) \in \mathbb{R}\,, \tag{2.1}$$

which depends smoothly on $p \in M$, is differentiable and is symmetric. We often use the notation

Or the metric is a $(2,0)$-tensor which eats two vectors and spits out a number.

$$\langle v, w\rangle_p = g_p(v, w), \quad v, w \in T_p(M). \tag{2.2}$$

The pair (M, g) is then called a **Riemannian manifold**.

Example $M = \mathbb{R}^n$ is a differentiable manifold with one global coordinate chart, which is just the identity. $T_p(\mathbb{R}^n)$ can be cannonically identified with pairwise disjoint copies of $\mathbb{R}^n$ via

$$c : (-\epsilon, \epsilon) \to \mathbb{R}^n, \quad c(0) = p, \quad c'(0) \in T_p(\mathbb{R}^n)$$

but also $c'(0) = (c_1'(0), \dots c_n'(0)) \in \mathbb{R}^n$. If one wants to stress the pairwise disjointness of the different $T_p(\mathbb{R}^n) \cong \mathbb{R}^n$, we would use $\{p\} \times T_p(\mathbb{R}^n)$.

A Reimannian metric $g_p : T_p(\mathbb{R}^n) \times T_p(\mathbb{R}^n) \to \mathbb{R}$ is given by the standard inner product $g_p(u, v) = v_i w_i$, which gives the standard Euclidean geometry.

Example Let $M \subset \mathbb{R}^3$ be a surface, then $T_p(M)$ can be canonically identifies with a two-dimensional subspace in $\mathbb{R}^3$. $T_p(M)$ inherits a natural inner product from $\mathbb{R}^3$, namely the first fundamental form (which is just that from the dot product). This inner produce defines a Riemannian metric on M.

For example, if $M = S^2$, then at $p = (0, 0, 1)$, we have $v = (v_1, v_2, 0)$ and $w = (w_1, w_2, 0) \in T_p(S^2)$, and $g_p(v, w) = v_1 w_1 + v_2 w_2$.

Let (M, g) be a Riemannian manifold. The **length** of a tangent vector $v \in T_p(M)$ is defined as

$$\|v\|_p = g_p(v, v) = \sqrt{\langle v, v\langle_p}\,. \tag{2.3}$$

Suppose $\phi : U \to V \subset \mathbb{R}^n$ be a co-ordinate chart with $\phi = (x_1, \dots x_n)$, then we can introduce functions that are components of the metric given by

$$g_{ij} : U \to \mathbb{R}, \quad g_{ij}(p) = g_p\left(\left.\frac{\partial}{\partial x_i}\right|_p, \left.\frac{\partial}{\partial x_j}\right|_p \right), \quad 1 \le i, j \le n\,. \tag{2.4}$$

Note that $g_{ij} = g_{ji}$ since the metric is symmetric.

Remark In the special case of a parameterised surface $M \subset \mathbb{R}^3$, the component functions g_{ij} of the associated co-ordinate chart coincide with the coefficients of the first fundamental form as $g_{11} = E$, $g_{12} = g_{21} = F$, $g_{22} = G$.

Example The n-dimensional (real) hyperbolic space has different models of the geometry

1. **Hyperboloid model**. Consider the indefinite symmetric form η on $\mathbb{R}^{n+1}$ given by

$$\eta(y, z) = y_i z_i - y_{n+1} z_{n+1}$$

Define $\mathbb{W}^n = \{y \in \mathbb{R}^{n+1} \mid \eta(y, y) = -1, y_{n+1} > 0\}$; see Fig. **??**. We can think of $\mathbb{W}^n$ as a n-submanifold of $\mathbb{R}^{n+1}$ and identify $T_p(\mathbb{W}^n)$ with a n-subspace of $\mathbb{R}^{n+1}$. We define

cf. spacetime, where the signature is $(-,+,+,+)$. The lightcone would be where $\eta(y,y) = 0$.

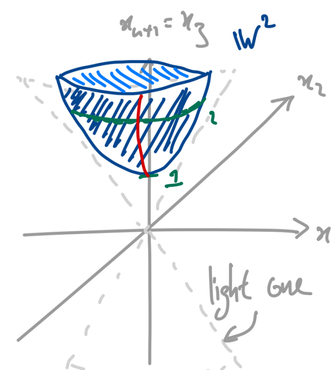

Figure 2.1: Illustration of the $\mathbb{W}^2$.

$$g_p(v, w) = \eta(v, w), \quad v, w \in T_p(\mathbb{W}^n).$$

For $n = 2$, an almost global co-ordinate chart of $\mathbb{W}^2$ is given by

$$\begin{aligned} \phi^{-1} &: (0, 2\pi) \times (0, \infty) \to \mathbb{W}^2, \\ (x_1, x_2) &\mapsto (\cos x_1 \sinh x_2, \sin x_1 \sinh x_2, \cosh x_2), \end{aligned}$$

since the image of ϕ^{-1} covers $\mathbb{W}^2$ except the curve obtained by intersection of $\mathbb{W}^2$ with the half-plane $\{x_1 \ge 0,\ x_2 = 0\}$. If we let $p = \phi^{-1}(x_1, x_2)$, then with the chart, we have

$$\begin{aligned} \left.\frac{\partial}{\partial x_1}\right|_p &= (-\sin x_1 \sinh x_2, \cos x_1 \sinh x_2, 0), \\ \left.\frac{\partial}{\partial x_2}\right|_p &= (\cos x_1 \cosh x_2, \sin x_1 \cosh x_2, \sinh x_2), \end{aligned}$$

so that

$$(g_{ij})(p) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} \sinh^2 x_2 & 0 \\ 0 & 1 \end{pmatrix},$$

so the metric is positive definite.

2. **Poincaré's ball model**. Let $\mathbb{B}^n = \{p \in \mathbb{R}^n \mid \|p\| < 1\}$ where $\|\cdot\|$ is the standard Euclean norm. Since $\mathbb{B}^n \subset \mathbb{R}^n$ is open, the tangent space $T_p(\mathbb{B}^n)$ can be canonically identified with $\mathbb{R}^n$. The metric on this ball however we take to be

$$g_p(v_1, v_2) = \frac{4}{(1 - \|p\|^2)^2} \langle v_1, v_2 \rangle$$

where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product. The distance gets progressively larger as we approach the boundary. See Fig. 2.2 for a rendering of $\mathbb{B}^2$, where we distance can be in units of fish, and the distance is increasingly stretched out as we approach the boundary of the disc.

3. **Upper half plane model**. Similar to above, $\mathbb{H}^n = \{p \in \mathbb{R}^n \mid x_n > 0\}$. Again, $T_p(\mathbb{H}^n)$ can be identified with $\mathbb{R}^n$. The metric we choose here is

$$g_p(v_1, v_2) = \frac{\langle v_1, v_2 \rangle}{x_n^2},$$

and distance is increasingly stretched out as we approach the boundary of the half-plane; see Fig. 2.2 for a rendering of $\mathbb{H}^2$.

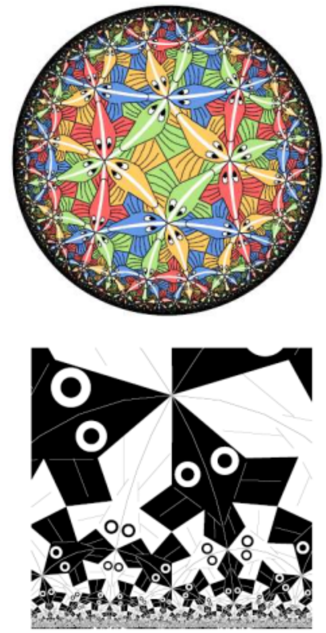

Figure 2.2: Computer rendering of M. C. Escher's *Circle Limit III* and *Circle Limit I*. From website of Douglas Dunham (UoM Duluth).

Let $V_{1,2}$ be two vector spaces with inner products $\langle \cdot, \cdot \rangle_{1,2}$. An isomorphism $T : V_1 \to V_2$ is called a **linear isometry** if

$$\langle v_1, v_2 \rangle_1 = \langle T(v_1), T(v_2) \rangle_2 \tag{2.5}$$

for all $v_{1,2} \in V_1$.

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds. A bijective differentiable map $f : M_1 \to M_2$ with differentiable inverse $f^{-1} : M_2 \to M_1$ is called a **diffeomorphism**. A diffeomorphism f above is called an **isometry** if $Df(p) : T_p(M_1) \to T_{f(p)}(M_2)$ is a linear isometry, i.e.

$$\langle (Df(p))(v), (Df(p))(w) \rangle_{2,f(p)} = \langle v, w \rangle_{1,p} \tag{2.6}$$

for all $v, w \in T_p(M_1)$.

Remark It is in face sufficient to check this for $v = w$, since

$$\langle v, w \rangle_p = \frac{1}{4} \left(\|v + w\|_p^2 - \|v - w\|_p^2 \right).$$

Example 1. Let $f : \mathbb{W}^2 \to \mathbb{B}^2$ as above. We can map each point p on the hyperboloid onto the a point on the disk that intersects the straight line through p and $(0, 0, -1)$ (e.g. the trough at $(0, 0, 1) \in \mathbb{W}^2$ is mapped to the origin), and vice-versa. Can do it in a way that preserves the inner product, so is an isometry.

2. For $\mathbb{B}^2, \mathbb{H}^2 \subset \mathbb{R}^2 \cong \mathbb{C}$, we can show that

$$f : \mathbb{H}^2 \to \mathbb{B}^2, \quad f(z) = \frac{z - \mathrm{i}}{z + \mathrm{i}}$$

is an isometry. We see that f is a diffeomorphism with

Note that this f is a Möbius map.

$$f^{-1} = \frac{z+1}{\mathrm{i}z - \mathrm{i}}.$$

Let $z = x + \mathrm{i}y \in \mathbb{H}^2$, $v = v_1 + \mathrm{i}v_2 \in T_z(\mathbb{H}^2)$. Then

$$g_z^{\mathbb{H}^2}(v, v) = \frac{v_1^2 + v_2^2}{y^2}.$$

Let $c(t) = z + tv$ represent v, namely $c(0) = z$, $c'(0) = v$, so

$$\begin{aligned}
(Df(z))(v) = (f \circ c)'(0) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\frac{c(t) - \mathrm{i}}{c(t) + \mathrm{i}}\right) \\
&= \frac{c'(0)(z + \mathrm{i}) - c'(0)(z - \mathrm{i})}{(z + \mathrm{i})^2} \\
&= \frac{2\mathrm{i}}{(z+1)^2} v \in T_z(\mathbb{B}^2).
\end{aligned}$$

Here,

$$g_z^{\mathbb{B}^2}(w, w) = \frac{4\left|\frac{2\mathrm{i}}{(z+1)^2}\right|^2}{\left(1 - \left|\frac{z-\mathrm{i}}{z+\mathrm{i}}\right|^2\right)^2} \langle v, v\rangle = \frac{16\langle v, v\rangle}{(|z+\mathrm{i}|^2 - |z - \mathrm{i}|^2)^2} = \frac{\langle v, v\rangle}{y^2},$$

and hence f is an isometry.

