Geometric Aspects of Harmonic Analysis

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Q1

$$\left.\begin{matrix} f : [a,b] \to \mathbb{R} \text{ integrable} \\ F(x) = \int_a^x f(t)\,\mathrm{d}t \end{matrix}\right] \underset{?}{\Rightarrow} F \text{ diff. (a.e. } x\text{)},\ F' = f$$

Q2 Conditions of F (on $[a,b]$) s.t.

- $F'(x)$ exists a.e.
- F' integrable
- $\int_a^b F'(x)\,\mathrm{d}x = F(b) - F(a)$

?

Q1 Differentiation of the integral

$$\frac{F(x+h)-F(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,\mathrm{d}t = \frac{1}{|I|}\int f = \mathrm{avg}_I f = \fint_I f$$

$I = (x, x+b)$, $|I|$ Lebesgue measure of I.

Q1 equivalent to averaging problem: Given $f \in L^1(\mathbb{R}^d)$, is it true, that

$$\lim_{|B|\to 0,\ x\in B} = \frac{1}{|B|}\int_B f = f(x) \quad (x\text{-a.e.})?$$

$B \subset \mathbb{R}^d$ open ball

Yes, if f continuous $\forall \varepsilon \exists \delta |x-y| < \delta \Rightarrow |f(x) - f(y) < \varepsilon$. $x \in B$

$$|f(x) = \fint_B f| = |\fint_B (f(y) - f(x))\,\mathrm{d}y| < \varepsilon \tag{1}$$

provided B is an opeb ball of radius $< \frac{\delta}{2}$ containing x

Yes, if f is integrable (not so easy). Hardy, Littlewood (1D, rearrangements; later Wiener for $d > 1$). $f \in L^1(\mathbb{R}^d)$

$$(Mf)(x) := \sup_{x\in B} \frac{1}{|B|}\int_B |f|$$

uncentered HL maximal function

Theorem. *Let f be integrable on $\mathbb{R}^d$. Then*

(i) Mf is measurable.

(ii) $(Mf)(x) < \infty$ *a.e.* x

(iii)

$$|\{x \in \mathbb{R}^d : (Mf)(x) > \alpha\}| < \frac{c}{\alpha}\|f\|_{L^1(\mathbb{R}^d)} \ (\forall x > 0). \tag{2}$$

$c = c_d = 3^d$, *independent of* f, α.

$f \neq f \in L^1 \Rightarrow Mf(x) \sim |x|^{-d}$ for large radius of x. So then $Mf \notin L^1$.

$$M : \begin{matrix} L^1 \not\to L^1 \\ L^1 \to L^{1,\infty} \end{matrix}$$

Proof. (i) easy $E_\alpha = \{x \in \mathbb{R}^d : (Mf)(x) > \alpha\}$ is open $(\forall x > 0)$ (because Mf is lowes semicontinuous)

(ii) $|\{x \in \mathbb{R}^d : (Mf)(x) = \infty\}| \subset |\{x \in \mathbb{R}^d : Mf(x) > \alpha\}|$, take $\alpha \to \infty$.

(iii) follows from an elemantary version of *Vitali covering*

□

Lemma. Let $B = \{B_1, B_2, \dots, B_N\}$ be a finite collection of open balls on $\mathbb{R}^d$. Then there exists a disjoint subcolletcino $B_{i_1}, B_{i_2}, \dots, B_{i_k}$ of B such that

$$|\bigcup_{j=1}^{n} B_j| \leq 3^d \sum_{j=1}^{k} |B_{ij}$$

Proof. (i) B_{i_1} = largest ball

(ii) Delete B_{i_1} and its neighbors

(iii) B_{i_2} = largest ball

(iv) repeat...

- Algorithm stops in at most N steps
- output has desired properties:
  - disjointness is clear
  - size $B \cap B' \neq \emptyset$, $r_{B'} \leq r_B$. B^* = ball with the same center as B but 3 times the radius. $\Rightarrow B' \subset B^*$. $|B^*| = 3^d|B|$

□

Back to (iii): Choose $\alpha > 0$, $E_\alpha = \{x \in \mathbb{R}^d : (Mf)(x) > \alpha\}$. Fr each

$$x \in E_\alpha \exists B = B_x := \frac{1}{|B_x|}\int_{B_x} |f(y)|\,\mathrm{d}y > \alpha$$

equivalent

$$|B_x| < \alpha^{-1}\int_{B_x} |f(y)|\,\mathrm{d}y$$

Fix $K \ll E_\alpha$ compact subset covered by $\bigcup_{x\in K} B_x$, $K \subset \bigcup_{l=1}]N B_l$

$$|K| \leq |\bigcup_{l=1}^{N} B_l| \underset{\text{Vitali}}{\leq} 3^d \sum_{j=1}^{k} |B_{ij}| \leq \frac{3^d}{\alpha} \in_{j=1}^{k} \int_{B_{i_j}} |f()| \, \mathrm{d}y = \frac{3^d}{\alpha} \int_{\bigcup_{j=1}^{k} B_{i_j}} |f(y)| \, \mathrm{d}y \leq \frac{3^d}{\alpha} \|f\|_{L^1(\mathbb{R}^d)}$$

Since K was choseen arbitrary (cpt.), it follows that

$$E_\alpha| \leq \frac{3^d}{\alpha} \|f\|_{L^1}$$

Can interpolate between weak type L^1-inequality and $L^\infty \to L^\infty$ (very easy).

Corollary (Lebesque differentiation theorem)**.** Let $f \in L^1(\mathbb{R}^d)$ Then

$$\lim_{|B|, \to 0, x \in B} \fint f = f(x) \quad x\text{-a.e.} \tag{3}$$

Proof.

$$E_\alpha = \{x \in \mathbb{R}^d : \limsup_{|B| \to 0, x \in B} |\fint_B f - f(x) > 2\alpha\}$$

ETS $|E_\alpha| = 0 \ \forall \alpha > 0$. Then $E = \bigcup_{n \in \mathbb{N}} E_{\frac{1}{n}} = 0$ and (3) holds on $E^\complement$.

Fix $\alpha > 0$, given $\varepsilon > 0$ choose $g \in C_0^0(\mathbb{R}^d)$ s.t. $\|f - g\|_{L^1} < \varepsilon$. Already seen

$$\lim_{|B| \Rightarrow 0, x \in B} \fint g = g(x) \ \forall x$$

$$\fint_B f - f(x) = \fint_B (f - g) + \fint_B g - g(x) + g(x) - f(x)$$

$$F_\alpha = \{x : M(f - r)(x) > \alpha\}$$
$$G_\alpha = \{x : |f(x) - g(x)| > \alpha\}$$

$E_\alpha \subset F_\alpha \cup G_\alpha$ since $u_1, u_2 > 0,\ u_1 + u_2 > 2\alpha \Rightarrow u_1 > \alpha \vee u_2 > \alpha$.

$$|G_\alpha| \leq \frac{1}{\alpha} \|f - g\|_{L^1} \quad \text{(Chebyshew)}$$

$$|F_\alpha| \leq \frac{c_d}{\alpha} \|f - g\|_{L^1} \quad \text{(weak type)}$$

$$|E_\alpha| \leq |F_\alpha| + |G_\alpha| \leq (\frac{c_d}{\alpha} + \frac{1}{\alpha}) \|f - g\|_{L^1} \leq \frac{c_d' \varepsilon}{\alpha}$$

Since $\varepsilon > 0$ was arbitrary $|E_\alpha| = 0$. □

$h \in L^1 \subset L^{1,\infty}$ by Chebyshew: $\infty > \|h\|_{l^1} = \int_{\mathbb{R}^d} |h(y)| \, \mathrm{d}y \geq \int_{h(y) \geq \alpha} |h(y)| \, \mathrm{d}y \geq \alpha |\{|h| > \alpha\}|$.
Would have been enough to replace $L^1(\mathbb{R}^d)$ by L^1_{loc}.

Sets $E \subset \mathbb{R}^d$ measurable, $x \in \mathbb{R}^d$ (not necc. in E) x is a point of Lebesque density of E if

$$\lim_{|B| \to 0, x \in B} \frac{|B \cap E|}{B} = 1$$

Corollary. Let $E \subset \mathbb{R}^d$ be measurable. Then

(i) Almost every $x \in E$ is a point of Lebesque density of E.

(ii) Almost every $x \notin E$ is not a point of Lebesque density.

