Complex Analysis 2H

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- *Last compiled: June 2022*
- Blended from notes of R. Gregory and J. Bolton, Durham
- This was part of the Durham core second year modules. Involves more things to do with analysis in the complex plane, involving holomorphic functions, contour integrals, residue theorems, conform mappings, etc.
- The original course does not have geometry of complex numbers since that was covered in Core A (Geometry 1A), but for consistency reasons this has been moved here.

1 Geometry of complex numbers

1.1 Complex numbers and the Argand diagram

We define $\sqrt{-1} = \mathrm{i}$, which is the basic unit imaginary number. A **complex number** is then a combination of real and imaginary parts $z = a + b\mathrm{i}$, with $a, b \in \mathbb{R}$. The complex numbers $\mathbb{C}$ then obeys the same axioms for addition and multiplication as $\mathbb{R}$ (both are **fields**).

Consider instead $\mathbb{C}$ as a vector space $z = (x, y)$, where multiplication is defined on $\mathbb{R}^2$ as

$$z_1 \times z_2 = (x_1 x_2 - y_1 y_2, x_1 y_2 - x_2 y_1),$$

and this is commutative. $1 = (1, 0)$ is the identity. So we see that $\mathbb{R}^2$ with this multiplication is a concrete visualisation of $\mathbb{C}$, and is called the **Argand diagram**.

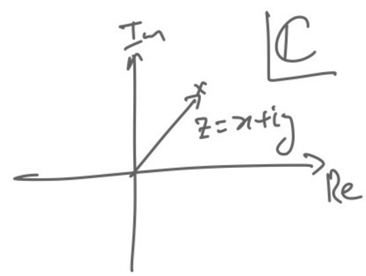

Figure 1.1: Argand diagram.

Given $z = x + \mathrm{i}y$, the **conjugate** of z is defined to be $\overline{z} = x - \mathrm{i}y$. Geometrically, this represents a reflection of z in the 'real' axis. The **real** and **imaginary** part of z is given respectively by

$$\mathrm{Re}(z) = \frac{z + \overline{z}}{2}, \qquad \mathrm{Im}(z) = \frac{z - \overline{z}}{2}.$$

In polar form, $z = r(\cos\theta + \mathrm{i}\sin\theta)$. r is called the **modulus** of z and is denoted $|z|$, whilst θ is called the **argument** of z, denoted $\arg(z)$.

1.2 Geometry of addition and multiplication in $\mathbb{C}$

Addition is as in $\mathbb{R}^2$. From this, we can deduce the **triangle inequality**.

Lemma 1.2.1 *For $z_1, z_2 \in \mathbb{C}$, $|z_1 + z_2| \leq |z_1| + |z_2|$, and we have an equality iff $\arg(z_1) = \arg(z_2)$. By corollary, we have $|z_2 + z_2| \geq ||z_1| - |z_2||$.*

Proof Without loss of generalisation, let $|z_1| > |z_2|$, then $|z_1| = |z_1 + z_2 + (-z_2)| \leq |z_1 + z_2| + |z_2|$ by the triangle inequality for real numbers. So $|z_1| - |z_2| \leq |z_1 + z_2|$, and since $|z_1| > |z_2|$, we have the corollary of the result as required. ∎

For multiplication, we observe that $|z_1 z_2| = |z_1||z_2|$ and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$. Geometrically, this is a spiral scaling.

We can use the $\mathbb{C}$-plane to describe various geometrical objects.

Example A circle may be described by $|z - z_0| = a$, where z_0 is the centre of the circle and a is the radius; expanding this accordingly, we see that $a^2 = (x - x_0)^2 + (y - y_0)^2$.

Example The equation $|z - x_0| + |z + x_0| = 2r$ describes an ellipse, where $r > |x_0|$. This may be done via expansion in (x, y). Alternatively, in polar form, we observe that, for $z = a + ib$, $|z \pm x_0|^2 = (a^2 - b^2)\cos^2\theta \pm 2ax_0\cos\theta + (x_0^2 + b^2)$. If $x_0^2 = (a^2 - b^2)$, then this may be simplified to $|z \pm x_0| = a \pm x_0 \cos\theta$ since $a > x_0$. With this, we obtain $|z - x_0| + |z + x_0| = 2a$, thus, with $x = a\cos\theta$ and $y = b\sin\theta$, this describes an ellipse.

Example The locus of $|z - z_1| = |z - z_2|$ describes the line that is equidistant to the points z_1 and z_2. To see this, expanding everything in x and y and we obtain the equality

$$x(x_2 - x_1) + y(y_2 - y_1) = \frac{y_2^2 - y_1^2}{2} + \frac{x_2^2 - x_1^2}{2},$$

and the normal to the line is $z_2 - z_1$.

1.3 de Moivre's theorem

Theorem 1.3.1 (de Moivre's theorem) *For all $n \in \mathbb{Z}^+$ and angle θ, $\cos n\theta + \mathrm{i}\sin n\theta = (\cos\theta + \mathrm{i}\sin\theta)^n$.*

Proof We do this by induction. The $n = 1$ case is trivial, so, assuming it is true for n, then we observe that

$$\begin{aligned}
&\cos(n+1)\theta + \mathrm{i}\sin(n+1)\theta \\
&\qquad = \cos n\theta\cos\theta + \mathrm{i}^2\sin\theta\sin n\theta + \mathrm{i}\sin n\theta\cos\theta + \mathrm{i}\sin\theta\cos n\theta \\
&\qquad = (\cos n\theta + \mathrm{i}\sin n\theta)(\cos\theta + \mathrm{i}\sin\theta) \\
&\qquad = (\cos\theta + \mathrm{i}\sin\theta)^{n+1}.
\end{aligned}$$

■

Example Since

$$\cos 2\theta + \mathrm{i}\sin 2\theta = (\cos\theta + \mathrm{i}\sin\theta)^2 = (\cos^2\theta - \sin^2\theta) + \mathrm{i}(2\sin\theta\cos\theta),$$

and remembering the double angle formulae, the equality agrees.

From de Moivre's theorem, we see that

$$\cos n\theta = \mathrm{Re}(\cos\theta + \mathrm{i}\sin\theta)^n, \qquad \sin n\theta = \mathrm{Im}(\cos\theta + \mathrm{i}\sin\theta)^n.$$

We can also use the theorem to find sin or cos of rational multiples of π.

Example Express $\sin 4\theta / \cos\theta$ as a polynomial in $\sin\theta$, and hence find $\sin(\pi/4)$.

$$\begin{aligned}\sin 4\theta = \operatorname{Im}(\cos\theta + \mathrm{i}\sin\theta)^4 &= 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta \\ &= 4\cos\theta(\sin\theta - 2\sin^3\theta),\end{aligned}$$

so $\sin 4\theta/\cos\theta = 4\sin\theta(1 - 2\sin^{\theta})$. Evaluating this $\pi/4$, we see that the LHS is zero. Now, $4\sin(\pi/4) > 0$, so we conclude that $\sin(\pi/4) = 1/\sqrt{2}$, as expected.

Example Find $\cos(k\pi/6)$ for $k = 1, 2, 3, 4, 5$.

Letting $c = \cos\theta$ and $s = \sin\theta$, observe that

$$\sin 6\theta = sc(6c^4 + 6s^4 - 20s^2c^2) = sc(32c^4 - 32c^2 + 6) = 2sc(4c^3 - 3)(4c^2 - 1).$$

Now, $\sin(k\pi) = 0$, so LHS is zero, and since $\sin(k\pi/6) \neq 0$, we have

$$\cos^2(k\pi/6) = 3/4, \quad \cos^2(k\pi/6) = 1/4, \quad \cos\theta = 0,$$

which implies that

$$\cos(k\pi/6) = \pm\sqrt{3}/2,\ \pm 1/2,\ 0.$$

Since $\cos\theta$ is a decreasing function in $[0, \pi]$, we have

$$\begin{aligned}\cos(\pi/6) = \sqrt{3}/2, \quad \cos(2\pi/6) = 1/2, \quad \cos(\pi/2) = 0, \\ \cos(2\pi/3) = -1/2, \quad \cos(5\pi/6) = -\sqrt{3}/2.\end{aligned}$$

1.4 *Imaginary exponentials*

de Moivre's theorem hints at a deeper geometric significance of cosine and sine functions and a way of encoding multiplication by imaginary numbers. Suppose $f(\theta) = \cos\theta + \mathrm{i}\sin\theta$, then we notice that $f'(\theta) = \mathrm{i}f(\theta)$, and, more generally, $f^{(n)}(\theta) = \mathrm{i}^n f(\theta)$. We know that also that the n-th derivative of $\mathrm{e}^{\lambda x}$ is $\lambda^n \mathrm{e}^{\lambda x}$, so this suggests a link with exponential functions; indeed, we have **Euler's formula**

$$\cos\theta + \mathrm{i}\sin\theta = \mathrm{e}^{\mathrm{i}\theta}. \tag{1.1}$$

By de Moivre's theroem then,

$$r(\cos n\theta + \mathrm{i}\sin n\theta) = r(\cos\theta + \mathrm{i}\sin\theta)^n = r\mathrm{e}^{\mathrm{i}n\theta}.$$

Lemma 1.4.1 (Euler identity) $\mathrm{e}^{\mathrm{i}\pi} + 1 = 0.$ □

Example Find all the roots of $z^6 + 4z^3 + 8 = 0$.

Factorising the above gives $z^3 = -2 \pm 2\mathrm{i}$. So since $|z^3| = 2\sqrt{2}$, we have $|z| = \sqrt{2}$. Now,

$$\arg(-2+2\mathrm{i}) = \frac{3\pi}{4}, \qquad \arg(-2-2\mathrm{i}) = \frac{5\pi/4}{,}$$

and the argument of the roots z satisfies

$$\arg(z) = \frac{3\pi/4 + 2n\pi}{3}, \qquad \arg(z) = \frac{5\pi/4 + 2n\pi}{3},$$

where the division by 3 is to take into account the cube root, and the $2n\pi$ factors is to account for all the roots. This eventually yields

$$z = \sqrt{2}(\mathrm{e}^{\mathrm{i}\pi/4}, \mathrm{e}^{5\mathrm{i}\pi/4}, \mathrm{e}^{11\mathrm{i}\pi/12}, \mathrm{e}^{13\mathrm{i}\pi/12}, \mathrm{e}^{19\mathrm{i}\pi/12}, \mathrm{e}^{21\mathrm{i}\pi/21}).$$

2 Basics of complex functions

A real function can for example be once differentiable, but not twice. One example is $f(x) = x|x|$, where $f'(x)$ is not differentiable at $x = 0$.

Theorem 2.0.1 *If a complex function is once differentiable, it is differentiable as many times as you like.*

It is possible for two real functions to agree on an interval but not everywhere, assuming they are differentiable. One example is $f(x) = x|x|$ and $g(x) = x^2$ for $x > 0$.

Theorem 2.0.2 *If two complex differentiable functions agree on any disc in the complex plane, then they agree everywhere (subject to certain conditions...)*

Recall that a real function assigns any real number x to at most one real number (i.e. it is injective). A **complex function** therefore assigns any complex number z to at most one complex number. These include standard polynomials, rational functions, transcendental functions, trigonometric functions, hyperbolic functions, where the argument is in z. Some examples have already been given above.

Example Solve $\mathrm{e}^z = 1$.

Writing $z = x + \mathrm{i}y$ and using Euler's formula,

$$\mathrm{e}^x(\cos y + \mathrm{i}\sin y) = 1,$$

and equating real and imaginary parts lead to

$$\mathrm{e}^x \cos y = 1, \qquad \mathrm{e}^x \sin y = 0.$$

Considering the imaginary part, since $\mathrm{e}^x \neq 0$, $y = n\pi$ for $n \in \mathbb{Z}$, but from the real part, since $\mathrm{e}^x > 0$ and $\cos n\pi = \pm 1$, we should only have $y = 2n\pi$ for $n \in \mathbb{Z}$. The real part then additionally implies that $x = 0$ since $\cos 2n\pi = 1$, so $z = 2\mathrm{i}n\pi$ for $n \in \mathbb{Z}$.

Note that $|\mathrm{e}^{\mathrm{i}z}| \geq 0$ for all $z \in \mathbb{C}$.

Example Solve $\sin z = 0$.

With the standard identity for sine with complex arguments, we have

$$\frac{e^{iz} - e^{-iz}}{2i} = 0.$$

Equating real and imaginary parts lead to $z = m\pi$, $m \in \mathbb{Z}$.

The (natural) **logarithm** we define by

$$\log z = \log |z| + i\arg z \tag{2.1}$$

to give a complex version of the log function that satisfies the usual rules of

$$\log z = \log re^{i\theta} = \log r + i\theta = \log |z| + i\arg z.$$

Here we need to choose a **branch**, and we take $\theta \in (-\pi, \pi)$ (the **principal branch**) to preserve the continuity property, so that $\log z$ is undefined on the negative real axis, coinciding with the real case.