Lemma 2.0.1 *Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and $f : M_1 \to M_2$ a diffeomorphism. Let $\phi : M_1 \to V \subset \mathbb{R}^n$ be a global co-ordinate chart with $\phi = (x_1, \dots x_n)$, then $\psi = \phi \circ f^{-1} : M_2 \to V$ is a global co-ordinate chart of M_2. For $\psi = (y_1, \dots y_n)$, i.e. $y_i = x_i \circ f$, f is an isometry if*

$$\left\langle \left.\frac{\partial}{\partial x_i}\right|_p, \left.\frac{\partial}{\partial x_j}\right|_p \right\rangle_{1,p} = \left\langle \left.\frac{\partial}{\partial y_i}\right|_{f(p)}, \left.\frac{\partial}{\partial y_j}\right|_{f(p)} \right\rangle_{2,f(p)}$$

for all $p \in M_1$.

Proof First of all, we have

$$(Df(p))\left(\left.\frac{\partial}{\partial x_i}\right|_p\right)(h) = \left.\frac{\partial}{\partial x_i}\right|_p (h \circ f) = \frac{\partial(h \circ f\phi^{-1})}{\partial x_i}(\phi(p)).$$

Noting that we have

$$f \circ \phi^{-1} = \psi^{-1}, \quad \psi(f(p)) = \phi \circ f^{-1}(f(p)) = \phi(p),$$

then

$$(Df(p))\left(\left.\frac{\partial}{\partial x_i}\right|_p\right)(h)=\frac{\partial(h\circ\psi^{-1})}{\partial x_i}(\psi(f(p)))=\left.\frac{\partial}{\partial y_i}\right|_{f(p)}.$$

h is arbitrary, so

$$(Df(p))\left(\left.\frac{\partial}{\partial x_i}\right|_p\right)=\left.\frac{\partial}{\partial y_i}\right|_{f(p)}.$$

Now, assuming that

$$\left\langle\left.\frac{\partial}{\partial x_i}\right|_p,\left.\frac{\partial}{\partial x_j}\right|_p\right\rangle_{1,p}=\left\langle\left.\frac{\partial}{\partial y_i}\right|_{f(p)},\left.\frac{\partial}{\partial y_j}\right|_{f(p)}\right\rangle_{2,f(p)}$$

Let $v\in T_p(M_1)$, then

$$v=a_i\left.\frac{\partial}{\partial x_i}\right|_p\quad\Rightarrow\quad Df(p)(v)=a_i\left.\frac{\partial}{\partial y_i}\right|_{f(p)},$$

which implies

$$\begin{aligned}\langle Df(p)(v),Df(p)(v)\rangle_{2,f(p)}&=\left\langle a_i\left.\frac{\partial}{\partial y_i}\right|_{f(p)},a_j\left.\frac{\partial}{\partial y_j}\right|_{f(p)}\right\rangle_{2,f(p)}\\&=a_ia_j\left\langle\left.\frac{\partial}{\partial y_i}\right|_{f(p)},\left.\frac{\partial}{\partial y_j}\right|_{f(p)}\right\rangle_{2,f(p)}\\&=a_ia_j\left\langle\left.\frac{\partial}{\partial x_i}\right|_p,\left.\frac{\partial}{\partial x_j}\right|_p\right\rangle_{1,p}\\&=\left\langle a_i\left.\frac{\partial}{\partial x_i}\right|_p,a_j\left.\frac{\partial}{\partial x_j}\right|_p\right\rangle_{1,p}\\&=\langle v,v\rangle_{1,p},\end{aligned}$$

and hence $Df(p)$ is a linear isometry and so f is an isometry. ∎

2.1 Integration on Riemannian manifolds

Let (M,g) be a n-Riemannanian manifold and $\phi=(x_1,\dots x_n):U\to V\subset\mathbb{R}^n$ be a co-ordinate chart. Recall that

$$g_{ij}(p)=\left\langle\left.\frac{\partial}{\partial x_i}\right|_p,\left.\frac{\partial}{\partial x_j}\right|_p\right\rangle:U\to\mathbb{R},\quad 1\le i,j\le n.$$

Let $f:M\to\mathbb{R}$ be a function with support in U only, and ϕ as above. Then we define the integral to be

$$\int_M f\,\mathrm{d(vol)}=\int_U f\,\mathrm{d(vol)}=\int_V\left(f\circ\phi^{-1}\right)(x)\cdot\sqrt{\det(g_{ij}\circ\phi^{-1}(x)}\,\mathrm{d}x\,. \tag{2.7}$$

To show that this definition is co-ordinate independent, let $F = \psi \circ \psi^{-1}$, $\psi = (y_1, \dots y_n) : U \to V'$ be another co-ordinate chart, so F is the change of co-ordinates. The transformation rule states that if $F : V \to V'$ is a diffeomorphism, then we need a Jacobian factor for the integral as

$$\int_V' h(y)\, \mathrm{d}y = \int_V (h \circ F)(x) |\det DF(x)|\, \mathrm{d}x. \tag{2.8}$$

for some function h. Recall that

$$\left.\frac{\partial}{\partial x_i}\right|_p = \frac{\partial y_j}{\partial x_i}(p) \left.\frac{\partial}{\partial y_j}\right|_p,$$

so the metric is

$$\begin{aligned}
\tilde{g}_{ij}(p) &= \left\langle \left.\frac{\partial}{\partial x_i}\right|_p, \left.\frac{\partial}{\partial x_j}\right|_p \right\rangle \\
&= \frac{\partial y_k}{\partial x_i}(p) \frac{\partial y_l}{\partial x_i}(p) \left\langle \left.\frac{\partial}{\partial y_k}\right|_p, \left.\frac{\partial}{\partial y_l}\right|_p \right\rangle \\
&= \left(\frac{\partial y_k}{\partial x_i}(p) \right) (g'_{kl}(p)) \left(\frac{\partial y_l}{\partial x_j}(p) \right)^T
\end{aligned}$$

for $p \in M$, where g'_{kl} is the metric expressed in the y co-ordinates. Let $x = \phi(p)$, then since $F_j = y_j \circ \phi^{-1}$, we have

$$DF(x) = \left(\frac{\partial (y_i \circ \phi^{-1})}{\partial x_j}(x) \right) = \left(\frac{\partial y_i}{\partial x_j}(p) \right),$$

so then $(g_{ij}(p)) = (DF(x))^T (g'_{ij}(p))(DF(x))$, and since the transpose does not alter the determinant,

$$\sqrt{\det g_{ij}(p)} = |\det DF| \sqrt{\det g'_{ij}(p)}.$$

Thus

$$\int_V (f \circ \phi^{-1}) \sqrt{\det(g_{ij} \circ \phi^{-1})}\, \mathrm{d}x = \int_V (f \circ \psi^{-1} \circ F) \sqrt{\det(g'_{ij} \circ \psi^{-1} \circ F)} |\det DF|\, \mathrm{d}x.$$

If we then define the function h above to be $h = f \circ \psi^{-1} \sqrt{\det(g'_{ij} \circ \psi^{-1})}$, then by the transformation rule we have that

$$\int_V (f \circ \phi^{-1}) \sqrt{\det(g_{ij} \circ \phi^{-1})}\, \mathrm{d}x = \int_{V'} (f \circ \psi^{-1}) \sqrt{\det(g'_{ij} \circ \psi^{-1})}\, \mathrm{d}y,$$

and we have co-ordinate independence as claimed.

Let (M, g) be a Riemanninan manifold with a global co-orindate chart $\phi = (x_1, \dots x_n) : M \to \mathbb{R}^n$. The **volume** of $A \subset M$ is

$$\mathrm{vol}(A) = \int_M \mathbb{1}_A\, \mathrm{d}(\mathrm{vol}) = \int_A \mathrm{d}(\mathrm{vol} = \int_{\phi(A)} \sqrt{\det(g_{ij} \circ \phi^{-1})}\, \mathrm{d}x, \tag{2.9}$$

where $\mathbb{1}_A$ is the indicator function supported over A.

Example Recall that $\mathbb{H}^2 = \{z \mid \mathrm{Im}(z) > 0\}$ with $\langle v, w\rangle = (v_1 w_1 + v_2 w_2)/y^2$ for $z = x + \mathrm{i}y$ has the global chart $\phi : \mathbb{H}^2 \to \mathbb{R}^2$ with $x + \mathrm{i}y \mapsto (x, y)$. Then

$$g_{11}(z) = \left\langle \frac{\partial}{\partial x_1}\bigg|_z, \frac{\partial}{\partial x_1}\bigg|_z \right\rangle = \frac{1}{y^2} = g_{22}(z), \quad g_{12}(z) = g_{21}(z) = 0.$$

For what would be visually the rectangular strip $A = \{z \in \mathbb{H}^2 \mid 0 < a \leq y \leq b, \ -n \leq x \leq n\}$, we would have

$$\mathrm{vol}(A) = \int_{-n}^{n} \int_{a}^{b} \sqrt{\det g_{ij}}\, \mathrm{d}y\, \mathrm{d}x = 2n\left(\frac{1}{a} - \frac{1}{b}\right),$$

as opposed to $2n(b - a)$ under the usual metric.

Let M be a differentiable manifold. A **partition of unity** on M is some $\{\psi_\alpha \in C^\infty(M)\}_{\alpha \in A}$, $\psi_\alpha : M \to [0, 1]$ such that

- for all $p \in M$, there exists an open neighbourhood $U_p \subset M$ with $\psi_\alpha|_{U_p} \neq 0$ for finite many $\alpha \in A$,
- $\sum_{\alpha \in A} \psi_\alpha = 1$.

Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of M. A partition of unity $\{\psi_\alpha\}_{\alpha \in A}$ is **subordinated** to $\{U_\alpha\}$ if, for all $\alpha \in A$, $\mathrm{supp}\psi_\alpha \subset U_\alpha$.

supp is support rather than supremum.

