How Fast Can You Escape a Compact Polytope?

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Abstract

The Continuous Polytope Escape Problem (CPEP) asks whether every trajectory of a linear differential equation initialised within a convex polytope eventually escapes the polytope. We provide a polynomial-time algorithm to decide CPEP for compact polytopes. We also establish a quantitative uniform upper bound on the time required for every trajectory to escape the given polytope. In addition, we establish iteration bounds for termination of discrete linear loops via reduction to the continuous case.

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1 Introduction

In ambient space $\mathbb{R}^d$, a *continuous linear dynamical system* is a trajectory $\mathbf{x}(t)$, where t ranges over the non-negative reals, defined by a differential equation $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t))$ in which the function f is *affine* or *linear*. If the initial point $\mathbf{x}(0)$ is given, the differential equation uniquely defines the entire trajectory. (Linear) dynamical systems have been extensively studied in Mathematics, Physics, and Engineering, and more recently have played an increasingly important role in Computer Science, notably in the modelling and analysis of cyber-physical systems; two recent and authoritative textbooks on the subject are [1, 14].

In the study of dynamical systems, particularly from the perspective of control theory, considerable attention has been given to the study of *invariant sets*, *i.e.*, subsets of $\mathbb{R}^d$ from which no trajectory can escape; see, *e.g.*, [7, 4, 2, 15]. Our focus in the present paper is on sets with the dual property that *no trajectory remains trapped*. Such sets play a key role in analysing *liveness* properties in cyber-physical systems (see, for instance, [1]): discrete progress is ensured by guaranteeing that all trajectories (*i.e.*, from any initial starting point) must eventually reach a point at which they 'escape' (temporarily or permanently) the set in question, thereby forcing a discrete transition to take place.

More precisely, given an affine function $f : \mathbb{R}^d \to \mathbb{R}^d$ and a convex polytope $\mathcal{P} \subseteq \mathbb{R}^d$, both specified using rational coefficients encoded in binary, we consider the *Continuous Polytope Escape Problem (CPEP)* which asks whether, for all starting points $\mathbf{x}_0$ in $\mathcal{P}$, the corresponding trajectory of the solution to the differential equation

$$\begin{cases} \dot{\mathbf{x}}(t) = f(\mathbf{x}(t)) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$

eventually escapes $\mathcal{P}$.[1]

CPEP was shown to be decidable in [13], in which an algorithm having complexity between **NP** and **PSPACE** was exhibited. It is worth noting that, when the polytope $\mathcal{P}$ is unbounded in space, the time taken for a given trajectory to escape may be unboundedly large. For example, consider the unbounded one-dimensional polytope $\mathcal{P} = \{x \in \mathbb{R} \mid x \geq 1\}$ and differential equation $\dot{x}(t) = -x(t)$. For any starting point x_0, the trajectory $x(t) = e^{-t}x_0$ converges to 0 and thus all trajectories eventually escape. However, the escape time is at least $\log(x_0)$ and hence is not bounded over all initial points in $\mathcal{P}$. Even if the polytope is bounded, there still need not be a uniform bound on the escape time. For example, consider the polytope $\mathcal{P} = (0, 1]$ and the equation $\dot{x}(t) = x(t)$. Given an initial point x_0, the trajectory $x(t) = e^t x_0$ necessarily escapes $\mathcal{P}$: but the escape time is at least $\log(1/x_0)$, which again is not bounded over $\mathcal{P}$.

Main contributions. We show that, for *compact* (*i.e.*, closed and bounded) polytopes, CPEP is decidable in polynomial time. Moreover, we show how to calculate uniform escape-time upper bounds; these bounds are exponential in the bit size of the descriptions of the differential equation and of the polytope, and doubly exponential in the ambient dimension. In the case of differential equations specified by invertible or diagonalisable matrices, we have singly exponential bounds.

[1] By "escaping" $\mathcal{P}$, we simply mean venturing outside of $\mathcal{P}$—we are unconcerned whether the trajectory might re-enter $\mathcal{P}$ at a later time or not.

In comparing the above with the results from [13], we note both a substantial improvement in complexity (from **PSPACE** to **PTIME**) as well as the production of explicit uniform bounds on escape times. It is worth pointing out that the mathematical approach pursued in [13] is non-effective, and therefore does not appear capable of yielding any quantitative escape-time bounds. The new constructive techniques used in the present paper, which originate mainly from linear algebra and algebraic number theory, are applicable owing to the fact that we focus our attention on *compact* polytopes. In practice, of course, this is usually not a burdensome restriction; in most cyber-physical systems applications, for instance, all relevant polytopes will be compact (see, *e.g.*, [1]).

Another interesting observation is that the seemingly closely related question of whether a given single trajectory of a linear dynamical system escapes a compact polytope appears to be vastly more challenging and is not known to be decidable; see, in particular, [3, 8, 9]. However, whether a given trajectory eventually hits a given single point is known as the *Continuous Orbit Problem* and can be decided in polynomial time [11].

Finally, we also consider in the present paper a discrete analogue of CPEP for discrete-time linear dynamical systems, namely the *Discrete Polytope Escape Problem (DPEP)*. This consists in deciding, given an affine function $f : \mathbb{R}^d \to \mathbb{R}^d$ and a convex polytope $\mathcal{P} \subseteq \mathbb{R}^d$, whether for all initial points $\mathbf{x}_0 \in \mathcal{P}$, the sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ defined by the initial point and the recurrence $\mathbf{x}_{n+1} = f(\mathbf{x}_n)$ eventually escapes $\mathcal{P}$. This problem—phrased as "termination of linear programs" over the reals and the rationals respectively—was already studied and shown decidable in the seminal papers [5, 17], albeit with no complexity bounds nor upper bounds on the number of iterations required to escape. By leveraging our results on CPEP, we are able to show that, for *compact* polytopes, DPEP is decidable in polynomial time, and moreover we derive upper bounds on the number of iterations that are singly exponential in the bit size of the problem description and doubly exponential in the ambient dimension.

2 Preliminaries

2.1 The Continuous Polytope Escape Problem

As noted in the previous section, the Continuous Polytope Escape Problem (CPEP) for continuous linear dynamical systems consists in deciding, given an affine function $f : \mathbb{R}^d \to \mathbb{R}^d$ and a convex polytope $\mathcal{P} \subseteq \mathbb{R}^d$, whether there exists an initial point $\mathbf{x}_0 \in \mathcal{P}$ for which the trajectory of the unique solution of the differential equation $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t)), \mathbf{x}(0) = \mathbf{x}_0, t \geq 0$, is entirely contained in $\mathcal{P}$. For $T \in \mathbb{R} \cup \{\infty\}$, we denote by $X(T)$ the set $\{\mathbf{x}(t) \mid t \in \mathbb{R}_{\geq 0}, t \leq T\}$. A starting point $\mathbf{x}_0 \in \mathcal{P}$ is said to be a *fixed point* if for all $t \geq 0$, $\mathbf{x}(t) = \mathbf{x}_0$, and it is *trapped* if the trajectory of $\mathbf{x}(t)$ is contained in $\mathcal{P}$ (*i.e.*, $X(\infty) \subseteq \mathcal{P}$); thus solving the CPEP amounts to deciding whether there is a trapped point.

We will represent a d-dimensional instance of the CPEP by a triple $(A, B, \mathbf{c})$, where $A \in \mathbb{R}^{d\times d}$ represents the linear function $f_A : \mathbf{x} \mapsto A\mathbf{x}$ [2] and $B \in \mathbb{R}^{n\times d}$, $\mathbf{c} \in \mathbb{R}^n$ represent the polytope $\mathcal{P}_{B,\mathbf{c}} = \{\mathbf{x} \in \mathbb{R}^d \mid B\mathbf{x} \leq \mathbf{c}\}$. Given such an instance and an initial point $\mathbf{x}_0$, the solution of the differential equation is $\mathbf{x}(t) = \exp(At)\mathbf{x}_0 \in \mathbb{R}^d$. For the computation of bounds, we

[2] We remark that by increasing the dimension by one, the general CPEP can be reduced to the homogeneous case, in which the function f is linear.

assume that all the coefficients of A, B and $\mathbf{c}$ are rational and encoded in binary. The decidability results and escape bounds computed in this paper can be adapted to the case of algebraic coefficients, but we don't pursue this here.