Functions $f \in L^1_{\text{loc}}(\mathbb{R}^d)$.

$$Leb(f) := \{x \in \mathbb{R}^d : f(x) < \infty \text{ and } \lim_{|B|\to 0, x\in B} \fint_B |f(y) - f(x)|\,\mathrm{d}y = 0\}$$

f continuous at $\bar{x} \Rightarrow \bar{x} \in \text{Leb}(f) \Rightarrow \fint_B f \underset{|B|\to 0, x\in B}{\rightarrow} f(\bar{x})$ (all the inverse implications are wrong)

Corollary. $f \in L^1_{\text{loc}}(\mathbb{R}^d) \Rightarrow$ Almost every point belongs to $\text{Leb}(f)$.

(By checking the proof again?)

These things also works with other sets that "shrink regularly to x than balls". It gets worse however when one takes all parallel rectangles and even worse when arbitrarily oriented rectangles are allowed.

Q.2 Key: bounded varioation (BV) $F : [a, b] \to \mathbb{R}$, $P = \{a = t_0 < t_1 < \ldots < t_N = b\}$

$$V_F^P = \sum_{j=0}^{N} |F(t_j) - F(t_{j+1})|$$

is the variation of f over P. F is of bounded variation if

$$T_F(a, b) = T_F \sup_P V_F^P < \infty$$

$P \subset \tilde{P}$ partitions $\Rightarrow V_F^P \leq V_P^{\tilde{P}}$

Example. (i) f monotonic (increasing) and bounded, $|F| \leq M \Rightarrow F \in \text{BV}$

$$V_F^P = \sum_{j=1}^{N} |F(t_j)T = Nt_{j-1})| = F(b) - F(a) \leq 2M$$

(ii) F differentiable with F' bounded, $|F'| \leq M$, then by mean value theorem $f \in \text{BV}$. Or F LIpschitz

(iii) F α-Hölder ($\alpha < 1$) $6 \implies$ $F \in \text{BV}$. Take $F : [0, 1] \to \mathbb{R}$, $x \mapsto \mathrm{d}(x, C)^\alpha$, where C is the cantor set. 2^{n-1} intervals of length 3^{-n}

$$\alpha > \frac{\log 2}{\log 3} \Rightarrow \sum_{n=1}^{\infty} 2^{n-1}(3^{-n})^\alpha < \infty$$

- *Total variation* of F on $[a, x]$ (where $a \leq x \leq b$) is

$$T_F(a, x) = \sup \sum_{j=0}^{N} |F(t_j) - F(t_{j-1})|$$

- *Positive variation* of F on $[a, 1]$ is

$$P_F(a, x) = \sup \sum_{(+)} (F(t_j) - F(t_{j-1})) \quad \text{all } j : F(t_j) \geq F(t_{j-1})$$

- *Negative variation* of F on $[a, 1]$ is

$$N_F(a, x) = \sup \sum_{(-)} -(F(t_j) - F(t_{j-1}))$$

Lemma. $f : [a,b] \to \mathbb{R}$. Then

(i) $F(x) = F(a) + P_F(a,x) - N_F(a,x)$

(ii) $T_F(a,x) = PF_(a,x) + N_F(a,x)$

$(\forall x \in [a,b])$

recall from measure theory: $f = f^+ - f^-$, $|f| = f^+ + f^-$

Proof. (i) given $\varepsilon > 0$, $\exists P = \{a = t_0 < t_1 < \ldots < t_N = x\}$

$$|PVF - \sum_{(+)}(F(t_j) - F(t_{j-1}))| < \varepsilon$$

$$|N_F - \sum_{(-)} -(F(t_j) - F(t_{j-1})))| < \varepsilon$$

Also

$$F(x) - F(a) = \sum_{(+)} F(t_j) - F(t_{j-1}) - \sum_{(-)} -(F(t_j) - F(t_{j-1}))$$

□

Corollary. $F : [a,b] \to \mathbb{R} \in \mathrm{BV}$ iff F is the differenco of two inclasing bounded functions

Theorem. $F : [a,b] \to \mathbb{R} \in \mathrm{BV} \Rightarrow F$ *differentiable a.e.*

Wlog f mononotic increasing, "Wlog" f continuous

Lemma (of the rising sun)**.** $G : \mathbb{R} \to \mathbb{R}$ continuous.

$$E = \{x \in \mathbb{R} : \exists h = h_x > 0\ G(x+h) > G(x)\}$$

Then

(i) E is open ($E = \bigcup_{n=1}^{\infty}(a_n, b_n)$)

(ii) $g(a_n) = G(b_n)$, provided $b_n - a_n < \infty$.

Proof. Let (a_n, b_n) be a finite interval in the decomposition. $a_k \notin E$ then $g(a_k) \geq G(b_k)$. Assume $G(a_k) > G(b_k)$. $\exists c \in (a_k, b_k)\ g(c) = \frac{g(a_k)+g(b_k)}{2}$. Choose rightmost such c. $\exists d \in (c, b_k)\ G(d) > G(c)$. But then by continuity c could not have been chosen rightmost, contradiction. □

Can replace $\mathbb{R}$ by $[a,b]$, but then only get for $a_0 = a$ that $G(a_0) \leq G(b_0)$

Proof. of theorem

$$\begin{aligned}
\Delta_h(F)(x) &= \frac{F(x+h) - F(x)}{h} \\
D^{\pm}(F)(x) &= \limsup_{h\to 0, h><0} \Delta_h(F)(x) \\
D_{\pm}(F)(x) &= \liminf_{h\to 0, h><0} \Delta_h(F)(x)
\end{aligned}$$

Dini numbers. Upshot: They are all the same and finite. $D_- \leq D^-$, $D_+ \leq D^+$ clear. ETS

(i) $D^+(F)(x) < \infty$ (a.e. x)

(ii) $D^+(F)(x) \le D_-(F)(x)$ (a.e. x)

(ii) is equivalent to $D^-(F)(x) \le D_+(F)(x)$ by replacing $F(x)$ by $-F(-x)$ somewhere. Then $D^+ \le D_- \le D^- \le D_+ \le D^+ < \infty$.

(i) relacc: F increasing ,bounded, continuous on $[a,b]$. Fix $\gamma > 0$,

$$E_\gamma := \{x : D^+(F)(x) > \gamma\}$$

- E_γ is measurable
- Apply rising sun to $G(x) = F(x) - \gamma x$

$$E_\gamma \subset E = \{x \in [a,b] : \exists h > 0 \ G(x+h) > G(x)\} = \bigcup_{k=1}^{\infty}(a_k, b_k)$$

The condition in the set is equivalent to

$$\begin{aligned} &\iff \exists h > 0 \ F(x+h) - \gamma x - \gamma h > F(x) - \gamma x \\ &\iff \exists h > 0 \ \frac{F(x+h) - F(x)}{h} > \gamma \\ &\Leftarrow D^+(F)(x) > \gamma \end{aligned}$$

$G(a_k) \le G(b_k) \iff F(a_k) - \gamma a_k \le F(b_k) - \gamma b_k \iff \gamma(b_k - a_k) \le F(b_k) = F(a_k)$. Therefore

$$|E_\gamma| \le |E| \le \sum_{k=1}^{\infty}(b_k - a_k) \le \frac{1}{\gamma}\sum_{k=1}^{\infty}(F(b_k) - F(a_k)) = \frac{1}{\gamma}(F(b) - F(a))$$

Take $\gamma \to \infty$, done.

(ii) see Stein-Shakarchi (vol 3)

□

Corollary. F increasing, continuous $\Rightarrow F'$ exists a.e., measurable, nonnegative and

$$\int_a^b F'(x)\,\mathrm{d}x \le F(b) - F(a).$$

Proof. Let

$$G_h(x) = \frac{F(x + \frac{1}{n}) - F(x)}{\frac{1}{h}}$$

By the theorem, $G_h(x) \to F'(x)$ $(h \to 0)$ pointwise a.e. By Fatou

$$\int_{[a,b]} F' \le \liminf_{n\to\infty} \int_a^b G_h(x)\,\mathrm{d}x = \liminf_{n\to\infty} ⨍_b^{b+\frac{1}{n}} F(x)\,\mathrm{d}x - ⨍_a^{a+\frac{1}{n}} F(x)\,\mathrm{d}x$$

□

Cannot do better than $\leq$: For the Devil's staircase the left hand side is 0 while the rightn hand side is 1.