Example $\log(1 - i) = \log\sqrt{2} - i(\pi/4)$

We use $\log z$ to define powers of complex numbers. Recall that for real numbers we have $x^a = e^{a\log a}$ for $a > 0$, so for $z, w \in \mathbb{C}$, we analogously define

$$z^w = e^{w\log z}, \tag{2.2}$$

choosing the principal branch unless otherwise stated.

Example

$$\begin{aligned}(1 + i\sqrt{3})^{1/2} &= \exp\left[\frac{1}{2}\log(1 + i\sqrt{3})\right] \\ &= \exp\left[\frac{1}{2}\left(\log 2 + i\frac{\pi}{3}\right)\right] \\ &= e^{\log\sqrt{2}}e^{i(\pi/6)} \\ &= \sqrt{2}e^{i(\pi/6)},\end{aligned}$$

which in this case is could have been gotten from $(1 + i\sqrt{3}) = 2e^{i(\pi/3)}$.

Example

$$(1 - i)^i = e^{i\log(1-i)} = e^{i(\log\sqrt{2} - i\pi/4)} = e^{\pi/4}e^{i\log\sqrt{2}}.$$

We say a complex function $f(z)$ is **complex differentiable at** $z = z_0$ if

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists, or that

$$\lim_{h\to 0} \frac{f(z + h) - f(z)}{h}$$

exists at $z = z_0$. The derivative is denoted $f'(z)$ as usual.

Example For $f(z) = z^2$,

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h\to 0} \frac{z^2 + 2hz + h^2 - z^2}{h} = \lim_{h\to 0} 2z + h = 2z.$$

$f(z)$ is differentiable everywhere.

The usual rules for differentiation hold (linearity, product rule, chain rule etc.)

Note that $f(x) = x|x|$ is real differentiable everywhere. $f(z) = z|z|$ on the other hand is differentiable on the real axis, and complex differentiable at the origin.

Complex differentiation is a much stronger condition. Recall that for the limit to exist in the real case, the limit only needs to be equal when approached from above or below on the real line. In the complex plane however there are an infinite numbers of cases the limit can be approach, and thus a infinite number of cases to check. We see that a necessary condition for complex differentiability is that the limit needs to exist when z_0 is approached in the lines parallel to the real and imaginary axis. If we set $f(z)$ to be

$$f(z) = u(x, y) + \mathrm{i}v(x, y)$$

for some real functions u and v, then it turns out that

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - \mathrm{i}\frac{\partial u}{\partial y},$$

when we take the limit in the direction parallel to the real and imaginary axis respectively. It follows that a *necessary* conditions for a function to be complex differentiable is that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{2.3}$$

These are known as the **Cauchy–Riemann equations**, and we actually have the following theorem.

Theorem 2.0.3 *If $f(z)$ is complex differentiable at $z = z_0$, then the Cauchy–Riemann equations hold at (x_0, y_0) for $z_0 = x_0 + \mathrm{i}y_0$, and that*

$$f'(z_0) = \left(\frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x}\right)\bigg|_{(x_0,y_0)} = \left(\frac{\partial v}{\partial y} - \mathrm{i}\frac{\partial u}{\partial y}\right)\bigg|_{(x_0,y_0)}.$$

Proof If we approach z_0 in a line parallel to the real axis, we have

$$\begin{aligned} f'(z_0) &= \lim_{x\to x_0} \frac{u(x, y_0) + \mathrm{i}v(x, y_0) - u(x_0, y_0) - \mathrm{i}v(x_0, y_0)}{x - x_0} \\ &= \lim_{x\to x_0} \left(\frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} + \mathrm{i}\frac{v(x, y_0) - v(x_0, y_0)}{x - x_0}\right) \\ &= \frac{\partial u}{\partial x}\bigg|_{(x_0,y_0)} + \mathrm{i}\,\frac{\partial v}{\partial x}\bigg|_{(x_0,y_0)}. \end{aligned}$$

We have the analogous result when approaching z_0 in a line parallel to the imaginary axis. ■

In actual fact, the Cauchy–Riemann equation holding is a necessary *and* sufficient condition for complex differentiability.

Theorem 2.0.4 *Let $f(z) = u(x,y) + \mathrm{i}v(x,y)$. If the partial derivatives of u and v exist in some disk centered on (x_0, y_0) and are continuous at $z_0 = x_0 + \mathrm{i}y_0$, and u and v satisfy the Cauchy–Riemann equation, then $f(z)$ is complex differentiable at z_0.* □

A function is said to be **holomorphic** (or **analytic**) at z_0 if it is complex differentiable on some disk centred at z_0. A function is holomorphic if it is analytic at all points where it is defined.

Example If $f(z) = y^3 - 3\mathrm{i}xy^2$, fine where $f(z)$ is complex differentiable, and compute $f'(z)$.

Note that for $u = y^3$ and $v = -3xy^2$,

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial v}{\partial x} = -3y^2, \quad \frac{\partial u}{\partial y} = -3y^2, \quad \frac{\partial v}{\partial y} = -6xy,$$

so it is differentiable if $-6xy = 0$ and $3y^2 = 3y^2$, which is only satisfied at $x = 0$ or $y = 0$, i.e. at the co-ordinate axes. In this case $f'(z) = -\mathrm{i}3y^2$, and that $f(z)$ is nowhere holomorphic.

Theorem 2.0.5 *Let $f(z)$ be holomorphic and $f(z) = u(x,y) + \mathrm{i}v(x,y)$. Then u and v are solutions to Laplace's equation in two dimensions.*

Proof By Cauchy–Riemann equations and the holomorphic property,

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x \partial y} = \frac{\partial v}{\partial y \partial x} = \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial y^2},$$

so that $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$. Similarly for v. ■

Recall that if $f(x)$ is an infinitely differentiable real function, that its Taylor series about $x = x_0$ is given by

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

The complex counterpart is then if $f(z)$ is an infinitely complex differentiable complex function, its Taylor series about $z = z_0$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k.$$

Since derivatives of standard functions are the same in the complex case, their Taylor series are the same too.

Theorem 2.0.6 *Let $f(z)$ be complex differentiable. Then its Taylor series converges to $f(z)$ for all z where it converges.* □

This implies any complex differentiable function is just a power series.

If we let $\sum_{n=0}^{\infty} b_n(z-z_0)^n$ be a power series centred on $z = z_0$, then there exists some $R \in [0, \infty]$ where the power series

- converges for $|z - z_0| < R$,
- diverges for $|z - z_0| > R$,
- inconclusive for $|z - z_0| = R$.

R is called the **radius of convergence**, and $\{z : |z - z_0| < R\}$ is the **disk of convergence**.

To find the disk of convergence we can often use the ratio test.

Example Find the radius of convergence for $f(z) = (1-z)^{-1}$ around $z_0 = 0$.

Recall that $f(z) =\sim_{n=0}^{\infty} z^n$, then we note that $\lim_{n\to\infty} |z^{n+1}/z^n| = |z|$, hence we have convergence if $|z| < 1$ by the ratio test, and the radius of convergence is $R = 1$.

Example For $f(z) = \sum_{n=0}^{\infty} n^2(z-\mathrm{i})^{2n}/2^n$ as a power series around $z_0 = \mathrm{i}$, by the ratio test,

$$\lim_{n\to\infty} \left| \frac{(n+1)^2(z-\mathrm{i})^{2n+2}}{2^{n+1}} \frac{2^n}{n^2(z-\mathrm{i})^{2n}} \right| = \frac{|z-\mathrm{i}|^2}{2},$$

so we have convergence if $|z - \mathrm{i}| < \sqrt{2}$, and the radius of convergence is $R = \sqrt{2}$.

Theorem 2.0.7 *If $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ has radius of convergence R and converges to $f(z)$ of its disk of convergence D, then $f(z)$ is complex differentiable, and $\sum_{n=0}^{\infty} a_n n(z-z_0)^{n-1}$ converges to $f'(z)$ in D.* □

Theorem 2.0.8 *If $\sum_{n=0}^{\infty} a_n(z-z_0)^n \to f(z)$ in its disk of convergence, then $f(z)$ is complex differentiable an infinite number of times, and $f^{(n)}(z_0) = n! a_n$.*

Proof By previous theorem, we have

$$\begin{aligned} f(z) &= a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots \\ f'(z) &= a_1 + 2a_2(z-z_0) + \dots \\ f''(z) &= 2 \cdot 1 a_2 + \dots \end{aligned}$$

and so on. Hence the function is infinitely complex differentiable, and $f^{(n)}(z_0)$ is as required. ■

Example Find the Taylor series of $(1-z)^{-2}$ about $z=0$.

We see that since $\mathrm{d}/\mathrm{d}z(1-z)^{-1} = (1-z)^{-2}$,

$$\frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} nz^{n-1}, \qquad |z| < 1.$$

Example Find the Taylor series for $\cosh(4z^3)$ about $z=0$.

Recall that

$$\cosh y = 1 + \frac{y^2}{2!} + \ldots = \sum_{n=0}^{\infty} \frac{y^{2n}}{(2n)!},$$

so that

$$\cosh(4z^3) = \sum_{n=0}^{\infty} \frac{(4z^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{16^n z^{6n}}{(2n)!}, \qquad \forall z \in \mathbb{C}.$$

Example Find the Taylor series of $z^3/(1-5z)^2$ about $z=0$.

Using the identity from two examples ago,

$$\frac{1}{(1-5z)^2} = \sum_{n=1}^{\infty} n(5z)^{n-1},$$

so that

$$\frac{z^3}{(1-5z)^2} = \sum_{n=1}^{\infty} n5^{n-1}z^{n+2}, \qquad |z| < \frac{1}{5}.$$

Example Find the Taylor series for $3z(z+1)^{-1}(z-2)^{-1}$ about $z=0$.

First note that the radius of convergence cannot be greater than 1. By partial fractions,

$$\frac{3z}{(z+1)(z-2)} = \frac{1}{z+1} + \frac{2}{z-2},$$

so that the Taylor series is

$$\sum_{n=0}^{\infty}(-z)^n - \sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = \sum_{n=0}^{\infty}\left[(-1)^n - \frac{1}{2^n}\right]z^n, \qquad |z| < 1.$$

3 Integration in the complex plane

Recall that in the real case we have the **indefinite integral** with

$$\int f(x)\,\mathrm{d}x = F(x),$$

where $F(x)$ is the primitive of f. We also have the **definite integral** where, by the fundamental theorem of calculus, gives

$$\int_b^a f(x)\,\mathrm{d}x = F(b) - F(a).$$

Although we can generalise the indefinite integral to the complex case, the definite integral doesn't generalise directly, because we are essentially trying to talk about a 2-dimensional surface in 4-space. So instead we integrate complex functions along curves, or contours, in the complex plane.

3.1 Curves in $\mathbb{C}$

A differentiable curve in $\mathbb{C}$ is a function $\gamma : [a,b] \to \mathbb{C}$ such that $\gamma(t) = \gamma_1(t) + \mathrm{i}\gamma_2(t)$, where γ_1 and γ_2 are real differentiable functions in t.

Example One way to generate the unit circle is with

$$\gamma : [0,1] \to \mathbb{C}, \qquad \gamma(t) = \mathrm{e}^{2\pi \mathrm{i} t}.$$

Notice here that γ is closed, and has a direction characterised by how t is parameterised (in this case it is in the positive sense, or in the anti-clockwise). In general, a circle centred at z_0 with radius r has the associated curve

$$\gamma : [0,1] \to \mathbb{C}, \qquad \gamma(t) = z_0 + r\mathrm{e}^{2\pi \mathrm{i} t}.$$

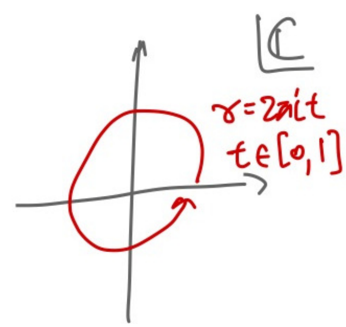

Figure 3.1: Unit circle transversed in the positive sense.

Example Consider two curves

$$\gamma(t) = t + \mathrm{i}t, \quad 0 \le t \le 1, \qquad \beta(t) = \begin{cases} t, & 0 \le t \le 1, \\ 1 + (t-1)\mathrm{i} & 1 \le t \le 2. \end{cases}$$

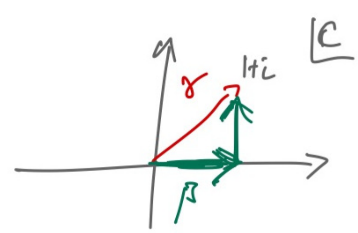

Figure 3.2: Two paths getting to the same point.

Both curves connect the origin to $z = 1 + \mathrm{i}$, but the path is different. $\beta(t)$ here is piecewise differentiable. One question of course is whether the path matters (see later). In general, a vector from z_0 to z_1 may be parameterised as $\gamma(t) = z_0 + t(z_1 - z_0)$, for $t \in [0, 1]$.

