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Theory and Methods Section

Power Function of the F-Test Under Non-Normal Situations

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The values of the power of the F-test employed in analysis-of-variance are calculated under non-normal situations and compared with the normal-theory values of the power. Non-normality seems to have little effect on the power of the F-test.

1. INTRODUCTION

Many investigations have been made to study the effect of non-normality on Type I error of the F-test employed in analysis-of-variance [1, 2, 5, 6, 7, 10, 16, 17], but there have been few attempts to investigate the effect on the power of the test. Donaldson [4] obtained values of the power for normal, exponential and log-normal distributions through Monte Carlo methods. David and Johnson [3] calculated the product moments of "between" and "within" sum of squares under most general assumptions of non-normality. Srivastava [13] obtained the values of the power basing his derivations on the first four terms of the Edgeworth series. In this article, an expression for the power function of the F-test under non-normal situations is obtained from Laguerre series expansions of "between" and "within" sum of squares. The values of the power are computed under non-normal situations and compared with the normal-theory values of the power. The non-normal theory power of F is found to differ from the normal-theory power by a correction term which decreases sharply with increasing sample size. A simple two moment F-approximation is also derived, and it is found to provide satisfactory values of the power in cases of near-normal populations.

2. NUMERICAL VALUES OF THE POWER

In one-way-classification for analysis of variance, let $x_{ij}(i=1, 2, \cdots, c; j=1, 2, \cdots, n)$ be the jth observation in the ith group. Assume the mathematical model

$$x_{ij} = a + g_i + e_{ij}, \qquad \textstyle\sum_i g_i = 0 \tag{2.1}$$

where a is a constant, g_i is the ith group effect and e_{ij} is a random error. Write

$$S_1 = n\textstyle\sum_i (x_{i.} - x_{..})^2 \text{ and } S_2 = \sum_i \sum_j (x_{ij} - x_{i.})^2 \tag{2.2}$$

where

$$x_{i.} = \textstyle\sum_j x_{ij}/n, \qquad x_{..} = \sum_i \sum_j x_{ij}/N, \quad N = nc.$$

To test the null hypothesis H_0: $g_1 = g_2 = \cdots = g_c = 0$ under the assumption that e_{ij}'s are independently normally distributed with mean zero and constant variance σ^2 for all i and j, the statistic

$$F = f_2 S_1/f_1 S_2, \qquad f_1 = c-1, \quad f_2 = N - c \tag{2.3}$$

is used [15, p. 46]. The hypothesis H_0 is rejected if the value of F is found to be greater than the tabulated value F_α for a preassigned level of significance α. The effect of moderate non-normality on the Type I error α is known to be not very serious (see [5], [10] and [16]). Let H_1 be the hypothesis that $g_i \neq 0$, for some i. If $p(F, H_1)$ denotes the probability density function of F when H_1 is true

$$1 - \beta = \int_{F_\alpha}^{\infty} p(F, H_1)dF \tag{2.4}$$

determines the normal-theory power of F. Tang [14] and Tiku [19] have tables of $1-\beta$.

Assume that e_{ij}'s are not normally distributed and have a common non-normal distribution. Specify this distribution by the cumulants $0, \sigma^2, \kappa_3, \cdots, \kappa_r, \cdots$. Define

$$\lambda_r = \kappa_r/\sigma^r, \qquad r = 3, 4, \cdots \tag{2.5}$$

and

$$\lambda = n\textstyle\sum_i (g_i/\sigma)^2, \quad \mu = n\sum_i (g_i/\sigma)^3, \quad \delta = n\sum_i (g_i/\sigma)^4 \tag{2.6}$$

and note that if H_0 is true, λ, μ and δ are all zero. Also note that for a normal distribution $\lambda_r = 0$, for $r = 3, 4, \cdots$. To study the effect of non-normality on the power, the test function F in the non-normal case will be denoted by F^*. An approximation to order N^{-2} for the power-function of F^* is obtained (see the mathematical appendix)

$$\begin{aligned} 1 - \beta^* &= \int_{F_\alpha}^{\infty} p(F^*, H_1)dF^* \\ &\simeq (1-\beta) - \lambda_3\mu A + \lambda_4(B + B_1\delta) - \lambda_3^2 C \\ &\quad + \lambda_5\mu D - \lambda_6 E + \lambda_4^2 H \end{aligned} \tag{2.7}$$

where $1-\beta$ is given by Equation (2.4), and A, B, etc., are non-normality corrective-functions. The numerical values of these functions, presented in Table 3, give an idea of the relative contributions of the population standard

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cumulants λ_r. Srivastava's [13] equation (21) is similar to (2.7) but he did not work out the corrective-terms due to λ_5, λ_6 and λ_4^2. His equation however contains terms in $\lambda_3^2\mu^2$ and $\lambda_3^2\delta$, besides the first four terms on the right-hand side of (2.7). The contributions of these two terms are very small (see [13, Table 1]).

