Effect of Non-Normality on the Power of the Analysis of Variance Test
Author(s): A. B. L. Srivastava
Source: *Biometrika*, Vol. 46, No. 1/2 (Jun., 1959), pp. 114-122
Published by: Biometrika Trust
Stable URL: http://www.jstor.org/stable/2332813
Accessed: 17/06/2014 19:53

EFFECT OF NON-NORMALITY ON THE POWER OF THE ANALYSIS OF VARIANCE TEST

By A. B. L. SRIVASTAVA
Indian Institute of Technology, Kharagpur

1. Introduction

A number of studies have been made of the effect of non-normality on the test functions used for the analysis of variance. The effect on Type I error, i.e. on the distribution under the null hypothesis, was studied by Pearson (1931), Geary (1947) and Gayen (1950), and that on Type II error, i.e. on the power function, has been considered in a study by David & Johnson (1951). Pearson showed that while both 'between-groups' and 'within-groups' mean squares, in the case of non-normal variation, provide unbiased estimates of population variance, they are no longer independently distributed; in fact, their variances and covariance contain a term in λ_4.* However, he reached the conclusion that non-normality would not have a serious effect on the distribution of their ratio w, in large samples. Considering the effect of kurtosis only, Geary (1947) gave an approximate formula for the probability correction for w, based on the large sample assumption. Gayen (1950) derived the distribution of w for non-normal populations specified by the first four terms of the Edgeworth series. He found it to consist of corrective terms in λ_4 and λ_3^2 in addition to the normal theory probability density function of w. Assuming that the observation x_{ij} contains an error term e_{ij} which has a distribution of any form whatever, with all the cumulants existing and varying from group to group, David & Johnson (1951) obtained the moments of the distribution of a certain function of the observations which makes the study of power possible in the non-normal case. They did not, however, make direct use of the distribution of a non-central variance-ratio as is done in the case of the 'normal theory' power function.

Tang (1938) and Patnaik (1949) studied the power of the normal theory χ^2- and F-tests, by deriving the corresponding non-central distributions. The non-central distribution arises when the hypothesis of equal means is not true. The effect of non-normality on the power of the analysis of variance test can likewise be studied by investigating the non-central distribution of the variance-ratio. This has been derived in the present paper, on the assumption that the distribution of the error term is represented by the first four terms of the Edgeworth series. In addition to the normal theory power (Tang, 1938), the corrective terms in λ_3, λ_4 and λ_3^2 have been determined. In deriving conclusions we have kept in view the fact that only the values of λ_3 and λ_4 within certain limits (Barton & Dennis, 1952) can be permitted if the Edgeworth series is to represent a positive definite and unimodal frequency function.

We have considered here the simple case of k groups of n observations. Gayen (1950) obtained the distribution of the variance-ratio assuming unequal number of observations in different groups, but we have avoided this assumption as it would necessitate introduction of more parameters of non-centrality and would make the results more complicated. The results obtained here enable one to calculate the effect of non-normality on the probability of the error of second kind, and hence on the power, when the standard F-table significance levels are used. The difficulty of having a systematic tabulation of the derived expression for

* Here and below we use λ_3 and λ_4 to denote the standardized third and fourth cumulants of the variables.

the power arises from the fact that it contains confluent hypergeometric functions, which do not appear to have been tabulated so far in detail.

2. DISTRIBUTION OF THE NON-CENTRAL VARIANCE-RATIO

Let us consider k groups of observations x_{ij} $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, k)$, and suppose that

$$x_{ij} = A + B_j + e_{ij}, \tag{1}$$

where A is the grand mean, B_j the deviation of the jth group from the grand mean so that $\Sigma B_j = 0$ and e_{ij} represents the random residual distributed non-normally with mean zero, variance σ^2 and standardized third and fourth cumulants $\lambda_3(=\sqrt{\beta_1})$ and $\lambda_4(=\beta_2-3)$. Assuming the higher order cumulants to be zero, we can express the distribution of the standardized variable $z_{ij}(=e_{ij}/\sigma)$, by the first four terms of the Edgeworth series as

$$f(z) = \phi(z) - \frac{\lambda_3}{6}\phi^{(3)}(z) + \frac{\lambda_4}{24}\phi^{(4)}(z) + \frac{\lambda_3^2}{72}\phi^{(6)}(z), \tag{2}$$

where $\phi(z)$ is the standard normal function, and $\phi^{(\nu)}(z)$ its νth derivative.

Now 'between-groups' and 'within-groups' sums of squares can be written as

$$X = \frac{n}{\sigma^2}\sum_{j=1}^{k}(\bar{x}_j - \bar{x})^2 = n\sum_{j=1}^{k}(\bar{z}_j - \bar{z} + \delta_j)^2, \quad \text{where} \quad \delta_j = B_j/\sigma \tag{3}$$

and

$$Y = \frac{1}{\sigma^2}\sum_{j=1}^{k}\sum_{i=1}^{n}(x_{ij} - \bar{x}_j)^2 = \sum_{j=1}^{k}\sum_{i=1}^{n}(z_{ij} - \bar{z}_j)^2. \tag{4}$$

In the analysis of variance, as is known, we test the hypothesis H_0 that $\delta_j = 0$ for all j, and use the distribution of

$$w = [(N-k)X]/[(k-1)Y]$$

(with $\delta_j = 0$ for all j) in order to determine the point w_0 such that the probability $P(w \geqslant w_0)$ has a predetermined value α. We reject the hypothesis H_0 only if $w > w_0$, and accept it otherwise. If H_0 is not true, we can determine from the distribution of w (with δ_j's not all zero), the probability of accepting H_0, that is, the probability of the error of second kind.