3.1.1 *Contour integrals*

To integrate along the curve $z = \gamma(t)$ with $t \in [a, b]$, we have from chain rule that $\mathrm{d}z = \gamma'(t)\,\mathrm{d}t$, so that

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t,$$

where the latter is as before since we are dealing with a function of a real variable.

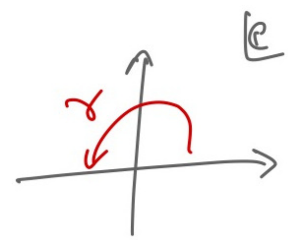

Figure 3.3: Upper semi-circle transversed in the positive sense.

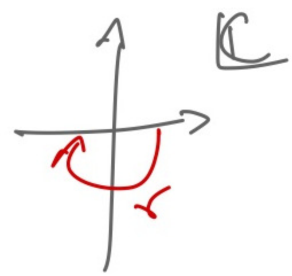

Figure 3.4: Lower semi-circle transversed in the *negative* sense.

Example Compute the contour integrals of the following:

1. $f(z) = z^2$, $\gamma(t) = \mathrm{e}^{\mathrm{i}\pi t}$, $t \in [0, 1]$
   The path is the upper unit semi-circle, and we have
   $$\int_\gamma f(z)\,\mathrm{d}z = \mathrm{i}\pi \int_0^1 \mathrm{e}^{3\mathrm{i}\pi t}\,\mathrm{d}t = -\frac{2}{3}.$$

2. $f(z) = z^2$, $\gamma(t) = \mathrm{e}^{-\mathrm{i}\pi t}$, $t \in [0, 1]$
   The path is the lower unit semi-circle, and we have
   $$\int_\gamma f(z)\,\mathrm{d}z = -\mathrm{i}\pi \int_0^1 \mathrm{e}^{-3\mathrm{i}\pi t}\,\mathrm{d}t = -\frac{2}{3}.$$
   Notice here the integral has the same value as the previous part, which in this case is not a coincidence.

3. $f(z) = \overline{z}$, $\gamma(t) = 1 + \mathrm{i}t$, $t \in [0, 2]$
   We have
   $$\int_\gamma f(z)\,\mathrm{d}z = \mathrm{i} \int_0^2 (1 - \mathrm{i}t)\,\mathrm{d}t = 2 + 2\mathrm{i}.$$

A **contour** is a continuous curve made up a finite number of differentiable curves. The contour itself does not need to be differentiable although the individual pieces should. The integral of $f(z)$ along a contour is then the sum of integrals along each individual differentiable curve.

Proposition 3.1.1 *We have the following properties for contour integrals:*

1. *Linearity, where*
   $$\int_\gamma (\alpha f(z) + \beta g(z))\,\mathrm{d}z = \alpha \int_\gamma f(z)\,\mathrm{d}z + \beta \int_\gamma g(z))\,\mathrm{d}z.$$

2. *If contours γ_1 and γ_2 have the same track in $\mathbb{C}$ and transverse it in the same direction, then*

$$\int_{\gamma_1} f(z)\ \mathrm{d}z = \int_{\gamma_2} f(z)\ \mathrm{d}z.$$

3. *If $\gamma : [a,b] \to \mathbb{C}$ and $\mu : [-b,-a] \to \mathbb{C}$ where $\mu(t) = \gamma(-t)$, i.e. μ is the 'reverse' of γ, then*

$$\int_{\mu} f(z)\ \mathrm{d}z = -\int_{\gamma} f(z)\ \mathrm{d}z.$$

4. *We have the inequality*

$$\left|\int_{\gamma} f(z)\ \mathrm{d}z\right| \leq \int_{\gamma} |f(\gamma(t))| \cdot |\gamma'(t)|\ \mathrm{d}t \leq \mathrm{length}(\gamma) \cdot \max_{\gamma} |f(\gamma(t))|.$$

5. *(Fundamental Theorem of Calculus) Let $F(z)$ be holomorphic on an open set $D \subset \mathbb{C}$, and $F'(z) = f(z)$. Then for any contour $\gamma : [a,b] \to D$ with end points $z_0 = \gamma(a)$ and $z_1 = \gamma(b)$, we have*

$$\int_{\gamma} f(z)\ \mathrm{d}z = F(z_1) - F(z_0).$$

Proof 1. Since we have linearity when the integrals are real, this one is just by definition:

$$\begin{aligned}
\int_{\gamma} (\alpha f(z) + \beta g(z))\ \mathrm{d}z &= \int_a^b [\alpha f(\gamma(t)) + \beta g(\gamma(t))]\gamma'(t)\ \mathrm{d}t \\
&= \alpha \int_a^b f(\gamma(t))\gamma'(t)\ \mathrm{d}t + \beta \int_a^b g(\gamma(t))\gamma'(t)\ \mathrm{d}t \\
&= \alpha \int_{\gamma} f(z)\ \mathrm{d}z + \beta \int_{\gamma} g(z))\ \mathrm{d}z.
\end{aligned}$$

2. Let $\gamma_k : [a_k, b_k] \to \mathbb{C}$ with $k = 1,2$, and assume that $\gamma_2(h(t)) = \gamma_1(t)$. Then taking a substitution $u = h(t)$ and judicious use of chain rule gives

$$\begin{aligned}
\int_{\gamma_2} f(z)\ \mathrm{d}z &= \int_{a_2}^{b_2} f(\gamma_2(u))\gamma_2'(u)\ \mathrm{d}u \\
&= \int_{a_1}^{b_1} f(\gamma_2(h(t)))\gamma_2'(h(t))h'(t)\ \mathrm{d}t \\
&= \int_{a_1}^{b_1} f(\gamma_1(t))\gamma_1'(t)\ \mathrm{d}t \\
&= \int_{\gamma_1} f(z)\ \mathrm{d}z.
\end{aligned}$$

3. As in previous case but use different limits.

4. Let $\theta = \arg \int_\gamma f(z)\ \mathrm{d}z$, then

$$\begin{aligned}
\left|\int_\gamma f(z)\ \mathrm{d}z\right| &= \mathrm{e}^{-\mathrm{i}\theta} \int_\gamma f(z)\ \mathrm{d}z \\
&= \int_\gamma \mathrm{e}^{-\mathrm{i}\theta} f(z)\ \mathrm{d}z \\
&= \mathrm{Re}\left(\int_\gamma \mathrm{e}^{-\mathrm{i}\theta} f(z)\ \mathrm{d}z\right) \\
&= \mathrm{Re}\left(\int_a^b \mathrm{e}^{-\mathrm{i}\theta} f(\gamma(t))\gamma'(t)\ \mathrm{d}t\right) \\
&\leq \int_a^b \left|\mathrm{e}^{-\mathrm{i}\theta} f(\gamma(t))\gamma'(t)\right|\ \mathrm{d}t \\
&= \int_a^b |f(\gamma(t))| \cdot |\gamma'(t)|\ \mathrm{d}t \\
&\leq \mathrm{length}(\gamma) \cdot \max_\gamma |f(\gamma(t))|.
\end{aligned}$$

5. Let $F(\gamma(t)) = u(t) + \mathrm{i}v(t)$, where u and v are real functions. By the chain rule, $u'(t) + \mathrm{i}v'(t) = F'(\gamma(t))\gamma'(t)$, so

$$\int_\gamma f(z)\ \mathrm{d}z = \int_a^b F'(\gamma(t))\gamma'(t)\ \mathrm{d}t = \int_a^b [u'(t) + \mathrm{i}v'(t)]\ \mathrm{d}t = F(b) - F(a).$$

Example Let $\gamma(t) = R\mathrm{e}^{\mathrm{i}t}$, $t \in [0, 2\pi]$, then

$$\int_\gamma \frac{1}{z}\ \mathrm{d}z = \int_0^{2\pi} \frac{1}{R\mathrm{e}^{\mathrm{i}t}} R\mathrm{i}\mathrm{e}^{\mathrm{i}t}\ \mathrm{d}t = 2\pi\mathrm{i},$$

and this is because the primitive is not well-defined at $z = 0$.

Theorem 3.1.2 (Path Independent Theorem) *Let f be continuous on an open connected set $D \subset \mathbb{C}$. Then the following statements are equivalent to each other:*

1. *integrals are path independent;*
2. *if γ is a closed curve in D, then $\oint_\gamma f(z)\ \mathrm{d}z = 0$;*
3. *there exists a primitive $F(z)$ of $f(z)$ where $F'(z) = f(z)$, defined globally on D.*

Proof We show that 1 is equivalent to 2, and 2 is equivalent to 3, so 1 is then equivalent to 3 by default.

($1 \Leftrightarrow 2$) Suppose Γ is a closed curve consisting of some arbitrary closed simple curves $\gamma_{0,1}$ as illustrated.

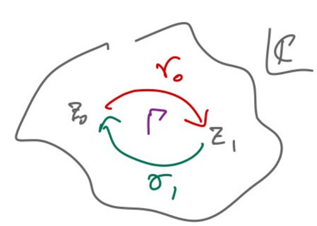

Figure 3.5: A joined path.

Then

$$\oint_\Gamma f(z)\ \mathrm{d}z = \left(\int_{\gamma_0} + \int_{-\gamma_1}\right) f(z)\ \mathrm{d}z = \left(\int_{\gamma_0} - \int_{\gamma_1}\right) f(z)\ \mathrm{d}z.$$

Since the integrals are path independent, we have $\int_\Gamma f(z)\ \mathrm{d}z = 0$. Conversely, if the integral is zero by assumption, since $\gamma_{0,1}$ are arbitrary, this implies path independence.

(2 ⇔ 3) Assuming there is a primitive, then the fundamental theorem of calculus implies that since we have the existence of the primitive, we have $\int_\gamma f(z)\ \mathrm{d}z = F(z_1) - F(z_0)$ regardless of path, so if $z_1 = z_0$ then $\oint_\gamma f(z)\ \mathrm{d}z = 0$.

Conversely, let z_0 be any fixed point on D, and z be any other point on D. Since D is open and connected, the contour γ joining z_0 to z exists. Defining then $F(z) = \int_\gamma f(\zeta)\ \mathrm{d}\zeta$, by the assumption of of path independence, $F(z)$ is well-defined, and by the estimation property, $F'(z) = f(z)$, so there exists a primitive. ■

3.1.2 *Cauchy's theorem and corollaries*

Cauchy's theorem is one of the centre pieces of complex analysis. Before the statement, we need an extra tool from topology regarding simple closed curves.

Theorem 3.1.3 (Jordan curve theorem) *Let γ be a simple closed contour, i.e. no self-intersections except at the end points. Then the compliment of γ in $\mathbb{C}$ is the disjoint union of exactly two sets, where exactly one is bounded.* □

Intuitively this says that a simple closed curve splits the space into an outside and inside (trivial as it may sound rigourously proofing this is not so obvious...)

Theorem 3.1.4 (Cauchy's theorem) *Let $f(z)$ be holomorphic on and inside a simple closed curve γ. Then $\oint_\gamma f(z)\ \mathrm{d}z = 0$.*

Proof Let $f = u + \mathrm{i}v$ for real u and v, then using Green's theorem (since the resulting integrands are real)

$$\begin{aligned}
\oint_\gamma f(z)\ \mathrm{d}z &= \oint_\gamma (u + \mathrm{i}v)(\mathrm{d}x + \mathrm{i}\mathrm{d}y) \\
&= \oint_\gamma [(u\ \mathrm{d}x - v\ \mathrm{d}y) + \mathrm{i}(u\ \mathrm{d}y + v\ \mathrm{d}x)] \\
&= \iint_A \left[\left(-\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + \mathrm{i}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\right] = 0.
\end{aligned}$$

The latter quality to zero is because $f(z)$ is holomorphic, so u and v satisfies the Cauchy–Riemann equations, and thus the partials are continuous and equal. ■

Example By Cauchy's theorem, $\oint_{|z|=1} z^n\ \mathrm{d}z = 0$ for all $n \geq 0$. On the other hand, Cauchy's theorem doesn't tell us anything if

$n < 0$ because $f(z)$ is then not holomorphic at $z = 0$. However, the above integral is zero for all $n \neq -1$ by the fundamental theorem of calculus since a primitive exists and is well-defined on the contour.

If γ is some contour not enclosing $z = 0$, then $\oint_\gamma z^n \, \mathrm{d}z = 0$ for all $n \geq 0$.

Example For $\oint_\gamma \exp[\cos z^3] \, \mathrm{d}z$, while we can't directly find a primitive of the integrand, since the integrand is a composite of holomorphic functions, the integrand is holomorhpic, and the integral is zero by Cauchy's theorem.