Theorem 2.1.1 *Let M be a manifold and $\{U_\alpha, \phi_\alpha\}_{\alpha \in A}$ be a countable atlas of M. Then there exists a partition of unity $\{\psi_\alpha\}$ subordinated to $\{U_\alpha\}$.* □

Let (M, g) be a Riemannian manifold with countable atlas $\{U_\alpha, \phi_\alpha\}_{\alpha \in A}$ and subordinated partition of unity $\{\psi_\alpha\}$. For any $f : M \to \mathbb{R}$, we define the **integral** of f as

$$\int_M f\, \mathrm{d}(\mathrm{vol}) = \sum_{\alpha \in A} \int_M f \psi_\alpha\, \mathrm{d}(\mathrm{vol}). \tag{2.10}$$

This definition is independent of atlas and partition of unity.

2.2 Riemannian manifolds as metric spaces

Let (M, g) be a Riemannian manifold, $c : [a, b] \to M$ be a differentiable curve. The **length** of c is

$$L(c) = \int_a^b \|c'(t)\|_{c(t)}\, \mathrm{d}t. \tag{2.11}$$

If c is piecewise smooth with cusps, then just chop it up in the the bits that can be integrated accordingly and then sum it up, i.e.

$$L(c) = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \|c'(t)\|_{c(t)}\, \mathrm{d}t. \tag{2.12}$$

Theorem 2.2.1 *Let $c : [0, T] \to M$ be a differentiable curve and $\gamma : [0, S]$ be an orientation preserving re-parameterisation of c (i.e. there exists a strictly monotonic differentiable function $\phi : [0, T] \to [0, S]$ with $\phi(0) = 0$, $\phi(T) = S$ such that $\gamma = c \circ \phi$. Then $L(c) = L(\gamma)$.*

Proof For $f \in D(M, \gamma(s))$,

$$\gamma'(s)(f) = (f \circ \gamma)'(s) = (f \circ c \circ \phi)'(s).$$

By the chain rule, we have

$$(f \circ c \circ \phi)'(s) = (f \circ c)'(\phi(s)) \cdot \phi'(s) = \phi'(s) c'(\phi(s))(f)$$

so that $\phi'(s)c'(\phi(s)) = \gamma'(s)$. Since ϕ was assumed to be orientation preserving and monotonic,

$$\begin{aligned} L(\gamma) &= \int_0^S \|\gamma'(s)\|_{\gamma(s)} \, \mathrm{d}s \\ &= \int_0^S \phi'(s) \|c'(\phi(s))\|_{c(\phi(s))} \, \mathrm{d}s \\ &= \int_0^T \|c'(t)\|_{c(t)} \, \mathrm{d}s = L(c) \end{aligned}$$

since $t = \phi(s)$. ■

Note the result is true even if re-parameterisation is orientation reversing, as there will be two minus signs arising in that case.

A differentiable curve $c : [a, b] \to M$ is **parameterised by arc length** if $\|c'(t)\|_{c(t)} = 1$ for all $t \in [a, b]$.

Lemma 2.2.2 *For an arc length parameterised curve $c : [a, b] \to M$, $L(c|_{[a,t]}) = t - a$ for all $t \in [a, b]$. (Just use the definition.)* □

Proposition 2.2.3 *Every differentiable curve $c : [a, b] \to M$ with $c'(t) \neq 0$ for all $t \in [a, b]$ has an arc length re-parameterisation $\gamma : [0, L(\gamma)] \to M$.*

Proof Let $\ell : [a, b] \to [0, L(\gamma)]$, $\ell(t) = L(c|_{[a,t]}) = \int_a^t \|c'(s)\|_{c(s)} \, \mathrm{d}s$. Clearly ℓ is differentiable and $\ell'(t) = \|c'(t)\|_{c(t)} > 0$, so ℓ is strictly monotonically increasing and hence bijective. Since $\ell(\ell^{-1}(s)) = s$,

$$\ell'(\ell^{-1}(s)) \cdot (\ell^{-1})'(s) = 1,$$

which implies

$$(\ell^{-1})'(s) = \frac{1}{\|c'(\ell^{-1}(s))\|_{c(\ell^{-1}(s))}}.$$

Let $\gamma = c \circ \ell^{-1} : [0, L(\gamma)] \to M$, then

$$\|\gamma'(s)\| = (\ell^{-1})'(s) \|c'(\ell^{-1}(s))\|_{\ell(\ell^{-1}(s))} = 1,$$

and hence we have an arc length re-parameterisation.

Example For $c : [a,b] \to \mathbb{H}^2$, $c(t) = \mathrm{i}t$, then

$$L(c) = \int_a^b \|c'(t)\|_{c(t)} \, \mathrm{d}t = \int_a^b \frac{|\mathrm{i}|}{t} \, \mathrm{d}t = \log \frac{b}{a}.$$

We show that for any other curve $\gamma : [0,T] \to \mathbb{H}^2$ with $\gamma(0) = a\mathrm{i}$, $\gamma(T) = b\mathrm{i}$, $L(\gamma) \geq L(c)$. Let $\gamma(t) = x(t) + \mathrm{i}y(t)$, then since $y(t) > 0$,

This is related to the fact this $c(t)$ is a **geodesic** in $\mathbb{H}^2$, and has the property that it is distance minimising.

$$\begin{aligned} L(\gamma) = \int_0^T \|\gamma'(t)\|_{\gamma(t)} \, \mathrm{d}t &= \int_0^T \frac{\sqrt{(x')^2 + (y')^2}}{y} \, \mathrm{d}t \\ &\geq \int_0^T \frac{\sqrt{(y')^2}}{y} \, \mathrm{d}t = \int_0^T \frac{|y'|}{y} \, \mathrm{d}t \\ &\geq \int_0^T \frac{y'}{y} \, \mathrm{d}t = \log \frac{y(T)}{y(0)} = \log \frac{b}{a} = L(c). \end{aligned}$$

Let (M,g) be a connected Riemannian manifold (i.e. for every $p,q \in M$, there is some $c : [a,b] \to M$ with $c(a) = p$, $c(b) = q$. We define a **distance function** $d_g : M \times M \to [0,\infty)$ as

$$d_g(p,q) = \inf\{L(c) \mid c \text{ piecewise differentiable, } c(a) = p, \ c(b) = q\}. \tag{2.13}$$

The distance function should satisfy

- $d_g(p,q) = 0$ iff $p = q$
- $d_g(p,q) = d(q,p)$
- $d_g(p_1,p_3) \leq d_g(p_1,p_2) + d_g(p_2,p_3)$

For a space equipped with a distance function d_g, (X,d) is called a **metric space**.

Example Let $M = \mathbb{R}^2 \setminus \{0\}$ equipped with the standard Riemannian metric. Taking two points p and $q = -p$, we see there is no curve $c : [a,b] \to \mathbb{R}^2\{0\}$ such that $L(c) = d_g(q-p)$ since such a curve passes through zero, which is not in the manifold.

Give $c : [a,b] \to M$ with $L(c) = d_g(c(a),c(b))$, c is called a **distance realising** curve.

Let (X,d) be a metric space. A subset $A \subset X$ is called **compact** if for all sequence $(x_n)_{n\in\mathbb{N}} \in A$, there is a sub-sequence $(x_{n_j})_{j\in\mathbb{N}}$ such that $d(x_{n_j}, x_\infty) \to 0$ as $j \to \infty$. The metric space here is **complete** if every Cauchy sequence in X is convergent, i.e. for all $(x_n)_{n\in\mathbb{N}} \in X$ and $\epsilon > 0$, there exists some N where $d(x_n, x_m) < \epsilon$ for all $n,m \in \mathbb{N}$, with limit point $x_\infty \in X$ where $d(x_n, x_\infty) \to 0$.

Example $\mathbb{R} \setminus \{0\}$ is not complete since $(1/n)_{n\in\mathbb{N}}$ is a Cauchy sequence but its limit is not in the space.

3 Levi-Civita connection and parallel transport

We aim to differentiate a vector field $X : M \to T(M)$ along a curve c. As an example, we consider directional derivatives of a vector field in $\mathbb{R}^n$. By identifying $T_p(\mathbb{R}^n) \cong \mathbb{R}^n$, a vector field of X on $\mathbb{R}^n$ can be considered as a map $X : \mathbb{R}^n \to \mathbb{R}^n$, i.e. $X = a_i(\partial/\partial x_i)$, so

$$X(p) = a_i(p) \left.\frac{\partial}{\partial x_i}\right|_p \cong (a_i(p))_i \in \mathbb{R}^n.$$

For a tangent vector $v \in T_p(\mathbb{R}^n)$, we can naturally define the derivative of x in the direction of v by

$$\nabla_v X = v(a_j)(\partial/\partial x_j)|_p\,, \tag{3.1}$$

since

$$\begin{aligned}
\nabla_v X = DX(p)(v) &= \lim_{t\to 0} \frac{X(p+tv) - X(p)}{t} \\
&= \left(\lim_{t\to 0} \frac{a_i(p+tv) - a_i(p)}{t}\right)_i \\
&= (v(a_i))_i \\
&= v(a_j)\boldsymbol{e}_j \\
&= v(a_j) \left.\frac{\partial}{\partial x_j}\right|_p .
\end{aligned}$$

Here, $\nabla_v X \in T_p(\mathbb{R}^n)$ is called the **covariant derivative** of X in the direction of v. Note that we define $\nabla_X Y \in \mathcal{X}(\mathbb{R}^n)$, with

$$\nabla_X Y(p) = \nabla_{X(p)} Y(p) \in T_p(\mathbb{R}^n)\,. \tag{3.2}$$

The **torsion** is defined as

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]\,, \tag{3.3}$$

and described how tangent spaces twist around a curve.

Proposition 3.0.1 *For $X, Y :\to \mathbb{R}^n \to \mathbb{R}^n$, $v, w \in T_p(\mathbb{R}^n)$, $\alpha, \beta \in \mathbb{R}$, $f \in C^\infty(\mathbb{R}^n)$, we have*

1. *linearity in vector field,* $\nabla_v(X+Y) = \nabla_v X + \nabla_v Y$
2. *product rule,* $\nabla_v(fX) = v(f) \cdot X(p) + f(p)\nabla_v X$
3. *linearity in directional vector,* $\nabla_{\alpha v+\beta w} X = \alpha\nabla_v X + \beta\nabla_w X$
4. *Riemannian property,* $v(\langle X, Y\rangle) = \langle \nabla_v X, Y(p)\rangle + \langle X(p), \nabla_v Y\rangle$
5. *torsion freeness,* $\nabla_X Y - \nabla_Y X = [X, Y]$.