Decidability of the CPEP was shown in [13]. In this paper we are interested in the following problem: given a positive instance of CPEP (*i.e.*, one in which every trajectory escapes), compute an upper bound on the time to escape that holds uniformly over all initial points in the polytope. In other words, we wish to compute $T \in \mathbb{R}_{\geq 0}$ such that for all points $\mathbf{x}_0 \in \mathcal{P}$ there exists $t_0 \in \mathbb{R}$ such that $t_0 \leq T$ and $\mathbf{x}(t_0) \notin \mathcal{P}$. We call such a T an *escape-time bound*.

As noted in the Introduction, such an escape-time bound need not exist in general. In the remainder of this paper, we therefore restrict our attention to *compact* polytopes.

2.2 Jordan Normal Forms

Let $A \in \mathbb{Q}^{d\times d}$ be a square matrix with rational entries. The *minimal polynomial* of A is the unique monic polynomial $m(x) \in \mathbb{Q}[x]$ of least degree such that $m(A) = 0$. By the Cayley-Hamilton Theorem, the degree of m is at most the dimension of A. The set $\sigma(A)$ of eigenvalues of A is the set of roots of m. The *index* of an eigenvalue λ, denoted by $\nu(\lambda)$, is defined as its multiplicity as a root of m.

For each eigenvalue λ of A we denote by $\mathcal{V}_\lambda$ the subspace of $\mathbb{C}^d$ spanned by the set of generalised eigenvectors associated with λ. We also denote by $\mathcal{V}^r$ the subspace of $\mathbb{C}^d$ spanned by the set of generalised eigenvectors associated with some real eigenvalue; we likewise denote by $\mathcal{V}^c$ the subspace of $\mathbb{C}^d$ spanned by the set of generalised eigenvectors associated with some non-real eigenvalue.

It is well known that each vector $\mathbf{v} \in \mathbb{C}^d$ can be written uniquely as $\mathbf{v} = \sum_{\lambda\in\sigma(A)} \mathbf{v}_\lambda$, where $\mathbf{v}_\lambda \in \mathcal{V}_\lambda$. It follows that $\mathbf{v}$ can also be uniquely written as $\mathbf{v} = \mathbf{v}^r + \mathbf{v}^c$, where $\mathbf{v}^r \in \mathcal{V}^r$ and $\mathbf{v}^c \in \mathcal{V}^c$. Moreover, we can write any matrix A as $A = Q^{-1}JQ$ for some invertible matrix Q and block diagonal Jordan matrix $J = \mathrm{diag}\,(J_1, \dots, J_N)$, with each block J_i, associated to the eigenvalue λ_i having the following form:

$$\begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda_i \end{pmatrix}.$$

Given a rational matrix A, its Jordan Normal Form $J = QAQ^{-1}$ can be computed in polynomial time, as shown in [6]. Note that each vector $\mathbf{v}$ appearing as a column of the matrix Q^{-1} is a generalised eigenvector. We also note that the index $\nu(\lambda)$ of some eigenvalue λ corresponds to the dimension of the largest Jordan block associated with it. Given J_i, a Jordan block of size k associated with some eigenvalue λ, the closed-form expression for its exponential is

$$\exp(J_i t) = \exp(\lambda t) \begin{pmatrix} 1 & t & \cdots & \frac{t^{k-1}}{(k-1)!} \\ 0 & 1 & \cdots & \frac{t^{k-2}}{(k-2)!} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Using this, for all $j \leq d$, the closed form of the j-th component of a trajectory is, $x^{(j)}(t) = \sum_{\lambda \in \sigma(A)} p_\lambda(t) \exp(\lambda t)$ where for all $\lambda \in \sigma(A)$, p_λ is a polynomial of degree at most $\nu(\lambda) - 1$.

2.3 The Discrete Polytope Escape Problem

We shall also consider the Discrete Polytope Escape Problem (DPEP). The DPEP consists in deciding, given an affine function $f : \mathbb{R}^d \to \mathbb{R}^d$ and a convex polytope $\mathcal{P} \subseteq \mathbb{R}^d$, whether there exists an initial point $\mathbf{x}_0 \in \mathcal{P}$ for which the sequence $(\mathbf{x}_n)_{n \in \mathbb{N}}$ defined by the initial point and the recurrence $\mathbf{x}_{n+1} = f(\mathbf{x}_n)$ is entirely contained in $\mathcal{P}$. The definitions of fixed and trapped points are immediately transposed to the discrete setting by considering the sequence instead of the trajectory.

As with the CPEP, a d-dimensional instance of the DPEP is represented by a triple $(A, B, \mathbf{c})$, where $A \in \mathbb{R}^{d \times d}$ represents the function $f_A : \mathbf{x} \in \mathbb{R}^d \mapsto A\mathbf{x} \in \mathbb{R}^d$ and $B \in \mathbb{R}^{n \times d}$ and $\mathbf{c} \in \mathbb{R}^n$ represent the polytope $\mathcal{P}_{B,\mathbf{c}} = \{\mathbf{x} \in \mathbb{R}^d \mid B\mathbf{x} \leq \mathbf{c}\}$. Using the Jordan Normal form, one can see that the general form of the j-th component of the sequence $(\mathbf{x}_n)_{n \in \mathbb{N}}$ is $\mathbf{x}_n^{(j)} = \sum_{\lambda \in \sigma(A)} p_\lambda(n) \lambda^n$, where for all $\lambda \in \sigma(A)$, p_λ is a polynomial of degree at most $\nu(\lambda) - 1$. We assume that all the coefficients of A, B and $\mathbf{c}$ are rational.

The examples showing one cannot build a bound when the polytope is open or unbounded for the CPEP can easily be carried over to the DPEP. Thus, when considering the DPEP, we also only consider compact polytopes.

3 Deciding the Polytope Escape Problem for Compact Polytopes

While the result of [13] allows us to decide the existence of a trapped point for continuous linear dynamical systems, the method is quite involved. When restricting ourselves to compact polytopes, however, we can use the following proposition, which shows that the existence of a trapped point is equivalent to the existence of a fixed point.

▶ **Theorem 1.** *Given a CPEP instance $(A, B, \mathbf{c})$, the polytope $\mathcal{P}_{B,\mathbf{c}}$ contains a trapped point iff it contains a fixed point.*

Proof. For the "if" direction, observe that a fixed point $\mathbf{x}_0 \in \mathcal{P}_{B,\mathbf{c}}$ is necessarily trapped.

Conversely, assume that there exists a trapped point $\mathbf{x}_0 \in \mathcal{P}_{B,\mathbf{c}}$. Let H be the closure of the convex hull of $X(\infty) = \{\mathbf{x}(t) \mid t \in \mathbb{R}_{\geq 0}\}$. Then H is convex, compact, and is contained in $\mathcal{P}_{B,\mathbf{c}}$. For each $n \in \mathbb{N}$ we define a function $s_n : H \to H$ by $s_n(\mathbf{x}) = e^{A2^{-n}}\mathbf{x}$. Note that this function is well-defined: clearly $X(\infty)$ is invariant under s_n; moreover, since s_n is linear, the convex hull of $X(\infty)$ is also invariant under s_n; finally, since s_n is continuous, the closure of the convex hull of $X(\infty)$ (i.e., H) is invariant under s_n.

For all $n \in \mathbb{N}$, as the function s_n is continuous, by Brouwer's fixed-point theorem s_n admits at least one fixed point on H. Let F_n be the non-empty set of fixed points of s_n in H. Since $s_n = s_{n+1} \circ s_{n+1}$ we have that $F_{n+1} \subseteq F_n$ for all $n \in \mathbb{N}$. Moreover, by continuity of the function f_A, F_n is a closed set for all $n \in \mathbb{N}$. Therefore, the intersection $F_\infty = \bigcap_{n\in\mathbb{N}} F_n$ is non-empty. By continuity of f_A, any point $\mathbf{y} \in F_\infty$ satisfies $f_A(\mathbf{y}) = \mathbf{0}$. Therefore, the CPEP instance admits at least one fixed point within $\mathcal{P}_{B,\mathbf{c}}$, which concludes the proof. ◀

Since the set $F = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\}$ of fixed points is easy to calculate, we simply need to check whether its intersection with the polytope is empty in order to decide CPEP. Since the latter can be formulated as a linear program, we can decide CPEP for compact polytopes in polynomial time.