Example It can be shown that if the curve γ encloses $z = 0$, then $\oint_\gamma z^{-1} \, \mathrm{d}z = 2\pi\mathrm{i}$. Consider the curve as illustrated. If the gap between the 'cut' is small enough, the line integrals going into the unit circle cancels each other, giving

$$\oint_\mu \frac{1}{z} \, \mathrm{d}z = \oint_\gamma \frac{1}{z} \, \mathrm{d}z - \oint_{|z|=1} \frac{1}{z} \, \mathrm{d}z.$$

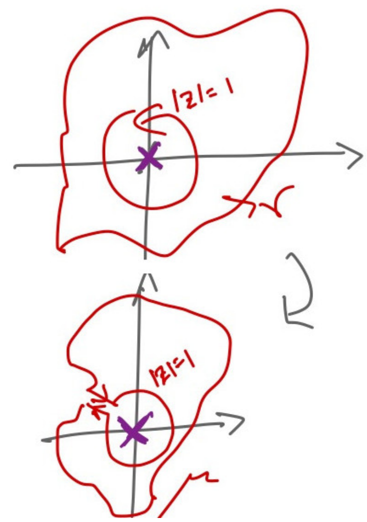

Figure 3.6: A keyhole curve.

By Cauchy's theorem, $\oint_\mu \frac{1}{z} \, \mathrm{d}z = 0$, so then

$$\oint_\gamma \frac{1}{z} \, \mathrm{d}z = \oint_{|z|=1} \frac{1}{z} \, \mathrm{d}z = 2\pi\mathrm{i}.$$

Note that Cauchy's theorem holding implies $\oint_\gamma f(z) \, \mathrm{d}z = 0$, which implies the primitive of $f(z)$ exists, and vice-versa. Although this of course does not mean we can write the primitive down in closed form.

Consider how many times a contour γ winds around a point z_0, as illustrated. More formally, let γ be a closed curve in $\mathbb{C}$ and $z_0 \in \mathbb{C}$ be a point not on γ. The **winding number** of γ with respect to z_0 is

$$I(\gamma; z_0) = \frac{1}{2\pi\mathrm{i}} \oint_\gamma \frac{\mathrm{d}z}{z - z_0}. \tag{3.1}$$

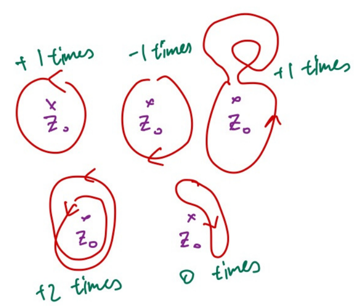

Figure 3.7: Illustrating the **winding number** of a curve around some point.

Theorem 3.1.5 *Let $\gamma : [a, b]$ be a (piece-wise) closed curve and z_0 not on γ. Then $I(\gamma; z_0) \in \mathbb{Z}$.*

Proof Consider.

$$g(t) = \int_a^t \frac{\gamma'(s)}{\gamma(s) - z_0} \, \mathrm{d}s.$$

At points where the integrand is continuous, by the fundamental theorem of calculus, we have

$$g'(t) = \frac{\gamma'(t)}{\gamma(t) - z_0} \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^{-g(t)}[\gamma(t) - z_0] = 0$$

for all t such that $g'(t)$ exists. Since the time-derivative of the continuous function is zero, the function is constant and equal to $\mathrm{e}^{-g(a)}[\gamma(a) - z_0]$. By a similar argument, we must have

$$\mathrm{e}^{-g(a)}[\gamma(a) - z_0] = \mathrm{e}^{-g(b)}[\gamma(b) - z_0],$$

but since $\gamma(a) = \gamma(b)$ as γ is closed, $e^{-g(a)} = e^{-g(b)}$. However, $g(a) = 0$, so $g(b) = 2\pi n i$ for $n \in \mathbb{Z}$, and so

$$I(\gamma; z_0) = \frac{1}{2\pi i} g(b) = n.$$

■

Theorem 3.1.6 (Cauchy's integral formula) *Let f be holomorphic on a region A which encloses γ, a closed curve in A (or,* homotopic *to a point). Let $z_0 \in A$ not be a point on γ, then*

$$f(z_0) I(\gamma; z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0} \, dz. \tag{3.2}$$

So if γ is in addition a simple closed curve, then

$$2\pi i f(z_0) = \oint_\gamma \frac{f(z)}{z - z_0} \, dz. \tag{3.3}$$

Proof Let

$$g(z) = \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0} & , z \neq z_0, \\ f'(z) & , z = z_0. \end{cases}$$

Since f is differentiable at z_0, it is continuous there. The function g is thus holomorphic maybe except at z_0, so by a version of Cauchy's theorem with z_0 deleted,

$$0 = \oint_\gamma g(z) \, dz = \oint_\gamma \frac{f(z)}{z - z_0} \, dz - \oint_\gamma \frac{f(z_0)}{z - z_0} \, dz,$$

so then

$$\oint_\gamma \frac{f(z)}{z - z_0} \, dz = f(z_0) \oint_\gamma \frac{1}{z - z_0} \, dz = f(z_0) I(\gamma; z_0) 2\pi i,$$

and the result follows. ■

Example Taking $z_0 = 0$, we have

$$\oint_{|z|=1} \frac{\cos z}{z} \, dz = 2\pi i \cos(0) = 2\pi i.$$

Example

$$\begin{aligned} \oint_{|z|=2} \frac{(z+1)\sin z}{(z-3)(z-1)} \, dz &= \oint_{|z|=2} \frac{(z+1)\sin z}{z-3} \frac{1}{z-1} \, dz \\ &= -2\pi i \sin 1. \end{aligned}$$

Example Making use of partial fractions, we have

$$\begin{aligned} \oint_{|z|=4} \frac{2e^z}{z^2 - 4z + 3} \, dz &= \oint_{|z|=4} \left(\frac{e^z}{z-3} - \frac{e^z}{z-1} \right) dz \\ &= 2\pi i (e^3 - 1). \end{aligned}$$

Note that, by defining $f(w) = (2\pi\mathrm{i})^{-1} \oint_\gamma f(z)/(z-w)\,\mathrm{d}z$ for all w inside the bounding curve γ, this shows that holomorphic functions $f(z)$ is completely determined inside γ by its value of the boundary curve γ. Additionally, note that if γ is the circle of radius $r > 0$ centred on w, then since we can parameterise the curve as $\gamma(t) = w + r\mathrm{e}^{\mathrm{i}t}$ for $t \in [0, 2\pi]$, so

$$\begin{aligned} f(w) = \frac{1}{2\pi\mathrm{i}} \oint_\gamma \frac{f(z)}{z-w}\,\mathrm{d}z &= \frac{1}{2\pi\mathrm{i}} \int_0^{2\pi} \frac{f(w + r\mathrm{e}^{\mathrm{i}t})}{w + r\mathrm{e}^{\mathrm{i}t} - w} \mathrm{i}r\mathrm{e}^{\mathrm{i}t}\,\mathrm{d}t \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(w + r\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}t, \end{aligned}$$

which is the average value of $f(z)$ on γ.

Taking $f(w)$ as above, and differentiating with respect to w, we get

$$\begin{aligned} f'(w) &= \frac{1}{2\pi\mathrm{i}} \oint_\gamma \frac{\mathrm{d}}{\mathrm{d}w} \frac{f(z)}{z-w}\,\mathrm{d}z \\ &= 1 \times \frac{1}{2\pi\mathrm{i}} \oint_\gamma \frac{f(z)}{(z-w)^2}\,\mathrm{d}z. \end{aligned}$$

It can be shown that, by induction,

$$f^{(n)} = \frac{n!}{2\pi\mathrm{i}} \oint_\gamma \frac{f(z)}{(z-w)^{n+1}}\,\mathrm{d}z. \tag{3.4}$$

Example For

$$\oint_{|z|=1} \frac{\mathrm{e}^{3z}\cos z}{z^2}\,\mathrm{d}z,$$

we note that we can define $f(z) = \mathrm{e}^{3z}\cos z$ and $w = 0$, which tells us

$$\begin{aligned} \oint_{|z|=1} \frac{\mathrm{e}^{3z}\cos z}{z^2}\,\mathrm{d}z &= \frac{2\pi\mathrm{i} f'(0)}{1!} \\ &= 2\pi\mathrm{i}\,(3\mathrm{e}^{3z}\cos z - \mathrm{e}^{3z}\sin z)\Big|_{z=0} \\ &= 6\pi\mathrm{i}. \end{aligned}$$

The following theorem is a converse to Cauchy's theorem.

Theorem 3.1.7 (Morera's theorem) *Let $f(z)$ be defined on an open subset $D \subset \mathbb{C}$ and is continuous in D. If $\oint_\gamma f(z)\,\mathrm{d}z = 0$ for all simple and closed γ in D, then $f(z)$ is holomorphic.*

Proof By the fundamental theorem of calculus, since $\oint_\gamma f(z)\,\mathrm{d}z = 0$ if and only if there exists a primitive $F'(z) = f(z)$ in D. By Cauchy's theorem for derivatives, $F(z)$ is twice differentiable (in fact infinitely differentiable) by Taylor's theorem (see next one), hence $f(z)$ itself is differentiable on D and therefore holomorphic. ■

Theorem 3.1.8 (Taylor's theorem) *Let $f(z)$ be holomorphic on $|z - z_0| < R$ (i.e. within the disk of convergence) for some $R > 0$. Then $f(z)$ is infinitely differentiable and*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n, \qquad |z - z_0| < R. \tag{3.5}$$

Proof By renaming the variable and choosing $z_0 = 0$ for simplicity, let

$$f(w) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} w^n.$$

By Cauchy's integral formula, for a curve γ within the disk of convergence, we have

$$\begin{aligned} f(w) &= \frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{f(z)}{z - w}\, \mathrm{d}z \\ &= \frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{f(z)}{z(1 - w/z)}\, \mathrm{d}z. \end{aligned}$$

The factor in the denominator is a sum of a geometric series, i.e.,

$$f(w) = \frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{f(z)}{z} \sum_{n=0}^{\infty} \left(\frac{w}{z}\right)^n \mathrm{d}z.$$

Since we are in the disk of convergence, we have uniform convergence, so we can swap the sum and integral, leading to

$$\begin{aligned} f(w) &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{n!}{2\pi \mathrm{i}} \oint_\gamma \frac{f(z)}{z} \left(\frac{w}{z}\right)^n \mathrm{d}z \\ &= \sum_{n=0}^{\infty} \frac{w^n}{n!} \frac{n!}{2\pi \mathrm{i}} \oint_\gamma \frac{f(z)}{z^{n+1}}\, \mathrm{d}z \\ &= \sum_{n=0}^{\infty} \frac{w^n}{n!} f^{(n)}(0) \end{aligned}$$

by Cauchy's formula for derivatives. Therefore $f(z)$ itself is clearly infinitely differentiable since it has a Taylor expansion within the disk of convergence. ∎

Corollary 3.1.9 *If $f(z)$ and $g(z)$ is holomorphic on some disc $D \subset \mathbb{C}$ and agree on some disk $K \subset D$, then they agree on the whole of D.*

Proof By Taylor's theorem, $f(z)$ and $g(z)$ are limits of the same power series, so $f(z) = g(z)$. ∎

3.2 *Residue theorem*

We say a function $h(z)$ has a **zero of order** r at $z = z_0$ if

$$h(z_0) = h'(z_0) = \ldots = h^{(r-1)}(z_0) = 0, \qquad h^{(r)}(z_0) \neq 0.$$

Example z^n clearly has a zero of order n at $z = 0$, which $z \sin z$ has zeroes of order 1 at $z = m\pi$ for $m \in \mathbb{Z} \setminus \{0\}$ and a zero of order 2 at $z = 0$.

Lemma 3.2.1 *If complex functions $h(z)$ and $m(z)$ have a zero of order r and s respectively at $z = z_0$, then $h(z)m(z)$ has a zero of order $r + s$ at $z = z_0$.* □

Try product rule followed by induction for example.

Lemma 3.2.2 *If $h(z)$ has a zero of order r at $z = z_0$, then $h(z) = (z - z_0)^r k(z)$ for some holomorphic function $k(z)$ where $k(z_0) \neq 0$.*

Proof By Taylor expansion,

$$
\begin{aligned}
h(z) &= \frac{h^{(r)}(z_0)}{r!}(z - z_0)^r + \frac{h^{(r+1)}(z_0)}{(r+1)!}(z - z_0)^{r+1} + \dots \\
&= (z - z_0)^r \left[\frac{h^{(r)}(z_0)}{r!} + \frac{h^{(r+1)}(z_0)}{(r+1)!} + \dots \right] \\
&= (z - z_0)^r k(z),
\end{aligned}
$$

with $k(z)$ defined according to the Taylor expansion, and it is clear that $k(z_0) \neq 0$ since the first term is the Taylor expansion is not zero. ■

Lemma 3.2.3 *Suppose we now have a function $f(z) = g(z)/h(z)$ where $g(z)$ is holomorphic at $z = z_0$, but $h(z_0) = 0$. If $h(z)$ has a zero of order r at $z = z_0$, then $f(z)$ may be written as*

$$
f(z) = \frac{c_{-r}}{(z - z_0)^r} + \frac{c_{-r+1}}{(z - z_0)^{r-1}} + \dots + c_0 + c_1(z - z_0) + \dots,
$$

with some constants c_i.