Proof 1. For $X = (a_i)_i$, $Y = (b_i)_i$,

$$\nabla_v(X+Y) = v(a_i + b_i)\boldsymbol{e}_i = v(a_i)\boldsymbol{e}_i + v(b_i)\boldsymbol{e}_i = \nabla_v X + \nabla_v Y.$$

2. As above,

$$\begin{aligned}\nabla_v(fX) = v(fa_i)\boldsymbol{e}_i &= v(f)a_i(p)\boldsymbol{e}_i + f(p)v(a_i)\boldsymbol{e}_i \\ &= v(f)X(p) + f(p)\nabla_v X.\end{aligned}$$

3. Also as above,

$$\begin{aligned}\nabla_{\alpha v+\beta w} X = (\alpha v + \beta w)a_i\boldsymbol{e}_i &= \alpha v(a_i)\boldsymbol{e}_i + \beta w(a_i)\boldsymbol{e}_i \\ &= \alpha\nabla_v X + \beta\nabla_w Y.\end{aligned}$$

4. For $\langle X, Y\rangle = a_i b_i$, we have

$$\langle \nabla_v X, Y(p)\rangle = \langle v(a_i)\boldsymbol{e}_i, b_j(p)\boldsymbol{e}_j\rangle = v(a_i)b_i(p).$$

Similarly, we have

$$\langle \nabla_v X(p), \nabla_v Y\rangle = v(b_i)a_i(p),$$

so their sum would be

$$v(a_i)b_i(p) + v(b_i)a_i(p) = v(a_i b_i)(p) = v\langle X(p), Y(p)\rangle.$$

5. We have

$$(\nabla_X Y)(p) = X(p)(b_i)\boldsymbol{e}_i = a_j(p)\frac{\partial b_i}{\partial x_j}(p)\ \frac{\partial}{\partial x_i}\bigg|_p,$$

while

$$(\nabla_Y X)(p) = Y(p)(a_i)\boldsymbol{e}_i = b_j(p)\frac{\partial a_i}{\partial x_j}(p)\ \frac{\partial}{\partial x_i}\bigg|_p,$$

so

$$\begin{aligned}(\nabla_X Y)(p) - (\nabla_Y X)(p) &= \left(a_j(p)\frac{\partial b_i}{\partial x_j}(p) - b_j(p)\frac{\partial a_i}{\partial x_j}(p)\right)\ \frac{\partial}{\partial x_i}\bigg|_p \\ &= [X, Y](p).\end{aligned}$$

■

Since

$$(\nabla_X Y)(p) = \lim_{t\to 0} \frac{Y(p+tX(p)) - Y(p)}{t},$$

we also have the following:

1. $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$,
2. $\nabla_X(fY) = [X(f)]Y + f\nabla_X Y$,
3. $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$,
4. $X(\langle Y, Z\rangle) = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$,
5. $\nabla_X Y - \nabla_Y X = [X, Y]$.

Let M be a differentiable manifold, and $\mathcal{X}(M)$ be the space of all vector fields on M. A map $\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ satisfying the first three properties above is called an **covariant derivative** (or **affine connection**).

An affine connection 'connects' nearby tangent spaces, allowing the notion of derivative to make sense.

Theorem 3.0.2 (Fundamental theorem of Riemannian geometry) *Let (M, g) be a Riemannian manifold. Then there exists a unique covariant derivative ∇ satisfying the Riemannian and torison freeness property (i.e. all properties above), and this connection is called the* ***Levi-Civita connection****.*

Proof For uniqueness, note that since ∇ is Riemannian, we have

$$\begin{aligned}
X\langle Y, Z\rangle &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, \\
Y\langle Z, X\rangle &= \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle, \\
Z\langle X, Y\rangle &= \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle,
\end{aligned}$$

so that (with torsion freeness and properties of the inner product)

$$\begin{aligned}
X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle &= \langle \nabla_X Y, Z\rangle + [\langle Y, \nabla_X Z\rangle - \langle \nabla_Z X, Y\rangle] \\
&+ [\langle \nabla_Y Z, X\rangle - \langle X, \nabla_Z Y\rangle] \\
&+ \langle Z, [Y, X] + \nabla_X Y\rangle \\
&= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z - \nabla_Z X\rangle \\
&+ \langle X, \nabla_Y Z - \nabla_Z Y\rangle \\
&+ \langle Z, [Y, X]\rangle + \langle Z, \nabla_X Y\rangle \\
&= 2\langle \nabla_X Y, Z\rangle + \langle Y, [X, Z]\rangle \\
&+ \langle X, [Y, Z]\rangle + \langle Z, [Y, X]\rangle,
\end{aligned}$$

which provides an explicit construction for $\nabla_X Y$ via

$$\begin{aligned}
\langle \nabla_X Y, Z\rangle = &\frac{1}{2}(X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\
&- \langle Y, [X, Z]\rangle - \langle X, [Y, Z]\rangle - \langle Z, [Y, X]\rangle).
\end{aligned} \tag{3.4}$$

For existence, we check that the above construction satisfies the properties. Clearly we have linearity since the inner product and Lie bracket are linear. The expression above satisfies the product rule, torsion freeness and Riemannian property follows essentially by brute force calculation. ■

3.1 *Christoffel symbols*

Let (M, g) be a Riemannian manifold, and ∇ the Levi-Civita connection on (M, g). For $\phi = (x_1, \dots x_n) : U \to V \subset \mathbb{R}^n$ a co-ordinate chart, we have

$$\left(\nabla_{\frac{\partial}{\partial x_i}\big|_p} \frac{\partial}{\partial x_j} \right)(p) = \Gamma^k_{ij} \frac{\partial}{\partial x_k}\bigg|_p, \tag{3.5}$$

where $\Gamma ij^k : U \to \mathbb{R}$ are the **Christoffel symbols** of the covariant derivative with respect to ϕ. Observing that $[\partial/\partial x_i, \partial/\partial x_j] = 0$, with Eq. (3.4),

$$\begin{aligned}\left\langle \nabla_{\frac{\partial}{\partial x_i}\big|_p} \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_l} \right\rangle (p) &= \frac{1}{2}\left[\frac{\partial}{\partial x_i} \left\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_l} \right\rangle + \frac{\partial}{\partial x_j} \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_l} \right\rangle - \frac{\partial}{\partial x_l} \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle \right](p) \\ &= \frac{1}{2}(g_{jl,i} + g_{il,j} - g_{ij,l})(p)\end{aligned}$$

where $g_{jl,i} = \partial g_{jl}/\partial x_i$. Since (g_{ij}) is symmetric and invertible by construction, we denote the entries of $(g_{ij})^{-1}$ by g^{ij}, and we thus have the equation

$$\Gamma^k_{ij} g_{kl} = \frac{1}{2}(g_{jl,i} + g_{il,j} - g_{ij,l}),$$

so that

$$\Gamma^k_{ij} = \frac{1}{2}(g_{jl,i} + g_{il,j} - g_{ij,l}) g^{kl}. \tag{3.6}$$

Remark • $g_{lm} = g_{ml}$ implies that $g^{lm} = g^{ml}$.

• $\Gamma^k_{ij} = \Gamma^k_{ji}$ since g_{ij} is symmetric.

The Christoffel symbols are not always symmetric by construction: in this case these originate from a metric connection so is symmetric.

Example Recall that for $\mathbb{H}^2$ we have $\langle v, w \rangle_z = \langle v, w \rangle / y^2$, $z = x + iy$, and that

$$g_{ij}(z) = \frac{1}{y^2} I \quad \Rightarrow \quad g^{ij}(z) = y^2 I.$$

The metric has no off-diagonal terms, and any $g_{ij,1}$ is going to be zero since there is no x dependence. For example,

$$\begin{aligned}\Gamma^1_{11} &= \frac{1}{2}(g_{1l,1} + g_{1l,1} - g_{11,l}) g^{l1} &&= \frac{1}{2}(g_{11,1} + g_{11,1} - g_{11,1}) g^{11} = 0, \\ \Gamma^2_{11} &= \frac{1}{2}(g_{1l,1} + g_{1l,1} - g_{11,l}) g^{l2} &&= \frac{1}{2}(g_{12,1} + g_{12,1} - g_{11,2}) g^{22} = -\frac{1}{2}\left(-2\frac{1}{y^3}\right) y^2 = \frac{1}{y}\end{aligned}$$

and so forth. To summarise, there should be eight entries, and

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^2_{21} = \Gamma^1_{22} = 0, \quad -\Gamma^2_{11} = \Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{22} = -\frac{1}{y}.$$

Then note that

$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = \Gamma^1_{11} \frac{\partial}{\partial x} + \Gamma^2_{11} \frac{\partial}{\partial y} = +\frac{1}{y} \frac{\partial}{\partial y}$$
$$\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = \Gamma^1_{22} \frac{\partial}{\partial x} + \Gamma^2_{22} \frac{\partial}{\partial y} = -\frac{1}{y} \frac{\partial}{\partial y},$$

while by the torison-free property and that $[\partial/\partial x, \partial/\partial y] = 0$,

$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = \Gamma^1_{12} \frac{\partial}{\partial x} + \Gamma^2_{12} \frac{\partial}{\partial y} = -\frac{1}{y} \frac{\partial}{\partial x}$$
$$\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial x} = \left[\frac{\partial}{\partial y}, \frac{\partial}{\partial x} \right] + \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial x} = -\frac{1}{y} \frac{\partial}{\partial x}.$$

Let $X(z) = y(\partial/\partial y)|_z$ be a vector field (and note that $\|X(z)\|_z = 1$ in the present metric). We have

$$\begin{aligned}(\nabla_X X)(z) &= y \left(\nabla_{\frac{\partial}{\partial y}} \left(y \frac{\partial}{\partial y} \right) \right) (z) \\ &= y \left[\frac{\partial y}{\partial y} \frac{\partial}{\partial y} + y \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} \right] (z) \\ &= y \left(1 \frac{\partial}{\partial y} - y \frac{1}{y} \frac{\partial}{\partial y} \right) (z) = 0.\end{aligned}$$

The covariant derivative of X in the direction X vanishes, which is perhaps not that surprising.

3.2 *Parallel transport*

Let (M, g) be a Riemannian manifold, ∇ the Levi-Civita connection, $c[a, b] \to M$ a curve, $X \in \mathcal{X}(M)$, then $\nabla_{c'(t)} X \in T_{c(t)}(M)$ only depends on X along c, i.e. if $\tilde{X} \in \mathcal{X}(M)$ with $\tilde{X}(c(t)) = X(c(t))$ for all $t \in [a, b]$ then $\nabla_{c'(t)} X = \nabla_{c'(t)} \tilde{X}$.

Let $c : [a, b] \to M$ be a differentiable curve on a Riemannian manifold. A map $X : [a, b] \to T(M)$ with $X(t) \in T_{c(t)}(M)$ is called a **vector field along** c. The space of all such vector fields is denoted by $\mathcal{X}_c(M)$.