The difference $\beta-\beta^*$ between the power $1-\beta^*$ and the normal-theory power $1-\beta$ represents the effect of non-normality on the power of the test function F. To calculate the values of this difference we note that with $\sum_i g_i=0$, the following two extreme situations can arise (see [14]):

1. The c groups form two distinct equal sets, with a single group falling mid-way between for odd c, the effect in one set cancelling the effect of the other set. In this situation the value of μ is zero and the value of δ is minimum and equal to λ^2/N or $c\lambda^2/N(c-1)$ according as c is even or odd.
2. Of the c groups, $c-1$ are similar and only one is different, its effect counterbalancing the rest. In this situation μ has an extreme positive or negative value $\pm(c-2)\lambda^{3/2}/\sqrt{\{N(c-1)\}}$, and δ is maximum and equal to $(c^2-3c+3)\lambda^2/N(c-1)$.

In practice, the situation may be anywhere between the two extremes (1) and (2).

Consider one of the three distributions described by E. S. Pearson [11, pp. 101–2] under his "Case B," namely, the Pearson Type IV curve

$$f(y \mid a, q, d) = \text{constant} \times (1 + y^2/a^2)^{-q} \cdot \exp\{-d \tan^{-1}(y/a)\}, \qquad -\infty \le y \le \infty,$$

the non-central t distribution

$$f(t \mid v, \tau) = \text{constant} \times (1 + t^2/v)^{-\frac{1}{2}(v+1)} \cdot \exp\{-\tfrac{1}{2}v\tau^2/(v+t^2)\} \cdot Hh_v\{-t\tau/\sqrt{(v+t^2)}\},$$

$$Hh_v(y) = \frac{1}{\Gamma(v+1)} \int_0^\infty u^v e^{-\frac{1}{2}(u+y)^2} du, \qquad -\infty \le t \le \infty,$$

and the Johnson S_U curve

$$f(y \mid \gamma_0, \gamma_1) = \text{constant} \times (\cosh^{-1} y) \cdot \exp\{-\tfrac{1}{2}(\gamma_0 + \gamma_1 \sinh^{-1} y)^2\}, \quad -\infty \le y \le \infty.$$

These are similar distributions with $\lambda_4=1.23$, $\lambda_3^2=0.58$ and have all values of λ_5 and λ_6 not far from 3.0 and 11.0, respectively. Suppose that the error has a distribution in one of these forms. We consider three extreme cases, namely,

a. $\mu=0$ and δ is minimum,
b. μ is maximum positive and δ is maximum and
c. μ is maximum negative and δ is maximum.

The values of the difference $\beta-\beta^*$ are given in Table 1, for various values of the normalized non-centrality parameter $\phi=\sqrt{\{\lambda/(f_1+1)\}}$, (see [14]). The values for case (a) generally lie between the values for cases (b) and (c), and the latter do not differ from each other by more than a few units in the third decimal place (see Table 1). All the values of the difference in Table 1 are small and are positive except for large values of ϕ. However, these negative differences may be due to the approximating nature of equation (2.7).

The values of the difference $\beta-\beta^*$ for case (a) for various values of λ_4 and λ_3^2 and $\lambda_6=0$ are given in Table 2. For other combinations of values of λ_r, the values of this difference may be calculated from Table 3. Calculations show that a value of λ_6 as high as twenty does not change these values by more than ± 0.009.