Proof By the previous lemma, since $h(z) = (z - z_0)k(z)$, then

$$
f(z) = \frac{1}{(z - z_0)^r} \left(\frac{g(z)}{k(z)} \right). \tag{3.6}
$$

Since $k(z_0) \neq 0$, $g(z)/h(z)$ is holomorphic, and can be represented as a power series about $z = z_0$ by Taylor's expansion. ■

The previous series is called a **Laurent series** of $f(z)$ about the singularity $z = z_0$. If $f(z)$ is represented by such a series, then z_0 is called a **pole** of $f(z)$, and the highest power of $1/(z - z_0)$ is called the **order** of the pole.

Example The function $f(z) = (3z - 1)/(z \sin z)$ has a pole of order 2 at $z = 0$, and poles of order 1 at $z = m\pi$, for $m \in \mathbb{Z}$.

On the other hand, the function $f(z) = \cos z/(z^4 \sin^2 z)$ has a pole of order 6 at $z = 0$ and poles of order 2 at $z = m\pi$ for $m \in \mathbb{Z} \setminus \{0\}$, since $\cos m\pi \neq 0$ for all $m \in \mathbb{Z}$.

In general, if $f(z) = g(z)/h(z)$, and if $g(z)$ has a zero of order r at $z = z_0$ but $h(z_0) \neq 0$, then $f(z)$ has a zero of order r at $z = z_0$. If $g(z)$ has a zero of order r and $h(z)$ has a zero of order s at $z = z_0$, and $s \leq r$, then z_0 is a **removable singularity**, in that the Laurent series of $f(z)$ about $z = z_0$ has no negative powers of $(z - z_0)$, so may be ignored. If on the other hand $r < s$, then $f(z)$ has a pole of order $s - r$ at $z = z_0$.

Lemma 3.2.4 *Assume γ is a simple and closed contour, and $f(z)$ has a singularity inside γ. Then by integrating its Laurent series we get, by Cauchy's theorem,*

$$\oint_\gamma f(z)\,\mathrm{d}z = 2\pi \mathrm{i} c_{-1}.$$

□

Here c_{-1} is called the **residue** of the pole of $f(z)$ at $z = z_0$, sometimes denoted as $\mathrm{Res}(f; z_0)$. There are two cases where the residues of $f(z)$ are relative easy to compute for:

- if $f(x) = g(z)/h(z)$, with $g(z_0) \neq 0$, and $f(z)$ has a pole of order 1 at $z = z_0$, then
$$\operatorname*{res}_{z \to z_0} f(z) = \frac{g(z_0)}{h'(z_0)}. \tag{3.7}$$
- For $f(z) = g(z)/(z - z_0)^r$ with $g(z_0) \neq 0$, then
$$\operatorname*{res}_{z \to z_0} f(z) = \frac{g^{(r-1)}(z_0)}{(r-1)!}. \tag{3.8}$$

Example For $f(z) = (z - 1)/\cos z$, $z_0 = \pi/2$ is a simple pole, and

$$\operatorname*{res}_{z \to \pi/2} f(z) = \frac{\pi/2 - 1}{-\sin \pi/2} = 1 - \frac{\pi}{2}.$$

Example For $f(z) = \sin z/(z - \pi/2)^3$, we have

$$\operatorname*{res}_{z \to \pi/2} f(z) = \frac{1}{2!} - \sin\frac{\pi}{2} = -\frac{1}{2}.$$

Theorem 3.2.5 (Residue theorem) *Let $f(z)$ be holomorphic on the simple closed curve γ and inside γ except at a finite amount of poles $z_1, \ldots z_n$. Then*

$$\oint_\gamma f(z)\,\mathrm{d}z = 2\pi\mathrm{i} \sum_{k=1}^{n} \operatorname*{res}_{z \to z_k} f(z). \tag{3.9}$$

Proof Consider something like the figure on the right. By using the previous lemma and Cauchy's theorem, we have

$$\oint_\gamma f(z)\,\mathrm{d}z = \sum_k \oint_{\gamma_k} f(z)\,\mathrm{d}z = 2\pi\mathrm{i} \sum_k \operatorname*{res}_{z \to z_k} f(z). \tag{3.10}$$

■

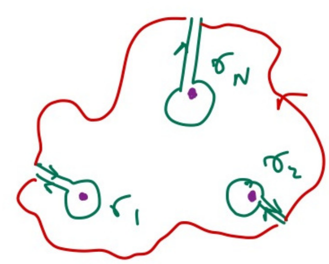

Figure 3.8: Curve with multiple keyhole cuts.

Example Compute

$$\oint_\gamma \frac{z-1}{(z^2+4)\sin z}\,\mathrm{d}z$$

where $z = 2\mathrm{i}$ and $z = 0$ lies inside γ, and γ is simple and closed.

Noting that $z^2 + 4 = (z+2\mathrm{i})(z-2\mathrm{i})$, we have two simple poles inside γ, with residues

$$\operatorname*{res}_{z\to 0} \frac{1}{\sin z}\frac{z-1}{z^2+4} = \frac{1}{\cos z}\frac{z-1}{z^2+4}\bigg|_{z=0} = -\frac{1}{4}$$

and

$$\operatorname*{res}_{z\to 2\mathrm{i}} \frac{z-1}{(z+2\mathrm{i})\sin z}\frac{1}{z-2\mathrm{i}} = \frac{z-1}{(z+2\mathrm{i})\sin z}\frac{1}{1!}\bigg|_{z=2\mathrm{i}} = \frac{1-2\mathrm{i}}{4\sinh 2},$$

so

$$\oint_\gamma \frac{z-1}{(z^2+4)\sin z}\,\mathrm{d}z = \frac{\pi\mathrm{i}}{2}\left(\frac{1-2\mathrm{i}}{\sinh 2} - 1\right).$$

3.3 *Applications for real integrals*

3.3.1 *Rational functions of* $\sin\theta$ *and* $\cos\theta$

Sometimes we have real integrals that we wish to evaluate that is actually quite easy to evaluate if we consider the integral as a complex integral.

Example Find

$$I = \int_0^{2\pi} \frac{\mathrm{d}\theta}{5 - 4\cos\theta}.$$

Let $z = \mathrm{e}^{\mathrm{i}\theta}$, so $\mathrm{d}z = \mathrm{i}z\,\mathrm{d}\theta$, $\cos\theta = (z+z^{-1})/2$. We have

$$I = \oint_{|z|=1} \frac{1}{\mathrm{i}z}\frac{\mathrm{d}z}{5-4(z+z^{-1})/2} = \oint_{|z|=1} \frac{-1}{\mathrm{i}}\frac{\mathrm{d}z}{5z-2z^2+2}.$$

Noting the denominator factorises to $(2z-1)(z-2)$, the only pole within the contour is $z = 1/2$, so by Cauchy's integral formula or the residue theorem, we have $I = (-2\pi\mathrm{i}/\mathrm{i})(1/2)(-3/2)^{-1} = 2\pi/3$.

Example What about for the above example if the integration domain is instead from 0 to π?

No new calculation really needs to be done here as we can rely on symmetries. Note that

$$\int_0^{\pi} \frac{\mathrm{d}\theta}{5-4\cos\theta} = \frac{1}{2}\int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{5-4\cos\theta}$$

since the integrand is an even function about $\theta = 0$. However, note also that

$$\frac{1}{2}\left(\int_{-\pi}^{0} + \int_0^{\pi}\right)\,\mathrm{d}\theta = \frac{1}{2}\left(\int_{\pi}^{2\pi} + \int_0^{\pi}\right)\,\mathrm{d}\theta$$

by linearity and periodicity, so the relevant integral evaluates to $\pi/3$.

3.3.2 *Some rational polynomial functions over the real line*

We could also consider some real integrals as segments of the analogous contour integral extended into complex space. The plan of attack is that we aim to compute the full contour integral (making use of Cauchy's theorem or the residue theorem accordingly), and generally aim to show that the segment that is extended into the complex plane goes to something with increasing distance from the real line, so the integral we actually wanted on the real line is then the difference between the full contour integral and the contour integral extended into the complex plane.

If the contour integral extended into the complex plane is zero as will be for the cases here, then the analogous real integral is just the full contour integral.

For this, we note the following two properties:

Proposition 3.3.1 • *(estimation) for all $\gamma : [a,b] \to \mathbb{C}$, we have*

$$\left| \int_\gamma f(z)\, \mathrm{d}z \right| \leq \|\gamma\| \max_\gamma |f(\gamma(t))| \tag{3.11}$$

• *(polynomial estimation) a general polynomial is bounded as*

Can show this by triangle inequality.

$$\frac{1}{2}|a_n z^n| \leq |a_n z^n + \ldots + a_0| \leq 2|a_n z^n|.$$

□

Example Find

$$I = \int_{-\infty}^{\infty} \frac{x+3}{(x^2+1)^2}\, \mathrm{d}x.$$

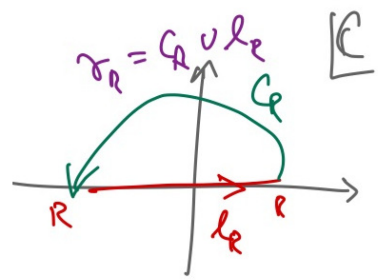

Figure 3.9: Semi-circle curve with extrusion into the complex plane, with $\gamma_R = C_R \cup \ell_R$.

Here we take $x \to z$ and consider the semi-circle contour $\gamma_R = \ell_R + C_R$ as in the figure to the right. The aim is to show that the semi-circle part C_R goes to zero as $R \to \infty$.

For $R > 1$, the residue is at $z = \mathrm{i}$, and we note that

$$\operatorname*{res}_{z\to\mathrm{i}} \frac{z+3}{(z-\mathrm{i})^2}\frac{1}{(z+\mathrm{i})^2} = \frac{1}{1!}\left(\frac{\mathrm{d}}{\mathrm{d}z}\frac{z+3}{(z+\mathrm{i})^2}\right)\bigg|_{z=\mathrm{i}} = \frac{3}{4\mathrm{i}},$$

so $\oint f(z)\, \mathrm{d}z = 3\pi/2$ by the residue theorem. Now, we have

$$|(z^2+1)^2| \geq \frac{1}{2}|z|^4, \qquad |z+3| \leq 2|z|,$$

so

$$\left|\int_{C_R} f(z)\, \mathrm{d}z\right| \leq \max_{C_R}\left|\frac{z+3}{(z^2+1)^2}\right| \pi R \leq \frac{2R}{R^4/2}\pi R = O\left(\frac{1}{R^2}\right),$$

thus the integral goes to zero as $R \to \infty$ by the absolute convergence theorem. By linearity of integrals,

$$\frac{3\pi}{2} = \oint_{\gamma_R} f(z)\, \mathrm{d}z = \left(\int_{\ell_R} + \int_{C_R}\right) f(z)\, \mathrm{d}z \to I,$$

so that

$$\int_{-\infty}^{\infty} \frac{x+3}{(x^2+1)^2}\, \mathrm{d}x = \frac{3\pi}{2}.$$

Example Find
$$I = \int_{-\infty}^{\infty} \frac{1}{x^2 - 2x + 5} \, \mathrm{d}x.$$

The function has a pole at $z_0 = 1 + 2\mathrm{i}$ (via the use of the quadratic formula for example), with residue $-\mathrm{i}/4$. Using the semi-circle contour as above, we have also
$$\left| \int_{C_R} f(z) \, \mathrm{d}z \right| = O\left(\frac{1}{R}\right),$$
so $I = 2\pi\mathrm{i}(-\mathrm{i}/4) = \pi/2$.

Example Find
$$I = \int_{-\infty}^{\infty} \frac{\cos x}{x^2 + 1} \, \mathrm{d}x.$$

We consider instead
$$I' = \mathrm{Re} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}x}}{x^2 + 1} \, \mathrm{d}x.$$
On the semi-circle extended into the complex plane, we have
$$\left| \int_{C_R} \frac{\mathrm{e}^{\mathrm{i}z}}{z^2 + 1} \, \mathrm{d}z \right| \leq |\mathrm{e}^{\mathrm{i}z}| O\left(\frac{1}{R}\right).$$
Since $|\mathrm{e}^{\mathrm{i}z}|$ is bounded by unity, the integral on C_R goes to zero as $R \to \infty$. On the other hand, the integrand has a simple pole at $z = \mathrm{i}$, where
$$\operatorname*{res}_{z \to \mathrm{i}} \frac{1}{z - \mathrm{i}} \frac{\mathrm{e}^{\mathrm{i}z}}{z + \mathrm{i}} = \left. \frac{\mathrm{e}^{\mathrm{i}z}}{z + \mathrm{i}} \frac{1}{1!} \right|_{z = \mathrm{i}} = \frac{\mathrm{i}}{2\mathrm{e}},$$
so $I' = \pi/\mathrm{e}$, which is purely real. So we have
$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + 1} \, \mathrm{d}x = \frac{\pi}{\mathrm{e}}, \qquad \int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 1} \, \mathrm{d}x = 0.$$

The sine one could have been anticipated since the integrand there is odd about $x = 0$.