Why is the sunrise Lemma a covering Lemma?

$$\left.\begin{aligned} f_+^* &= \sup \tfrac{1}{h} \textstyle\int_x^{x+h} |f(y)| \, \mathrm{d}y \\ E_\alpha^+ &= \{x \in \mathbb{R} : f_+^*(x) > \alpha\} \end{aligned}\right\} |E_\alpha^+| = \frac{1}{\alpha} \int_{E_\alpha^+} |f|$$

Why? Let

$$G(x) = \int_0^x |f(y)| \, \mathrm{d}y - \alpha x$$

$$x \in E_\alpha^+ \iff f_+^*(x) > \alpha \iff \exists h > 0 \ \frac{1}{h} \int_x^{x+h} |f(y)| \, \mathrm{d}y > \alpha \iff \exists h > 0 \ G(x+h) > G(x)$$

$$\{x \in \mathbb{R} : \exists h > 0 \ G(x+h) > G(x)\} = \dot{\bigcup_{k \in \mathbb{N}}} (a_k, b_k), \quad G(a_k) = G(b_k)$$

$$|E_\alpha^+| = \sum_k (b_k - a_k) = \frac{1}{\alpha} \sum_k \int_{(a_k, b_k)} |f| = \frac{1}{\alpha} \int_{\bigcup_k (a_k, b_k)} |f| = \frac{1}{\alpha} \int_{|E_\alpha^+|} |f|.$$

Definition. $F : [a, b] \to \mathbb{R}$ is *absolutely continuous* if

$$\forall \varepsilon > 0 \exists \delta > 0 : \sum_{k`1}^{N} (B_k - a_k) < s$$

intervals $(a_k b_k)$ disjoint $(k = 1, ..., N) \Rightarrow$

$$\sum_{k=1}^{N} |F(b_k) - F(a_h)| < \varepsilon$$

Remark. (i) On a bounded Inetrval $I \subset \mathbb{R}$

$$C^1(I) \subset \mathrm{Lip}(I) \subset AC(I) \subset BV(I)$$

So they are diff. a.e.. All the inclusions are strict.

(ii) abs cont $\Rightarrow$ unif. con. $\Rightarrow$ cont.

(iii) $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ $F(x) = \int_0^x f(t) \, \mathrm{d}t$ Then F is absolutely continuous. $(\forall \varepsilon \exists \delta |E| < \delta \Rightarrow \int_E |f| < \varepsilon)$

Upshot: AC functions are the ones which re diff a..e. and vrify FTC.

Theorem. $F \in AC(a, b) \Rightarrow F'$ *exists a.e.,* $F' = 0$ *a.e.* $\Rightarrow F$ *constant*

- Existence of F' clear $\surd$
- $F' = 0$ a.e. $\Rightarrow F$ constant: refinement of Vitali

Definition. A collection $\mathscr{B} = \{B\}$ of (open) balls ön $\mathbb{R}^d$. is a *Vitali covering* of a set E if

$$\forall x \in E \forall \eta > 0 \exists B \in \mathscr{B} : x \in B, |B| < \eta$$

Lemma. $E \subset \mathbb{R}^d$ meas. $|E| < \infty$, $\mathscr{B}$ Vitali covering of E, $\delta > 0$. Then there exist finitely many disjoint balls $B_1, \ldots, bvB_N \in \mathscr{B}$

$$\sum_{j=1}^{N} |B_j| \geq |E| - \delta$$

Recall elementary Vitali: $\mathscr{B} = \{B_1, \dots, B_N\}$ finite collection of pen balls in $\mathbb{R}^d \Rightarrow \exists$ disjoint subcollection $B_{i_1}, \dots, B_{i_k}$ with

$$|\bigcup_{j=1}^{B} B_j| \leq 3^d \sum_{j=1}^{k} |B_{i_j}|$$

Proof of Lemma. wlog $\delta > |E|$. Vitali $\Rightarrow \exists$ disjoint subcollection $B_1, \dots, B_N \in \mathscr{B}$

$$\sum_{i=1}^{N_1} |B_i| \geq 3^{-d}\delta$$

Sequence of balls $B_1, \dots, B_N$. question: Is $\sum_{j=1} |B_j| \geq |E| - \delta$? Yes: done with $N = N_1$. No: work harder.

$$\sum_{j=1}^{N_1} |B_j| < |E| - \delta$$

$$E_2 = E \setminus \bigcup_{j=1}^{N_1} \bar{B}_j$$

$$|E_2| \geq |E| - \sum_{j=1}^{N_1} |B_j| > |E| - (|E| - \delta) = \delta$$

$\mathscr{B}$ Vitali covering $\Rightarrow$ balls in $\mathscr{B}$ disjoint frm $\bigcup_{i=1}^{N_1} \bar{B}_i$ still covers E_2. Vitali $\Rightarrow \exists$ finite disjoint subcollection of these balls $B_{N_1+1}, \dots, B_{N_2}$

$$\sum_{N_1 < j < N_2} |B_j| \geq 3^{-d}\delta.$$

After k steps, $B_1, \dots, B_{N_1}, \dots B_{N_1}, \dots, B_{N_k}$ with

$$\sum_{j=1}^{N_k} \geq h 3^{-d}\delta \geq |E| - \delta$$

iff $k \geq 3^d \frac{|E|-\delta}{\delta}$, stop. $\square$

need to approximate with compact from inside somewhere and with open from outside somewhere else

Corollary. The balls can be arranged in such a way that

$$E \setminus \bigcup_{i=1}^{N} B_i| < 2\delta$$

Proof. Choose open $O \supset E : |O \setminus E| < \delta$. $\mathscr{B}$ Vitali covering $\Rightarrow$ wlog all balls in $\mathscr{B}$ are contained in O.

$$(E \setminus \bigcup_{i=1}^{N} B_i) \cup \bigcup_{i=1}^{}]N B_i \subset O. : |E \setminus \bigcup_{i=1}^{N} B_i| \leq |O| - \sum_{i=1}^{N} |B_i| \leq |E| + \delta - (|E| - \delta) = 2\delta$$

$\square$

$F \in AC$ *Back to the real lin:* Goal: $F' = 0$ a.e. $\Rightarrow F$ constant. ETS $F(a) = F(b)$

$$E = \{x \in (a,b) : F'(x) \text{ exists and } = 0\} \quad |E| = b - a\}$$

Fix $\varepsilon > 0$. For $x \in E$, $\lim_{h\to 0} |\frac{F(x+h)-F(x)}{h}| = 0$. $\forall \eta > 0 \exists$ open interval $I = (a_x, b_x) \subset [a,b]$ containing x. $F(b_x) - F(a_x)| \le \varepsilon(b_x - a_x)$ and $b_x - a_x < \eta$. The collection of these intervals (over all $\eta > 0$) forms a Vitali covering of E. Lemma $\Rightarrow$ Given $\delta > 0$ can select finitely many, disjoint $I_j = (a_j, b_j)_{j=1}^N$ such that

$$\sum_{j=1}^N |I_j| \ge |E| - \delta = b - a - \delta$$

But

$$\sum_{j=1}^N |F(b_j) T F(a_j)| \le \varepsilon \sum_{j=1}^N (b_j - a_j) \le \varepsilon(b-a)$$

$$[a,b] \setminus \bigcup_{j=1}^N I_j = \bigcup_{k=1}^M [\alpha_k, \beta_k]$$

with total length $\le \delta$. $\therefore F \in AC$

$$\sum_{k=1}^M |F(\beta_k) - F(\alpha_k)| < \varepsilon$$

$$|F(b) - F(a)| \le \sum |F(b_j) - F(a_j)| + \sum |F(\beta_k) - F(\alpha_k)| \le \varepsilon(b-a) + \varepsilon,$$

done. □

Theorem. $F \in AC(a,b)$. *Then*

(i) F' *exists a.e. and is integrable*

(ii)

$$F(x) - F(a) = \int_a^x F'(t)\,\mathrm{d}t \quad (\forall a \le x \le b)$$

Conversely, if $f \in L^1(,b)$, *then there exists* $F \in AC(a,b) : F' = f$ *a.e..*

Proof. $\Rightarrow$

(i) seen last lecture.

(ii)

$$G(x) := \int_a^x F'(t)\,\mathrm{d}t$$

$\therefore G \in AC$ $\therefore F - G \in AC$. Lebesgue diff. $\Rightarrow G'(x) = F'(x)$ (a.e. x) $\therefore$ $(F-G)' = 0$ a.e.. Therefore $(F-G)(x) = (F-G)(a)$, $F(x) - G(x) = F(a)$, equivalent to (*)

$\Leftarrow$

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

AC $\surd$. Leb. diff $\Rightarrow F' = f$ a.e. □

Next: Monotone functions which are not nec. continuous. Wlog. F increasing, bounded on $[a,b]$.

$$F(x^-)F\lim_{y\to x, y<x} F(y) \quad F(x^+) := \dots$$

$F(x^-) \le F(x) \le F(x^+)$, F cont. at x if $F(x^-) = F(x^+)$. Otherwise F has a jump discontinuity at x.