The proof of Theorem 1 carries over with very small changes (considering the function f_A directly, instead of the family $(s_n)_{n\in\mathbb{N}}$) to prove an analogous result for DPEP:

▶ **Theorem 2.** *Given a DPEP instance $(A, B, \mathbf{c})$, $\mathcal{P}_{B,\mathbf{c}}$ has a trapped point iff it contains a fixed point.*

4 Bounding the Escape Time for a Positive CPEP Instance

The goal of this section is to establish a uniform bound on the escape time of a positive CPEP instance. The main result is as follows:

▶ **Theorem 3.** *Given a d-dimensional positive instance of the CPEP, described by a tuple of bit size b, the time to escape the polytope is bounded by*

$$T = 4\exp\left(640bd^{4d+10}\right) = e^{bd^{O(d)}}.$$

We prove this bound in four steps. First, in Subsection 4.1, we show that one can ignore the component of the initial vector lying in the complex eigenspace $\mathcal{V}^c$ after a certain amount of time. Intuitively speaking, this stems from the fact that a convex polytope that contains a spiral must contain the centre of that spiral. Thus whenever we have a complex eigenvalue we can ignore the effects of the rotation by focusing on the axis of the helix formed by the trajectory.

We could then try to find a bound on escape time by looking at positivity of expressions of the form $\mathbf{b}^T \exp(At)y_0$, where $\mathbf{b}$ is the normal to a hyperplane supporting a face of the polytope. Unfortunately, these expressions contain terms corresponding to many different eigenvalues, which significantly complicates the analysis. We get around this problem in Subsection 4.2 by bounding the distance of the polytope to the origin and to the set of fixed points of the differential equation using hypercubes in the Jordan basis. This allows us to disentangle the effects of the different eigenvalues. We prove that the trajectories of the system escape the enclosing hypercube, and use the escape time of the hypercube as an upper bound on the escape time of the polytope.

Our next step is then, in Subsection 4.3, to compute a uniform escape bound for our hypercube. Finally, Subsection 4.4 combines the results from the previous sections to get the desired bound on the escape time of the original polytope.

4.1 Removing the Complex Eigenvalues

Let $(A, B, \mathbf{c})$, be a positive CPEP instance. Assume for now that A is given in Jordan normal form. This assumption is not without cost as we will see in the next subsection. In this subsection, we consider a single block J_i of A corresponding to a non-real eigenvalue λ_i. Considering only the dimensions associated to the Jordan block J_i (*i.e.*, the space $\mathcal{V}_\lambda$) and writing $k = \nu(\lambda_i)$, we have that given an initial point $\mathbf{x}_0 = [x^{(1)}, \ldots, x^{(k)}]$, the components of the trajectory $\mathbf{x}(t)$ are

$$\begin{bmatrix} x^{(1)}(t) \\ x^{(2)}(t) \\ \vdots \\ x^{(k)}(t) \end{bmatrix} = \exp(\lambda_i t) \begin{bmatrix} x^{(1)} + x^{(2)}t + x^{(3)}t^2/2 + \cdots + x^{(k)}t^{k-1}/(k-1)! \\ x^{(2)} + x^{(3)}t + \cdots + x^{(k)}t^{k-2}/(k-2)! \\ \vdots \\ x^{(k)} \end{bmatrix}.$$

In order to compute the escape times in the presence of non-real eigenvalues we use the fact that if a convex set contains a spiralling or helical trajectory, it must contain the axis of that trajectory. A trajectory starting on this axis is not affected by the eigenvalue that generates the rotation, moreover, if the trajectory starting in the axis escapes, then the original trajectory also escapes (albeit, potentially a bit later). This allows us to reduce to the case where we only have real eigenvalues. The following lemma formalizes this intuition.

▶ **Lemma 4** (Zero in convex hull)**.** *Let*

$$\mathbf{x}(t) = (p_{1,0}(t)e^{\lambda_1 t}, \ldots, p_{1,\nu(\lambda_1)-1}(t)e^{\lambda_1 t}, \ldots, p_{r,0}(t)e^{\lambda_r t}, \ldots, p_{r,\nu(\lambda_r)-1}(t)e^{\lambda_r t})^T$$

be a trajectory where, for all j, $\lambda_j = \eta_j + i\theta_j$, θ_j is non-zero, and $p_{j,k}$ is the Taylor polynomial corresponding to the factor $e^{\lambda_j t}$ of degree k. Then there exists a time T such that $\mathrm{Conv}(X(T))$ *contains the origin (where* Conv *represents the convex hull). In particular, this T satisfies*

$$T \leq \sum_{j=1}^{r} \nu(\lambda_j) \frac{\pi}{\theta_j}.$$

Proof Sketch. The basic idea is to take an initial point parametrized by t, travel along the trajectory to the point of opposite phase for a particular component, and create a new point where this component is equal to 0 by adding together a suitable convex combination of the opposite-phase point and the initial one. Since both these points were parametrized by t, we can take the trajectory starting in the newly created point (which lies in the convex hull of the original trajectory) and repeat for the other dimensions until every component corresponding to the $\mathcal{V}^c$ subspace is equal to 0. ◀

4.2 Replacing the Polytopes with Hypercubes

Let $(A, B, \mathbf{c})$ be a d-dimensional positive CPEP instance, $J \in \mathbb{R}^{d \times d}$ a matrix in Jordan normal form, and $Q \in \mathbb{R}^{d \times d}$ be such that $A = Q^{-1}JQ$.

Let us assume that all eigenvalues of A are real. Our approach is to work in the Jordan basis. To this end we note that the trajectory $\mathbf{x}(t) = \exp(At)\mathbf{x}_0$ escapes the polytope $\mathcal{P}_{B,\mathbf{c}}$ for all $\mathbf{x}_0 \in \mathbb{R}^d$ if and only if the trajectory $\mathbf{y}(t) = \exp(Jt)\mathbf{y}_0$ escapes the polytope $\mathcal{P}_{BQ^{-1},\mathbf{c}}$

for all $\mathbf{y}_0 \in \mathbb{R}^d$. (Note that all entries of Q^{-1} are real algebraic.) Below we analyse the latter version of CPEP, i.e., with a matrix J in Jordan form with real algebraic entries.

The key intuition is that for every initial vector $\mathbf{y}_0 \in \mathbb{R}^d$ the trajectory $\mathbf{y}(t) = \exp(Jt)\mathbf{y}_0$ will either converge to a fixed point of the system or otherwise will diverge to infinity in some component. In either case the trajectory must exit the polytope since the polytope is bounded and does not meet the set $F := \{\mathbf{y} \in \mathbb{R}^d \mid J\mathbf{y} = \mathbf{0}\}$ of fixed points. We are thus led to define constants $C, \varepsilon > 0$ such that every trajectory $\mathbf{y}(t) = \exp(Jt)\mathbf{y}_0$ that either exits the hypercube $[-C, C]^d$ or comes within distance ε of the set F of fixed points will necessarily have left the polytope $\mathcal{P}_{BQ^{-1},\mathbf{c}}$. More precisely, we seek $C > 0$ and $\varepsilon > 0$ such that:

1. $\mathcal{P}_{BQ^{-1},\mathbf{c}} \subseteq [-C, C]^d$,
2. For all $\mathbf{y} \in F$ the hypercube $\{\boldsymbol{y} + \boldsymbol{x} \mid \boldsymbol{x} \in [-\varepsilon, \varepsilon]^n\}$ does not meet $\mathcal{P}_{BQ^{-1},\mathbf{c}}$.