Now, in order to obtain the power $\beta = 1 - P_{\mathrm{II}}$, we first derive the non-central distribution of w. Let us put

$$S_{1j} = \sum_{i=1}^{n} z_{ij} + n\delta_j, \quad S_{2j} = \sum_{i=1}^{n}(z_{ij} - \bar{z}_j)^2 \tag{5}$$

and

$$X_1 = \sum_{j=1}^{k} S_{1j}, \quad X_2 = \frac{1}{n}\sum_{j=1}^{k} S_{1j}^2, \tag{6}$$

so that $X = X_2 - X_1^2/N$, where $N = nk$.

Using the joint distribution of S_{1j} and S_{2j} which is obtained by replacing S_{1j} by $S_{1j} - n\delta_j$ in the result (2·11) of Gayen (1949), the joint characteristic function of X_1, X_2 and Y has been derived as

$$\begin{aligned}
\phi(t_1, t_2, u) = e^{\lambda\tau_{22}}\phi_0(t_1, t_2, u) &+ \frac{\exp\{-\frac{1}{2}(Nt_1^2 - 2it_2\lambda)/(1-2it_2)\}\,\tau_{22}^2}{(1-2it_2)^{\frac{1}{2}k}(1-2iu)^{\frac{1}{2}(N-k)}}\Bigg[\frac{2\lambda_3}{3}\{3\lambda\tau_{12} + 2\mu\tau_{22}\} \\
&+ \frac{\lambda_4}{3}\left\{3\lambda\left(\tau_{12}^2 - 1 + \frac{k}{N(1-2it_2)} + \frac{N-k}{N(1-2iu)}\right) + 4\mu\tau_{12}\tau_{22} + 2\nu\tau_{22}^2\right\} \\
&+ \frac{\lambda_3^2}{9}\Bigg\{3\lambda\Bigg([N\tau_{12}^4 - 3(N+3)\tau_{12}^2 + 6] + \frac{3}{(1-2it_2)}[(k+3)\tau_{12}^2 - (N+3k)/N] \\
&+ \frac{6k}{N(1-2it_2)^2} + \frac{3(N-k)}{N(1-2iu)}(N\tau_{12}^2 - 3) + \frac{3(N-k)}{N(1-2iu)(1-2it_2)} + \frac{3(N-k)}{N(1-2iu)^2}\Bigg) \\
&+ 2\mu\tau_{12}\tau_{22}\left([N\tau_{12}^2 - 3(N+6)] + \frac{3(k+6)}{(1-2it_2)} + \frac{3(N-k)}{(1-2iu)}\right) \\
&+ 18\tau_{22}^2(\lambda^2\tau_{12}^2 + 2\nu\tau_{22}) + 24\lambda\mu\tau_{12}\tau_{22}^3 + 8\mu^2\tau_{22}^4\Bigg\}\Bigg].
\end{aligned} \tag{7}$$

where $$\tau_{12} = \frac{it_1}{(1-2it_2)}, \quad \tau_{22} = \frac{it_2}{(1-2it_2)}; \tag{8}$$

$$\lambda = n\sum_1^k \delta_j^2, \quad \mu = n\sum_1^k \delta_j^3 \quad \text{and} \quad \nu = n\sum_1^k \delta_j^4 * \tag{9}$$

and where $\phi_0(t_1, t_2, u)$ denotes Gayen's (1950) result (2·6)†, in which we have taken $k'^2 = k^2$.

Obtaining the joint frequency function of X_1, X_2 and Y from (7) by Fourier's inversion theorem, and integrating out X_1, we find the joint frequency function of X and Y can be written in the form

$$g(.,X,Y) = g_0(.,X,Y) + \lambda_3 g_{\lambda_3}(.,X,Y) + \lambda_4 g_{\lambda_4}(.,X,Y) + \lambda_3^2 g_{\lambda_3^2}(.,X,Y), \tag{10}$$

the typical term in it being $$\phi(Y,r_1)\,\psi(X,r_2), \tag{11}$$

where $$\phi(Y,r_1) = \frac{Y^{\frac{1}{2}r_1-1}e^{-\frac{1}{2}Y}}{2^{\frac{1}{2}r_1}\Gamma(\frac{1}{2}r_1)} \tag{12}$$

and $$\psi(X,r_2) = \frac{e^{-\frac{1}{2}\lambda}e^{-\frac{1}{2}X}}{2^{\frac{1}{2}r_2}}\sum_{j=0}^{\infty}\frac{X^{\frac{1}{2}r_2+j-1}\lambda^j}{2^{2j}\Gamma(\frac{1}{2}r_2+j)\,j!}. \tag{13}$$

We can now deduce the frequency function of $w = [(N-k)X]/[(k-1)Y]$ from (10) in the form

$$g(w) = g_0(w) + \lambda_3 g_{\lambda_3}(w) + \lambda_4 g_{\lambda_4}(w) + \lambda_3^2 g_{\lambda_3^2}(w). \tag{14}$$