Obs: A (bounded) increasing finction F on $[a,b]$ has at most countable many jumps. There exists an injective map $\mathrm{Disc}(F) \to \mathbb{Q}$

.: $\mathrm{Disc}(F) = \{x_n\}_{n=1}^\infty$ $\alpha_n = F(x_n^+) - F(x_n^-) =$ jump of F at x_n. $F(x_n^+) = F(x_n^-) + \alpha_n$ $F(x_n) = F(x_n^-) + \theta_n\alpha_n$, $\theta_n \in [0,1]$. $F(x) = \mu((-\infty, x])$. Corresponds to singular + abs. cont measures.

$$j_n(x) = \begin{cases} 0 & x < x_n \\ \phi_n & x = x_n \\ 1 & x > x_n \end{cases}$$

Jump function associated to F is

$$J_F(x) = \sum_{n=1}^\infty j_n(x)$$

$$\sum_{n=1}^\infty \alpha_n = \sum_{n=1}^\infty F(x_n^+) - F(x_n^-) \le F(b) - F(a) < \infty$$

because F incr, and F bounded.

Lemma. F increasing, bounded on $[a,b]$, $\mathrm{Disc}(F) = \{x_n\}_{n=1}^\infty$

(i) $J_F(x)$ is discontinuous precisely at $\{x_n\}_{n=1}^\infty$, has a jump at x_n equal to that if F.

(ii) Th function $F - J_F$ is increasing and continuous.

Proof. (i) $x \ne x_n (\forall n) \Rightarrow$ each j_n is continuous at $x \Rightarrow J_F$ is continuous at x because of uniform convergence. $x = x_N (\exists N) \Rightarrow J_F = \sum_{n=1}^N \alpha_n j_n(x) + \sum_{n>N} \alpha_n j_n(x)$. First sum has jump discontinuity x_N of size α_N

(ii) $F - J_F$ is continuous

$$F(x) - J_F(x) \le F(y) T J_F(y) \iff J_F(y) - J_F(y) \le F(y) - F(x)$$

, where

$$J_F(y) = \sum_{x<x_n\le y} \alpha_n = \sum_{x<x_n\le y} F(x_n^+) - F(x_n^-) \le F(y) - F(x)$$

□

Since $F = (F - J_F) + J_F$ ETS J_F is diff a.e.. This was essential step of

$$\mu = \mu_{AC} + \mu_S + \mu_{PP}$$

$z(t) = (x(t), y(t))$. curve γ. $x, y : [a,b] \to \mathbb{R}$ continuous.
γ rectifiable if length

$$L(\gamma) = \sup \sum_{j=1}^N |z(t_j) - z(t_{j-1})| < \infty$$

sup over all partitions $P = \{a = t_0 < t_1 <_< t_N = b\}$ of $[a,b]$

When is

$$L(\gamma) = \int_a^b |z'(t)|\,\mathrm{d}t?$$

Lemma. γ is rectifiable iff x, y are of bounded variation (and cont.).

see $F = x + iy$
Assume γ rectifiable, let $L(A, B)$ length of $\gamma(A, B)$, $(a \leq A \leq B \leq b)$

(i) $L(A, B) = T_F(A, B)$ (where $F(t) = z(t)$)

(ii) $L(A, C) + L(C, B) = L(A, B)$ $(A \leq C \leq B)$

(iii) $A \mapsto L(A, B)$ (fix B) is continuous
$B \mapsto L(A, B)$ (fix A)
seen: $F \in \mathrm{BV}(a, b)$, cont. $\Rightarrow T_F$ cont.

Warning: $[0, 1] \ni t \mapsto (F(t), F(t))$, F Cantor. F cont. incr. $F(0) = 0$, $F(1) = 1$, $F' = 0$ a.e.

Theorem. *$z : [a, b] \to \mathbb{R}^2, t \mapsto (x(t), y(t)) \sim$ curve γ. $x, y \in \mathrm{AC}(a, b)$. $\Rightarrow \gamma$ rectifiable and*

$$L(\gamma) = \int_a^b |z'(t)| \, \mathrm{d}t$$

why? $F : [a, b] \to \mathbb{C}$ is $\mathrm{AC}(a, b)$

$$\Rightarrow T_F(a, b) = \int_a^b |F'(t)| \, \mathrm{d}t.$$

$$\sum_{j=1}^N |F(t_j) - F(t_{j-1})| = \sum_{j=1}^N \left| \int_{t_{j-1}}^{t_j} F'(t) \, \mathrm{d}t \right| \leq \sum_{j=1}^N \int_{t_{j-1}}^{t_j} |F'(t)| \, \mathrm{d}t = \int_a^b |F'(t)| \, \mathrm{d}t$$

First inequality by FTC. For $\geq$, write $F' = g + h$, g step function, h small in L^1 $G, H =$ def. integrals of g, h. Check $T_F \geq T_G, T_H$, T_H small, $T_G \geq \int_a^b |g(t)| \, \mathrm{d}t$.

Minkowski content of a curve simple, simple closed, quasi-simple curves.
trace of γ: $\Gamma = \{z(t) \in \mathbb{R}^2 : t \in [a, b]\}$.
Given $K \subset\subset \mathbb{R}^2$ and $\delta > 0$ define

$$K^\delta = \{x \in \mathbb{R}^2 : \mathrm{d}(x, K) < \delta\}$$

where $\mathrm{d}(x, K) = \inf_{k \in K} \mathrm{d}(x, k)$

Definition. The set K has (1D) Minkowski content if

$$\lim_{\delta \to 0} \frac{|K^\delta|}{2\delta}$$

exists (in $\mathbb{R}$), denoted $M(K)$.

Theorem. *Let $\Gamma = \{z(t) : a \leq t \leq b\}$ be (the trace of) a quasi-simple curve γ. Then Γ has Minkowski content iff γ is rectifiable (in which case $M(\Gamma) = L(\gamma)$).*

Upper Mink: content.

$$\limsup_{\delta \to 0^+} \frac{|K^\delta|}{2\delta} =: M^*(K)$$

lower

$$\liminf_{\delta \to 0^+} \frac{|K^\delta|}{2\delta} =: M_*(K).$$

Proposition. $T = \{z(t) : a \leq t \leq b\}$ quasi simple. If $M_*(\Gamma) < \infty$, then γ is rectifiable and $L(\gamma) \leq M_*(\Gamma)$.

Proposition. $\Gamma = \{z(t) : a \leq t \leq b\}$ rectifiable γ. Then $M^*(\Gamma) \leq L(\gamma)$.

Proof of Prop. 1, for simple curves. Obs: $\Gamma = \{z(t) : a \leq t \leq b\}$ any curve. $\Delta = |z(b) - z(a)|$. $|\Gamma^\delta| \geq 2\delta\Delta$.

Take any partition P of $[a, b)$. $L_P = \sum_{j=1}^N |z(t_j) - z(t_{j-1})|$. Given $\varepsilon > 0, \exists N$ proper closed subintervals $I_j = [a_j, b_j] \subset (t_{j-1}, t_j)$:

$$\sum_{j=1}^N |z(b_j) - z(a_j)| \leq L_P - \varepsilon$$

$I_1, \ldots, I_N$ disjoint $\Rightarrow \Gamma_1, \ldots, \Gamma_N$ disjoint because Γ is simple. $\Leftrightarrow \Gamma_1^\delta, \ldots, \Gamma_N^\delta$ disjoint, provided $\delta > 0$ small enough.

$$\bigcup_{j=1}^N \Gamma_j^\delta \subset \Gamma^\delta$$

$$|\Gamma^\delta| \geq \sum_{j=1}^N |\Gamma_j^\delta| \geq 2\delta \sum_{j=1}^N |z(t_j) - z(t_{j-1})| \geq 2\delta(L_p - \varepsilon)$$

□

Isoperimetric inequality (soft) $\gamma : [a, b] \to \mathbb{R}^2$, $\gamma \in C^1(a, b) : \gamma'(s) \neq 0 \forall s$, $\gamma(a) = \gamma(b)$. Arclength parametrization: $\gamma : [0, L] \to \mathbb{R}^2$, $|\gamma'(s)| = 1 \forall s$.

Theorem. $\Gamma \subset \mathbb{R}^2$ *simple colsed* C^2 *curve of length* L. A *area of the region enclosed by* Γ.

$$A = \frac{1}{2}| \int_\Gamma > (x\,\mathrm{d}y - y\,\mathrm{d}x)| = \frac{1}{2}| \int_0^L (x(s)y'(s) - x'(s)y(s))\,\mathrm{d}s.$$

Then $4\pi A \leq L^2$. *Equality iff* Γ *is a circle.*

Proof. wlog (rescale) $L = 2\pi$.: WTS $A \leq \pi$, equality iff Γ is a circle of radius 1.