Note that such a positive ε must exist since, $\mathcal{P}_{BQ^{-1},\mathbf{c}} \cap F = \emptyset$, $\mathcal{P}_{BQ^{-1},\mathbf{c}}$ is compact, and F is closed. Having computed C and ε, we obtain the escape bound for the polytope $\mathcal{P}_{BQ^{-1},\mathbf{c}}$ by computing the time to either exit the hypercube in Item 1 or enter one of the hypercubes mentioned in Item 2.

In order to compute the escape bound, we only need the upper bound on the ratio C/ε given in the following lemma.

▶ **Lemma 5.** *Let $(A, B, \mathbf{c})$, be a d-dimensional positive CPEP instance involving rationals, each of at most $b \in \mathbb{N}$ bits. One can select $C \in \mathbb{R}$ and $\varepsilon > 0$ satisfying Conditions 1 and 2, above, and such that*

$$\frac{C}{\varepsilon} \leq \exp\left(640bd^{3d+8}\right).$$

Sketch of proof. The proof relies on Liouville's inequality, which states that the size of an algebraic number can be upper- and lower-bounded in terms of the degree and height (coefficient size) of its minimal integer polynomial, and an arithmetic complexity lemma which bounds the logarithmic height of the output of an arithmetic circuit in terms of the heights of the inputs. We apply these bounds to the vertices of the polytope in the Jordan basis (which are computed using the entries of B, $\mathbf{c}$ and Q^{-1}). ◀

Let us illustrate how the change of basis can lead to an exponential size polytope. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1.01 \end{bmatrix},$$

its associated Jordan decomposition

$$A = Q^{-1}JQ = \begin{bmatrix} 1 & 0 & 10000 \\ 0 & 1 & 100 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1.01 \end{bmatrix} \begin{bmatrix} 1 & 0 & -10000 \\ 0 & 1 & -100 \\ 0 & 0 & 1 \end{bmatrix}$$

and the polytope $\mathcal{P} = \{(0, 1, x_3) \in \mathbb{R}^3 \mid 0 \leq x_3 \leq 1\}$. This polytope is contained in the hypercube of size $C = 1$ and every point is at least at distance $\varepsilon = 1$ from any fixed point. However, in the Jordan basis, this polytope becomes equal to the set $\{(-10000x_3, 1 - 100x_3, x_3) \in \mathbb{R}^3 \mid (0, 1, x_3) \in \mathcal{P}\}$, which forces a choice of C and ε such that $\frac{C}{\varepsilon} \geq 10000$.

In general, using the same reasoning on the matrix of dimension d

$$A = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1+1/2^b \end{bmatrix},$$

leads to a blowup in the value for C/ε of $2^{b(d-1)}$, thus exponential in the dimension.

The bound obtained in Lemma 5 is however doubly exponential in the dimension. Analysing the proof of the lemma, in order to obtain an example for which the bound is tight, one would need to build a family of polynomials with splitting fields of degree exponential in the degree of the polynomial. Such polynomials unfortunately seem hard to find.

4.3 Computing an Upper Bound on the Escape Time for each Eigenspace

Consider a real eigenvalue λ of the Jordan matrix J associated with a Jordan block of size k. Let $\mathbf{x}_0 = (x^{(1)}, x^{(2)}, \dots, x^{(k)})$ be a point in the polytope. By construction of C, we know that $\forall i \leq k, x^{(i)} \leq C$. The trajectory $\mathbf{x}(t)$, in that generalized eigenspace is

$$\begin{bmatrix} x^{(1)} \\ x^{(2)} \\ \vdots \\ x^{(k)} \end{bmatrix}(t) = \exp(\lambda t) \begin{bmatrix} x^{(1)} + x^{(2)}t + \frac{x^{(3)}t^2}{2} + \cdots + \frac{x^{(k)}t^{k-1}}{(k-1)!} \\ x^{(2)} + x^{(3)}t + \cdots + \frac{x^{(k-1)}t^{k-2}}{(k-2)!} \\ \vdots \\ x^{(k)} \end{bmatrix}.$$

The trajectory, limited to this Jordan block, will either escape the hypercube $[-C, C]^d$ that encloses $\mathcal{P}_{BQ^{-1},\mathbf{c}}$, or will become so small that it will be at distance less than ε from the fixed point $\mathbf{0}$. We therefore consider three cases: $\lambda = 0$ and $\lambda > 0$ for which the trajectory will grow, and $\lambda < 0$ which decreases the coefficients. Once we have an escape bound for each eigenvalue, we will deduce a uniform bound for the entire trajectory.

Note that escaping the hypercube or converging to a fixed point do not give symmetric results: If we find a single component that grows larger than C, this is enough to escape the polytope, but *all* dimensions need to become smaller than ε in order to escape via entering the ε-region around the fixed point.

Case $\lambda < 0$.

For all $j \leq k$, $x^{(j)}(t) = \exp(\lambda t) \sum_{i=j}^{k} x^{(i)} \frac{t^{i-j}}{(i-j)!}$. Using the bounds on the coefficients, we thus have when $t > 1$

$$|x^{(j)}(t)| = |\exp(\lambda t) \sum_{i=j}^{k} x^{(i)} \frac{t^{i-j}}{(i-j)!}| \leq \exp(\lambda t) kCt^k \quad \text{for } j \in \{1, \dots, k\}$$

In order to have $|x^{(j)}(t)| < \varepsilon$, it is enough to have $\exp(\lambda t)kCt^k < \varepsilon$, which is equivalent to $\frac{kCt^k}{\varepsilon} < \exp(-\lambda t)$, and $t > \frac{1}{-\lambda} \log\left(\frac{kC}{\varepsilon}\right) + \frac{k}{-\lambda} \log t$

Here we need a small technical lemma.

▶ **Lemma 6** (Lemma A.1 and A.2 from [16]). *Suppose $a \geq 1$ and $b > 0$, then $t \geq a \log t + b$ if $t \geq 4a \log(2a) + 2b$.*

Applying this lemma with $a = \max\{1, \frac{k}{-\lambda}\}$ (we assume $\frac{k}{-\lambda} > 1$ in the following in order not to overload the formulas) and $b = \frac{1}{-\lambda} \log\left(\frac{kC}{\varepsilon}\right)$, we get a bound T_λ such that for all $j \leq k$, $x^{(j)}(T) < \varepsilon$, namely

$$\boxed{T_\lambda \leq \frac{4k}{-\lambda} \log\left(\frac{2k}{-\lambda}\right) + \frac{2}{-\lambda} \log\left(\frac{kC}{\varepsilon}\right).}$$

Case $\lambda = 0$.

In this case, the trajectory restricted to this eigenspace is

$$x^{(j)}(t) = \sum_{i=j}^{k} x^{(i)} \frac{t^{i-j}}{(i-j)!} \quad \text{for } j \in \{1, \dots, k\}.$$

Assume that there exists $j \geq 2$ such that $|x^{(j)}| > \varepsilon$. This holds because by the definition of ε a point of the polytope is at distance at least ε from a fixed point. In particular, the line $\{x_j = 0 \mid j \neq 1\}$ is a line of fixed points of the differential equation. Now we require a time T_λ such that at least one of these components is larger in magnitude than $|C|$. We construct an upper bound on this time iteratively, using the fact that at least one coefficient $x^{(j)}$ is greater than ε, and all of them are less than C, giving the following bound on T_λ:

$$\boxed{T_\lambda \leq \frac{1}{k} \left(\frac{k^2 C}{\varepsilon}\right)^{2^{k-1}}.}$$

Case $\lambda > 0$.