Using $p(w; r_1, r_2)$ to denote

$$\sum_{j=0}^{\infty}\frac{e^{-\frac{1}{2}\lambda}(\frac{1}{2}\lambda)^j}{j!\,B(\frac{1}{2}r_1+j,\frac{1}{2}r_2)}\left(\frac{\nu_1}{\nu_2}\right)^{\frac{1}{2}r_1+j}\frac{w^{\frac{1}{2}r_1+j-1}}{\left(1+\frac{\nu_1}{\nu_2}w\right)^{\frac{1}{2}(r_1+r_2)+j}}, \tag{15}$$

where $\nu_1 = k-1$, $\nu_2 = N-k$ are the degrees of freedom of X and Y, we find

$$g_0(w) = p(w;\nu_1,\nu_2), \tag{16}$$

$$g_{\lambda_3}(w) = \tfrac{1}{6}\mu p_3(w;\nu_1,\nu_2) \tag{17}$$

$$\begin{aligned} g_{\lambda_4}(w) = \frac{1}{24N}[3\{&(\nu_1+\nu_2)^2 p(w;\nu_1,\nu_2) - 2\nu_1(\nu_1+\nu_2)p(w;\nu_1+2,\nu_2) \\ &+\nu_1^2 p(w;\nu_1+4,\nu_2) - 2\nu_2(\nu_1+\nu_2)p(w;\nu_1,\nu_2+2) + 2\nu_1\nu_2 p(w;\nu_1+2,\nu_2+2) \\ &+\nu_2^2 p(w;\nu_1,\nu_2+4)\} + 6\lambda\{-(\nu_1+\nu_2)p_2(w;\nu_1,\nu_2) + \nu_1 p_2(w;\nu_1+2,\nu_2) \\ &+\nu_2 p_2(w;\nu_1,\nu_2+2)\} + N\nu p_4(w;\nu_1,\nu_2)], \end{aligned} \tag{18}$$

$$\begin{aligned} g_{\lambda_3^2}(w) = \frac{1}{72N}[6\{&-(\nu_1+\nu_2)(\nu_1+\nu_2-1)p(w;\nu_1,\nu_2) + 3\nu_1(\nu_1+\nu_2-1)p(w;\nu_1+2,\nu_2) \\ &-3\nu_1(\nu_1-1)p(w;\nu_1+4,\nu_2) + \nu_1(\nu_1-1)p(w;\nu_1+6,\nu_2) \\ &+3\nu_2(\nu_1+\nu_2-1)p(w;\nu_1,\nu_2+2) - 6\nu_1\nu_2 p(w;\nu_1+2,\nu_2+2) \\ &-3\nu_2(\nu_2-1)p(w;\nu_1,\nu_2+4) + 3\nu_1\nu_2 p(w;\nu_1+2,\nu_2+4) \\ &+\nu_2(\nu_2-\nu_1-1)p(w;\nu_1,\nu_2+6)\} + 18\lambda\{(\nu_1+\nu_2-1)p_2(w;\nu_1,\nu_2) \\ &-2(\nu_1-1)p_2(w;\nu_1+2,\nu_2) + (\nu_1-1)p_2(w;\nu_1+4,\nu_2) - 2\nu_2 p_2(w;\nu_1,\nu_2+2) \\ &+\nu_2 p_2(w;\nu_1,\nu_2+4)\} + 9(\lambda^2-N\nu)\{p_4(w;\nu_1,\nu_2) - p_4(w;\nu_1+2,\nu_2)\} \\ &+N\mu^2 p_6(w;\nu_1,\nu_2)], \end{aligned} \tag{19}$$

* The conventional λ, or non-central parameter, used here should not be confused with the standardized cumulants, λ_3 and λ_4, nor should ν be confused with the degrees of freedom, ν_1 and ν_2, defined below.

† In the first line of formula (2.6) of Gayen (1950), the factor $(1-2it_2)^{\frac{1}{2}}$ in the denominator should read as $(1-2it_2)^{\frac{1}{2}k}$.

where $$p_r(w; r_1, r_2) = p(w; r_1+2r, r_2) - rp(w; r_1+2r-2, r_2) + \frac{r(r-1)}{2!} p(w; r_1+2r-4, r_2) - \ldots + (-1)^r p(w; r_1, r_2). \tag{20}$$

3. POWER FUNCTION

We are now in a position to evaluate the probability of the error of second kind P_{II} by integrating $g(w)$ over the region of acceptance. If w_0 be the point which determines the critical region, we shall have

$$P_{\mathrm{II}}(w_0) = \int_0^{w_0} g(w)\,dw = P_0(w_0) + \lambda_3 P_{\lambda_3}(w_0) + \lambda_4 P_{\lambda_4}(w_0) + \lambda_3^2 P_{\lambda_3^2}(w_0). \tag{21}$$