Example For $c : [a, b] \to M$ a curve, then $t \mapsto c'(t)$ is a vector field along c.

Proposition 3.2.1 *Let ∇ be the Levi-Civita connection of (M, g), $c : [a, b] \to M$ a curve. Then there is a unique map called the* ***covariant derivative*** *along c*

$$\frac{\mathrm{D}}{\mathrm{d}t} : \mathcal{X}_c(M) \to \mathcal{X}_c(M) \tag{3.7}$$

satisfying

1. $(D/dt)(aX + bY) = a(DX/dt) + b(DY/dt)$ *for* $a, b \in \mathbb{R}$,

2. $(D/dt)(fX) = f'X + f(DX/dt)$ *for* $f \in C^\infty([a,b])$,

3. *for* $\tilde{X} \in \mathcal{X}(M)$ *a local extension of* X, *i.e. there exists* $t_0 \in [a,b]$ *where* $\tilde{X}(c(t)) = X(t)$ *for all* $t \in [t_0 - \epsilon, t_0 + \epsilon]$ *for some* $\epsilon > 0$, *we have*

$$\frac{DX}{dt}(t_0) = \nabla_{c'(t_0)}\tilde{X}. \tag{3.8}$$

Proof To proof uniqueness, let $\phi = (x_1, \dots x_n) : U \to V \subset \mathbb{R}^n$ be a co-ordinate chart with $c([t_0 - \epsilon, t_0 + \epsilon]) \subset U$, $X \in \mathcal{X}_c(M)$. Then, locally,

$$X(t) = a_i(t) \left.\frac{\partial}{\partial x_i}\right|_{c(t)},$$

so that

$$\frac{DX}{dt}(t) = a_i'(t) \left.\frac{\partial}{\partial x_i}\right|_{c(t)} + a_i(t)\frac{D}{dt}\left(\frac{\partial}{\partial x_i} \circ c\right)(t).$$

Note that $(\partial/\partial x_i \circ c) \in \mathcal{X}_c(M)$ and is a natural extension of X, so

$$\frac{DX}{dt}(t) = a_i'(t) \left.\frac{\partial}{\partial x_i}\right|_{c(t)} + a_i(t)\nabla_{c'(t)}\frac{\partial}{\partial x_i} \in T_{c(t)}(M),$$

and hence we have uniqueness. Since above is an explicit construction, we can check that it satisfies all the desired properties, so we also have existence. ■

Example Let $M \subset \mathbb{R}^3$ be a surface, D/dt the covariant derivative along a curve c, and $X \in \mathcal{X}_c(t)$ with $X(t) = [a_i(t)]_i \in \mathbb{R}^3 \cong T_{c(t)}(\mathbb{R}^3)$. Then

$$\frac{DX}{dt}(t) = \pi_{c(t)}[a_i'(t)]_i \in T_{c(t)}(M)$$

where $\pi_{c(t)}$ is the orthogonal projection onto M. In particular, if $c(t) \in M$, then $c(t) \in T_{c(t)}(M)$, and

$$\frac{Dc'}{dt}(t) = \pi_{c(t)}[c_i''(t)]_i.$$

For $Dc'/dt \equiv 0$, this is equivalent to $c''(t)$ being normal to $T_{c(t)}(M)$.

This is the condition revisited later that c is a **geodesic**.

A vector field $X : [a,b] \to T(M)$ along c is called **parallel along** c iff $DX/dt \equiv 0$.

A geodesic is defined to be a curve whose tangent vectors remain parallel if they are transported along it. Requires the notion of an affine connection.

Theorem 3.2.2 *Let* $c : [a,b] \to M$ *be a curve on a Riemannian manifold* (M, g) *and* $v \in T_{c(a)}(M)$. *Then there exists a unique parallel vector field* $X \in \mathcal{X}_c(M)$ *with* $X(a) = v$. *Also, for* $\dim M = n$, *the space of all parallel vector fields in* $\mathcal{X}_c(M)$ *is a n-vector space over* $\mathbb{R}$.

Proof For simplicity, assume there exists a co-ordinate chart $\phi = (x_1, \dots x_n) : U \to V \subset \mathbb{R}^n$ with $c([a,b]) \subset U$. Let

$$X(t) = a_i \left.\frac{\partial}{\partial x_i}\right|_{c(t)} \in \mathcal{X}_c(M),$$

and $(\phi \circ c) = (c_1(t), \dots c_n(t)) : \mathbb{R} \to \mathbb{R}^n$, then

$$c'(t) = c_i'(t) \left.\frac{\partial}{\partial x_i}\right|_{c(t)}.$$

So we have

$$\begin{aligned}
\frac{\mathrm{D}X}{\mathrm{d}t}(t) &= a_i'(t) \left.\frac{\partial}{\partial x_i}\right|_{c(t)} + a_i(t)\nabla_{c'(t)}\frac{\partial}{\partial x_i} \\
&= a_i'(t) \left.\frac{\partial}{\partial x_i}\right|_{c(t)} + a_i(t)c_j'(t)\nabla_{\left.\frac{\partial}{\partial x_j}\right|_{c(t)}}\frac{\partial}{\partial x_i} \\
&= a_i'(t) \left.\frac{\partial}{\partial x_i}\right|_{c(t)} + a_i(t)c_j'(t)\Gamma_{ij}^k(c(t)) \left.\frac{\partial}{\partial x_k}\right|_{c(t)} \\
&= \left[a_k'(t) + a_i(t)c_j'(t)\Gamma_{ij}^k(c(t))\right] \left.\frac{\partial}{\partial x_k}\right|_{c(t)}.
\end{aligned}$$

Since $\partial/\partial x_j$ form a basis, if $\mathrm{D}X/\mathrm{d}t \equiv 0$, this implies that we have

$$a(t) = A(t)a(t), \quad A(t) = [A_{ki}(t)] = \left[-c_j'(t)\Gamma_{ij}^k(c(t))\right]$$

for all t and k. The theory of ODEs tells us that there is a unique solution throughout the domain for any choice fo initial conditions. For $v = \alpha_i(\partial/\partial x_i)|_{c(a)} \in T_{c(a)}(M)$, this proves uniqueness and existent of the parallel vector field

$$X = a_i \left(\frac{\partial}{\partial x_i} \circ c\right), \quad X(a) = v.$$

Since parallel vector fields form a real vector space and are uniquely determined by their initial vector $v \in T_{c(a)}(M)$, we have

$$\dim \left\{ X \in X_c(M) \;\middle|\; \frac{\mathrm{D}X}{\mathrm{d}t} \equiv 0 \right\} = \dim T_{c(a)}(M) = n.$$

∎

Example For $c(t) = \mathrm{i}y + t$ in the hyperbolic plane with $g_z(v,w) = \langle v,w\rangle/y^2$, $z = x + \mathrm{i}y$, recall we have the Christoffel symbols from before:

$$\Gamma_{11}^1 = \Gamma_{12}^2 = \Gamma_{21}^2 = \Gamma_{22}^1 = 0, \quad -\Gamma_{11}^2 = \Gamma_{12}^1 = \Gamma_{21}^1 = \Gamma_{22}^2 = -\frac{1}{y}.$$

The vector field $X(t) = a_1(t)(\partial/\partial x)|_{c(t)} + a_2(t)(\partial/\partial y)|_{c(t)}$ is parallel along c with $X(0) = (\partial/\partial y)|_{\mathrm{i}y}$, and we have

$$c_1'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Re}(c) = 1, \quad c_2'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Im}(c) = 0.$$

Using the notation as in the theorem, we have $A_{ki}(t) = -c'_j\Gamma^k_{ij}(c) = -\Gamma^k_{i1}(c)$, so

$$a'(t) = -\begin{bmatrix}\Gamma^1_{11} & \Gamma^1_{12}\\ \Gamma^2_{11} & \Gamma^2_{12}\end{bmatrix} a(t) = \begin{bmatrix}0 & 1/y\\ -1/y & 0\end{bmatrix} a(t), \quad a(0) = \begin{pmatrix}0\\1\end{pmatrix}.$$

If we can find solutions to the system, then the solution corresponds to $\mathrm{D}X/\mathrm{d}t = 0$ and we have our parallel vector field along c. By the usual theory of differential equations, we have

$$a(t) = \mathrm{e}^{At}\begin{pmatrix}0\\1\end{pmatrix} = \left(\sum_k \frac{t^k}{k!}\begin{bmatrix}0 & 1/y\\ -1/y & 0\end{bmatrix}\right)\begin{pmatrix}0\\1\end{pmatrix}.$$

Note that

$$\begin{bmatrix}0 & 1/y\\ -1/y & 0\end{bmatrix}^{2k} = \begin{bmatrix}-1/y^2 & 0\\ 0 & -1/y^2\end{bmatrix}^k = \begin{bmatrix}(-1)^k/y^{2k} & 0\\ 0 & (-1)^k/y^{2k}\end{bmatrix},$$

$$\begin{bmatrix}0 & 1/y\\ -1/y & 0\end{bmatrix}^{2k+1} = \begin{bmatrix}0 & (-1)^k/y^{2k+1}\\ (-1)^k/y^{2k+1} & 0\end{bmatrix}^{2k+1},$$

so we have

$$a(t) = \sum_k\left(\frac{t^k}{k!}\begin{bmatrix}(-1)^k/y^{2k} & (-1)^k/y^{2k+1}\\ (-1)^k/y^{2k+1} & (-1)^k/y^{2k}\end{bmatrix}\right)\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\sin t/y\\ \cos t/y\end{pmatrix}.$$

Therefore

$$X(t) = \left(\sin\frac{t}{y}\right)\frac{\partial}{\partial x}\bigg|_{\mathrm{i}y+t} + \left(\cos\frac{t}{y}\right)\frac{\partial}{\partial y}\bigg|_{\mathrm{i}y+t}$$

has $X(0) = (\partial/\partial y)|_{\mathrm{i}y}$ and $X(t)$ is parallel along c.

Let (M, g) be a Riemannian manifold, $c : [a, b] \to M$ a curve. The **parallel transport** is a linear map $\rho_c : T_{c(a)}(M) \to T_{c(b)}(M)$, and for $c \in T_{c(a)}(M)$, we have $\rho_c(v) = X(b)$, where $X \in \mathcal{X}_c(M)$ is the unique parallel vector field along c with $X(a) = v$.

Transport along a vector field preserving the parallelism with respect to the connection.