This case proceeds similarly to the $\lambda = 0$ case, although the presence of an exponential factor gives us a much better bound T_λ:

$$\boxed{T_\lambda \leq \frac{2^{k-1}}{\lambda} \log\left(\frac{kC}{\varepsilon}\right).}$$

4.4 Constructing a Uniform Bound

We can now combine the results of the previous sections to get a uniform escape bound, considering all eigenvalues (real or not) simultaneously. Let the complex eigenvalues of A be $\{\eta_1 + i\theta_1, \eta_1 - i\theta_1 \dots, \eta_r + i\theta_r, \eta_r - i\theta_r\}$ and the real eigenvalues be $\{\lambda_1, \dots, \lambda_s\}$. Consider an arbitrary trajectory $\mathbf{x}(t)$ satisfying the differential equation $\dot{\mathbf{x}}(t) = A\mathbf{x}(t)$. By Lemma 4 we know that for $T_c := \sum_{j=1}^{r} \nu(\eta_j + i\theta_j)\frac{\pi}{\theta_j}$ there exists a point in the convex hull of $\{\mathbf{x}(t) \mid 0 \leq t \leq T_c\}$ that lies in the real eigenspace of A. This allows us to derive a bound on the escape time of the polytope $\mathcal{P}$ from a bound on the escape time of $\mathcal{P} \cap \mathcal{V}^r$. Indeed, let T_r be such that every 'real' trajectory escapes the polytope in time T_r. Then any 'complex' trajectory of duration $T_c + T_r$ contains in its convex hull a 'real' trajectory of duration T_r

which thus must have escaped the polytope. As the polytope is convex, this means that the complex trajectory itself escaped.

As for the subspace $\mathcal{V}^r$, we can derive from the escape bounds T_λ on each eigenspace computed in Subsection 4.3 a time bound beyond which every real point has escaped the polytope.

▶ **Lemma 7** (Real Time Bound). *Given an initial point* $\mathbf{x}_0 \in \mathbb{R}^n$ *with zero components in* $\mathcal{V}^c$*, the trajectory* $\mathbf{x}(t)$ *escapes within time* $T_r = 2\max_\lambda T_\lambda$.

Proof. Within a time $T_r/2 = \max_\lambda T_\lambda$, thanks to the analysis of subsection 4.3, there are three possibilities:

- the trajectory escapes the hypercube of size C, this occurs if there was a coefficient associated to a non-negative eigenvalue that was larger than ε;
- all coefficients are now smaller than ε, entering the hypercube of size ε and escaping the polytope since all the purely imaginary coefficients are zero;
- some component corresponding to a positive or zero eigenvalue originally less than ε has become greater than ε. In this case, waiting another $T_r/2$ amount of time puts the trajectory in the first case, ensuring it escapes.

Thus in all cases the trajectory has escaped by time T_r. ◀

From the above, we can deduce that every trajectory escapes within time $T_r + T_c$. We finally obtain Theorem 3 by analysing the complexity of this time bound in terms of the number of bits of the instance and its dimension.

The magnitude of the resulting escape bound is singly exponential in the bit size of the matrix entries and doubly exponential in the dimension of the matrix. However, if the matrix is diagonalizable or invertible, we can ignore the case where the eigenvalue is zero. Then the bound becomes $O(4^{bd^2})$ which is singly exponential in the bit size and dimension.

In Subsection 4.2 we showed how the change of basis explained the exponential factor in the number of dimensions. It is clear that the escape time can also be exponential in the bit size of the matrix.

For a very simple example, consider a 1-dimensional case where the polytope is the interval $[1, 2]$ and the differential equation is $\dot{x}(t) = 2^{-b}x(t)$ (which obviously can be written using constants of bit size at most b). Then the initial point $x_0 = 1$ yields a trajectory $x(t) = \exp(2^{-b}t)x_0$ whose escape time is $2^b \log 2$, which is exponential in b.

5 The Discrete Case

Tiwari [17] and Braverman [5] have shown decidability for the DPEP over the rationals and reals. In general, even if every trajectory is known to be escaping, it is not possible to place a uniform bound on the number of steps. However if the polytope is compact, we can use techniques similar to those used for the CPEP in order to provide a bound.

▶ **Theorem 8.** *Given a* d*-dimensional positive DPEP instance* $(A, B, \mathbf{c})$ *where the rational numbers use at most* $b \in \mathbb{N}$ *bits and an initial point* $\mathbf{x}_0$*, then for* $N = e^{bd^{O(d)}}$*, we have* $\mathbf{x}_N \notin \mathcal{P}_{B,\mathbf{c}}$.

Sketch of proof. We reduce the problem to the continuous case. Assuming every eigenvalue is positive, the matrix logarithm G of A is well defined. The trajectory of a continuous linear dynamical sysems generated by G is of the form $\mathbf{x}(t) = \exp(Gt)\mathbf{x}(0)$. In particular, for an initial point x_0 and $n \in \mathbb{N}$, we have

$$\mathbf{x}(n) = \exp(Gn)\mathbf{x}_0 = \exp(G)^n\mathbf{x}_0 = A^n\mathbf{x}_0 = \mathbf{x}_n$$

Therefore, we can relate the escape time of the CPEP instance $(G, B, \mathbf{c})$ to the escape time of the DPEP instance $(A, B, \mathbf{c})$.

The eigenvalues that are not positive are dealt with using a variant of the convex hull Lemma 4. ◀

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A Proof of Section 4

A.1 Proof of Lemma 4

We establish this result by induction over r, the number of distinct eigenvalues.

Base case. Assume $r = 1$, we have

$$\mathbf{x}(t) = e^{\eta_1 t} e^{i\theta_1 t}(p_{1,0}(t), p_{1,1}(t), \ldots, p_{1,\nu(1)-1}(t)) \in \mathbb{C}^{\nu(1)}.$$

We define a new starting point belonging to the convex hull of the trajectory $\mathbf{x}(t)$ by

$$\mathbf{z}_1(0) = \frac{p_{1,0}(\frac{\pi}{\theta_1})e^{\eta_1 \frac{\pi}{\theta_1}}}{p_{1,0}(0) + p_{1,0}(\frac{\pi}{\theta_1})e^{\eta_1 \frac{\pi}{\theta_1}}}\mathbf{x}(0) + \frac{p_{1,0}(0)}{p_{1,0}(0) + p_{1,0}(\frac{\pi}{\theta_1})e^{\eta_1 \frac{\pi}{\theta_1}}}\mathbf{x}(\frac{\pi}{\theta_1}).$$

Now observe that

$$\begin{aligned} z_1(0) &= \frac{p_{1,0}(\frac{\pi}{\theta_1})e^{\eta_1 \frac{\pi}{\theta_1}}}{p_{1,0}(0) + p_{1,0}(\frac{\pi}{\theta_1})e^{\eta_1 \frac{\pi}{\theta_1}}}(\, p_{1,0}(0), p_{1,1}(0), \ldots, p_{1,\nu(1)-1}(0)\,) \\ &\quad + \frac{p_{1,0}(0)}{p_{1,0}(0) + p_{1,0}(\frac{\pi}{\theta_1})e^{\eta_1 \frac{\pi}{\theta_1}}} e^{\eta_1 \frac{\pi}{\theta_1}} e^{i\theta_1(\frac{\pi}{\theta_1})}(\, p_{1,0}(\frac{\pi}{\theta_1}), \ldots, p_{1,\nu(1)-1}(\frac{\pi}{\theta_1})\,) \\ &= \frac{e^{\eta_1 \frac{\pi}{\theta_1}}}{p_{1,0}(0) + p_{1,0}(\frac{\pi}{\theta_1})e^{\eta_1 \frac{\pi}{\theta_1}}}(0, \ldots, p_{1,0}(\frac{\pi}{\theta_1})p_{1,\nu(1)-1}(0) - p_{1,0}(0)p_{1,\nu(1)-1}(\frac{\pi}{\theta_1})) \\ &= (0, q_{1,0}(0), \ldots, q_{1,\nu(1)-2}(0)), \end{aligned}$$

where $q_{1,k}(t)$ is a polynomial of degree at most k.

Iterating this process, we build the family of points $(\mathbf{z}_k)_{k \leq \nu(\lambda_1)}$ such that the k first coordinates of $\mathbf{z}_k$ are null. Thus, we have $\mathbf{z}_{\nu(\lambda_1)}(t) = \mathbf{0} \in \mathbb{C}^{\nu(\lambda_1)}$. Each step of this process requires an additional $\frac{\pi}{\theta_1}$ time units to ensure the constructed point belongs to the convex hull of the trajectory $\mathbf{x}(t)$. Thus after $T \geq \nu(\lambda_1)\frac{\pi}{\theta_1}$, we have $\mathbf{0} \in \mathrm{Conv}(X(T))$ as required.