1. VALUES OF $\beta-\beta^*$, THE DIFFERENCE BETWEEN THE POWER $1-\beta^*$ AND THE NORMAL-THEORY POWER $1-\beta$ FOR PEARSON'S DISTRIBUTION, FOR DEGREES OF FREEDOM f_1 AND f_2 AND NORMALIZED NONCENTRALITY PARAMETER ϕ: $\alpha=0.05$

		$f_1=1$			$f_1=4$			$f_1=9$		
ϕ	(μ,δ)	f_2								
		10	20	60	10	20	60	10	20	60
0.0	1-β	.050	.050	.050	.050	.050	.050	.050	.050	.050
	(a)	-.001	-.001	-.000	.000	-.001	-.001	.003	-.000	-.001
0.5	1-β	.098	.103	.107	.095	.106	.116	.097	.115	.134
	(a)	.002	.002	.001	.001	.001	.000	.003	.001	.000
	(b)	.002	.002	.001	.001	.001	.000	.003	.000	.000
	(c)	.002	.002	.001	.001	.001	.000	.003	.001	.000
1.0	1-β	.242	.261	.274	.262	.319	.367	.290	.392	.496
	(a)	.014	.011	.005	.006	.008	.006	.008	.009	.007
	(b)	.014	.011	.005	.003	.004	.003	.006	.004	.004
	(c)	.014	.011	.005	.010	.013	.010	.011	.016	.014
1.5	1-β	.471	.507	.531	.550	.655	.730	.625	.789	.896
	(a)	.026	.016	.005	.017	.013	.002	.011	.006	.004
	(b)	.026	.016	.005	.013	.012	.006	.009	.008	.003
	(c)	.026	.016	.005	.021	.014	-.001	.014	.003	-.010
2.0	1-β	.716	.760	.788	.820	.907	.950	.890	.974	.996
	(a)	.016	.003	.000	.009	-.003	-.003	-.001	-.007	-.002
	(b)	.016	.003	.000	.013	.005	.002	.003	-.001	-.000
	(c)	.016	.003	.000	.005	-.011	-.009	-.005	-.014	-.004
2.5	1-β	.890	.923	.942	.958	.989	.997	.984	.999	1.000
	(a)	-.007	-.009	-.004	-.007	-.006	-.001	-.007	-.002	-.000
	(b)	-.007	-.009	-.004	-.006	-.002	.000	-.004	-.001	-.000
	(c)	-.007	-.009	-.004	-.009	-.009	-.002	-.011	-.003	-.000

2. VALUES OF THE DIFFERENCE $\beta-\beta^*$ FOR $\alpha=0.05$ AND 0.01: DEGREES OF FREEDOM $f_1=4$ AND $f_2=20$: $\lambda_6=0$: $\mu=0$, $\delta=c\lambda^2/N(c-1)$

		$\lambda_3^2=0.0$				$\lambda_3^2=1.5$				$\lambda_3^2=3.0$			
	1-β	λ_4											
		-1.5	0	1.5	4	-1.5	0	1.5	4	-1.5	0	1.5	4
0.0	.050	.002	0	-.002	-.016	.004	.001	-.003	-.014	.006	.003	-.002	-.013
	.010	.001	0	-.001	-.007	.003	.002	.000	-.005	.006	.004	.002	-.003
0.5	.106	-.001	0	.000	.000	-.002	-.001	-.001	-.001	-.003	-.002	-.001	-.002
	.027	.001	0	-.001	-.003	.002	.001	.000	-.002	.004	.003	.002	.000
1.0	.319	-.017	0	.020	.059	-.021	-.004	.016	.056	-.024	-.007	.013	.052
	.123	-.008	0	.012	.039	-.011	-.002	.009	.037	-.013	-.005	.007	.034
1.5	.655	-.017	0	.015	.042	-.016	.001	.016	.043	-.015	.002	.017	.044
	.530	-.029	0	.030	.080	-.033	-.003	.026	.077	-.029	.001	.024	.073
2.0	.907	.010	0	-.010	-.027	.012	.002	-.008	-.025	.013	.003	-.007	-.024
	.708	-.012	0	.009	.014	-.009	.003	.012	.017	-.005	.007	.016	.021
2.5	.989	.008	0	-.007	-.018	.007	-.001	-.008	-.019	.007	-.001	-.008	-.019
	.924	.017	0	-.016	-.039	.019	.002	-.014	-.037	.020	.003	-.013	-.036
3.0	.999	.001	0	-.001	-.002	.001	.000	-.001	-.002	.001	.000	-.001	-.002
	.990	.004	0	-.004	-.008	.005	.001	-.005	-.009	.005	.001	-.005	-.009

From Tables 1 and 2 it is clear that the effect of moderate non-normality on the power of the F-test is unimportant. However, λ_4 has a greater effect on the power than λ_3^2.