In this, $P_0(w_0)$ as obtained from (16), is given by

$$P_0(w_0) = \sum_{j=0}^{\infty} \frac{e^{-\frac{1}{2}\lambda}(\frac{1}{2}\lambda)^j}{j!} I_{u_0}(\tfrac{1}{2}\nu_1 + j, \tfrac{1}{2}\nu_2), \tag{22}$$

where $u_0 = (k-1)\,w_0/[(N-k)+(k-1)\,w_0]$ and I_{u_0} denotes the incomplete β-function. This is the normal-theory expression for $P_{\mathrm{II}}(w_0)$ and the methods of its evaluation have been discussed by Tang (1938) and Patnaik (1949). Here the expressions for $P_{\lambda_3}(w_0)$, $P_{\lambda_4}(w_0)$ and $P_{\lambda_3^2}(w_0)$ as obtained by integrating (17), (18) and (19), respectively, are expressible in terms of the confluent hypergeometric functions

$$F(a, b, x) = \sum_{j=0}^{\infty} \frac{\Gamma(a+j)\,\Gamma(b)}{\Gamma(b+j)\,\Gamma(a)} \frac{x^j}{j!}.$$

Writing $$F_{(0)}(a, b, \tfrac{1}{2}\lambda u_0) = \frac{\Gamma(a)}{\Gamma(a-b+1)\,\Gamma(b)} F(a, b, \tfrac{1}{2}\lambda u_0), \tag{23}$$

and
$$\left.\begin{aligned} F_{(r)}(a, b, \tfrac{1}{2}\lambda u_0) &= F_{(0)}(a, b, \tfrac{1}{2}\lambda u_0) - {}^rC_1 u_0 F_{(0)}(a+1, b+1, \tfrac{1}{2}\lambda u_0) \\ &\quad + \ldots + (-1)^r u_0^r F_{(0)}(a+r, b+r, \tfrac{1}{2}\lambda u_0), \\ F_{(r)'}(a, b, \tfrac{1}{2}\lambda u_0) &= F_{(0)}(a, b, \tfrac{1}{2}\lambda u_0) - {}^rC_1(1-u_0)\, F_{(0)}(a+1, b, \tfrac{1}{2}\lambda u_0) \\ &\quad + \ldots + (-1)^r (1-u_0)^r F_{(0)}(a+r, b, \tfrac{1}{2}\lambda u_0), \end{aligned}\right\} \tag{24}$$

we get $$P_{\lambda_3}(w_0) = -\tfrac{1}{6}\mu\, e^{-\frac{1}{2}\lambda} u_0^{\frac{1}{2}\nu_1} (1-u_0)^{\frac{1}{2}\nu_2} F_{(2)}\left(\frac{\nu_1+\nu_2}{2}, \frac{\nu_1+2}{2}, \tfrac{1}{2}\lambda u_0\right), \tag{25}$$

$$\begin{aligned} P_{\lambda_4}(w_0) = \frac{1}{8N} e^{-\frac{1}{2}\lambda} u_0^{\frac{1}{2}\nu_1}(1-u_0)^{\frac{1}{2}\nu_2} \Bigg[& \nu_1^2 F_{(1)}\left(\frac{\nu_1+\nu_2}{2}, \frac{\nu_1+2}{2}, \tfrac{1}{2}\lambda u_0\right) \\ & - \nu_2^2 F_{(1)'}\left(\frac{\nu_1+\nu_2}{2}, \frac{\nu_1}{2}, \tfrac{1}{2}\lambda u_0\right) - 2\nu_1\nu_2 F_{(1)}\left(\frac{\nu_1+\nu_2}{2}, \frac{\nu_1}{2}, \tfrac{1}{2}\lambda u_0\right) \\ & - 2\lambda \left\{ \nu_1 F_{(2)}\left(\frac{\nu_1+\nu_2}{2}, \frac{\nu_1+2}{2}, \tfrac{1}{2}\lambda u_0\right) - \nu_2 F_{(2)}\left(\frac{\nu_1+\nu_2}{2}, \frac{\nu_1}{2}, \tfrac{1}{2}\lambda u_0\right) \right\} \\ & + \frac{N}{3} \nu F_{(3)}\left(\frac{\nu_1+\nu_2}{2}, \frac{\nu_1+2}{2}, \tfrac{1}{2}\lambda u_0\right) \Bigg], \end{aligned} \tag{26}$$

$$\begin{aligned} P_{\lambda_3^2}(w_0) = \frac{1}{12N} e^{-\frac{1}{2}\lambda} u_0^{\frac{1}{2}\nu_1}(1-u_0)^{\frac{1}{2}\nu_2} \Bigg[& -\nu_1(\nu_1-1)\, F_{(2)}\left(\frac{\nu_1+\nu_2}{2}, \frac{\nu_1+2}{2}, \tfrac{1}{2}\lambda u_0\right) \\ & + \nu_2(\nu_2-1)\, F_{(2)}\left(\frac{\nu_1+\nu_2}{2}, \frac{\nu_1}{2}, \tfrac{1}{2}\lambda u_0\right) - \nu_1\nu_2 \left\{ 3F_{(2)'}\left(\frac{\nu_1+\nu_2}{2}, \frac{\nu_1+2}{2}, \tfrac{1}{2}\lambda u_0\right) \right. \\ & \left. + F_{(2)'}\left(\frac{\nu_1+\nu_2}{2}, \frac{\nu_1}{2}, \tfrac{1}{2}\lambda u_0\right) \right\} + 3\lambda \left\{ (\nu_1-1)\, F_{(3)}\left(\frac{\nu_1+\nu_2}{2}, \frac{\nu_1+2}{2}, \tfrac{1}{2}\lambda u_0\right) \right. \\ & \left. + \nu_2 F_{(3)}\left(\frac{\nu_1+\nu_2}{2}, \frac{\nu_1-2}{2}, \tfrac{1}{2}\lambda u_0\right) \right\} + \tfrac{3}{2}(\lambda^2 - N\nu)\, F_{(4)}\left(\frac{\nu_1+\nu_2}{2}, \frac{\nu_1+2}{2}, \tfrac{1}{2}\lambda u_0\right) \\ & + \frac{N}{6} \mu^2 F_{(5)}\left(\frac{\nu_1+\nu_2}{2}, \frac{\nu_1+2}{2}, \tfrac{1}{2}\lambda u_0\right) \Bigg]. \end{aligned} \tag{27}$$