$\gamma : [0, 2\pi] \to \mathbb{R}^2$, $s \mapsto \gamma(s) = (x(s), y(s))$ arclength par. $x'(s)^2 + y'(s)^2 = 1 \forall s$.:

$$\frac{1}{2\pi} \int_0^{2\pi} (x'(s)^2 + y'(s)^2)\,\mathrm{d}s = 1$$

Γ closed $\Rightarrow x(s), y(s)$ 2η-periodic.

$$x'(s) = \sum_n a_n i n e^{ins}$$
$$y'(s) = \sum_n b_n i n e^{ins}$$

Parseval $\Rightarrow$

$$\sum_{n\in\mathbb{Z}} n^2(|a_n|^2 + |b_n|^2) = 1$$

$$A = \frac{1}{2} \int_0^{2\pi} (x(s)y'(s) - x'(s)y(s))\,\mathrm{d}s| = \pi| \sum_{n\in\mathbb{Z}} n(a_n\bar{b}_n - b_n\bar{a}_n)|$$

by bilinear Parseval

$$|a_n\bar{b}_n - b_n\bar{a}_n| \leq 2|a_n||b_n| \leq |a_n|^2 + |b_n|^2$$
$$A \leq \pi \sum_{n\in\mathbb{Z}} n^2(|a_n|^2 + |b_n|^2) \leq \pi$$

□

Cases of equality: $A = \pi \Rightarrow$

$$x(s) = a_{-1}e^{-is} + a_0 + a_1 e^{is}$$

$$y(s) = b_{-1}e^{-is} + b_0 + b_1 e^{is}$$

x, y real-valued $\Rightarrow a_1 = \bar{a}_{-1},\ b_1 = \bar{b}_{-1}$. (**) $\Rightarrow 2(|a_1|^2 + |b_1|^2) = 1$.
(***) $\Rightarrow$

$$|a_1| = |b_1| = \frac{1}{2}. : \quad a_1 = \frac{1}{2}e^{i\alpha} \quad b_1 = \frac{1}{2}e^{i\beta} \quad (\alpha, \beta \in \mathbb{R})$$

$$1 = 2|a_1\bar{b}_1 - \bar{a}_1 b_1| = \sin(\alpha - \beta)|. : \quad \alpha - \beta = \frac{k\pi}{2} \quad (\text{odd } k)$$

$$x(s) = a_0 + \cos(s + \alpha)$$

$$y(s) = b_0 \pm \sin(s + \alpha)$$

$\pm$ dep. on parity of $\frac{k-1}{2}$.

Isoperimetric inequality (hard) $\Omega \subset \mathbb{R}^2$ bounded, open, $\partial\Omega = \bar{\Omega} - \Omega =: \Gamma$ rectifiable curve (not nec. simple) with length $l(\Gamma)$.

Theorem.

$$4\pi|\Omega| \leq l(\Omega)^2$$

Proof. inner:

$$\Omega_-^\delta = \{x \in \mathbb{R}^2 \,:\, \mathrm{d}(x, \mathbb{R}^2 \setminus \Omega) \geq \delta\}$$

outer:

$$\Omega_+^\delta = \{x \in \mathbb{R}^2 \,:\, \mathrm{d}(x, \bar{\Omega}) < \delta\}$$

$$\Gamma^\delta = \{x \,:\, \mathrm{d}(x, \Gamma) < \delta\}$$

$$\Omega_+^\delta = \Omega_-^\delta \dot{\cup} \Gamma >^\delta$$

$A, B \subset \mathbb{R}^d,\ A + B = \{a + b \,:\, a \in A, b \in B\}$ Note: $\Omega + B_\delta \subset \Omega_+^\delta,\ \Omega_-^\delta + B_\delta \subset \Omega$.
Brunn-Minkowski: $A, B \subset \mathbb{R}^2$ meas., $A + B$ meas.

$$|A + B|^{\frac{1}{2}} \geq |A|^{\frac{1}{2}} + |B|^{\frac{1}{2}}$$

$$|\Omega_-^\delta| \geq (|\Omega|^{\frac{1}{2}} + |B_\delta|^{\frac{1}{2}})^2 \geq |\Omega| + 2|\Omega|^{\frac{1}{2}} \underbrace{|B_\delta|^{\frac{1}{2}}}_{=(\pi\delta^2)^{\frac{1}{2}}}$$

$$|\Omega| \geq (|\Omega_-^\delta|^{\frac{1}{2}} + |B_\delta|^{\frac{1}{2}})^2 \geq |\Omega_-^\delta| + 2|\Omega_-^\delta|^{\frac{1}{2}}|B_\delta|^{\frac{1}{2}}$$

$$|\Gamma^\delta| \geq |\Omega| + 2|\Omega|^{\frac{1}{2}}\sqrt{\pi} - |\Omega| + |\Omega_-^\delta|^{\frac{1}{2}}\sqrt{\pi}$$

$$\limsup_{\delta \to 0^+} \frac{|\Gamma^\delta|}{2\delta} \geq 2|\Omega|^{\frac{1}{2}}\sqrt{\pi}$$

$$4\pi|\Omega| \leq M^*(\Gamma)^2 \leq l(\Gamma)^2$$

Note, that only in the very last inequality did we use the rectifiability of Γ. $\square$

Brunn-Minkowski ineq. ($\mathbb{R}^d$) $A, b \subset \mathbb{R}^d$ measurable. $A + B = \{a + b : a \in A, , b \in B\}$. $\lambda A = \{\lambda a : a \in A\}$ $(\lambda > 0)$.

Q.: Can $|A + B|$ be controlled in terms of $|A|, |B|$? No! There exist sets A, B $|A| = |B| = 0$ with $|A + B| > 0$. Example $[0, 1] \times [0, 1]$. Another example $A = B = C \subset [0, 1]$ Cantor set. Then $A + B = [0, 2]$.

Q.: Can $|A + B|^\alpha \geq c_\alpha(|A|^\alpha + |B|^\alpha)$ hold? (for some $\alpha > 0$ with $c_\alpha < \infty$, indep of A, B) Best possible $c_\alpha = 1$.

What about α? Convex sets play a role. A = convex, $B = \lambda A$. $|B| = |\lambda A| = \lambda^d |A|$. $|A + B| = |A + \lambda A| = |(1 + \lambda)A| = (1 + \lambda)^d |A|$ because A is convex.

$(\lambda_1 A + \lambda_2 A = (\lambda_1 + \lambda_2)A$ iff A is convex.)

$|A + B|^\alpha \geq |A|^\alpha + |B|^\alpha$ iff $(1 + \lambda)^{d\alpha} \geq 1 + \lambda^{d\alpha} \Rightarrow \alpha \geq \frac{1}{d}$.

$(a + b)^\gamma \geq a^\gamma + b^\gamma \forall a, b \geq 0,\ \gamma \geq 1$.

Candidate inequality:

$$|A + B|^{\frac{1}{d}} \geq |A|^{\frac{1}{d}} + |B|^{\frac{1}{d}}$$

(BM)

A, B measurable $6 \implies$ $A + B$ measurable. Take $[0, 1] \times$ nonmeasurable.

(i) A, B closed $\Rightarrow$ $A + B$ measurable

(ii) A, B compact $\Rightarrow$ $A + B$ compact

(iii) A, B open $\Rightarrow$ $A + B$ open

Theorem. *(BM) holds if $A, B, A + B$ measurable.*

(i) A, B rectangles with sidelengths $\{a_j\}_{j=1}^\infty$, $\{b_j\}_{j=1}^\infty$

(ii) A, B unions of fifinely many rectangles with disjoint interiors.

(iii) A, B open sets of finite measure

(iv) A, B compact

(v) $A, B, A + B$ measurable.

Proof. (i) (BM) becomes

$$\prod_{j=1}^d (a_j + b_j)^{\frac{1}{d}} \geq \prod_{j=1}^d a_j^{\frac{1}{d}} + \prod_{j=1}^d b_j^{\frac{1}{d}}$$

$a_j \to \lambda_l a_j,\ b_j \to \lambda_j b_j$. Both sides are multiplied by $(\lambda_1 \lambda_2 \ldots \lambda_d)^{\frac{1}{d}}$: wglog can assume $a_j + b_j = 1 \forall j$ (Choose $\lambda_j = a_j + b_j$)

AMGM:

$$\prod_{j=1}^d a_j^{\frac{1}{d}} \leq \frac{1}{d} \sum_{j_1}^d a_j$$

$$\prod_{j=1}^d b_j^{\frac{1}{d}} \leq \frac{1}{d} \sum_{j_1}^d b_j$$

$$\prod a_j^{\frac{1}{d}} + \prod b_j^{\frac{1}{d}} \leq \frac{1}{d} \sum_{j=1}^d (a_j + b_j) = 1$$

(ii) Induction on n = numeber of rectangles in A and B. Choose pair of disjoint rectangles R_1, R_2 in A. Can rotate s.t. R_1 and R_2 are separated by hyperplane $\{x_j = 0\}$. R_1 lies in $A_+ = A \cap \{x_j \geq 0\}$, $A_I = A \cap \{x_j \leq 0\}$.