Inductive case. Let $r \geq 1$ and

$$\mathbf{x}(t) = (p_{1,0}(t)e^{\lambda_1 t}, \ldots, p_{1,\nu(\lambda_1)-1}(t)e^{\lambda_1 t}, \ldots, p_{r+1,0}(t)e^{\lambda_{r+1} t}, \ldots, p_{r+1,\nu(\lambda_{r+1})-1}(t)e^{\lambda_{r+1} t})^T.$$

By induction hypothesis, for $T_1 = \sum_{j=1}^{r} \nu(\lambda_j)\frac{\pi}{\theta_j}$, there exists a point $\mathbf{z}_0$ in $\mathrm{Conv}(X(T_1))$ such that the components corresponding to the first r eigenvalues remain equal to 0. Therefore, the trajectory starting in $\mathbf{z}_0$ is of the form

$$\mathbf{z}(t) = (0, \ldots, 0, q_{r+1,0}(t)e^{\lambda_{r+1}t}, \ldots, q_{r+1,\nu(\lambda_{r+1})-1}(t)e^{\lambda_{r+1}t})^T$$

where the $q_{r,j}$ are polynomials.

Applying the process used in the base case, one gets that the zero vector belongs to the set $\mathrm{Conv}(Z(T_2))$ for $T_2 = \nu(\lambda_{r+1})\frac{\pi}{\theta_{r+1}}$. Moreover, as $\mathbf{z}_0 \in \mathrm{Conv}(X(T_1))$, we have that $\mathbf{0} \in \mathrm{Conv}(X(T_1 + T_2))$.

A.2 Proof of Lemma 5

We first recall some known results on the heights of algebraic numbers. More details can be found in [18].

▶ **Definition** (Naive height). Given an algebraic number α, its *naive height* $H(\alpha)$ is the largest absolute value of any coefficient of its minimal polynomial in the ring $\mathbb{Z}[x]$. The *degree* of α is the degree of this polynomial.

▶ **Lemma 9** (Liouville's inequality). *Given $\alpha \neq 0$ an algebraic number,*

$$\frac{1}{H(\alpha)+1} < |\alpha| < H(\alpha) + 1.$$

One can also define a logarithmic height of an algebraic number. It satisfies the following lemma.

▶ **Lemma 10** (logarithmic height). *Given an algebraic number α of degree n, its* absolute logarithmic height $h(\alpha)$ *satisfies the following relations.*

$$\frac{1}{n}\log H(\alpha) - \log 2 < h(\alpha) < \frac{1}{n}\log H(\alpha) + \frac{1}{2n}\log(n+1)$$

Moreover, for algebraic numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$, we have

$$h(\prod_{i=1}^{k} \alpha_i) \leq \sum_{i=1}^{k} h(\alpha_i), \text{ and } h(\sum_{i=1}^{k} \alpha_i) \leq \log k + \sum_{i=1}^{k} h(\alpha_i).$$

This lemma directly implies the following result.

▶ **Corollary 11** (arithmetic complexity). *Given algebraic numbers $\{\alpha_i\}_{i=1,\ldots,k}$, such that $h(\alpha_i) \leq h_{max}$ and a function f that computes an arithmetic circuit involving at most m operations of addition, multiplication, subtraction and division,*

then

$$h\left(f(\alpha_1, \ldots, \alpha_k)\right) \leq (m+1)h_{max} + m\log 2.$$

We aim to give an upper bound on the ratio C/ε. If we can find a maximum height H_{max} of any component of a vertex of the polytope in the Jordan basis, then, using Lemma 9,

$(H_{max}+1)$ is an upper bound of the (component-wise) distance of any point of the polytope to the origin, and $1/(H_{max}+1)$ is a lower bound. Thus $(H_{max}+1)^2$ is an upper bound for C/ε.

Recall that $J = QAQ^{-1}$ is the Jordan normal form of A, so we work in the basis $y = Qx$. Note that each vector $\mathbf{v}$ appearing as a column of the matrix Q^{-1} is a generalised eigenvector.

Vertices of the original polytope are vectors of the form $B'^{-1}c'$ for B' an invertible square submatrix of B. Let x be a vertex. In the Jordan basis, this vertex has numerical coordinates corresponding to the components of Qx. We wish to bound the height of each component, call it $(Qx)^{(i)}$, for every x in $\mathcal{P}_{B,\mathbf{c}}$. For all $i \leq d$, we have $h((Qx)^{(i)}) = h(\sum_j q_{ij}x^{(j)}) \leq 2d(\max_{i,j\leq d}\{h(q_{ij}), h(x^{(i)})\} + \log 2)$.

Now $h(x_i)$ is easy to compute: The elements x_i are solutions to the linear system $B'x = c$. Using Gaussian elimination, which has less than $3d^3$ arithmetic complexity, we can compute the entries of x. Since the entries of B and c are at most b-bit rationals, they have logarithmic height b. This implies that $h(x_i) \leq 3d^3(b + \log 2)$.

The entries q_{ij} of Q can be computed in $3d^3$ operations from the entries of Q^{-1}, which is the matrix of generalised eigenvectors of A. For q and q^{-1} representing the maximum logarithmic height of an entry of Q and Q^{-1} respectively, via Gaussian elimination, we have:

$$q \leq 3d^3(q^{-1} + \log 2) \tag{1}$$

Let λ be an eigenvalue of A, and let $M_\lambda = A - \lambda I$. Then the columns of Q^{-1} are vectors v, where v is a generalized eigenvector that satisfies $M_\lambda^d v = 0$ for some eigenvalue λ. Note that the equation $M_\lambda^d v = 0$ is underdetermined, but this does not matter for our purposes, since we just need one valid eigenvector to compute a bound on the height. Note that by the definition of Jordan normal form, $M_\lambda^d v = 0$ has at least one non-zero solution. Again by Gaussian elimination, for m_λ^d the maximum logarithmic height of an entry of M_λ^d, we have:

$$q^{-1} \leq 3d^3(m_\lambda^d + \log 2) \tag{2}$$

Computing an element of M_λ^d from M_λ is more complicated and gives height:

$$m_\lambda^d \leq (2d)^d(m_\lambda + \log d) \tag{3}$$

Since $M_\lambda = A - \lambda I$, for a the maximum logarithmic height of an entry of A,

$$m_\lambda \leq a + h(\lambda) + \log 2 \leq b + h(\lambda) + \log 2 \tag{4}$$

The height of any eigenvalue λ is determined by the coefficients of the characteristic polynomial of A. For this, we rely on the following lemma given in [10].

▶ **Lemma 12** (characteristic polynomial bound). *Let $A \in \mathbb{C}^{n\times n}$, with $n \geq 4$, whose coefficients are bounded in absolute value by $B > 1$. The coefficients of the characteristic polynomial C_A of A are denoted by $c_j, j = 0, ..., n$. and $||C_A||_\infty = \max_j\{|c_j|\}$.*

Then $||C_A||_\infty \leq (2nB^2)^{n/2}$.

Note this is only a factor of $2^{n/2}$ larger than the Hadamard bound on the determinant, which is the zeroth coefficient.

Since the entries of A are b-bit rationals encoded in binary, we can use 2^b as the bound B of this lemma. Thus the coefficients of the characteristic polynomial over $\mathbb{Q}$ are bounded by $(2d2^{2b})^{d/2}$. Multiplying by 2^{bd} (the largest possible denominator) ensures an integer polynomial, we obtain the following bound on $h(\lambda)$:

$$h(\lambda) \leq \log_2[(2d2^{2b})^{d/2}2^{bd}] \leq 2bd + \frac{d}{2}\log(2d) \leq 3bd^2. \tag{5}$$

Gathering (1), (2), (3), (4) and (5), we obtain:

$$q \leq 80bd^{d+7}2^d.$$

Thus, for all $i \leq d$, we have

$$h((Qx)^{(i)}) \leq 2d(\max\{h(q_{ij}), h(x_i)\} + \log 2) \leq 160bd^{d+8}2^d.$$

Using the log height lemma, we get the bound

$$H((Qx)^{(i)}) \leq 2^n \exp(nh((Qx)^{(i)})),$$

where n is the degree of $(Qx)^{(i)}$ as an algebraic number.