If ν_2 is an even integer, the above expressions can be reduced to the much simpler form of finite series by using the formula

$$F(a+r,a,x) = e^x F(-r,a,-x) = e^x \sum_{i=0}^{r} \frac{B(a,r+1)}{B(a+i,r+1-i)} \frac{x^i}{i!}. \tag{28}$$

For the normal-theory term, $P_0(w_0)$, such an expression in the form of a finite series when ν_2 is even, has already been given by Tang (1938). It is found that by using (28) the $F_{(r)}$ and $F_{(r)'}$ functions occurring in the corrective terms yield expressions which are not difficult to evaluate for ν_2 as large as 20. For example, in order to evaluate $P_{\lambda_3}(w_0)$ we have to calculate the value of $F_{(2)}\{\frac{1}{2}(\nu_1+\nu_2), \frac{1}{2}(\nu_1+2), \frac{1}{2}\lambda u_0\}$ which can be easily done by using the formula

$$F_{(2)}\left(\frac{\nu_1+\nu_2}{2}, \frac{\nu_1+2}{2}, \tfrac{1}{2}\lambda u_0\right) = e^{\frac{1}{2}\lambda u_0} \sum_{i=0}^{\frac{1}{2}\nu_2-1} \left[\binom{\frac{1}{2}(\nu_1+\nu_2+2)}{\frac{1}{2}(\nu_2-2)-i} u_0^2 - 2\binom{\frac{1}{2}(\nu_1+\nu_2)}{\frac{1}{2}(\nu_2-2)-i} u_0 + \binom{\frac{1}{2}(\nu_1+\nu_2-2)}{\frac{1}{2}(\nu_2-2)-i}\right] \frac{(\frac{1}{2}\lambda u_0)^i}{i!},$$

where
$$\binom{a}{b} = \frac{\Gamma(a+1)}{\Gamma(b+1)\,\Gamma(a-b+1)}.$$

Similar expressions can be worked out for $F_{(1)}, F_{(1)'}, F_{(2)'}, F_{(3)}$, etc., which occur in the equations (26) and (27), in order to simplify the computation of $P_{\lambda_4}(w_0)$ and $P_{\lambda_3^2}(w_0)$ when ν_2 is even.

4. Numerical evaluation of results and discussion

We shall now consider numerical examples to illustrate the nature of the effects of non-normality on the power. It should be mentioned here that the formulae given above are useful only in determining the effects of such non-normality as is not of very serious type. However, a considerable range of values of λ_3 and λ_4 can be covered within the limits given by Barton & Dennis (1952) (see § 1 above).

The values of $P_{\lambda_3}(w_0), P_{\lambda_4}(w_0)$ and $P_{\lambda_3^2}(w_0)$ have been calculated for $\nu_1 = 4, \nu_2 = 10$ and $\nu_1 = 4$, $\nu_2 = 20$ when the critical region is determined by the 'normal theory' upper 5 % values of F and the alternatives are given by $\phi = \sqrt{(\lambda/k)} = 1{\cdot}0$, $1{\cdot}5$, $2{\cdot}0$, $2{\cdot}5$, etc. This critical region is, of course, based on an erroneous F-value; the actual probability of Type I error, however, can be obtained by adding to 0·05 the corrections due to non-normality given by Gayen (1950)*. In the examples considered, since ν_2 is even, it has been possible to use the finite series expansions for calculation. The values are given in Table 1.

It is first necessary to examine what values μ and ν can take when $\lambda = k\phi^2$ has some fixed positive value. We have seen that the null-hypothesis H_0 assumes that δ_j's, the deviations of the group means from the grand mean, are all zero. If differences between groups exist, the δ_j's will not be all zero and their set of values will define an alternative to H_0. In practice, the true values of group differences are not known, and one would depend on the parameters λ, μ and ν determined by δ_j's to define an alternative hypothesis. As indicated by Tang (1938) in an example, two extreme forms of the set of δ_j's under the restriction $\sum_1^k \delta_j = 0$ will be as follows:

(*a*) the k groups form two distinct equal sets (with a single group midway between if k is odd), the effects in one set being equal and opposite in sign to those in the other;

(*b*) of the k groups, $k-1$ are quite similar and only one is divergent, with its effect counter-balancing the rest.

* There are a few slips in Gayen's (1950, pp. 242) formulae (2·30) to (2·33 *bis*). There should be a *minus* sign before λ_4 in (2·30) instead of the *plus* sign and in all the formulae from (2·31) to (2·33 *bis*) x_0 should be replaced by $1-x_0$.