The parallel transport along a curve c yields a linear isomorphism $\rho_c : T_p(M) \to T_q(M)$ where $p = c(a)$ and $q = c(b)$, i.e., a connection between two disjoint tangent spaces. Note the parallel transport is defined via the Levi-Civita connection, and hence the Levi-Civita connection induces a connection between disjoint tangent spaces along curves connecting their footpoints. On the other hand, the isomorphism ρ_c depends in general on a curve c, i.e., if $\gamma : [\alpha, \beta] \to M$ with $\gamma(\alpha) = c(a)$ and $\gamma(\beta) = c(b)$, it is not necessarily true that $\rho_c = \rho_\gamma$.

Proposition 3.2.3 *The parallel transport $\rho_c T_p(M) \to T_q(M)$ is a linear isometry, i.e.*

$$g_p(v_1, v_2) = g_q(\rho_c(v_1), \rho_c(v_2))$$

for all $v_{1,2} \in T_p(M)$.

Proof For $X, Y \in \mathcal{X}_c(M)$, assume there is a global co-ordinate chart $\phi = (x_1, \dots x_n) : U \to V$ with $c([a,b]) \subset U$. Then

$$X(t) = a_j(t) \left.\frac{\partial}{\partial x_j}\right|_{c(t)}, \quad Y(t) = b_j(t) \left.\frac{\partial}{\partial x_j}\right|_{c(t)},$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle X, Y\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\left(a_j b_k \left[\left\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right\rangle \circ c\right]\right).$$

The Riemannian property implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right\rangle \circ c\right) = \left\langle \nabla_{c'(t)}\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right\rangle + \left\langle \frac{\partial}{\partial x_j}, \nabla_{c'(t)}\frac{\partial}{\partial x_k}\right\rangle.$$

Together, this implies

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle X, Y\rangle &= (a_j' b_k + a_j b_k')\left[\left\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right\rangle \circ c\right] \\
&\quad + a_j b_k \left(\left\langle \nabla_{c'(t)}\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right\rangle + \left\langle \frac{\partial}{\partial x_j}, \nabla_{c'(t)}\frac{\partial}{\partial x_k}\right\rangle\right) \\
&= \left\langle a_j' \left.\frac{\partial}{\partial x_j}\right|_{c(t)} + a_j \nabla_{c'(t)}\frac{\partial}{\partial x_j}, b_k \left.\frac{\partial}{\partial x_k}\right|_{c(t)}\right\rangle \\
&\quad + \left\langle a_j \left.\frac{\partial}{\partial x_j}\right|_{c(t)}, b_k' \left.\frac{\partial}{\partial x_k}\right|_{c(t)} + b_k \nabla_{c'(t)}\frac{\partial}{\partial x_k}\right\rangle \\
&= \left\langle \frac{\mathrm{D}}{\mathrm{d}t}\left(a_j \frac{\partial}{\partial x_j} \circ c\right), Y\right\rangle + \left\langle X, \frac{\mathrm{D}}{\mathrm{d}t}\left(b_k \frac{\partial}{\partial x_k} \circ c\right)\right\rangle \\
&= \left\langle \frac{\mathrm{D}X}{\mathrm{d}t}, Y\right\rangle + \left\langle X, \frac{\mathrm{D}Y}{\mathrm{d}t}\right\rangle = 0.
\end{aligned}$$

Since X and Y are parallel, $\langle X, Y\rangle$ is the constant function, so

$$\langle \rho_c X(a), \rho_c X(a)\rangle = \langle X(b), Y(b)\rangle = \langle X(a), Y(a)\rangle,$$

and we therefore have an isometry. ■

3.3 Geodesics

Let (M, g) be a Riemannian manifold. A curve $c : [a,b] \to M$ is a **geodesic** if c' is parallel along the curve c for all t, i.e.,

$$\boxed{\frac{\mathrm{D}}{\mathrm{d}t}c'(t) \equiv 0} \tag{3.9}$$

for all $t \in [a,b]$, and $\mathrm{D}/\mathrm{d}t$ denotes the covariant derivative along c.

Lemma 3.3.1 *Let $c : [a,b] \to M$ be a geodesic. Then c is parameterised proportional to arc length.*

Proof We just need to prove that there exists $k > 0$ where $\|c'(t)\| = k$ for all $t \in [a,b]$. Note that we have

$$\|c'\|^2 = k^2 \quad \Leftrightarrow \quad \langle c', c' \rangle \quad \Leftrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\langle c', c' \rangle = 0.$$

But

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle c', c' \rangle = \left\langle \frac{\mathrm{D}}{\mathrm{d}t}c', c' \right\rangle + \left\langle c', \frac{\mathrm{D}}{\mathrm{d}t}c' \right\rangle = 0$$

since we have a geodesic, so we have what we need. ∎

Theorem 3.3.2 *Let (M,g) be a Riemannian manifold. Then for all $v \in T_p(M)$, there exists some $\epsilon > 0$ and a unique geodesic $c : (-\epsilon, \epsilon) \to M$ where $c(0) = p$ and $c'(0) = v$.*

Proof Let $v \in T_p(M)$ and $\phi = (x_1, \dots x_n) : U \to V$ be a local chart, and $p \in U$. Then $v = v_i(\partial/\partial x_i)|_p$. Let $c : (-\epsilon, \epsilon) \to U$ be some curve with $c(0) = p$ and $(\phi \circ c)(t) = (c_1(t), \dots c_n(t))$. Then $c'(t) = c_i'(t)(\partial/\partial x_i)|_{c(t)}$. Then

$$\begin{aligned}
\left(\frac{\mathrm{D}}{\mathrm{d}t}c'\right)(t) &= c_i'' \left.\frac{\partial}{\partial x_i}\right|_c + c_i' \nabla_{c'} \frac{\partial}{\partial x_i}(c(t)) \\
&= c_i'' \left.\frac{\partial}{\partial x_i}\right|_c + c_i' c_j' \left(\nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i}\right)(c(t)) \\
&= \left[c_k'' + c_i' c_j' \Gamma_{ij}^k(c(t))\right] \left.\frac{\partial}{\partial x_k}\right|_c .
\end{aligned}$$

For a geodesic the above is equal to zero, and since $\partial/\partial x_k$ is a basis, we have the **geodesic equations**

$$c_k'' + c_i' c_k' \Gamma_{ij}^k(c(t)) = 0, \qquad c_k(0) = x_k(p), \quad c_k'(0) = v_k$$

for all k. This is a system of second order ODEs, and by the theory of ODEs there exists a unique solution in a neighbourhood $(-\epsilon, \epsilon)$, and so we have existence and uniqueness. ∎

Let $c : [a,b] \to M$ be a differentiable curve. A differentiable map $F : (-\epsilon, \epsilon \times [a,b] \to M$ is called the (differentiable) **variation** of c if $F(0,t) = c(t)$ for all $t \in [a,b]$. The variation is called **proper** if we have $F(s,a) = c(a)$ and $F(s,b) = c(b)$. The **variational vector field** X of a variation F of c is

End points are pinned and only the interior can be varied.

$$X(t) = \frac{\partial F}{\partial s}(0,t) . \tag{3.10}$$

If F is proper then $X(a) = X(b) = 0$ since $s \mapsto F(s,a)$ and $s \mapsto F(s,b)$ are constant maps. The **length** and **energy** is given by

$$\ell(s) = \int_a^b \left\| \frac{\partial F}{\partial t}(s,t) \right\| \, \mathrm{d}t \tag{3.11}$$

and

$$E(s) = \frac{1}{2}\int_a^b \left\| \frac{\partial F}{\partial t}(s,t) \right\|^2 \, \mathrm{d}t \,. \tag{3.12}$$

Lemma 3.3.3 *Symmetry lemma Let $W \subset \mathbb{R}^2$ be open and $F : W \to M$ be the variation. Let $\mathrm{D}/\mathrm{d}t$ denote the covariant derivative along $t \mapsto F(s,t)$ and $\mathrm{D}/\mathrm{d}s$ denote the same but along $s \mapsto F(s,t)$. Then*

$$\frac{\mathrm{D}}{\mathrm{d}t}\frac{\partial F}{\partial s} = \frac{\mathrm{D}}{\mathrm{d}s}\frac{\partial F}{\partial t}. \tag{3.13}$$

Proof Without loss of generality, assume that there is a co-ordinate chart $\phi = (x_1, \ldots x_n) : U \to V \subset \mathbb{R}^n$, $U \subset M$ and $F(W) \subset U$. Let $(\phi \circ F)(s,t) = (\alpha_1(s,t), \ldots a_n(s,t))$, then

$$\begin{aligned}
\frac{\mathrm{D}}{\mathrm{d}t}\frac{\partial F}{\partial s} &= \frac{\mathrm{D}}{\mathrm{d}t}\left(\frac{\partial \alpha_j}{\partial s}\frac{\partial}{\partial x_j}\right) \\
&= \frac{\partial^2 \alpha_j}{\partial t \partial s}\frac{\partial}{\partial x_j} + \frac{\partial \alpha_j}{\partial s}\nabla_{\frac{\partial \alpha_k}{\partial t}\frac{\partial}{\partial x_k}}\frac{\partial}{\partial x_j} \\
&= \frac{\partial^2 \alpha_j}{\partial t \partial s}\frac{\partial}{\partial x_j} + \frac{\partial \alpha_k}{\partial t}\frac{\partial \alpha_j}{\partial s}\left(\nabla_{\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_k} + \left[\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right]\right) \\
&= \frac{\mathrm{D}}{\mathrm{d}s}\frac{\partial F}{\partial t}
\end{aligned}$$

by the torsion definition, but also that the connection is torsion free.
■

Theorem 3.3.4 (First variational formula for length) *Let F be a variation of c with $c'(t) \neq 0$ for all $t \in [a,b]$, and X be the variational vector field. Let $\ell(s)$ denote the associated length, then*

$$\ell'(0) = \int_a^b \frac{1}{\|c'(t)\|}\left[\frac{\mathrm{d}}{\mathrm{d}t}\langle X(t), c'(t)\rangle - \left\langle X(t), \frac{\mathrm{D}}{\mathrm{d}t}c'(t)\right\rangle\right] \, \mathrm{d}t \tag{3.14}$$

If c is parameterised by proportional to arc length, then $\|c'(t)\| = k$, and

$$k\ell'(0) = \langle X(b), c'(b)\rangle - \langle X(a), c'(a)\rangle - \int_a^b \left\langle X(t), \frac{\mathrm{D}}{\mathrm{d}t}c'(t)\right\rangle \, \mathrm{d}t. \tag{3.15}$$

Proof Recall that an equivalent condition for a connection ∇ to be Riemannian is that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle Y, Z\rangle = \left\langle \frac{\mathrm{D}}{\mathrm{d}t}Y, Z\right\rangle + \left\langle Y, \frac{\mathrm{D}}{\mathrm{d}t}Z\right\rangle.$$