Rem.: Both A_+, A_- contain at leas one less rectangle than A, $A = A_+ \subset A_-$ and $A_B \cap A_-$ has measure zero.

Now: translate B s.t. B_- and B_+ satisfy

$$\frac{|B_\pm|}{|B|} = \frac{|A_\pm|}{|A|}$$

$(A_+ + B_+) \cup (A_- + B_-) \subset A + B$ Number of rectangles in A_+ and B_+, number of rectangles in A_- and B_- is $< n$.

$$\begin{aligned}|A+B| &\geq |A_+ + B_-| + |A_- B +_-| \geq (|A_+|^{\frac{1}{d}} + |B_+|^{\frac{1}{d}})^d + (|A_-|^{\frac{1}{d}} + |B_-|^{\frac{1}{d}})^d \\ &= (|A_+|(1 + (\frac{|B_+|}{|A_+|})^{\frac{1}{d}})^d + |A_-|(1 + (\frac{|B_-|}{|A_-|})^{\frac{1}{d}})^d = (|A_+| + |A_-|)(1 + (\frac{|B|}{|A|})^{\frac{1}{d}})^d \\ &= (|A|^{\frac{1}{d}} + |B|^{\frac{1}{d}})^d.\end{aligned}$$

(iii) Open sets of finite measure A, B. $\forall \varepsilon > 0 \exists A_\varepsilon, B_\varepsilon$ finet unions of parallel rectangles with disjoint interiors. $A_\varepsilon \subset A, B_\alpha \subset B$, $|A| \leq |A_\varepsilon| + \varepsilon$, $|B| \leq |B_\varepsilon| + \varepsilon$.

$|A + B| \geq |A_\varepsilon + B_\varepsilon| \geq (|A_\varepsilon|^{\frac{1}{d}} + |B_\varepsilon|^{\frac{1}{d}})^d \geq ((|A| - \varepsilon)^{\frac{1}{d}} + (|B| - \varepsilon)^{\frac{1}{d}})^d$. Let $\varepsilon \to 0^+$, done.

(iv) A, B compact. Let $A^\varepsilon = \{x : d(x, A) < \varepsilon\}$. $A + B \subset A^\varepsilon + B^\varepsilon \subset (A+B)^{2\varepsilon}$

(v) $A, B, A + B$ measurable: usi inner regularity of Lebesque measure.

$\square$

Remark. A, B open sets of finite positive measure. Equality in (BM) iff A, B convex and similar. $\exists \delta > 0 \exists h \in \mathbb{R}^d : A = \delta B + h$ (A convex iff $\lambda_i A + \lambda_2 A = (\lambda_1 + \lambda_2) A$)

Consequences for isoperimetric inequality $A \subset \mathbb{R}^d$ bounded open with smooth boundary. (∂A, $B \subset \mathbb{R}^d$ ball $|B| = |A|$)

$$|\partial A| = \lim_{\varepsilon \to 0^+} \frac{|A + \varepsilon B| - |A|}{\varepsilon}$$

Isoper ineq.: $|\partial A| \geq |\partial B|$.

Proof.

$$\frac{|A + \varepsilon B| - |A|}{\varepsilon} \geq \frac{(|A|^{\frac{1}{d}} + |\varepsilon B|^{\frac{1}{d}})^d - |A|}{\varepsilon} = \frac{(1+\varepsilon)^d - 1}{\varepsilon} |B| \to d|B| = |\partial B|$$

for $\varepsilon \to 0$. $\square$

Better: $A \subset \mathbb{R}^d$ has finite perimeter ($\iff$ $1_A \in \mathrm{BV}(U)$, $U \subset \mathbb{R}^d$ bdd open)

$$\frac{\mathscr{H}^{d-1}(\partial A)}{|A|^{\frac{d-1}{d}}} \geq \frac{\mathscr{H}^{d-1}(S^{d-1})}{|B^d(0,1)|^{\frac{d-1}{d}}}$$

Hausdorff measure Q: How does a set replicate under scaling? $E \to nE = E_1 \cup \ldots \cup E_m$ disjoint congruent copies of E. Examples: line $m = n^1$, square $m = n^2$, cube $m = n^3$, Cantor set $3C = C_1 \cup C_2$ $2 = 3^\alpha \iff \alpha = \frac{\log 3}{\log 2}$

$\#(\varepsilon)$ =least # of segments that arise from such poygonal lines. Γ rectifiable iff $\#(\varepsilon) \sim \varepsilon^{-1}$ as $\varepsilon \to 0^+$. If $\#(\varepsilon) \sim \varepsilon^{-\alpha}$ $(\alpha > 1)$ In this case, say "Γ has dim α". Snowflake has $\alpha = \frac{\log 4}{\log 3} > 1$.

Upshot: E $\alpha > 1$. $m_\alpha(E) = \alpha$-dimensional mass of E among sets of "dimension" α.

- $\alpha > \dim(E) \Rightarrow m_\alpha(E) = 0$
- $\alpha < \dim(E) \Rightarrow m_\alpha(E) = \infty$
- $\alpha = \dim(E)$ interesting

R. Gardener Bulletin AMS more about Brunn-Minkowski, geometrically, including more proofs, e.g. with induction of the dimension.

Hausdorff measure $E \subset \mathbb{R}^d$ any subset.

$$m^*_\alpha(E) := \lim_{\delta \to 0^+} \underbrace{\inf\{\sum_k (\operatorname{diam} F_k)^\alpha : E \subset \bigcup_{k=1}^{\infty} F_k \ \ \operatorname{diam}(F_k) \le \Delta\}}_{H^\delta_\alpha(E)}$$

exterior/outer α-dim Hausdorff measure.

Remark. $H^\delta_\alpha(E) \le H^\delta_\alpha(E) \le m^*_\alpha(E) (\forall \delta > 0)$. $H^\delta_\alpha(E)$ increases when δ ecreases. $\therefore m^*_\alpha(E) = \lim_{\delta \to 0^+} H^\delta_\alpha(E)$ exists

Remark. Coverings must be by sets of arb. small measure. (If we allowed the δ to be arbitrary then two parallel lines would get the same 1d-measure as one of them.)

Remark (Skaling). "The measure of a set should scale like its dimension". E.g.: $\Gamma \subset \mathbb{R}^d$ smoot cureve of length L *sim* $\lambda\Gamma$ has length λL. $Q \subset \mathbb{R}^d$ cube *sum* λQ has measure $\lambda^d |Q|$. $|F|$ scaled by $\lambda \Rightarrow (\operatorname{diam} F)^\alpha$ scaled by λ^α

Properties

(i) $E_1 \subset E_2 \Rightarrow m^*_\alpha(E_1) \le m^*_\alpha(E_2)$

(ii) $\{E_j\} \subset \mathbb{R}^d$ countable family of sets $\Rightarrow m^*_\alpha(\bigcup_{j=1}^\infty E_j) \le \sum_{j=1}^\infty m^*_\alpha(E_j)$

(iii) (Finite additility) $\inf_{x \in E_1, y \in E_1} |x - y| = \operatorname{d}(E_1, E_2) > 0 \Rightarrow m^*_\alpha(E_1 \cup E_2) = m^*_\alpha(E_1) + m^*_\alpha(E_2)$

Proof. ETS $\ge$. Fix $0 < \varepsilon < \operatorname{d}(E_1, E_2)$. Given any cover of $E_1 \cup E_2$ with sets $F_1, F_, \ldots$ of diam $\le \delta < \varepsilon$, let $F'_j = F_j \cap E_1$, $F''_j = F_j \cap E_2$.

$$\sum (\operatorname{diam}_j F'_j)^\alpha + \sum_j (\operatorname{diam} F''_j)^\alpha \le \sum_k \operatorname{diam}(F_k)^\alpha$$

Take inf over all covers, let $\delta \to 0^+$, done. □

m^*_α satisfies all properties of a Caratheodory outer measure $\therefore m^*_\alpha$ is a countabley additive maisure when restricted to Borel sets, call it m_α= α-dim Hausdorff measure.