If there are singularities on the real line itself, then we could in principle 'indent' the contour on the real line to bypass the singularities in the way.

Lemma 3.3.2 (Indentation lemma) *Let f have a simple pole at $z = a$, then*
$$\lim_{z \to a} \int_{C_R} f(z) \, \mathrm{d}z = \pi\mathrm{i} \operatorname*{res}_{z \to a} f(z), \tag{3.12}$$
where the pole is assumed to be bypassed in the positive (anti-clockwise) sense.

The $\pi\mathrm{i}$ factor comes from going halfway around the full circle, although to show this properly is non-trivial. This also only works for simple poles (essentially assumes small R and there is a Laurent expansions where terms can be bound accordingly).

□

Example Find the integral of the (unnormalised) **sinc function**
$$I = \int_{-\infty}^{\infty} \frac{\sin x}{x} \, \mathrm{d}x.$$

Here what we do is take an indented semi-circle contour as in the figure on the right, and compute each parts separately.

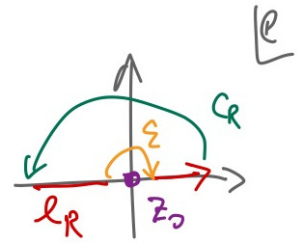

Figure 3.10: Curve with indentation in the *negative* sense around a pole, where $\gamma_R = C_R \cup \ell_R \cup \epsilon$.

Starting with the upper semi-circle arc C_R, the standard estimate tells us the integral on the semi-circle goes as $O(1)$ even if we bound $\sin x$ by unity, so we need to work a bit harder. Note that by doing integral by parts, we have

$$\int_{C_R} \frac{1}{z} \mathrm{e}^{\mathrm{i}z} \, \mathrm{d}z = \left[\frac{1}{z} \frac{\mathrm{e}^{\mathrm{i}z}}{\mathrm{i}} \right]_{-R}^{R} + \frac{1}{\mathrm{i}} \int_{C_R} \frac{1}{z^2} \mathrm{e}^{\mathrm{i}z} \, \mathrm{d}z.$$

Since the exponential terms are bounded by unity, both the boundary as well as the integral in this case behaves as $O(1/R^2)$ as $R \to \infty$, so we have what we need from the integral extruded into the complex plane.

For the indented part, we note here that $z = 0$ is a pole with residue 1, and noting that we are going clockwise *above* the pole thus giving as an extra minus sign, the integral is $-\pi\mathrm{i}$. Now, by Cauchy's theorem, the closed integral over the indented semi-circle since there are no singularities within the contour, and so we have

$$\begin{aligned} 0 = \oint_{\gamma_R} f(z) \, \mathrm{d}z &= \left(\int_{\epsilon} + \int_{C_R} + \int_{-R}^{R} \right) f(z) \, \mathrm{d}z \\ &\to -\pi\mathrm{i} + 0 + \int_{-\infty}^{\infty} f(z) \, \mathrm{d}z \end{aligned}$$

as $R \to \infty$. Together, we have

$$I = \mathrm{Im} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}z}}{z} \, \mathrm{d}z = \mathrm{Im}(\pi\mathrm{i}) = \pi.$$

4 More analysis topics

This chapter contains a collection of results that utilises some previous integration techniques.

4.1 Some theorems

4.1.1 Liouville's theorem

Theorem 4.1.1 *A bounded entire function is constant.*

Proof Applying Cauchy's integral formula, we get

$$f(a) - f(b) = \frac{1}{2\pi \mathrm{i}} \oint_\gamma \left(\frac{f(z)}{z-a} - \frac{f(z)}{z-b} \right) \mathrm{d}z,$$

with γ chosen such that γ is a circle of radius R centred on b, with R large enough to contain a. We see then

$$f(a) - f(b) = \frac{a-b}{2\pi \mathrm{i}} \oint_\gamma \frac{f(z)}{(z-a)(z-b)} \mathrm{d}z.$$

Since $z \in \gamma$, we note that $|z-b| = R$, while

$$|z-a| = |z-b+b-a| \geq |z-b| - |b-a| \geq \frac{R}{2}.$$

Thus we have (since $f(z) \leq K$ is assumed to be bounded)

$$\begin{aligned}
|f(a) - f(b)| &= \left| \frac{b-a}{2\pi \mathrm{i}} \right| \cdot \left| \oint_\gamma \frac{f(z)}{(z-a)(z-b)} \mathrm{d}z \right| \\
&\leq \frac{|b-a|}{2\pi} \oint_\gamma \frac{K}{R \cdot R/2} |\mathrm{d}z| \\
&= \frac{|b-a|}{2\pi} \frac{2K}{R^2} 2\pi R \\
&\to 0
\end{aligned}$$

as $R \to \infty$, hence $f(a) = f(b)$, i.e. a constant entire function. ■

4.1.2 Cauchy's inequality

Lemma 4.1.2 *Let $f : D(a;r) \subset \mathbb{C} \to \mathbb{C}$ be holomorphic, then*

$$\left| f^{(n)}(a) \right| \leq K \frac{n!}{r^n}, \tag{4.1}$$

where $D(a;r) = \{z \in \mathbb{C} \mid |z-a| < r\}$ is the disk centred at $z = a$ of radius r, for $n = 0, 1, \ldots,$ and such that $|f(z)| \leq K$ for $z \in D(a;r)$.

Proof By Cauchy's integral formula, we have

$$f^{(n)}(a) = \frac{n!}{2\pi \mathrm{i}} \oint_{\partial D(a;r)} \frac{f(z)}{(z-a)^{n+1}} \, \mathrm{d}z,$$

so that

$$\left| f^{(n)}(a) \right| \leq \frac{n!}{2\pi} \frac{K}{r^{n+1}} 2\pi r = K \frac{n!}{r^n}.$$

■

Corollary 4.1.3 *Applying Cauchy's inequality to some $a \in \mathbb{C}$ we see that $|f'(a)| \to 0$ as $R \to \infty$, showing that f is an entire constant function, and provides a simpler proof to Liouville's theorem.*

4.1.3 Fundamental theorem of algebra

Theorem 4.1.4 *A polynomial of degree $n \geq 1$ has n roots in $\mathbb{C}$ if the leading coefficient is non-zero.*

Proof We suppose otherwise, and that the polynomial $p(z)$ has no roots at all in $\mathbb{C}$. So then since $p(z) \neq 0$, $p^{-1}(z)$ is holomorphic on $\mathbb{C}$. The leading coefficient of $p(z)$ is non-zero, so $f(z) = p^{-1}(z)$ is not constant. So, by triangle inequality,

$$|p(z)| \geq |a_n||z|^n - |a_0| - |a_1||z| - \ldots |a_{n-1}||z|^{n-1}.$$

Let

$$a = \sum_{i=0}^{n-1} |a_i|,$$

then for $|z| > 1$, we have

$$|p(z)| \geq |z|^{n-1} \left(|a_n||z| - \frac{|a_0|}{|z|^{n-1}} - \ldots \frac{|a_{-1}|}{1} \right) \geq |z|^{n-1} \left(|a_n||z| - a \right).$$

Let $K = \max\{1, (M+a)/|a_n|\}$. We see that if $|z| > K$ then $|p(z)| > M$, and hence $|p(z)|^{-1} < M^{-1}$.

But for $|z| \leq K$, $f(z) = p^{-1}(z)$ is bounded in the absolute value because it is holomorphic, hence it is continuous. If this bound is L, then for all z, $|f(z)| < \max\{M^{-1}, L\}$ and hence we have a bounded entire function, which is a contradiction according to Liouville's theorem. Thus $p(z)$ has a root in $\mathbb{C}$, and we factorise the root out and continue by induction until the remaining polynomial has zero degree, i.e. a constant. ■

4.1.4 Maximum principle

Theorem 4.1.5 (Maximum modulus principle) *Let $f : U \subset \mathbb{C} \to \mathbb{C}$ be holomorphic with ∂U a simple closed curve. Assuming f is non-constant and that $|f(z_0)| = \max_{z\in\overline{U}} |f(z)|$, then $z_0 \in \partial U$, i.e. occurs on the boundary.* □

Example Find all $f : D(0,1) \to \mathbb{C}$ that are holomorphic and satisfy $f(0) = 1$, $|f| \geq 1$ for all $z \in D(0,1)$.

The inequality is the wrong way round so we consider $g = 1/f$. Note that $g(0) = 1$, g is holomorphic, and $|g| \leq 1$ in the unit disk by construction. Then assuming g is non-constant, by the maximum principle we have

$$1 = |g(0)| < \max_{\partial D} |f| \leq \max_{D} |f| = 1,$$

which gives a contradiction, so g and thus f is a constant, which in this case means $f(z) \equiv 1$.

Example Find the maximum value of $|f(z)|$ in the unit disk for $f(z) = z^2 - 3z + 2$.

f is holomorphic on the unit disk and the boundary is simple and closed, so by the maximum principle, for $z \in \overline{D(0,1)}$,

$$\begin{aligned} |f(z)| \leq \max_{|w|\leq 1} |f(w)| &= \max_{\theta\in[0,2\pi]} \left|f(e^{i\theta})\right| \\ &\max_{\theta\in[0,2\pi]} \sqrt{(e^{2i\theta} - 3e^{i\theta} + 2)e^{-2i\theta} - 3e^{-i\theta} + 2)} \\ &= \max_{\theta\in[0,2\pi]} \sqrt{14 - 18\cos\theta + 4\cos 2\theta}. \end{aligned}$$

The extrema are at $\theta = 0, \pi$, and the latter turns out to be the maximiser, and $|f(z)| \leq 6$.

4.1.5 Argument theorem

Theorem 4.1.6 (Argument theorem) *Let $f : U \subset \mathbb{C} \to \mathbb{C}$ be holomorphic on U $\{w_1, \ldots w_\ell\}$ and poles of order $p_i \in \mathbb{N}$ at w_j, and assume that ∂U is a simple closed curve. Let $z_1, \ldots z_m \in U$ be the zeroes of f with order n_j. If*

e.g. keyhole indented curves.

$$\sum_1^{\ell} p_j = P, \quad \sum_1^{m} n_j = N,$$

then

$$\frac{1}{2\pi i} \oint_{\partial U} \frac{f'}{f} \, dz = N - P. \tag{4.2}$$

Proof We see that

$$\frac{1}{2\pi i} \oint_{\gamma_j} \frac{f'}{f} \, dz = -p_j, \quad \frac{1}{2\pi i} \oint_{\Gamma_j} \frac{f'}{f} \, dz = n_j,$$

where γ_j are the indented curves circling the poles and Γ_j are the indented curves circling the zeroestransversing in the negative sense. Then, constructing the curve that is the union of ∂U, Γ_j and γ_j such that f'/f contain no zeros or poles, we have by Cauchy's theorem (and transversing the indented curves in the negative sense)

Both following on from residue theorem.

$$\begin{aligned}0 &= \frac{1}{2\pi i}\oint_{\partial U}\frac{f'}{f}\,dz + \frac{1}{2\pi i}\oint_{\gamma_j}\frac{f'}{f} + \frac{1}{2\pi i}\oint_{\Gamma_j}\frac{f'}{f}\,dz \\ &= \frac{1}{2\pi i}\oint_{\partial U}\frac{f'}{f}\,dz + \sum_1^m(-n_j) + \sum_1^\ell p_j,\end{aligned}$$

from which the result follows. ■

Example Let $f = z^5 - 3iz^3 + z - 20$ and let γ be a curve which encloses all zeroes of f. Evaluate

$$I = \oint_\gamma \frac{5z^4 - 2iz^2 + 1}{z^5 - 3iz^3 + z - 20}\,dz.$$

By the argument theorem, note that f has five zeroes and no poles, so $I = 2\pi i(5-0) = 10\pi zi$.