For a fixed λ, in the case (*a*) the value of μ will be zero and that of ν will be minimum ($= \lambda^2/N$ or $k\lambda^2/[N(k-1)]$ according as k is even or odd). In the case (*b*), μ will have an extreme positive or negative value ($= \pm(k-2)\lambda^{\frac{3}{2}}/[N(k-1)]^{\frac{1}{2}}$) and ν will be maximum ($= (k^2-3k+3)\lambda^2/[N(k-1)]$). In practice the situation may be anywhere between these two extreme cases.

Table 1. *Showing the values of the corrective functions for* $\alpha = 0\cdot05$ *required in equation* (21) *when* (*i*) $\nu_1 = 4$, $\nu_2 = 10$ *and* (*ii*) $\nu_1 = 4$, $\nu_2 = 20$

D.F.	ϕ	$P_0(w_0)$	$P_{\lambda_3}(w_0)$	$P_{\lambda_4}(w_0)$	$P_{\lambda_3^2}(w_0)$
$\nu_1 = 4$	0·0	0·950	0	0·00376	$-0\cdot00429$
$\nu_2 = 10$	1·0	0·738	$0\cdot0010750\,\mu$	$-0\cdot01273-0\cdot0000744\,\nu$	$0\cdot00480-0\cdot0000213\,\nu + 0\cdot00000604\,\mu^2$
	1·5	0·449	$0\cdot0002396\,\mu$	$-0\cdot01916-0\cdot0000489\,\nu$	$0\cdot00300+0\cdot0000422\,\nu - 0\cdot00000007\,\mu^2$
	2·0	0·179	$-0\cdot0001670\,\mu$	$-0\cdot00156-0\cdot0000045\,\nu$	$-0\cdot00390+0\cdot0000157\,\nu - 0\cdot00000067\,\mu^2$
	2·5	0·043	$-0\cdot0001089\,\mu$	$0\cdot00944+0\cdot0000047\,\nu$	$-0\cdot00174-0\cdot0000008\,\nu - 0\cdot00000008\,\mu^2$
	3·0	0·006	$-0\cdot0000239\,\mu$	$0\cdot00495+0\cdot0000015\,\nu$	$0\cdot00085-0\cdot0000010\,\nu + 0\cdot00000002\,\mu^2$
$\nu_1 = 4$	0·0	0·950	0	0·00241	$-0\cdot00096$
$\nu_2 = 20$	1·0	0·681	$0\cdot0018003\,\mu$	$-0\cdot01122-0\cdot0001695\,\nu$	$0\cdot00273-0\cdot0000038\,\nu + 0\cdot00001832\,\mu^2$
	1·5	0·341	$0\cdot0000834\,\mu$	$-0\cdot00929-0\cdot0000818\,\nu$	$-0\cdot00159+0\cdot0001133\,\nu - 0\cdot00000190\,\mu^2$
	2·0	0·093	$-0\cdot0003425\,\mu$	$0\cdot00050+0\cdot000008\dot{3}\,\nu$	$-0\cdot00182+0\cdot0000169\,\nu - 0\cdot00000148\,\mu^2$
	2·5	0·012	$-0\cdot0000978\,\mu$	$0\cdot00458+0\cdot0000074\,\nu$	$0\cdot00084-0\cdot0000083\,\nu + 0\cdot00000006\,\mu^2$

N.B. The parameter λ, μ and ν have been defined in equations (9).

In Table 2, the values of $P_{\lambda_3}(w_0)$, $P_{\lambda_4}(w_0)$ and $P_{\lambda_3^2}(w_0)$ (of Table 1) are given for the following cases: (*a*) $\mu = 0$, ν maximum; (b_1) μ maximum positive, ν minimum; and (b_2) μ maximum negative, ν minimum. The sign of μ, which distinguishes case (b_2) from (b_1), only affects $P_{\lambda_3}(w_0)$. Where there are two signs with a value for case (*b*), the upper sign relates to (b_1). The values tabled are expected to be correct to five places of decimals.

In the examples, the effects of λ_3, λ_4 and λ_3^2 are observed generally in the third place of decimals but sometimes in the second place. The effect of λ_4 is, in general, higher than that of λ_3 or λ_3^2. Also it appears that the values of $P_{\lambda_4}(w_0)$, as well as of $P_{\lambda_3^2}(w_0)$, are not much different in the cases (*a*) and (*b*).

We must now compute the power of the test in the case of non-normal populations with given values of λ_3 and λ_4. In Table 3, such comparison is given for the examples considered above. Here the two entries before any value of ϕ in a column with $\lambda_3 = 0$ correspond to the cases (*a*) and (*b*), respectively, and the three entries for $\lambda_3 \neq 0$, correspond to the casess (*a*), (b_1) and (b_2). It is to be noted that the values for the cases (b_1) and (b_2) will interchange as λ_3 changes sign.