Since $(Qx)^{(i)}$ is obtained by performing arithmetic operations with all the eigenvalues of A, it lies in the splitting field of the characteristic polynomial of A, which may have degree $d!$. Using Lemma 9, this gives us a quantitative bound on C/ε, which is

$$C/\varepsilon \leq (H((Qx)^{(i)}) + 1)^2 \leq 4H((Qx)^{(i)})^2 \leq 2^{2d!+2}\exp\left(320bd^{d+8}2^d(d!)\right) \leq \exp\left(640bd^{3d+8}\right).$$

A.3 Proofs of Section 4.3

Case $\lambda = 0$.

Proof. Given the set of equations

$$x_j(t) = \sum_{i=j}^{k} x_i \frac{t^{i-j}}{(i-j)!},$$

we want a T such that there exists j such that $|x_j(T)| > C$.

By construction of ε, there exists $j_1 \geq 2$ such that $|x_{j_1}| > \varepsilon$. Consider the component

$$x_{j_1-1}(t) = \sum_{i=j_1-1}^{k} x_i \frac{t^{i-j_1+1}}{(i-j_1+1)!}.$$

We set $T_{j_1} = kC/\varepsilon$. Observe that

$$|x_{j_1-1}(T_{j_1})| \geq |x_{j_1}kC/\varepsilon| - |x_{j_1-1}| - \sum_{i=j_1+1}^{k} \left| x_i \frac{T_{j_1}^{i-j_1+1}}{(i-j_1+1)!} \right|$$

Since the first term is larger than kC and the second term is smaller than C, the only way $|x_{j_1-1}(T_{j_1})|$ could be less than C (and thus not escape the polytope) is if one of the later terms is larger than C. Let j_2 be the highest index such that $\left| x_{j_2} \frac{T_{j_1}^{j_2-j_1+1}}{(j_2-j_1+1)!} \right| \geq C$. Note that

$j_2 > j_1$. We now have a lower bound on a higher index coefficient, namely $|x_{j_2}|\frac{T_{j_1}^{j_2-j_1+1}}{(j_2-j_1+1)!} > C$. We now repeat the process with the component

$$x_{j_2-1}(t) = x_{j_2-1} + x_{j_2}t + \sum_{i=j_2+1}^{k} x_i \frac{t^{i-j_2+1}}{(i-j_2+1)!}$$

We have $|x_{j_2}|\frac{T_{j_1}^{j_2-j_1+1}}{(j_2-j_1+1)!} > C$, thus setting $T_{j_2} > k\frac{T_{j_1}^{j_2-j_1+1}}{(j_2-j_1+1)!}$ ensures that $|x_{j_2}T_{j_2}| > kC$.

Continuing this process, we will either find a component that escapes the polytope or move on to a component with higher index, which can happen at most $k-1$ times, because we have the constraints $j_1 \geq 2, \forall i, j_i \leq k$, and $j_i > j_{i-1}$. This gives us a recursive definition for the bound, which is

$$T_{j_n} > k\frac{T_{j_{n-1}}^{j_n-j_{n-1}+1}}{(j_n-j_{n-1}+1)!}$$

We wish to find an upper bound on $T = T_N$, the time by which we are guaranteed that at least one component escapes, subject to the constraints $j_1 \geq 2, j_N \leq k$, and $j_n > j_{n-1}$. We can solve the recursive inequality by weakening it (since we only need an upper bound on T) to

$$T_{j_n} > (kT_{j_{n-1}})^{j_n-j_{n-1}+1}.$$

Note that pulling the constant k into the exponentiated part is valid because $j_n - j_{n-1}+1 > 2$ always. Setting $S_{j_n} = kT_{j_n}$, we get $S_{j_n} > S_{j_{n-1}}^{j_n-j_{n-1}+1}, S_{j_1} = k^2C/\varepsilon$, which reduces to

$$S_{j_N} > \left(\frac{k^2C}{\varepsilon}\right)^{\prod_{i=2}^{N}(j_i-j_{i-1}+1)}$$

The term $\prod_{i=2}^{N}(j_i - j_{i-1}+1)$ is maximised when for all i, $j_i = j_{i-1}+1$, thus in the worst case we have

$$S_{j_N} > \left(\frac{k^2C}{\varepsilon}\right)^{2^{k-1}}$$

Thus we have a bound for a zero-eigenvalue component to escape, which is

$$\boxed{T \leq \frac{1}{k}\left(\frac{k^2C}{\varepsilon}\right)^{2^{k-1}}.}$$

◀

Case $\lambda > 0$.

Proof. The proof is very similar in structure to the zero eigenvalue case, though the presence of an exponential factor gives us a much better bound.

By construction of ε, there exists $j_1 \geq 2$ such that $|x_{j_1}| > \varepsilon$. Consider the component

$$x_{j_1}(t) = \sum_{i=x_1}^{k} \exp(\lambda t) x_i \frac{t^{i-j_1}}{(i-j_1)!}.$$

Set $T_{j_1} = \frac{1}{\lambda}\log(kC/\varepsilon)$ and observe that

$$|x_{j_1}(T_{j_1})| \geq \exp(\lambda\frac{1}{\lambda}\log(kC/\varepsilon))|x_{j_1}| - \sum_{i=j_1+1}^{k} \exp(\lambda T_{j_1})|x_i|\frac{T_{j_1}^{i-j_1}}{(i-j_1)!}.$$

Since the first term is larger than kC, the only way $|x_{j_1}(T_{j_1})|$ can be less than C (and thus not escape the polytope) is if one of the later terms is larger than C. Let j_2 be the highest index such that $\exp(\lambda T_{j_1})|x_{j_2}|\frac{T_{j_1}^{j_2-j_1}}{(j_2-j_1)!} \geq C$. Note that $j_2 > j_1$. We now have a lower bound on a higher index coefficient, namely $|x_{j_2}|\frac{T_{j_1}^{j_2-j_1}}{(j_2-j_1)!}\exp(\lambda T_{j_1}) > C$. Now we repeat the process with the component

$$x_{j_2}(t) = \exp(\lambda t)x_{j_2} + \sum_{i=j_2+1}^{k} \exp(\lambda t)x_i\frac{t^{i-j_2}}{(i-j_2)!}$$

We want $|x_{j_2}|\exp(\lambda T_{j_2}) > kC$, so it is enough to set

$$\exp(\lambda T_{j_2})\frac{(j_2-j_1)!}{T_{j_1}^{j_2-j_1}}\exp(-\lambda T_{j_1}) > k.$$

Ignoring the factorial term for simplicity, we get the constraint

$$T_{j_2} > T_{j_1} + \frac{j_2-j_1}{\lambda}\log(T_{j_1}) + \frac{1}{\lambda}\log k.$$

Continuing the process, we will either find a component that escapes the polytope or move on to a component with higher index, which can happen at most $k-1$ times, because we have the constraints $j_1 \geq 2, \forall i, j_i \leq k$, and $j_i > j_{i-1}$. This process gives us a recursive definition for the bound, which is

$$T_{j_n} > T_{j_{n-1}} + \frac{j_n - j_{n-1}}{\lambda}\log(T_{j_{n-1}}) + \frac{1}{\lambda}\log k.$$

We wish to find an upper bound on $T = T_N$, the time by which we are guaranteed that at least one component escapes, subject to the constraints $j_1 \geq 1, j_N \leq k$, and $j_n > j_{n-1}$. We can solve the recursive inequality by weakening it (since we only need an upper bound on T), observing that $T_{j_{n-1}} > \frac{k}{\lambda}\log(T_{j_{n-1}}) \Rightarrow T_{j_{n-1}} > \frac{j_n-j_{n-1}}{\lambda}\log(T_{j_{n-1}})$ and the lefthand side of this implication holds if $T_{j_{n-1}} > \frac{4k}{\lambda}\log\left(\frac{2k}{\lambda}\right)$ (using Lemma 6). Thus if we ensure that $T_{j_1} > \frac{4k}{\lambda}\log\left(\frac{2k}{\lambda}\right)$, we can work with the much simpler recurrence

$$T_{j_n} > 2T_{j_{n-1}} + \frac{1}{\lambda}\log k,$$

which is easily solved to get

$$T_N > 2^{N-1}T_{j_1} - \frac{1}{\lambda}\log k.$$

As $N \leq k$ and assuming $\frac{1}{\lambda}\log(kC/\varepsilon) > \frac{4k}{\lambda}\log\left(\frac{2k}{\lambda}\right)$ (which is valid as the order of magnitude of the first one is greater than the second one), we have a bound for a positive-eigenvalue component to escape, which is

$$\boxed{T \leq \frac{2^{k-1}}{\lambda}\log\left(\frac{kC}{\varepsilon}\right).}$$