Following on from the argument theorem, the local version is as follows. Let $f : U \subset \mathbb{C} \to \mathbb{C}$ be holomorphic and let f have a zero of order $n \ \geq 1$ at $a \in U$. Taylor's theorem says that

$$f(z) = \sum_{k=n}^{\infty} a_k(z-a)^k = (z-a)^n \sum_{k=n}^{\infty} a_k(z-a)^{k-n} = (z-a)^n g(z),$$

where $g(z)$ is holomorphic and $g(a) = a_n \neq 0$ by assumption. We see that we have

$$\frac{f'}{f} = \frac{d}{dz}\log f(z) = \frac{d}{dz}\log(z-a)^n + \frac{d}{dz}\log g(z) = \frac{n}{z-a} + \frac{g'}{g}.$$

Since $g(a) \neq 0$, we have g holomorphic in some $D(a,r)$ for small r. So then

$$\frac{1}{2\pi i}\oint_{\partial D}\frac{f'}{f}\,dz = \frac{1}{2\pi i}\oint_{\partial D}\left(\frac{n}{z-a} + \frac{g'}{g}\right)\,dz = n$$

by Residue theorem and Cauchy's theorem. Hence if f has a zero of order n at $z = a$, then

$$\oint_{\partial D}\frac{f'}{f}\,dz = 2\pi i n. \tag{4.3}$$

By a similar argument, if f has a pole of order $p \geq 1$ at $z = b$, then the Laurent series will be

$$f(z) = \sum_{k=-p}^{\infty} a_k(z-b)^k = (z-b)^{-p}\sum_{k=-p}^{\infty} a_k(z+p)^{k-n} = (z-b)^{-p}g(z),$$

with $a_{-p} \neq 0$, so that

$$\oint_{\partial D}\frac{f'}{f}\,dz = -2\pi i p. \tag{4.4}$$

4.1.6 Rouche's theorem

Theorem 4.1.7 *Let $f, g : U \subset \mathbb{C} \to \mathbb{C}$ be holomorphic and ∂U be simple and closed. If $|g| < |f|$ on ∂U, then $f(z)$ and $f(z) + g(z)$ have the same number of solutions in U.*

From Wikipedia: if a person were to walk a dog on a leash around and around a tree, such that the distance between the person and the tree is always greater than the length of the leash, then the person and the dog go around the tree the same number of times.

Proof Consider $F(z) = g(z)/f(z)$, and let N_1 and N_2 be the number of zeroes of $f + g$ and f in U respectively. By the argument theorem, we have

$$N_1 = \oint_{\partial U} \frac{f' + g'}{f + g}\,\mathrm{d}z, \quad N_2 = \oint_{\partial U} \frac{f'}{f}\,\mathrm{d}z,$$

and there are no poles since the functions are holomorphic. We note, then

$$\begin{aligned} N_1 - N_2 &= \frac{1}{2\pi \mathrm{i}}\left[\oint_{\partial U} \frac{f' + g'F + fF'}{f + fF}\,\mathrm{d}z - \oint_{\partial U} \frac{f'}{f}\,\mathrm{d}z\right] \\ &= \frac{1}{2\pi \mathrm{i}} \oint_{\partial U} \frac{F'}{f(1+F)}\,\mathrm{d}z. \end{aligned}$$

Since $|F| = |g/f| < 1$ by assumption, we can Taylor expand and obtain

$$N_1 - N_2 = \frac{1}{2\pi \mathrm{i}} \oint_{\partial U} F'(1 - F + F^2 + \ldots)\,\mathrm{d}z.$$

Notice however that $F'F^k \sim (F^{k+1})'$ for sensible values of integer k, so by fundamental theorem of calculus, we conclude that the integral is zero, so that $N_1 = N_2$. ∎

Alternate proof of the Fundamental Theorem of Algebra We can actually use Rouche's theorem. Let $f = a_n z^n$ and $g = \sum_i^{n-1} a_i z^i$, and make use of the fact that f has n zeroes in the domain (the roots of unity). Then on $|z| = r > 1$, we have

$$\left|\frac{g}{z}\right| = \frac{|\sum a_i z^i|}{|a_n z^n|} \leq \frac{\sum_i |a_i| r^i}{|a_n| r^n} \leq \frac{\sum_i |a_i| r^{n-1}}{|a_n| r^n}$$

since $r > 1$. So since $|g/f| \to 0$ as $r \to \infty$, we can invoke Rouche's theorem, so that $f + g = \sum_i^n a_i z^i$ has the same amount of zeroes as $f = a_n z^n$ within $\mathbb{C}$. ∎

Example Show that all roots of $z^7 - 5z^3 + 12 = 0$ lie in the annulus $1 \leq |z| < 2$.

We aim to show there are no roots in the unit circle and seven roots in the circle of radius two, so by fundamental theorem of calculus we account for all the roots within the annulus. Let $f = 12$ and $g =^7 -5z^3$ on $|z| < 1$. For this choice we have

$$|f| = 12, \quad |g| = |z^7 + 5z^3| < |z|^7 + 5|z|^3 = 6,$$

so $|f| > |g|$, and since f has no roots in the unit circle, neither does $f + g$.

Within the circle of radius two, we let $f = z^7$ and $g = 12 - 5z^3$. Then

$$|f| = 128, \quad |g| = |12 - 5z^3| < 12 + 5|z|^3 = 52,$$

so $|f| > |g|$, and since f has seven roots here, so does $f + g$. By fundamental theorem of algebra, $f + g$ can only have seven roots, so all roots occur in the target annulus.

4.1.7 *Zeroes of holomorphic functions*

Let $A \subset \mathbb{C}$ and $a \in A$, then a is **isolated** in A if there exists some $\epsilon > 0$ such that the intersection of the disk $D(a, \epsilon)$ with A is simply the point a.

Theorem 4.1.8 *Let $f : U \subset \mathbb{C} \to \mathbb{C}$ be holomorphic and denoting $Z(f)$ to be the set of $z \in U$ that are zeroes of f. Assuming $f(z)$ is not identically zero, then $Z(f)$ is discrete.* □

Or, there are no 'lines' or 'areas' of zeroes for non-trivial functions.

Theorem 4.1.9 (Uniqueness theorem) *Let $f : U \subset \mathbb{C} \to \mathbb{C}$ be holomorphic and U is connected. If $Z(f)$ is not discrete, then $f(z) \equiv 0$.* □

Example Find all holomorphic functions $f : D(0, 1) \to \mathbb{C}$ where $f(1/n) = 1/n$ for $n \in \mathbb{N}$.

Consider the holomorphic function $g(z) = f(z) - z$, then clearly $\{0, 1/n\} \subseteq Z(g)$ by construction. However, $z = 0$ is no isolated since $1/n \to 0$ for $n \to \infty$, so $g(z) \equiv 0$, and so $f(z) = z$ is the only choice.

4.2 *Pointwise and uniform convergence*

A sequence of functions $f_j(z)$ for $z \in U$ **converges pointwise** to $f(z)$ if for all $\epsilon > 0$, there exists some N where for all $N > j$, we have $|f_j(z) - f(z)| < \epsilon$, where ϵ can be a function of z.

On the other hand, $f_j(z)$ **converges uniformly** to $f(z)$ if ϵ is independent of the choice of z, or, more formally, we have instead

$$\sup_{z \in U} |f_j(z) - f(z)| < \epsilon, \tag{4.5}$$

i.e. the supremum tends to zero, rather than just the differences tend to zero. Note that uniform convergence implies pointwise convergence but not vice-versa.

Example $f_n = z^n$ for $|z| < 1$ converges pointwise to $f(z) = 0$ but not uniformly, because the points with smaller modulus goes to zero 'faster'. To formalise this, we note that

$$|f_n(z) - f(z)| = |z|^n \to 0 \quad \text{as} \quad n \to \infty$$

so we have pointwise convergence. However, taking $z_n = (1/2)^{1/n}$,

$$\max_{|z|<1} |f_n(z) - f(z)| = \max_{|z|<1} |f_n(z)| \geq |f_n(z_n)| = \frac{1}{2},$$

which does not tend to zero as $n \to \infty$, and we do not have uniform convergence.

Note that if we consider $|z| \leq 1$ then we don't even have pointwise convergence.

A sum $\sum_{j=0}^{\infty} f_j(z)$ converges uniformly to $S(z)$ if $s_n(z) = \sum_{j=1}^{n} f_j(z) \to s(z)$ as $n \to \infty$.

Theorem 4.2.1 *Let $f_j(z)$ converge uniformly to $f(z)$, then*

1. *$f_j(z)$ continuous for all j implies $f(z)$ is continuous*
2. $\int_\gamma \lim_{j\to\infty} f_j \, \mathrm{d}z = \lim_{j\to\infty} \int_\gamma f_j \, \mathrm{d}z$
3. *$f_j(z)$ holomorphic for all j implies f is holomorphic.*

Proof Only doing the first one. Since sequences converges uniformly, we have for all $\epsilon > 0$, there exists N such that for all $n \geq N$, $|f_n(z) - f(z)| < \epsilon/3$. Choosing some $n_1 \geq N$, since f_{n_1} is continuous by assumption, we have that there exists some $\delta > 0$ for all $\epsilon/3$ where

$$|z - z_1| < \delta \quad \Rightarrow \quad |f_n(z) - f_{n_1}(z_1)| < \frac{\epsilon}{3}.$$

So then

$$\begin{aligned} |f(z) - f(z_1)| &= |f(z) - f_{n_1}(z) + f_{n_1}(z) - f_{n_1}(z_1) + f_{n_1}(z_1) - f(z_1)| \\ &\leq |f(z) - f_{n_1}(z)| + |f_{n_1}(z) - f_{n_1}(z_1)| + |f_{n_1}(z_1) - f(z_1)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

for all $|z - z_1| < \delta$, and we have continuity of f. □

Theorem 4.2.2 *Let $\sum_{j=1}^{\infty} f_j(z)$ converge uniformly to $S(z)$, then*

$$\int_\gamma \sum_{j=1}^{\infty} f_j \, \mathrm{d}z = \sum_{j=1}^{\infty} \int_\gamma f_j \, \mathrm{d}z$$

and

$$\frac{\mathrm{d}}{\mathrm{d}z} \sum_{j=1}^{\infty} f_j = \sum_{j=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}z} f_j.$$

So with uniform convergence we are allowed to swap the ordering of limits and other operations, which is not an obvious property. Compare this with absolute convergence.

The below provides a useful test for uniform convergence for series.

Theorem 4.2.3 (Weierstrauss M-test) *Assume* $|f_n(z)| \leq M_n$ *for all* $z \in U$ *and* $\sum_{n=1}^{\infty} M_n \leq \infty$, *then* $\sum_{n=1}^{\infty} f_n(z)$ *converges uniformly.*

Proof Assuming the sum of M_n converges so for all $\epsilon > 0$, then by Cauchy's criterion there exists some N where for $n \geq N$,

$$\sum_{k=n+1}^{m} M_k < \epsilon$$

for all choices of $m > n > N$. Thus, denoting $S_m(z)$ as the partial sums, for $n \geq N$, we have

$$|S_m(z) - S_n(z)| = \left| \sum_{k=n+1}^{m} f_k(z) \right| \leq \sum_{k=n+1}^{m} |f_k(z)|$$

by the triangle inequality. Further, by assumption,

$$\sum_{k=n+1}^{m} |f_k(z)| \leq \sum_{k=n+1}^{n+p} M_k < \epsilon.$$

So since

$$|S_m(z) - S_n(z)| < \epsilon,$$

S_m is a Cauchy sequence, and by completeness of $\mathbb{C}$ we have

$$|S(z) - S_n(z)| = | \lim_{m\to\infty} S_m(z) - S_n(z)| = \lim_{m\to\infty} |S_m(z) - S_n(z)| < \epsilon.$$

The choice of N is independent of choice of z, we have uniform convergence. ∎

Example 1. Find the domain U where

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!} z^n$$

converges uniformly.

We have

$$\left| \frac{(-1)^n}{(2n+1)!} z^n \right| = \frac{|z|^n}{(2n-1)!} \leq \frac{|z|^n}{n!} = M_n.$$

Now,

$$\sum_{n=1}^{\infty} M_n = \sum_{n=0}^{\infty} \frac{|z|^n}{n!} - 1 = \mathrm{e}^{|z|} - 1 < \infty$$

for all $z \in \mathbb{C}$, so the above series converges uniformly for all $z \in \mathbb{C}$ by the Weierstrauss M-test.

2. Show that

$$\sum_{n=0}^{\infty} \frac{1}{n+1} \frac{1}{z^n}$$

is uniformly convergent on $|z| \leq R > 1$.

We have

$$\left|\frac{1}{n+1}\frac{1}{z^n}\right| = \frac{1}{n+1}\frac{1}{|z|^n} \leq \frac{1}{n+1}\frac{1}{R^n} \leq \frac{1}{R^n}.$$

Since sum of $1/R^n$ converges for $R > 1$, the above function is uniformly convergent on $|z| \leq R > 1$ by the Weierstrauss M-test.

3. Prove that

$$\sum_{n=0}^{\infty} \frac{1}{n^2+z^2}$$

converges uniformly on $U = \{z : 1 < |z| < 2\}$.

Note that by the reverse triangle inequality,

$$|n^2 + z^2| \geq n^2 - |z|^2 \geq n^2 - 4 \geq \frac{n^2}{2}$$

for $n \geq 3$. We can patch up the shortfall by noting that

$$\frac{1}{z^2} + \frac{1}{1+z^2} + \frac{1}{4+z^2}$$

is holomorphic on U and hence finite, so that

$$\sum_{n=0}^{\infty} \frac{1}{n^2+z^2} \leq \frac{1}{z^2} + \frac{1}{1+z^2} + \frac{1}{4+z^2} + \sum_{n=3}^{\infty} \frac{2}{n^2} = \sum_n M_n < \infty,$$

so by Weierstrauss M-test the above function is uniformly convergent on U.