(iv) $\{E_j\}$ countable family of disjoint Borel sets $\Rightarrow$

$$m_\alpha(\dot{\bigcup}_{j=1}^\infty E_j) = \sum_{j=1}^\infty m_\alpha(E_j)$$

(v) Hausdorff masure is invariant under translation and rotations. It scales like:

$$m_\alpha(\lambda E) = \lambda^\alpha m_\alpha(E)$$

(vi) $m_0(E) = \# E,\ m_1(E) = |E|$ (=1D LEbesgue measure of E), $E \subset \mathbb{R}$ Borel.

(vii) $E \subset \mathbb{R}^d$ Borel, $m_\alpha(E) \simeq |E|$

Proof. (i) Isodiametric inequality: $|E| \leq v_d(\frac{\operatorname{diam} E}{2})^d$, v_d volume of the unit ball in $\mathbb{R}^d$. Prove first for sets $E = -E$ and then something hard.

(ii) Covering argument: Given $\varepsilon, \delta > 0$, there exists a covering of E by balls $\{B_j\}$: $\operatorname{diam} B_j < \delta,\ \sum_j |B_j| \leq |E| + \varepsilon$

$$H_d^\delta(E) \leq \sum_j (\operatorname{diam} B_j)^d = c_d \sum_j |B_j| \leq c_d(|E| + \varepsilon),$$

let $\delta, \varepsilon \to 0^+$, get one of the inequalities.

□

(viii) if $m_\alpha^*(E) < \infty$ and $\beta > \alpha$, then $m_\beta^*(E) = 0$. If $m_\alpha^*(E) > 0$ and $\beta < \alpha$, then $m_\beta^(E) = \infty$.

Proof. $\operatorname{diam} F < \delta, \beta > \alpha \Rightarrow (\operatorname{diam} F)^\beta = (\operatorname{diam} F)^{\beta-\alpha}(\operatorname{diam} F)^\alpha < \delta^{\beta-\alpha}(\operatorname{diam} F)^\alpha$ □

Consequence: Given $E \subset \mathbb{R}^d$ Borel, $\exists!\, \alpha$ such that

$$m_\beta(E) = \begin{cases} \infty & \beta < \alpha \\ 0 \beta > \alpha \end{cases}$$

$$\alpha = \sup\{\beta : m_\beta(E) = \infty\} = \inf\{\beta : m_\beta(E) = 0\} := \text{Hausdorff dimension of } E = \dim E$$

At the critical value $\alpha = \dim E$ $0 \leq m_\alpha(E) \leq \infty$. If E is bounded and the enequalities are strict, we say that E has strict Hausdorff dimension α.

Theorem. *The Cantor set $C \subset [0, 1)$ has strict Hausdorff dimensios $\frac{\log 2}{\log 3}$.*

ETS: $0 < m_\alpha(C) \leq 1$

Proof. $m_\alpha(C) \leq 1$*:* $C = \bigcap C_k$ where each C_k is a finite unionof 2^k inetrvals of length 3^{-k}.. Given $\delta > 0$ coose k large enough tuch tht $3^{-k} < \delta$. C_k covers C and Consists of 2^k intervals of diameter $3^{-k} < \delta$. $H_\alpha^\delta(C) \leq 2^k(3^{-k})^\alpha = 1$, let $\delta \to 0^+$, done.

$m_\alpha(C) > 0$:

Lemma. $E \subset\subset \mathbb{R}^d$ compact, $f : E \to \mathbb{R}$ γ-Hölder,

$$|f(x) - f(y)| \leq m|x - y|^\gamma \quad (\forall x, y \in E) \quad 0 < \gamma \leq 1$$

Then

(i) $m_\beta(f(E)) \le M^\beta m_\alpha(E)$ if $\beta = \frac{\alpha}{\gamma}$.

(ii) $\dim f(E) \le \frac{1}{\gamma}\dim(E)$

Proof. $\{F_k\}$ countable family of sets that overs E $\therefore \{f(F_k \cap E)\}$ covers $f(E)$. $\operatorname{diam} f(F_k \cap E) \le M(\operatorname{diam} F_k)^\gamma$.

$$\sum_k (\operatorname{diam} f(E \aleph F_k))^{\frac{\alpha}{\gamma}} \le M^{\frac{\alpha}{\gamma}} \sum_k (\operatorname{diam} F_k)^\alpha,$$

done. and 1 implies 2. □

Lemma. The Cantor-Lebesgue function $F : C \to [0,1]$ is $\gamma = \frac{\log 2}{\log 3}$-Hölder.

Proof. Goal: $|F(x) - F(y)| \le c|x-y|^\gamma$ $\forall x, y \in C$.

F_n increases at most 2^{-n} on an interval of length 3^{-n}. $\therefore$ slope $\le (\frac{3}{2})^n$ $\therefore |F_n(x) - F_n(y)| \le (\frac{3}{2})^n|x-y|$. $|F_n(x) - F(x)| \le 2^{-n}$. Given x, y chose n: $3^n|x-y| \sim 1$, $3^\gamma = 2$.

$$|F(x)-F(y)| \le |F_n(x)-F_n(y)|+|F_n(x)-F(x)|+|F_n(y)-F(y)| \le (\frac{3}{2})^n|x-y|+2\cdot 2^{-n} \le c2^{-n} = c(3^{-n})^\gamma \le c'|x-y|^\gamma$$

□

Apply LEmma 1 with $E = C$, $f = F$, $\gamma = \frac{\log 2}{\log 3} \Rightarrow 1 = m_1([0,1]) \le M m_\alpha(C)$, $\dim C = \frac{\log 2}{\log 3}$ □

Rectifiable curves

Theorem. *$\gamma : [a,b] \to \mathbb{R}^d$ continuous and simple. Then γ is rectifiable iff $\Gamma = \{\gamma(t) : a \le t \le b\}$ has strict Hausdorff dimension equal to 1. $m_1(\Gamma) = l(\gamma)$.*

Proof. $\Rightarrow$: Let γ be rectifiable of length L. Consider acrlength parametrization $\tilde\gamma$. $\Gamma = \{\tilde\gamma(s) : 0 \le s \le L\}$.

$$|\tilde\gamma(s_1) - \tilde\gamma(s_2)| \le |s_1 - s_2|$$

By Lemma 1 (i) $m_1(\Gamma) \le L$. Wh $m_1(\Gamma) \ge L$?

$$\Gamma_j = \{\gamma(t) : t_j \le t \le t_{j+1}\}$$

$$\Gamma = \bigcup_{j=1}^{N-1} \Gamma_j \quad m_1(\Gamma) = \sum_{j=0}^{N-1} m_1(\Gamma_j)$$

Claim: $m_1(\Gamma_j) \ge l_j := |\gamma(t_j) - \gamma(t_{j+1})|$

Proof. $\pi : \mathbb{R}^2 \infty \mathbb{R}$ $(x,y) \mapsto x$ Lipschitz, $\pi(\Gamma_j) \subset [0, l_j]$ Lemma 1 (i) implies the claim. □

$\therefore m_1(\Gamma) \ge \sum m_1(\Gamma_j) \ge \sum l_j$, $L := \sup_p \sum l_j \therefore m_1(\Gamma) \ge L$, done. □

Theorem. $f \in C_0^0(\mathbb{R}^2)$, $0 < \delta \le \frac{1}{2}$. *Then*

$$\int_{S^1} R_\delta^*(f)(\gamma)\,\mathrm{d}\sigma_\gamma \le\sim (\log\frac{1}{\delta})^{\frac{1}{2}}(\|f\|_{L^1(\mathbb{R}^2)} + \|f\|_{L^2(\mathbb{R}^2)})$$

Proof of Theorem. Modified version of lemma 3: Setting

$$F_\delta(t) = \int_{-\infty}^{\infty} \hat{F}(\lambda)(\frac{e^{2\pi i(t+\delta)\lambda} - e^{2\pi i(t-\delta)\lambda}}{2\pi i\lambda(2\delta)})\,\mathrm{d}\lambda$$

Suppose $\sup_{\lambda\in\mathbb{R}} |\hat{F}(\lambda)| \leq A$ and $\int_{-\infty}^{\infty} |\hat{F}(\lambda)|^2|\lambda|^{d-1}\,\mathrm{d}\lambda \leq B^2$.