Table 2. *Giving the values of* $P_{\lambda_3}(w_0)$, $P_{\lambda_4}(w_0)$ *and* $P_{\lambda_3^2}(w_0)$ (*of Table* 1) *in the cases* (a), (b_1) *and* (b_2)

ϕ	Case	$\nu_1 = 4$, $\nu_2 = 10$			$\nu_1 = 4$, $\nu_2 = 20$		
		$P_{\lambda_3}(w_0)$	$P_{\lambda_4}(w_0)$	$P_{\lambda_3^2}(w_0)$	$P_{\lambda_3}(w_0)$	$P_{\lambda_4}(w_0)$	$P_{\lambda_3^2}(w_0)$
0·0		0	0·00376	−0·00429	0	0·00241	−0·00096
1·0	(a)	0	−0·01289	0·00476	0	−0·01143	0·00273
	(b)	± 0·00465	−0·01313	0·00479	± 0·00604	−0·01177	0·00293
1·5	(a)	0	−0·01968	−0·00345	0	−0·00981	−0·00087
	(b)	± 0·00350	−0·02050	−0·00415	± 0·00094	−0·01064	0·00003
2·0	(a)	0	−0·00171	−0·00338	0	−0·00067	−0·00148
	(b)	∓ 0·00579	−0·00195	−0·00334	∓ 0·00919	0·00093	−0·00201
2·5	(a)	0	0·00982	−0·00181	0	0·00494	0·00058
	(b)	∓ 0·00737	0·01043	−0·00228	∓ 0·00502	0·00552	0·00032
3·0	(a)	0	0·00520	0·00068	—	—	—
	(b)	∓ 0·00279	0·00561	0·00068	—	—	—

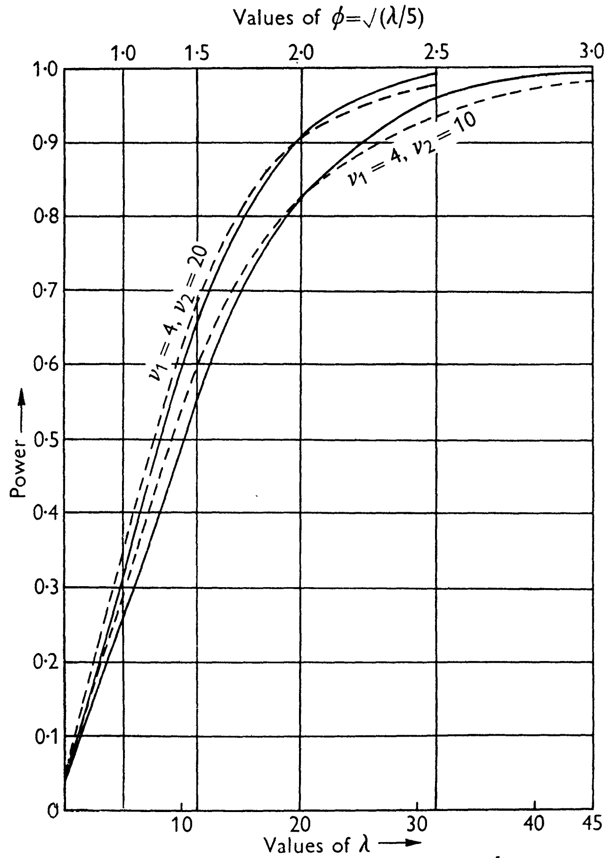

Fig. 1. Showing comparison of powers. —, normal; - - - -, non-normal ($\lambda_3 = 0$, $\lambda_4 = 2{\cdot}4$).

It is again seen that when $\lambda_3 = 0$ there is practically no difference between the values of power in the situations (a) and (b). When $\lambda_3 \neq 0$, in the three situations (a), (b_1) and (b_2) that arise, the powers differ to some extent though not very significantly from the practical point of view. The situation (a), in this case, gives rise to values which are generally intermediate between those of (b_1) and (b_2).

Table 3. *Showing comparison of powers of the F-test in normal and non-normal cases when using the normal theory with* 5 % *significance level*