It may be noted that (2.7) includes corrections only due to the first few population standard cumulants. For moderately non-normal populations it is expected that the contributions due to higher order standard cumulants will not be important especially for large N (see also Table 3). This might not be true for extremely non-normal populations in which case these cumulants could be very large in magnitude and approximations like (2.7) which ignore these cumulants might not be very useful (see [11]). It is clear that the accuracy of such approximations also depends on the relative magnitude of the successive standard cumulants λ_r. If (as in the case of chi-square and certain other distributions) $\lambda_r = 0\ (q^{1-\frac{1}{2}r})$, $q > 1$ being some constant, (2.7) will provide very accurate approximations.

3. VALUES OF THE NON-NORMALITY CORRECTIVE TERMS IN EQUATION (2.7) FOR DEGREES OF FREEDOM f_1 AND f_2 AND NORMALIZED NON-CENTRALITY PARAMETER ϕ: $\alpha = 0.05$

$f_1 = 4$, f_2	10^5A	10^5B	10^5B_1	10^5C	10^5D	10^5E	10^5H
				$\phi = 0.0$			
10	146	-377	- 3	-429	32	-35	-98
20	248	-240	-14	- 97	37	-15	-40
30	302	-172	-21	- 19	33	- 7	-19
60	373	91	-30	18	23	- 2	- 5
				$\phi = 0.5$			
10	154	-61	6	-171	21	-11	-16
20	281	53	12	70	21	3	15
30	349	74	16	81	18	4	13
60	439	65	23	50	11	2	6
				$\phi = 1.0$			
10	104	1324	6	410	0	72	150
20	168	1209	12	236	-4	40	63
30	191	981	15	122	-5	22	30
60	212	594	19	30	-4	7	7
				$\phi = 1.5$			
10	10	2086	2	262	-6	40	-17
20	-21	1063	1	-69	-6	-11	-43
30	-42	620	0	-72	-4	-12	-26
60	-70	228	-8	-31	-2	- 5	- 8
				$\phi = 2.0$			
10	-22	95	-0	-291	-17	-65	-94
20	-38	-658	-1	-107	0	-25	- 1
30	-41	-592	-2	- 30	1	- 9	6
60	-41	-352	-2	2	1	- 1	3
				$\phi = 2.5$			
10	-10	-1055	-0	-108	0	-24	35
20	- 7	- 482	-3	42	0	7	10
30	- 5	- 249	-0	26	0	4	2
60	- 3	- 83	-2	7	0	1	0

MATHEMATICAL APPENDIX

A.1 Distribution of F^*

To obtain the probability density function of F^*, write

$$x = \frac{1}{2}\,\frac{1}{\rho}\,S_1/\sigma^2 \quad\text{and}\quad y = \frac{1}{2}\,S_2/\sigma^2, \qquad \rho = (f_1 + 2\lambda)/(f_1 + \lambda) \tag{A.1}$$

and consider the approximation (see [16, p. 84])

$$p(x, y) \simeq \left\{ \textstyle\sum_{r=0}^{4} \sum_{s=0}^{4} \beta_{rs} L_r^{(m)}(x) L_s^{(k)}(y) \right\} p_m(x) p_k(y). \tag{A.2}$$

Here

$$\begin{cases} m = \frac{1}{2}p = \frac{1}{2}(f_1+\lambda)^2/(f_1+2\lambda), \quad k = \frac{1}{2}f_2, \quad p_n(u) = \dfrac{1}{\Gamma(n)}\, e^{-u} u^{n-1}, \\ \text{and} \\ L_r^{(n)}(u) = \dfrac{1}{r!} \sum_{j=0}^{r} (-1)^j \dbinom{r}{j} \dfrac{\Gamma(n+r)}{\Gamma(n+j)} u^j \quad (j = 0, 1, 2, \cdots) \end{cases} \tag{A.3}$$

is the rth Laguerre polynomial associated with $p_n(u)$.

Note that under normality assumption of the error distribution

$$p(x, y) \simeq p_m(x) p_k(y) \tag{A.4}$$

that is, x and y are independently distributed; $p_m(x)$ being Patnaik's [8] two-moment approximation to the distribution of $\frac{1}{2}\chi'^2/\rho = \frac{1}{2}(1/\rho)S_1/\sigma^2$, where χ'^2 is a non-central chi-square variate having f_1 degrees of freedom and non-centrality parameter λ.