5 Conformal mapping and harmonic functions

5.1 Conformal mapping

A mapping $f : A \to B$ is called **conformal** if for each $z_0 \in A$, f rotates tangent vectors of curves through z_0 by a definite angle θ and a stretch factor r. A conformal map preserves angles.

Example We have a few simple examples of conform maps:

- *translations* $f(z) = z + \alpha$, $\alpha \in \mathbb{C}$
- *rotations* $f(z) = \mathrm{e}^{\mathrm{i}\theta}$, $\theta \in \mathbb{R}$
- *stretching* $f(z) = \beta z$, $\beta \in \mathbb{R}^+$
- *inversion* $f(z) = 1/z$

Theorem 5.1.1 *Let $f : U \to \mathbb{C}$ be holomorphic, then f is conformal if and only if and $f'(z) \neq 0$ for all $z \in U$.*

Proof We see that, by definition, for any curve $\gamma(t) \in U$ with $\gamma(0) = z_0$ where $\gamma'(0) \neq 0$, the transformed curve $\sigma(t) = f(\gamma(t))$ is differentiable at $t = 0$. Let $u = \sigma'(0)$ and $v = \gamma'(0)$, we have, assuming f is conformal,

$$|u| = r|v|, \qquad \arg v + \theta \pmod{2\pi} = \arg u.$$

By chain rule, $u = \sigma'(0) = f'(z_0)\gamma'(0) = f'(z_0)v$, which implies

$$r = |f'(z_0)|, \qquad \theta = \arg f'(z_0),$$

which for non-zero curves is only true if $f'(z_0) \neq 0$. ■

The aforementioned maps are conformal (with special circumstances for the inversion map).

Theorem 5.1.2 (Inverse function theorem) *Let $f : U \to \mathbb{C}$ be conformal and ∂U be simple and closed. f by definition is invertible, and $f^{-1} : \mathrm{Im}(f) \to U$ is such that f^{-1} is also conformal.* □

Theorem 5.1.3 *All conformal maps preserve the angles of two curves at their respective intersections.*

Proof for a conformal map f, suppose curve τ_j is mapped to γ_j. Then by the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma_j = \frac{\mathrm{d}}{\mathrm{d}t}f(\tau_j) = \frac{\mathrm{d}f}{\mathrm{d}z}\frac{\mathrm{d}\tau_j}{\mathrm{d}t}.$$

Since f is conformal, $\mathrm{d}f/\mathrm{d}z \in \mathbb{C} \setminus \{0\}$, so we have either a rotation or dilation, but the angle of intersection is preserved.

5.1.1 Möbius transformations

A **Möbius transformation** is a map of the form

$$T(z) = \frac{az+b}{cz+d} \tag{5.1}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

Note $ad - bc$ looking like the determinant of a 2×2 matrix is not a coincidence.

Theorem 5.1.4 *We have the following properties for Möbius maps:*

- *$T(z)$ is holomorphic for $z \neq -d/c$*
- *$T'(z)$ is not zero for $z \neq -d/c$*
- *for every $T(z)$ there exists an inverse $T^{-1}(z)$ that is also a Möbius map*
- *the **cross ratios***

$$[z_1, z_2 : z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} \tag{5.2}$$

are invariant under Möbius transformations, i.e.

$$[z_1, z_2 : z_3, z_4] = [T(z_1), T(z_2) : T(z_3), T(z_4)]$$

So Möbius maps are conformal except at $z = d/c$.

Related to a non-vanishing determinant.

□

Example Find a Möbius transformation T which maps $z = 0, -\mathrm{i}, -1$ to $w = \mathrm{i}, 1 0$ respectively.

There are two ways to do this:

- shove the relevant values in to obtain

$$\mathrm{i} = \frac{b}{d}, \quad 1 = \frac{-a\mathrm{i}+b}{-c\mathrm{i}+d}, \quad 0 = -a+b,$$

which results in

$$a = b, \quad d = -\mathrm{i}a, \quad c = \mathrm{i}a,$$

from which

$$T(z) = \frac{-\mathrm{i}z - \mathrm{i}}{z-1}.$$

- by invariance of the cross ratio, we have

$$\frac{(z+\mathrm{i})(0+1)}{(z+\mathrm{i})(0+\mathrm{i})} = \frac{(w-1)(\mathrm{i}-0)}{(w-0)(\mathrm{i}-1)},$$

  which upon making w the argument results in

$$w = T(z) = \frac{-\mathrm{i}z - \mathrm{i}}{z-1}.$$

Example Let $z_0 \in \mathbb{C}$, $\mathrm{Im} z_0 > 0$, and $\theta_0 \in [0, 2\pi]$. Show that the Möbius transformation

$$T(z) = \mathrm{e}^{\mathrm{i}\theta_0} \frac{z - z_0}{z - \overline{z}_0}$$

maps the upper half plane into the interior of the unit circle.

Note that

$$|T(z)| = \left| \mathrm{e}^{\mathrm{i}\theta_0} \frac{z - z_0}{z - \overline{z}_0} \right| = \frac{|z - z_0|}{|z - \overline{z}_0|}.$$

Now, since we are considering the domain $z \in U = \{z : \mathrm{Im}(z) > 0\}$, and since $\mathrm{Im} z_0 > 0$, $|z - z_0| \leq |z - \overline{z}_0|$, so the image satisfies $|T(z)| \leq 1$, which is the unit circle.

Example Find a Möbius transformation that maps the upper half plane into the unit circle with i being mapped to 0, and $\lim_{z\to+\infty} T(z) = -1$.

Using the above example, we should choose $z_0 = \mathrm{i}$, and the limiting behaviour tells us we should see choose $\theta_0 = -\pi$, resulting in

$$T(z) = \mathrm{e}^{-\mathrm{i}\pi} \frac{z - \mathrm{i}}{z + \mathrm{i}}.$$

Note that the set of Möbius transformation form the **Möbius group**.

Which, given the suggested link with 2×2 matrices with non-vanishing determinant, is maybe expected.

Theorem 5.1.5 • *The equation $Az\overline{z} + Bz + \overline{B}\overline{z} + C = 0$, determines a circle or line in the z-plane for $A, C \in \mathbb{R}$, $B \in \mathbb{C}$.*

- *The Möbius transformation maps a circle/line to a circle/line (but not necessarily respectively).*

□

5.2 *Harmonic functions*

Denoting ∇^2 to be the Laplacian operator (operator of second derivatives), a real function u is **harmonic** if $\nabla^2 u = 0$. Harmonic functions show up in Laplace and/or Poisson equations that are very prevalent in physical applications, and they have a very strong link with complex analysis.

Theorem 5.2.1 *Let $f(z) = u + iv$ for $z = x + iy$. If f is holomorphic, then u and v are harmonic.*

Proof If $f = u + iv$ is holomorphic, by definition the Cauchy–Riemann equations are satisfied, so

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0, \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0,$$

and assuming the relevant functions are sufficiently continuous and ordering of differentiation can be swapped, cross differentiation gives

$$\nabla^2 u = 0, \quad \nabla^2 v = 0.$$

■

The converse to the above is not true: u and v harmonic does not necessarily imply they satisfy the Cauchy–Riemann equations, so $f(z) = u + iv$ does not imply holomorphic. However, u and v are known as **harmonic conjugates** if they satisfy the Cauchy–Riemann equations, and then it does imply $f(z) = u + iv$ is holomorphic.

Example Let $u = 8xy - 3x$. Find a real function v where u and v are harmonic conjugates.

From one of the Cauchy–Riemann equations, we have

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \quad \Rightarrow \quad v = \int \frac{\partial u}{\partial x}\, dy = 4y^2 - 3y + C(x),$$

while the other gives

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \quad \Rightarrow \quad 8x + C'(x) = 0,$$

which leaves

$$v = 4y^2 - 3y - 4x^2 + c.$$

Let $U \subset \mathbb{R}^2$, ∂U be a simple closed curve, and $h, g : \partial U \to \mathbb{R}$. We might be interested in

1. finding $u : U \to \mathbb{R}$ with $\nabla^2 u = 0$ and $u = h$ on ∂U,
2. finding $u : U \to \mathbb{R}$ with $\nabla^2 u = 0$ and $\partial u / \partial n = g$ on ∂U, where n is the outward pointing normal to ∂U.

Both problems can be shown to admit unique solutions of u, and are known respectively as the **Dirichlet** and **Neumann problems**. These arise naturally in various physical problems (e.g. electrostatics).

5.3 *An example: Joukowsky transform*

In two-dimensional inviscid flow, assuming irrotational flow ($\nabla^{\perp}\boldsymbol{u} = \boldsymbol{e}_z \cdot \nabla \times \boldsymbol{u}$) and incompressible flow ($\nabla \cdot \boldsymbol{u} = 0$), we can introduce a potential ϕ and streamfunction ψ such that

$$\boldsymbol{u} = -\nabla\phi = \nabla^{\perp}\psi,$$

where $\nabla^{\perp} = (-\partial/\partial y, \partial/\partial x)$. Now, note that

$$\nabla^2\phi = 0, \quad \nabla^2\psi = 0,$$

so we seek ϕ and ψ that are harmonic functions. Additionally, from $-\nabla\phi = \nabla^{\perp}\psi$, so ϕ and ψ satisfy the Cauchy–Riemann equations, and are thus harmonic conjugates, and the resulting complex potential

$$F = \phi + \mathrm{i}\psi$$

is holomorphic. In this case, one can show that because ϕ and ψ are harmonic conjugates, the lines of constant ϕ and ψ are orthogonal to each other.

The idea here now is that if we can solve ϕ and ψ for special cases, we can construct the potential F that is holomorphic perhaps except at isolated locations. Via a suitable conformal map, F can be mapped to other complex potentials in other configurations, and because of the angle preserving quantity, the transformed ϕ and ψ will be still orthogonal and harmonic conjugates of each other, and are thus analogous solutions for the transformed problem.

One such case is the case of a uniform flow past a disk. For simplicity, we consider the unit disk, and the domain is thus $\mathbb{C} \setminus D(1,0)$. For the harmonic problems, it is easier to work in polar coordinates, and the Laplace equation for the potential ϕ is given by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0,$$

which in this case is subject to boundary conditions at infinity and Neumann boundary conditions at the boundary of the unit disk

$$\phi \sim Ur\cos\theta \quad \text{as} \quad r \to \infty, \qquad \frac{\partial\phi}{\partial r} = 0 \quad \text{on} \quad r = 1.$$

By separation of variables, one can show that

$$\phi = U\left(r + \frac{1}{r}\right)\cos\theta,$$

where U is the free-stream uniform flow, directed in the x direction. A similar argument leads to

$$\psi = U\left(r - \frac{1}{r}\right)\sin\theta.$$

The complex potential is then

$$F = U\left[\left(r + \frac{1}{r}\right)\cos\theta + \mathrm{i}\left(r - \frac{1}{r}\right)\sin\theta\right]$$
$$= U\left(z + \frac{1}{z}\right),$$

since $z = re^{\mathrm{i}\theta}$.

Note this potential leads to no lift, since there is no circulation $\Gamma = \oint_\gamma \boldsymbol{u}\cdot \mathrm{d}\boldsymbol{l}$. The application of the **Kutta condition** adds a point vortex term $\mathrm{i}\Gamma/(2\pi)\log z$ so that the closed contour integral to compute the circulation is non-zero.

A particular choice of practical relevance is the **Joukowsky map**

$$w = f(z) = z + \frac{1}{z}.$$

The function is holomorphic except at infinity, and noting that $f'(z) = 1 - 1/z^2 \neq 0$ except at $z = \pm 1$ (the leading and trailing edge of the unit disk) and $z = 0$ (which is not actually in the domain anyway), thus is it conformal away from those special points. In this instance, the unit circle is mapped onto the line segment going from $w = -2$ to $w = 2$, and the potential is conformally mapped onto the domain away from the line segment, and thus the transformed potential and streamfunction are still valid potential and streamfunction for the irrotational and incompressible flow problem. New solutions can be generated accordingly (e.g. a unit disk not centered at the origin), which has applications in aerofoil theory in aeronautics.

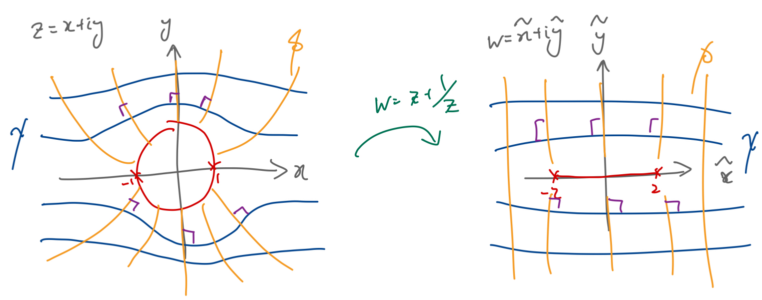

Figure 5.1: Sketch of constant potential and streamlines for the unit disk, and the image under the Joukowsky transform to a line segment. An off centered disk in x would could be sent to something that resembles an aerofoil.