Then since

$$\ell(s) = \int_a^b \left\| \frac{\partial F}{\partial t}(s,t)\right\| \, \mathrm{d}t = \int_a^b \left\langle \frac{\partial F}{\partial t}, \frac{\partial F}{\partial t}\right\rangle^{1/2} \, \mathrm{d}t$$

we have

$$
\begin{aligned}
\ell'(0) &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \int_a^b (\cdots)\,\mathrm{d}t \\
&= \int_a^b \left.\frac{\partial}{\partial s}\right|_{s=0} (\cdots)\,\mathrm{d}t \\
&= \int_a^b \frac{1}{2}\frac{1}{\|\partial F/\partial t|_{s=0}\|} 2\left\langle \frac{\mathrm{D}}{\mathrm{d}s}\left.\frac{\partial F}{\partial t}\right|_{s=0}, \left.\frac{\partial F}{\partial t}\right|_{s=0}\right\rangle \mathrm{d}t \\
&= \int_a^b \frac{1}{\|c'(t)\|}\left\langle \frac{\mathrm{D}}{\mathrm{d}s}\left.\frac{\partial F}{\partial t}\right|_{s=0}, c'(t)\right\rangle \mathrm{d}t \\
&= \int_a^b \frac{1}{\|c'(t)\|}\left\langle \frac{\mathrm{D}}{\mathrm{d}s} X, c'(t)\right\rangle \mathrm{d}t \\
&= \int_a^b \frac{1}{\|c'(t)\|}\left[\frac{\mathrm{d}}{\mathrm{d}t}\langle X, c'(t)\rangle - \left\langle X, \frac{\mathrm{D}}{\mathrm{d}s}c'\right\rangle\right] \mathrm{d}t.
\end{aligned}
$$

■

Lemma 3.3.5 *Let $c : [a, b] \to M$ be a differentiable curve, and X a vector field along c, with $X(a) = X(b) = 0$. Then there exists a proper variation F with X as its variational vector field.* □

Theorem 3.3.6 *Let $c : [a, b] \to M$ be a differentiable curve. It is a geodesic iff c is parameterised proportional to arc length, and $\ell'(0) = 0$ for any proper variation of c.*

Or, as we might expect intuitively, geodesics can be locally length minimising (we have only shown it is an extrema at the moment).

Proof If c is a geodesic then clearly it has to be parameterised proportional to arc length, so $\|c'(t)\| = k > 0$ assuming c is non-singular. By the first variational formula for length,

$$
\ell'(0) = \int_a^b \frac{1}{k}\left[\frac{\mathrm{d}}{\mathrm{d}t}\langle X, c'(t)\rangle - 0\right] \mathrm{d}t = \frac{1}{k}\left[\langle X, c'(t)\rangle\right]_a^b = 0
$$

since X is a proper variational vector field and using the above lemma.

Suppose now $\ell'(0)$ ad $\|c'\| = k > 0$. If c is not a geodesic, then $\mathrm{D}c'/\mathrm{d}t(t_0) \neq 0$ for some $t_0 \in [a, b]$. By continuity, we assume $t_0 \in (a, b)$. We choose a smooth function ϕ where $\phi : [a, b] \to [0, 1]$ with $\phi(a) = \phi(b) = 0$, $\phi(t_0) = 1$, $\phi(t) \geq 0$, and we define a vector field along c by

$$
X(t) = \phi(t)\frac{\mathrm{D}c'}{\mathrm{d}t}.
$$

Since $X(a) = X(b) = 0$, X can represent the variational vector field of a proper variation, and $X(t) \neq 0$ by the lemma. Then, by the hypothesis,

$$\begin{aligned} 0 &= k\ell'(0) \\ &= -\int_a^b \left\langle X(t), \frac{\mathrm{D}}{\mathrm{d}t} c'(t) \right\rangle \, \mathrm{d}t \\ &= -\int_a^b \phi(t) \left\| \frac{\mathrm{D}c'}{\mathrm{d}t} \right\|^2 \, \mathrm{d}t < 0, \end{aligned}$$

so we have a contradiction, and c is a geodesic. ■

3.4 Geodesic flow

Lemma 3.4.1 (Scaling lemma) *Let $c : [a, b] \to M$ be a geodesic and $k > 0$. Let $\gamma : [a/k, b/k] \to M$ with $\gamma(t) = c(kt)$ for all t. Then γ is also a geodesic, and $\gamma'(t) = kc'(kt)$.*

Note the reparameterisation is still in terms of an **affine** parameter (and all scale factors in the geodesic equations drop out).

Proof Note that

$$\gamma'(t)(f) = \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \gamma)(t) = \frac{\mathrm{d}}{\mathrm{d}t}(f \circ c)(kt) = k(f \circ c)'(kt) = kc'(kt)(f),$$

so that $\gamma'(t) = kc'(kt)$ for all $f \in D(M, \gamma(t))$.

Let $\phi = (x_1, \ldots x_n) : U \to V \subset \mathbb{R}^n$ be a chart with $\gamma(t) = c(kt) \in U$. Let $(\phi \circ \gamma) = (\gamma_1, \ldots \gamma_n)$ and $(\phi \circ c) = (c_1, \ldots c_n)$, then

$$\begin{aligned} &\frac{\mathrm{D}}{\mathrm{d}t}\gamma' \\ &= \nabla_{\gamma'}\gamma' \\ &= \left(\gamma_l'' + \gamma_i'\gamma_j'\Gamma_{ij}^l(\gamma)\right) \left.\frac{\partial}{\partial x_l}\right|_\gamma \\ &= \left(k^2 c_l''(kt) + k^2 c_i'(kt) c_j'(kt) \Gamma_{ij}^l(c(kt))\right) \left.\frac{\partial}{\partial x_l}\right|_\gamma \\ &= k^2 \frac{\mathrm{D}}{\mathrm{d}t} c' = 0, \end{aligned}$$

so γ is a geodesic.

Corollary 3.4.2 *For $(p, v) \in T(M)$, let $c_v : I_v \to M$ denote the unique geodesic with maximal interval $I_v \in \mathbb{R}$ such that $c_v(0) = p$, $c_v'(0) = v$. Let $I_v = (a, b)$ and $k > 0$, then $I_{kv} = (a/k, b/k)$ with $c_{kv}(t) = c_v(kt)$, and we can extend to cover $\mathbb{R}$ for $k \to 0$.*

Theorem 3.4.3 *Let X be a smooth vector field on an open set $V \subset M$ with $p \in V$. Then there exists an open set $V_0 \subset V$, $p \in V_0$, $\delta > 0$ and smooth $\phi : (-\delta, \delta) \times V_0 \to V$ such that the curve $t \mapsto \phi(t, q)$ is the unique trajectory of X with $\phi(0, q) = q$ for all $q \in V_0$, i.e.,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(p, q) = X(\phi(t, q)).$$

□

Here, $\phi(t) = \phi(t, q)$ is called the **flow** of X in V.

Lemma 3.4.4 *Let M be a Riemannian manifold. There exists a unique vector field $G \in T(M)$ whose trajectories are of the form $t \mapsto (\gamma(t), \gamma'(t))$ where γ is a geodesic on M.* □

The vector field G is called the **geodesic field** on $T(M)$, and its flow is the **geodesic flow**.

Proposition 3.4.5 *Give $p \in M$, there exists a neighbourhood V of $p \in M$, $\delta > 0$, $\epsilon_1 > 0$, and the differentiable map $\gamma : (-\delta, \delta) \times M \to U$ with $U = \{(q, v) \in T(M), q \in V, \|v\| < \epsilon_1\}$ such that $t \mapsto \gamma(t, q, v)$ is the unique geodesic of M, with $\gamma(0, q, v) = q$, $\mathrm{d}/\mathrm{d}t\gamma(t, q, v)|_{t=0} = v$ for all $q \in V$, $v \in T_p(M)$ and $\|v\| < \epsilon_1$.*

The geodesic $\gamma(t, q, v)$ is defined for $|t| < \delta$ and $\|v\| < \epsilon_1$. Using the rescaling lemma, letting $k = \delta/2$, we have $\gamma(t, q, (\delta/2)v)$ with $t < \delta/k = 2$. Note also $\|(\delta/2)v\| = |\delta/2| \cdot \|v\| < (\delta/2)\epsilon_1$. Taking $\epsilon < (\delta/2)\epsilon_1$, the above proposition may have γ and U replaced by

$$\gamma : (-2, 2) \times \tilde{U} \to M, \qquad \tilde{U} = \{(q, v) \in T(M), q \in V_1, \|v\| < \epsilon\}.$$

Let $p \in M$, and $\tilde{U}$ be the open set of $T(M)$ given as above. The **exponential map** at p is given by

$$\boxed{\exp_p(v) = c_v(1) = \gamma(1, p, v)}\,, \tag{3.16}$$

where γ is a geodesic as above. Note that $\exp_p : B_\epsilon(0_p) \subset T_p(M) \to M$ with $B_\epsilon(0_p) = \{v \in T_p(M) \mid \|v\| < \epsilon\}$ is the appropriate ball of radius ϵ, with ϵ given as above. The exponential map may or may not be defined on the whole of $T_p(M)$ in this case.

Example For $S^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$, recall all geodesics are parts of great circles. Let $p = (0, 0, 1)$ be the norther pole. For $v_0 \in T_p(S^2)$ with $\|v_0\| = 1$, the geodesic c_{v_0} is given by

$$c_{v_0}(t) = (\cos t)p + (\sin t)v_0$$

from a previous example. Indeed, $c_{v_0}(0) = p$ and $c'_{v_0} = v$ for $t \in \mathbb{R}$, so $\exp_p(v_0) = c_{v_0}(1)$ in this case. For arbitrary $v \in T_p(M)$, $v = tv_0$, and for $\|v_0\| = 1$, we have

$$\exp_p(v) = \exp_p(tv_0) = c_{tv_0}(1) = c_{v_0}(t),$$

thus the exponential map is in face well defined on the whole of $T_p(S^2)$ for this case.

Example Suppose we remove the south pole $p = (0, 0, -1)$. Using the same arguments as in the previous example, we see the maximal geodesic $c_{v_0}(t)$ is only defined on $t \in (-\pi, \pi)$. Thus $\exp_p$ is only given on $B_\pi(0, p) = \{v \in T_p(M) \mid \|v\| < \pi\}$, since $t \in (-\pi, \pi)$ means $v = tv_0 \in (-\pi v_0, \pi v_0)$.