Using the tables of the product moments of S_1 and S_2 [3] and proceeding on the same lines as Tiku [16], we obtain

$$\begin{aligned} &\beta_{00} = 1, \qquad \beta_{10} = \beta_{01} = 0 \\ &\beta_{20} = (1/N)(f_1^2/(p+2)(f_1+2\lambda))\lambda_4, \\ &\beta_{11} = (1/N)(f_1/(f_1+\lambda))\lambda_4, \\ &\beta_{02} = (1/N)(f_2/(f_2+2))\lambda_4 \end{aligned} \tag{A.5}$$

and so on. For $\lambda = 0$, (A.5) agree with the corresponding expressions in [16, p. 85]. Note that β_{rs}, $r+s \le 4$, are all the coefficients in terms of the population standard cumulants of order N^{-2} (see [3]).

From (A.2) and (A.5) the probability density function $p(w)$ of

$$w = kx/my = F^*/(1 + \lambda/f_1) \tag{A.6}$$

is obtained as

$$p(w) \simeq p_{00}(w) + \textstyle\sum \sum_{2 \le r+s \le 4} \beta_{rs} p_{rs}(w); \tag{A.7}$$

$p_{oo}(w)$ and $p_{rs}(w)$ are given by Tiku's [16] Equation (4.2) with v_1 replaced by p and v_2 by f_2.

A.2 Mean and Variance

From (A.7) we obtain the following expressions for the mean and variance of F^* (see also [16, p. 86]):

$$\begin{aligned} \mu_1' &= \left(1 + \frac{\lambda}{f_1}\right) \frac{f_2}{f_2 - 2} \left\{ 1 - \frac{1}{N}\,\frac{f_1}{(f_1+\lambda)}\lambda_4 + \frac{1}{N}\,\frac{f_2}{(f_2+2)}\lambda_4 + 0\left(\frac{1}{N^2}\right) \right\} \\ &\simeq \left(1 + \frac{\lambda}{f_1}\right)\left(1 + \frac{2}{N}\right) + \frac{1}{f_1 N}\lambda_4\lambda + 0\left(\frac{1}{N^2}\right) \end{aligned} \tag{A.8}$$

and

$$\begin{aligned} \mu_2 &= \left(1 + \frac{\lambda}{f_1}\right)^2 2f_2^2 \frac{(p + f_2 - 2)}{p(f_2-2)^2(f_2-4)} \\ &\quad \cdot \left\{ 1 + \frac{1}{2N}\,\frac{f_1^2(f_2-2)}{(p+f_2-2)(f_1+2\lambda)}\lambda_4 \right. \\ &\quad - \frac{1}{N}\,\frac{f_1(pf_2 + 4f_2 - 8)}{(p+f_2-2)(f_1+\lambda)}\lambda_4 \\ &\quad + \frac{1}{2N}\,\frac{f_2(pf_2 + 6f_2 + 2p - 12)}{(p+f_2-2)(f_2+2)}\lambda_4 \\ &\quad \left. - \frac{2}{N}\,\frac{f_2 - 2}{(p+f_2-2)(f_1+\lambda)(f_1+2\lambda)}\lambda_4\lambda^2 + 0\left(\frac{1}{N^2}\right) \right\} \\ &\simeq \frac{2}{p}\left[\frac{p+2}{f_1+2}\left(1 + \frac{f_1+6}{N}\right)\left(1 + \frac{\lambda}{f_1}\right)^2 \right. \\ &\quad + \frac{1}{2}\,\frac{p+2}{N}\left\{ \frac{p}{p+2} - \left(1 + \frac{\lambda}{f_1}\right) \right\}\lambda_4 \\ &\quad - \frac{1}{N}\,\frac{p+2}{f_1+2}\left(1 + \frac{\lambda}{f_1}\right)\lambda_4\lambda \\ &\quad \left. - \frac{2}{N(f_1+2)(f_1+2\lambda)}\left(1 + \frac{\lambda}{f_1}\right)\lambda_4\lambda^2 + 0\left(\frac{1}{N^2}\right) \right]. \end{aligned} \tag{A.9}$$

For $\lambda=0$, (A.8) and (A.9) agree with the corresponding expressions in [5] and [16]. In Section A.4 a simple two-moment F-approximation is derived from the above expressions. In cases of moderately non-normal populations, (A.8) and (A.9) should give accurate values for the mean and variance.

