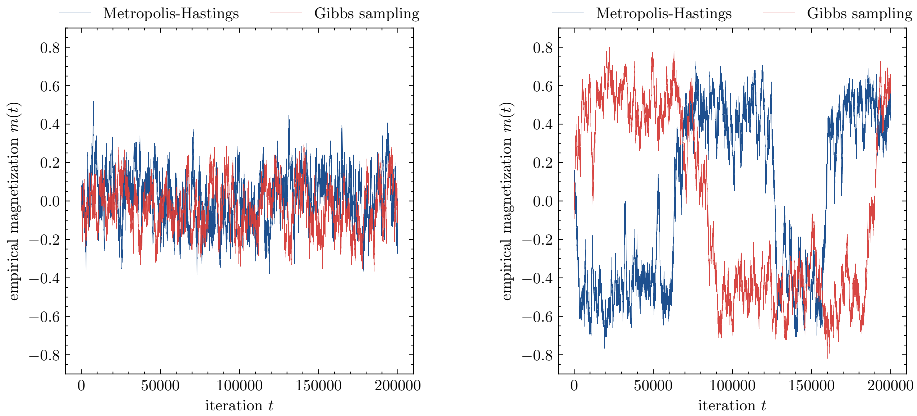
2.6.3 Numerical simulations

The before mentioned MCMC algorithms were implemented in Python. The code for the numerical simulation can be found in appendix D. For the simulation, the initial configuration $\sigma^{(0)}$ was drawn uniformly random from the state space Ω.



(a) $\beta = 0.8$

(b) $\beta = 1.1$

Figure 1: The empirical magnetization $m(t)$ as a function of the time step t of the Curie-Weiss model. In a) we set $\beta = 0.8$ and in b) $\beta = 1.1$. The other parameters were in both cases $n = 300, h = 0, J = 1, N = 200000$.

We see that for each β the empirical magnetization of the Gibbs chain and the Metropolis-Hastings chain show a similar behavior. For $\beta = 1.1$ chains remains more often in configurations where most of the spin have the same value.

The temperature (and therefore also β) has a strong influence on the *Gibbs distribution*. For high temperatures ($\beta \to 0$) the *Gibbs distribution* is similar to the uniform distribution over Ω. For low temperatures and $J > 0$ one expects that the chain stays for longer periods in one of two typical configuration, where either most of the spins are 1 or most of the spins are -1.

In figure 2 (b) it is observed that the Gibbs chain with $\beta = 1.1$ favors configurations where most of the spins are either positive or most of the spins are negative.

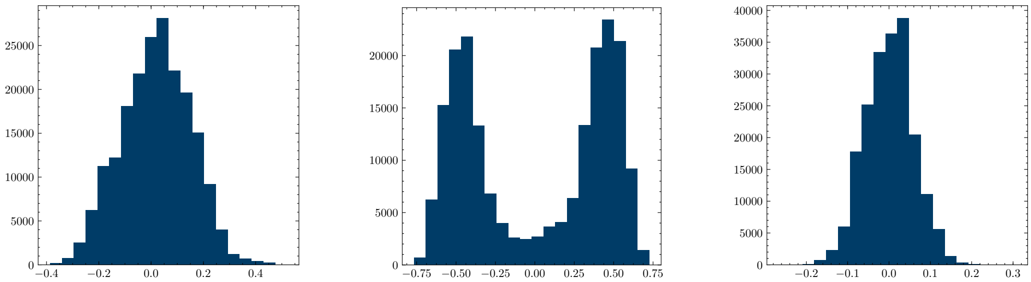
2.6.4 Mixing time of the Curie-Weiss model

The mixing time measures the number of steps that a chain has to take until it is sufficiently close to the stationary distribution π. It is given by

$$t_{mix}(\varepsilon) = \min\{t : \max_{x \in \Sigma} \|K^t(x, \cdot) - \pi\|_{TV} < \varepsilon\}.$$

Let $\hat{t} := t_{mix}(1/2)$. Then, for all $x \in \Sigma$ and for a positive integer k

$$\|K^{k\hat{t}}(x, \cdot) - \pi\|_{TV} < 2^{-k}.$$



(a) $\beta = 0.8$ (b) $\beta = 1.1$ (c) uniform distribution on Ω

Figure 2: Histogram of the empirical magnetization $m(t)$. Figure (a) and (b) display the histogram of the empirical magnetization of the Gibbs chain of the Curie-Weiss model. In (a) $\beta = 0.8$ and and in (b) $\beta = 1.1$. The other parameters were in both cases $n = 300, h = 0, J = 1, N = 200000$. In (c) we draw $N = 200000$ samples from a uniform distribution over $\{-1, 1\}^n$ with $n = 300$ and computed the empirical magnetization for each draw.

Let us consider Gibbs sampling for a Curie-Weiss model with n spins. Then, theorem 15.3 from [12] states that:

(i) If $\beta < 1$, then
$$\hat{t} \leq \frac{n(\log n + \log(2))}{1 - \beta}.$$

(ii) If $\beta > 1$, then there exists a positive function $r(\beta)$ such that
$$\hat{t} \geq \mathcal{O}(\exp(r(\beta)n)).$$

This means that fast mixing is only possible for $\beta < 1$ and the mixing time grows exponentially in n if $\beta > 1$. For $\beta > 1$ there is a bottleneck in the state space. This means that there are regions in the state space that are difficult to reach from some starting configurations.

In figure 1 b) it can be observed that if a chain starts in region with a positive empirical magnetization, then it is difficult to reach regions with a negative empirical magnetization.