◀

A.4 Analysis of the complexity of $T_c + T_r$

The escape time is bounded by

$$T_c + T_r \leq \sum_{j=1}^{r} \nu(\eta_j + i\theta_j)\frac{\pi}{\theta_j} +$$
$$2\max_{\lambda}\left\{\frac{4\nu(\lambda)}{|\lambda|}\log\frac{2\nu(\lambda)}{|\lambda|} + \frac{2}{|\lambda|}\log\frac{\nu(\lambda)C}{\varepsilon},\ \frac{2^{\nu(\lambda)-1}}{\lambda}\log\frac{\nu(\lambda)C}{\varepsilon},\ \frac{1}{\nu(0)}\left(\frac{\nu(0)^2 C}{\varepsilon}\right)^{2^{\nu(0)-1}}\right\}.$$

In terms of magnitude, the worst case occurs for a zero eigenvalue, in which case $T_r \leq \frac{2}{\nu(0)}(\nu(0)^2 C/\varepsilon)^{2^{\nu(0)}} \leq \frac{2}{d}(d^2C/\varepsilon)^{2^d}$.

We need to bound $\frac{C}{\epsilon}$, $\frac{1}{\theta}$, and $\frac{1}{\lambda}$. The bound on $\frac{C}{\epsilon}$ was given in Lemma 5:

$$\exp\left(640.b.d^{3d+8}\right).$$

For $\frac{1}{\theta}$ and $\frac{1}{\lambda}$, we can use the Mignotte root separation bound :

▶ **Proposition** (Mignotte bound [12]). Let $f \in \mathbb{Z}[x]$. If α_1 and α_2 are distinct roots of f, then

$$|\alpha_1 - \alpha_2| > \frac{\sqrt{6}}{d^{(d+1)/2}H^{d-1}}$$

where d and H are respectively the degree and height (maximum absolute value of the coefficients) of f.

We apply this result on the polynomial $xP(x)$ where P is the characteristic polynomial of A. This gives a distance between the root 0 and λ. It also gives a bound for θ as it is obtained as the difference between two conjugate roots of A. Using the bounds on the height of P computed for Lemma 5, we get

$$\frac{1}{\theta}, \frac{1}{\lambda} \leq \frac{1}{\sqrt{6}}\left((2d)^{d/2} \cdot 2^{2bd}\right)^{d-1} d^{(d+1)/2} \leq 4^{3bd^3}.$$

Thus, T_c is very small compared to the bound obtained for T_r in the zero eigenvalue case. Therefore, the escape time is bounded by

$$T \leq \frac{4}{d}(d^2\exp\left(640bd^{3d+8}\right))^{2^d} \leq 4\exp\left(640bd^{4d+10}\right) = e^{bd^{O(d)}}.$$

B Proof of Theorem 8

Given a d-dimensional instance $(A, B, \mathbf{c})$ of the DPEP, the proof of the bound has two steps. First we deal with the negative and zero eigenvalues, showing how they can be ignored in a manner similar to the continuous case in Subsection 4.1 (*i.e.* by showing that the axis of the symmetries created by the negative eigenvalues is in the polytope and if the trajectories starting on the axis escape, then every trajectory escapes in a small additional number of steps). Second, we reduce the problem (with only non-negative eigenvalues) to the continuous case using the matrix logarithm.

Let us start with the negative eigenvalues. We state a lemma that is essentially a discrete version of Lemma 4.

▶ **Lemma 13** (Zero in convex hull (discrete case)). *Suppose there are r negative real eigenvalues, $\lambda_1, \ldots, \lambda_r$ with λ_j of multiplicity $\nu(j)$. Let $N = \sum_{j=1}^{r} \nu(j)$.*

Given a vector v where every component is equal to 0 outside the negative real eigenspaces, the convex hull of the set $\{v, Av, A^2v, \ldots, A^N v\}$ contains the origin.

Proof. The point $A^n v \in \mathbb{R}^N$ (considering only the negative real eigenspace, since other coordinates are zero) can be written as

$$\mathbf{x}_n = (p_{1,0}(n)\lambda_1^n, \ldots, p_{1,\nu(1)-1}(n)\lambda_1^n\, , p_{2,0}(n)\lambda_2^n, \ldots, p_{r,\nu(r)-1}(n)\lambda_r^n)^T,$$

where $p_{j,k}$ is a polynomial corresponding to the j'th eigenvalue of degree k, as can be seen from the Jordan normal form.

Now observe that $\lambda_j^n = (-|\lambda_j|)^n = e^{(\log|\lambda_j| + i\pi)n}$. Thus we can apply the method of Lemma 4, with each step being done in one time unit, giving a bound $N \leq \sum_{j=1}^{r} \nu(j)$ on the number of steps to contain the origin. ◀

We observe that this bound on the number of steps is always smaller or equal to the dimension of the instance. For the eigenspaces associated to zero eigenvalues, note that the corresponding Jordan blocks are nilpotent, so in at most d iterations, the component in this eigenspace goes to zero.

Now, let us assume the components associated to negative or zero eigenvalues are null. Let J be the Jordan Normal Form of A. Let $\mathcal{V}_{-}$ be the eigenspace of negative and zero eigenvalues, and let $\mathcal{V}_{cp}$ be the eigenspace of all other eigenvalues. Let J' be the submatrix of J restricted to $\mathcal{V}_{cp}$. Similarly, define $\mathcal{P}'$ to be the polytope obtained by projecting $\mathcal{P}_{BQ^{-1},\mathbf{c}}$ onto this subspace. We define $G' = \operatorname{Log} J'$. As there are no negative and zero eigenvalue, this operation is well defined. G' translates J' into the continuous setting, in particular, for $n \in \mathbb{N}$, we have (for $\mathbf{y}_n$ the sequence $\mathbf{x}_n$ in the Jordan basis and restricted to the complex and positive eigenspaces) $\mathbf{y}_n = (J')^n \mathbf{y}_0 = \exp(G')^n \mathbf{y}_0 = \exp(G'n)\mathbf{y}_0 = \mathbf{y}(n)$. Let T_c be the escape time bound given by Theorem 3 for the instance defined by the polytope $\mathcal{P}'$ and the dynamics $\dot{\mathbf{x}} = G'\mathbf{x}$. In a similar fashion to the analysis of the continuous case, we have that the number of steps needed to escape the polytope is bounded by

$$N = \lceil T_{cp} \rceil + d.$$

This value is dominated by the term T_{cp}, for which the worst case is due to the Jordan blocks with eigenvalue 1, since this corresponds to the zero eigenvalue case after we take the logarithm of the matrix.

The only difference from the continuous case is that we need to estimate $\frac{1}{\log(\lambda)}$ and $\frac{1}{\log(\theta)}$ (where λ and $\eta + i\theta$ are eigenvalues of the original matrix A), since the bound is defined in terms of A, not G'.

We observe that $\frac{1}{\log(\lambda)}$ is bounded by how small $\log(\lambda)$ can get, but $\log(\lambda)$ is only small in the neighbourhood of 1, where $\log(\lambda) \approx \lambda - 1$, and this can be bounded (for $\lambda \neq 1$) in exactly the same way we previously used the Mignotte bound, this time with $(x-1)P(x)$. A similar analysis applies to $\frac{1}{\log(\theta)}$, confirming that the dominant term is the eigenvalue-1 Jordan block.

Thus we obtain that $N = e^{bd^{O(d)}}$, as in the continuous case.