A.3 Power Function of F-Test

The power function

$$\begin{aligned} 1-\beta^* &= \operatorname{Prob}(F^* \geq F_\alpha) \\ &= \operatorname{Prob}\{w \geq F_\alpha/(1+\lambda/f_1)\} \end{aligned} \tag{A.10}$$

of the above F-test (equation (A.6)) is obtained from (A.7) as

$$1-\beta^* \simeq P_{00}(z) + \sum \sum\nolimits_{2\leq r+s\leq 4} \beta_{rs} \cdot \frac{\Gamma(\frac{1}{2}p+r)\Gamma(\frac{1}{2}f_2+s)}{r!s!\Gamma(\frac{1}{2}p)\Gamma(\frac{1}{2}f_2)} I_{r,s}(z) \tag{A.11}$$

where

$$z=1/\{1+f_1(f_1+\lambda)F_\alpha/f_2(f_1+2\lambda)\}, \quad P_{00}(z)=I_z(\tfrac{1}{2}f_2, \tfrac{1}{2}p) \tag{A.12}$$

and

$$I_{r,s}(z) = \sum\nolimits_{i=0}^{r} (-1)^i \binom{r}{i} \cdot \left\{ \sum\nolimits_{j=0}^{s} (-1)^j \binom{s}{j} I_z(\tfrac{1}{2}f_2+j, \tfrac{1}{2}p+i) \right\},$$

$I_z(a, b)$ being incomplete β-function [9]. The alternative forms of $I_{r,s}$ can be obtained by the method of effecting "index changes" [12] on I_z (a, b). These are given by Tiku's [16] equations (6.3) with v_1 replaced by p and v_2 by f_2, and are easy to compute.

Retaining only the terms of order N^{-2} in (A.5), we obtain from (A.11) the explicit equation (2.7) given in Section 2. The terms on the right hand side of this equation can be easily obtained from (A.5) and (A.11). For example

$$\begin{aligned} \operatorname{Prob}(F \geq F_\alpha) &= 1-\beta \simeq I_z(\tfrac{1}{2}f_2, \tfrac{1}{2}p) \\ &\quad + \frac{1}{24}\frac{\lambda^2}{\rho^3(f_1+\lambda)}\left\{4I_{3,0} + 3\frac{(f_1+4\lambda)}{(f_1+\lambda)}I_{4,0}\right\} \end{aligned} \tag{A.13}$$

$$A = \frac{1}{6}\frac{1}{\rho^3}\left\{I_{3,0} + 3\frac{\lambda}{\rho(f_1+\lambda)}I_{4,0}\right\} \tag{A.14}$$

and so on. Incidentally, note that (A.13) is Tiku's [18, p. 423] approximation to the power of the F-test in the normal situation.

A.4 Two-Moment Central F Approximation

Assume that F^*/h is distributed as normal-theory F-distribution having degrees of freedom (f, f_2). Equating the mean μ_1'/h and variance μ_2/h^2 given by (A.8) and (A.9) with the mean and variance of this distribution we obtain (see also [20])

$$\begin{aligned} h &= (f_2-2)\mu_1'/f_2 \quad \text{and} \\ f &= (f_2-2)/\{\tfrac{1}{2}(\mu_2/\mu_1'^2)(f_2-4)-1\} \qquad (f_2>4) \end{aligned} \tag{A.15}$$

and therefore the equation

$$1-\beta^* \simeq I_u(\tfrac{1}{2}f_2, \tfrac{1}{2}f), \qquad u = 1/(1+fF_\alpha/hf_2). \tag{A.16}$$

Calculations show that even for "error" degrees of freedom f_2 as small as ten the equation (A.16) gives substantially the same values as Srivastava's equation (21). For example, for $\lambda_3=0.5$, $\lambda_4=2.0$, $f_1=4$, $f_2=10$, and $\phi=1.0$ and $\mu=0$, the values of $1-\beta^*$ calculated from these two equations are 0.281 and 0.285, respectively. In fact, the equation (A.16) gives accurate approximations in cases of near-normal populations represented by the first four terms of Edgeworth series ([5] and [13]).

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