Proposition 3.4.6 *Let (M, g) be a Riemannian manifold, and $p \in M$. Then there exists $\epsilon > 0$ such that $\exp_p : B_\epsilon(0_p) \to \exp_p(B_\epsilon(0_p))$ is a diffeomorphism.*

Proof We have

$$\mathrm{d}(\exp_p)(v) = \frac{\mathrm{d}}{\mathrm{d}t}\exp_p(tv)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}(c_{tv}(1))\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}(c_v(t))\Big|_{t=0} = c_v'(0) = v,$$

so that $\mathrm{d}(\exp_p) = \mathrm{Id}$ on $T_p(M)$. By the inverse function theorem, the exponential map is a local diffeomorphism. ∎

Lemma 3.4.7 (Gauss' lemma) *Let (M, g) be a Riemannian manifold, and $p \in M$. Let $\epsilon > 0$ be small enough that $\exp_p$ is a diffeomorphism. Then the readial geodesic $t \mapsto \exp_p(tv)$ for $t \geq 0$ intersects the hypersurface $A_\delta = \{\exp_p(w) \mid \|w\| = \delta, 0 < \delta < \epsilon\}$ orthogonally.*

Or, any sufficiently small sphere centred on a point in a Riemannian manifold is perpendicular to every geodesic through that point.

Proof Let $w \in T_p(M)$, $\|w\| = \delta$ and $c_w : [0,1] \to M$ with $c_w(t) = \exp_p(tw)$ be the corresponding radial geodesic. Let $v : (-\alpha, \alpha) \to T_p(M)$, $\|v\| = \delta$ for all $s \in (-t, t)$, $v(0) = w$. The for any $0 < b \leq 1$, $F : (-\alpha, \alpha) \times [0, b] \to M$ with $F(s,t) = \exp_p(tv(s))$ is a variation of c_w. Since $t \mapsto F(s,t) = \exp_p(tv(s)) = c_{v(s)}(t)$ are geodesics, they are parameterised by arc length. So $\|\partial F / \partial(s,t)\| = \|v(s)\| = \delta$, therefore

$$\ell(s) = \int_a^b \left\| \frac{\partial F}{\partial t}(s,t) \right\| \, \mathrm{d}t = b\delta$$

and thus $\ell'(0) = 0$. By the first variational formula for length,

$$0 = c\ell'(0) = \left\langle \frac{\partial F}{\partial s}(0, b), c_w'(0) \right\rangle - 0 - 0,$$

which implies

$$\left\langle \frac{\partial F}{\partial s}(0, b), c_v'(b) \right\rangle = 0,$$

and hence we have orthogonal intersection. ∎

Corollary 3.4.8 *Let (M, g), $p \in M$ and $\epsilon > 0$ as above, and $B_\epsilon(p) = \exp_p(B_\epsilon(0_p))$. Let $c : [0,1] \to B_\epsilon(p)$ be a geodesic with $c(0) = p$. If $\gamma : [0,1] \to M$ is a differentiable curve joining $c(0)$ and $c(1)$, then $\ell(c) \leq \ell(\gamma)$. If equality holds, then $\gamma([0,1]) = c([0,1])$.*

Proof Suppose $\gamma([0,1]) \subset B_\epsilon(p)$. As the exponential map is a diffeomorphism, γ is the image of a unique curve β on $B_\epsilon(0_p) \subset T_p(M)$. Expression β is polar co-ordinates, we have $\beta(s) = r(s)v(s)$, $\|v(s)\| = 1$, and this is allowed since $\beta \in T_p(M)$ is in Euclidean space. We assume that $r(s) > 0$ on $(0, 1]$ or $r(s) = 0$, because we are only interested in showing $\ell(c) \leq \ell(\gamma)$.

If $F(s,t) = \exp_p(tv(s))$, then

$$\gamma(s) = \exp_p(\beta(s)) = \exp_p(r(s)v(s)) = F(s, r(s)),$$

so that

$$\gamma' = \frac{\partial F}{\partial s}(s, r(s)) + \frac{\partial F}{\partial t}(s, r(s))r'(s).$$

By Gauss' lemma, we have $\langle \partial F/\partial s, \partial F/\partial t\rangle = 0$. Also, we have

$$\frac{\partial F}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}t}\exp_p(tv(s)) = \frac{\mathrm{d}}{\mathrm{d}t}\tilde{c}'_{v(t)}(t),$$

thus $\|\partial F/\partial t(s, r(s))\| = \|v(s)\| = 1$. So

$$\|\gamma'(s)\| = \sqrt{\left\|\frac{\partial F}{\partial s}(s, r(s))\right\|^2 + |r'(s)|^2} \geq |r'(s)|,$$

and therefore

$$\ell(\gamma) = \int_\delta^1 \|\gamma'(s)\|\ \mathrm{d}s \geq \int_\delta^1 |r'(s)|\ \mathrm{d}s \geq \left|\int_\delta^1 r'(s)\ \mathrm{d}s\right| = r(1) - r(\delta).$$

As $\delta \to 0$, we have $\ell(\gamma) \geq r(1) - 0$. Note that $c(1) = \gamma(1) = \exp_p(r(1)v(1)) = c_{v(1)}(r(1))$, so

$$\ell(c) = \int_{r(0)=0}^{r(1)} \left\|c'_{v(1)}(t)\right\|\ \mathrm{d}t = \int_0^{r(1)}\ \mathrm{d}t = r(1),$$

and thus $\ell(\gamma) \geq \ell(c)$. We have equality iff $\partial F/\partial s(s, r(s)) = 0$, r monotone and $v(s) = v = \text{const}$, i.e. $\gamma(s) = \exp_p(r(s)v)$ is a geodesic.

If $\gamma([0,1])$ is not contained in $B_\epsilon(p)$, then let t_1 be the first time such that $\gamma(t_1) \in \partial B_\epsilon(p)$, which would imply that $\ell(\gamma) \geq \ell_{[0,t_1]}(\gamma) \geq \epsilon > \ell(c)$. ∎

As a consequence, for all $p \in M$, there exists some $\epsilon > 0$ such that for all $q \in B_\epsilon(p)$ there is a unique curve connecting p and q, satisfying $\ell(c) = d_g(p,q)$. Thus $B_\epsilon(p)$ coincides with

$$B_\epsilon = \{q \in M \mid d_g(p,q) < \epsilon\} \subset (M, g),$$

and geodesics are length minimising at least locally. A geodesic $c : [a,b] \to M$ is called **minimal** if $\ell(c) = d_g(c(a), c(b))$. A geodesic $c : \mathbb{R} \to M$ is also called minimal if all its restrictions to $[a,b]$ are minimal.

A Riemannian manifold (M, g) is called **geodesically complete** if every geodesic $c : [a,b] \to M$ can be extended to a geodesic $\tilde{c} : \mathbb{R} \to M$.

Example Every arc of great circles in S^2 with angle less than or equal to π is minimal. S^2 is geodesically complete, but $S^2 \setminus (0,0,1)$ is not.

Theorem 3.4.9 *Let (M, g) be a Riemannian manifold that is geodesically complete. then any two points $p, q \in M$ can be joined by a minimal geodesic.*

Proof Let $r = d_g(p, q)$. We choose $0 < \epsilon < r$ such that $\exp_p : B_\epsilon(0_p) \to B_\epsilon(p)$ is a diffeomorphism. Let $0 < \delta < \epsilon$. Since

$$S_\delta(0_p) = \{v \in T_p(M) \mid \|v\| = \delta\} \subset B_\epsilon(0_p)$$

is compact and the exponential map is continuous, the image $S_\delta(p) = \exp_p(S_\delta(0_0))$ is also compact. Therefore, there exists a point $q' \in S_\delta(p)$ which is closest to p, i.e.

$$d_g(p, p') = \inf_{x \in \partial S_\delta(p)} d_g(x, q).$$

Let $q' = \exp_p(\delta v)$, $v \in T_p(M)$ and $\|v\| = 1$. By geodesic completeness, the geodesic c with $c(0) = 0$, $c'(0) = v$ is defined for all $\mathbb{R}$. Since e is the arc length, and geodesics are parameterised by arc length, we aim to show that $c(r) = q$ and $r = \ell(c) = d_g(p, q)$, so then $c : [0, r] \to M$ is a minimal geodesic connection p and q.

Let $A = \{t \in [0, r] \mid d_g(c(t), q) = r - t\}$. Then we have:

1. $\delta \in A$. Then

   $$r = d_g(p, q) = \delta + \inf_{x \in \partial S_\delta(p)} d_g(x, q) = \delta + d_g(q', q) = \delta + d_g(c(\delta), q),$$

   so $d_g(c(\delta), q) = r - \delta$.

2. $t_0 = \sup A$ and $t_0 \in A$. Clearly $t_0 \geq \delta$. That $t_0 \in A$ implies that $d_g(c(t_0), q) = r - t_0$. Let $(t_j) \in A$ and $t_j \to t_0$ be some sequence. By continuity, $c(t_j) \to c(t_0)$. We have

   $$d_g(c(t_0), q) = \lim_{j \to \infty} d_g(c(t_j), q) = \lim_{j \to \infty} (r - t_j) = r - t_0,$$

   so that $t_0 = \max A$.

3. $t_0 = r$. Suppose $t_0 < r$. Let $z = c(t_0$. Choose $\epsilon' > 0$ such that $\exp_z : B_{\epsilon'}(0_p) \to B_\epsilon(z)$ is a diffeomorphism, and ϵ' small enough that $q \notin B_{\epsilon'}(z)$. Choose $0 < \delta' < \epsilon'$ and $q'' \in S_{\delta'}(z)$ such that

   $$d_g(q'', q) = \inf_{x \in S_{\delta'}(z)} d_g(x, q).$$

   By a similar argument, $r - t_0 - \delta' = d_g(q'', q)$, so that $d_g(q'', q) = r - (t_0 + \delta')$. By the triangle inequality,

   $$\begin{aligned} d_g(p, q'') &\geq d_g(p, q) - d(q, q'') \\ &= r - (r - (t_0 + \delta') = t_0 + \delta'. \end{aligned}$$

   But $d_g(p, q'') = t_0 + \delta$, so the curve connecting p and q'' is continuous of the minimal geodesic c by uniqueness of geodesics. So $(t_0 + \delta') \in A$, contradicting $t_0 = \max A$, hence $r = t_0$.

Together, these imply that $c(r) = q$ and $r = \ell(c) = d_g(p, q)$ as required. ■

Theorem 3.4.10 (Hopf–Rinow theorem) *Let (M, g) be a Riemannian manifold. Then the following statements are equivalent:*

- *(M, g) is geodesically complete,*
- *every closed and bounded subset of (M, g) is compact,*
- *(M, d_g) is a complete metric space.*

□

4 Curvature