Claim:

$$\sup_t |F_\delta(t)| \leq\sim (\log\frac{1}{\delta})^{\frac{1}{2}}(A+B)$$

$$F_\delta(t) = \int_{-\infty}\infty = \int_{|\lambda|\leq 1} + \int_{|\lambda|>1} \leq cA + \int_{1<|\lambda|\leq\frac{1}{\delta}} |\hat{F}(\lambda)|\,\mathrm{d}\lambda + \frac{c}{\delta}\int_{|\lambda|>\frac{1}{\delta}} |\hat{F}(\lambda)||\lambda|^{-1}\,\mathrm{d}\lambda = I + II$$

CS:

$$I \leq\sim (\int_{\mathbb{R}} |\hat{F}(\lambda)|^2|\lambda|\,\mathrm{d}\lambda)^{\frac{1}{2}}(\int_{1<|\lambda|\leq\frac{1}{\delta}} |\lambda|^{-1}\,\mathrm{d}\lambda)^{\frac{1}{2}} \leq B(\log\frac{1}{\delta})^{\frac{1}{2}}$$

$$II \leq\sim \frac{c}{\delta}(\int_{\mathbb{R}^2} |\hat{F}(\lambda)|^2|\lambda|\,\mathrm{d}\lambda)^{\frac{1}{2}}(\int_{|\lambda|>\frac{1}{\delta}} |\lambda|^{-3}\,\mathrm{d}\lambda)^{\frac{1}{2}} \leq\sim B$$

□

Theorem. *There exists a subset $K \subset \mathbb{R}^2$ such that*

(i) K is compact

(ii) K has Lebesgue measure zero

(iii) K contains a transatle of every unit line segment

Theorem. *Suppose F is any set that satisfis cnditions (i) and (iii) from Theorem 1. Then F has Hausdorff dimension 2.*

Proof of Theorem 2. Let F be a Kakeya set. Fix $0 < \alpha < 2$. Lett $F \subset \bigcup_{i=1}^{\infty} B_i$ be a covering with balls B_i of diameter $\leq \delta$. It is enough to show

$$\sum(\mathrm{diam}B_i)^\alpha \geq c_\alpha > 0$$

Case 1: Assume $\mathrm{diam}B_1 = \delta \leq \frac{1}{2}$ and let $N < \infty$ be the number of balls in the covering. WTS $N\delta^\alpha \geq c_\alpha$. $B_i^* =$ doubel of B_i. $F^* = \bigcup_i B_i^*$. $|F^*| \leq \sum|B_i^*| = cN\delta^2$. F Kakeya $\Rightarrow \forall\gamma \in S^1 \exists s_\gamma \perp \gamma$ unit lime segment: $s_\gamma \subset F$. $s_\gamma^\delta \subset F^*$. $\therefore R_\delta^*(\chi_{F^*})(\gamma) \geq 1$ $(\forall\gamma \in S^1)$. Take $f = \chi_{F^*}$ in (*). Since $L^2 \subset L^1$,

$$\|\chi_{F^*}\|_{L^1} \sim\leq \|\chi_{F^*}\|_{L^2} = |F^*|^{\frac{1}{2}} \sim\leq N^{\frac{1}{2}}\delta.$$

(*) $\Rightarrow 0 < c \leq (\log\frac{1}{\delta})^{\frac{1}{2}}N^{\frac{1}{2}}\delta$. This implies $N\delta^\alpha \geq c_\alpha > 0$.

Case 2: General case. $F \subset \bigcup_{i=1}^{\infty} B_i$ with each ball B_i of diameter ≤ 1. For each $k \in \mathbb{N}$, let N_k be the number of balls ön $\{B_i\}$ with diameter $B_k \sim 2^{-k}$, i.e. $\in [2^{-k-1}, 2^{-k}]$. WTS

$$\sum_{k=0}^{\infty} N_k 2^{-k\alpha} \geq c_\alpha > 0.$$

ETS $\exists K' : N_{k'}2^{-k'\alpha} \geq c_\alpha$.

$$F_k = F \cap (\bigcup_{\mathrm{diam}B_i\sim 2^{-k}} B_i)$$

$$F_k^* = \bigcup_{\operatorname{diam} B_v \sim 2^{-k}} B_i^*$$

$$|F_k^*| \leq c N_k 2^{-2k} \quad \forall k$$

F Kakeya $\Rightarrow \forall \gamma \in O^2 \exists s_\gamma \perp \gamma : s_\gamma \subset F$ (in particular $m_1(s_\gamma \cap F) = 1$).

Key: For some k, a large proportion of s_γ belongs to F_k. Pick $\{a_k\}_{k=0}^\infty$ such that $0 \leq a_k < 1$, $\sum a_k = 1$, (a_k) dos not nend to 0 too quickly, e.g. $a_k = c_\varepsilon 2^{-k\varepsilon}$ (for sufficiently small ε.

Claim:

$$\exists k : m_1(s_\gamma \cap F_k) \geq a_k.$$

Otherwise $m_1(s_\gamma \cap F) \leq \sum_k m_1(s_\gamma \cap F_k) < \sum a_k = 1$, contradicts (**)

For this value of k,

$$R^*_{2^{-k}}(\chi_{F_k^*})(\gamma) \geq a_k.$$

Since this choice of k depends on γ, let

$$E_k = \{\gamma \in S^1 : R^*_{2^{-k}}(\chi_{F_k^*})(\gamma) \geq a_k\}.$$

$S^1 = \bigcup_{k_1}^\infty E_k$. Therefore $\exists k' : |E_{k'}| \geq 2\pi a_{k'}$.

$$2\pi a_{k'}^2 = 2\pi a_{k'} a_{k'} \leq \int_{E_{k'}} a_{k'} \, \mathrm{d}\sigma \leq_{S^1} R^*_{2^{-k'}}(\chi_{F^*_{k'}})(\gamma) \, \mathrm{d}\sigma_\gamma$$

$$2^{-2k^\varepsilon} \sim a_{k'}^2 \leq c(\log 2^{k'})^{\frac{1}{2}} |F^*_{k'}|^{\frac{1}{2}} \leq c(\log 2^{k'})^{\frac{1}{2}} N_{k'}^{\frac{1}{2}} 2^{-k'}$$

$\Rightarrow N_{k'} 2^{-\alpha k'} \geq c_\alpha$, provided $4\varepsilon < 2 - \alpha$. □

Construction of a Kakeya set I (Stein-Shakarch, III)

Thinner Cantor set, always taking away the half.

Take two of them, E_0, E_1, where E_1 has twice the length. Put E_0 on $y = 1$ and E_1 on $y = 0$. Let F be the union of all line segments that join a point in E_0 with one in E_1.

Construction of an ε-Kakeya set (Stein)

Theorem. *Given $\varepsilon > 0$, $\exists N = N_\varepsilon$ and 2^N rectangles $R_1, \ldots, R_{2^N}$ wth sidelengths 1×2^{-N} sucht that*

(i)

$$|\bigcup_{j=1}^{2^N} R_j| < \varepsilon$$

(ii) the reaches $\tilde{R}_j$ are mutually disjoint

$$|\bigcup_{j=1}^{2^N} \tilde{R}_j| = 1$$

Proof. Fix $\alpha \in (\frac{1}{2}, 1)$. Symmetric triangle ABC with M opposite C. Push the right part into the left part call resulting image $\Phi(T)$. It constists of heart $\Phi_h(T)$ and arms $\Phi_a(T)$. Then

$$|\Phi_h(T)| = \alpha^2 |T|$$

$$|\Phi_a(T)| = 2(1-\alpha)^2|T|$$

Conclusion

$$|\Phi(T)| = (\alpha^2 + 2(1-\alpha)^2)|T|$$

n-fold iteration (Peron trees): Split not into two but 2^n parts and do everything pairwise. Key: right side of $\Phi_h(A_0A_2C)$ // left side of $\Phi_n(A_2A_4C)$ // CA_2

Then look at heart/arms again. $|$arms of $\Psi_1(ABC)| \leq 2(1-\alpha)^2|T|$. $|$heart of $\Psi_1(ABC)| = \alpha^2|T| \therefore |\Psi_1(ABC)| = (\alpha^2 + 2(1-\alpha)^2)|T|$.

Iterate: Carry out this process on the heart of $\Psi_1(ABC)$ with ne replaced by $n-1$, given are the union of 2^{n-1} triangles.

Then retranslate all 2^n original triangles to obtain figure $\Psi_2(ABC)$.

$$|\text{heart of } \Psi_2(ABC) = \alpha^2\alpha^2|T|$$

$$|\text{addition arms of } \Psi_2(ABC)| \leq 2(1-\alpha)^2\alpha^2|T|$$

$$|\Psi_n(ABC)| \leq (\alpha^{2n}+2(1-\alpha)^2+2(1-\alpha)^2\alpha^2+\ldots+2(1-\alpha)^2\alpha^{2n-2}) \leq \alpha^{2n}+2(1-\alpha)^2+\underbrace{\sum_{n=0}^{\infty}\alpha^{2n}} \leq \alpha^{2n}+2(1-\alpha)$$

$\square$