ϕ	Symmetrical distributions, $\lambda_3 = 0{\cdot}0$						Skew distributions, $\lambda_3 = 0{\cdot}5$				$\lambda_3 = 0{\cdot}7$
	$\lambda_4 = -1{\cdot}0$	$\lambda_4 = 0{\cdot}0$	$\lambda_4 = 0{\cdot}5$	$\lambda_4 = 1{\cdot}0$	$\lambda_4 = 2{\cdot}0$	$\lambda_4 = 2{\cdot}4$	$\lambda_4 = -1{\cdot}0$	$\lambda_4 = 0{\cdot}0$	$\lambda_4 = 0{\cdot}5$	$\lambda_4 = 2{\cdot}0$	$\lambda_4 = 2{\cdot}4$
	Case (i) $\nu_1 = 4,\ \nu_2 = 10$										
0·0	0·054	**0·050**	0·048	0·046	0·042	0·041	0·055	0·051	0·049	0·044	0·043
1·0	0·249	**0·262**	0·268	0·275	0·288	0·290	0·248	0·261	0·267	0·287	0·291
	·249	—	·269	·275	·288	·294	·245	·258	·265	·285	·288
							·250	·263	·270	·289	·294
1·5	0·531	**0·551**	0·561	0·571	0·590	0·598	0·532	0·552	0·562	0·591	0·600
	·531	—	·561	·572	·592	600	·530	·550	·566	·591	·600
							·533	·554	·564	·595	·605
2·0	0·819	**0·821**	0·822	0·823	0·824	0 825	0·820	0·822	0·823	0·825	0·827
	·819	—	·822	·823	·825	·826	·823	·825	·826	·829	·831
							·817	·819	·820	·823	·823
2·5	0·967	**0·957**	0·952	0·947	0·937	0·933	0·967	0·957	0·953	0·938	0·934
	·967	—	·952	·947	·936	·932	·972	·961	·956	·940	·938
							·964	·954	·949	·933	·928
3·0	0·999	**0·994**	0·991	0·989	0·984	0·982	0·999	0·994	0·991	0·983	0·981
	—	—	·991	·988	·983	·981	·999	·995	·992	·984	·982
							·998	·992	·990	·981	·978
	Case (ii) $\nu_1 = 4,\ \nu_2 = 20$										
0·0	0·052	**0·050**	0·049	0·048	0·045	0·044	0·053	0·050	0·049	0·045	0·045
1·0	0·308	**0·319**	0·325	0·330	0·342	0·346	0·307	0·318	0·324	0·341	0·345
	·307	—	·325	·331	·343	·347	·303	·315	·321	·339	·342
							·310	·321	·327	·345	·350
1·5	0·649	**0·659**	0·664	0·669	0·679	0·683	0·649	0·659	0·664	0·679	0·683
	·648	—	·664	·670	·680	·685	·648	·658	·664	·680	·684
							·649	·659	·665	·681	·685
2·0	0·908	**0·907**	0·907	0·906	0·906	0·905	0·908	0·907	0·907	0·906	0·906
	·908	—	·907	·906	·905	·905	·913	·912	·912	·910	·912
							·904	·903	·902	·901	·899
2·5	0·993	**0·988**	0·986	0·983	0·978	0·976	0·993	0·988	0·985	0·978	0·976
	·994	—	·985	·982	·977	·975	·996	·990	·988	·979	·978
							·991	·985	·983	·974	·971

Power curves relating to the symmetric lepto-kurtic population with $\lambda_4 = 2{\cdot}4$ ($\beta_2 = 5{\cdot}4$) have been drawn in Fig. 1 along with the 'normal theory' power curves for the cases $\nu_1 = 4$, $\nu_2 = 10$ and $\nu_1 = 4$, $\nu_2 = 20$. This diagram illustrates the nature and magnitude of the effect of such a departure from normality in the distribution of the random residuals. Power curves in the case of the other non-normal populations considered in Table 3 would differ from normal theory to a less extent.

5. Conclusion

Obtaining the expressions for the power function of the analysis of variance test for a one-way classification in the case where the sampled population is represented by the first four terms of an Edgeworth series, we have considered certain numerical examples to see what is the nature of effects of non-normality on the power. The populations considered are only moderately non-normal, as the terms higher than the fourth in the Edgeworth series are assumed to be zero. Also the values of λ_3 and λ_4 have not been allowed to exceed the Barton & Dennis (1952) limits, ensuring thereby that the frequency function represented is positive definite and unimodal.

It is found that the effect of skewness is not much on the power of the analysis of variance test, at least when the values of λ_3 are confined to Barton & Dennis limits. In practice, however, higher values of λ_3 may occur and then the effect of skewness may be somewhat larger and comparable to that of kurtosis which is, in general, high. The presence of a fair degree of kurtosis, as is not uncommon in practice, leads to a noticeable change in the power curve particularly in the case of small samples. But a small departure from normality in respect of kurtosis (say, of the order $\lambda_4 = \pm 0{\cdot}5$) again does not cause any significant deviation in the power. When the population is leptokurtic ($\lambda_4 > 0$), the power increases in the beginning (for example, up to the point for which the power is approximately $0{\cdot}8$ in the case when $\nu_1 = 4$, $\nu_2 = 10$ (Fig. 1)), but in the region of very high power, it subsequently decreases in comparison with the normal-theory power. The reverse happens when λ_4 is negative.

The above conclusions derived from numerical results are expected to be valid in general. There is a good indication that the effect of non-normality on the power diminishes with increasing sample size, as expected. In practice the effect of kurtosis is likely to be more on the power than that of skewness. A general conclusion in this regard cannot, however, be given, for it is difficult to take into account all the ways in which the values of λ_3 and λ_4 of the distributions met in practice will vary. But on the whole it may be said that from the practical point of view, the effect of non-normality on power will not be of much consequence in the case of near-normal populations.

My thanks are due to Dr A. K. Gayen for his help and guidance in the course of preparation of this work. I am also grateful to Prof. E. S. Pearson for certain comments and suggestions leading to some improvement of the paper.

References

Barton, D. E. & Dennis, K. E. (1952). *Biometrika*, **39**, 425–7.
David, F. N. & Johnson, N. L. (1951). *Biometrika*, **38**, 43–57.
Gayen, A. K. (1949). *Biometrika*, **36**, 353–69.
Gayen, A. K. (1950). *Biometrika*, **37**, 236–55.
Geary, R. C. (1947). *Biometrika*, **34**, 209–42.
Patnaik, P. B. (1949). *Biometrika*, **36**, 202–32.
Pearson, E. S. (1931). *Biometrika*, **23**, 114–33.
Tang, P. C. (1938). *Statist. Res. Mem.* **2**, 126–57.