Geometric Aspects of Harmonic Analysis

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Q1

$$\left.\begin{array}{c} f:[a,b]\to\mathbb{R} \text{ integrable} \\ F(x)=\int_a^x f(t)\,\mathrm{d}t \end{array}\right] \underset{?}{\Rightarrow} F \text{ diff. (a.e. } x\text{)},\ F'=f$$

Q2 Conditions of F (on $[a,b]$) s.t.

- $F'(x)$ exists a.e.
- F' integrable
- $\int_a^b F'(x)\,\mathrm{d}x = F(b)-F(a)$

?

Q1 Differentiation of the integral

$$\frac{F(x+h)-F(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,\mathrm{d}t = \frac{1}{|I|}\int f = \mathrm{avg}_I f = {}_I f$$

$I=(x,x+b)$, $|I|$ Lebesgue measure of I.

Q1 equivalent to averaging problem: Given $f\in L^1(\mathbb{R}^d)$, is it true, that

$$\lim_{|B|\to 0,\ x\in B} = \frac{1}{|B|}\int_B f = f(x) \quad (x\text{-a.e.})?$$

$B\subset\mathbb{R}^d$ open ball

Yes, if f continuous $\forall\varepsilon\exists\delta|x-y|<\delta\Rightarrow|f(x)-f(y)<\varepsilon$. $x\in B$

$$|f(x) = \fint | = |\fint_B (f(y)-f(x))\,\mathrm{d}y| < \varepsilon \tag{1}$$

provided B is an opeb ball of radius $<\frac{\delta}{2}$ containing x

Yes, if f is integrable (not so easy). Hardy, Littlewood (1D, rearrangements; later Wiener for $d>1$). $f\in L^1(\mathbb{R}^d)$

$$(Mf)(x) := \sup_{x\in B}\frac{1}{|B|}\int_B |f|$$

uncentered HL maximal function

Theorem. *Let f be integrable on $\mathbb{R}^d$. Then* *(i)* Mf *is measurable.*

(ii) $(Mf)(x) < \infty$ *a.e.* x

(iii)
$$\{x \in \mathbb{R}^d : (Mf)(x) > \alpha\}| < \frac{c}{\alpha}\|f\|_{L^1(\mathbb{R}^d)} \ (\forall x > 0). \tag{2}$$

$c = c_d = 3^d$, *independent of* f, α.

$f \neq f \in L^1 \Rightarrow Mf(x) \sim |x|^{-d}$ for large radius of x. So then $Mf \notin L^1$.

$$M : \begin{matrix} L^1 \not\to L^1 \\ L^1 \to L^{1,\infty} \end{matrix}$$

Proof. (i) easy $E_\alpha = \{x \in \mathbb{R}^d : (Mf)(x) > \alpha\}$ is open $(\forall x > 0)$ (because Mf is lowes semicontinuous)

(ii) $|\{x \in \mathbb{R}^d : (Mf)(x) = \infty\}| \subset |\{x \in \mathbb{R}^d : Mf(x) > \alpha\}|$, take $\alpha \to \infty$.

(iii) follows from an elemantary version of *Vitali covering*

□

Lemma. Let $B = \{B_1, B_2, \dots, B_N\}$ be a finite collection of open balls on $\mathbb{R}^d$. Then there exists a disjoint subcolletcino $B_{i_1}, B_{i_2}, \dots, B_{i_k}$ of B such that

$$|\bigcup_{j=1}^{n} B_j| \leq 3^d \sum_{j=1}^{k} |B_{ij}$$

Proof. (i) B_{i_1} = largest ball

(ii) Delete B_{i_1} and its neighbors

(iii) B_{i_2} = largest ball

(iv) repeat…

- Algorithm stops in at most N steps
- output has desired properties:
  - disjointness is clear
  - size $B \cap B' \neq \emptyset$, $r_{B'} \leq r_B$. B^* = ball with the same center as B but 3 times the radius. $\Rightarrow B' \subset B^*$. $|B^*| = 3^d|B|$

□

Back to (iii): Choose $\alpha > 0$, $E_\alpha = \{x \in \mathbb{R}^d : (Mf)(x) > \alpha\}$. Fr each

$$x \in E_\alpha \exists B = B_x := \frac{1}{|B_x|} \int_{B_x} |f(y)|\,\mathrm{d}y > \alpha$$

equivalent

$$|B_x| < \alpha^{-1} \int_{B_x} |f(y)|\,\mathrm{d}y$$

Fix $K \ll E_\alpha$ compact subset covered by $\bigcup_{x \in K} B_x$, $K \subset \bigcup_{l=1}]NB_l$

$$|K| \le |\bigcup_{l=1}^{N} B_l| \underset{\text{Vitali}}{\le} 3^d \sum_{j=1}^{k} |B_{ij}| \le \frac{3^d}{\alpha} \in_{j=1}^{k} \int_{B_{i_j}} |f()|\,\mathrm{d}y = \frac{3^d}{\alpha} \int_{\bigcup_{j=1}^{k} B_{i_j}} |f(y)|\,\mathrm{d}y \le \frac{3^d}{\alpha} \|f\|_{L^1(\mathbb{R}^d)}$$

Since K was choseen arbitrary (cpt.), it follows that

$$E_\alpha| \le \frac{3^d}{\alpha} \|f\|_{L^1}$$

Can interpolate between weak type L^1-inequality and $L^\infty \to L^\infty$ (very easy).

Corollary (Lebesque differentiation theorem)**.** Let $f \in L^1(\mathbb{R}^d)$ Then

$$\lim_{|B|, \to 0, x \in B} \fint f = f(x) \quad x\text{-a.e.} \tag{3}$$

Proof.

$$E_\alpha = \{x \in \mathbb{R}^d : \limsup_{|B| \to 0, x \in B} |\fint_B f - f(x) > 2\alpha\}$$

ETS $|E_\alpha| = 0 \ \forall \alpha > 0$. Then $E = \bigcup_{n \in \mathbb{N}} E_{\frac{1}{n}} = 0$ and (3) holds on $E^{\complement}$.

Fix $\alpha > 0$, given $\varepsilon > 0$ choose $g \in C_0^0(\mathbb{R}^d)$ s.t. $\|f - g\|_{L^1} < \varepsilon$. Already seen

$$\lim_{|B| \Rightarrow 0, x \in B} \fint g = g(x) \ \forall x$$

$$\fint_B f - f(x) = \fint_B (f - g) + \fint_B g - g(x) + g(x) - f(x)$$

$$F_\alpha = \{x : M(f - r)(x) > \alpha\}$$
$$G_\alpha = \{x : |f(x) - g(x)| > \alpha\}$$

$E_\alpha \subset F_\alpha \cup G_\alpha$ since $u_1, u_2 > 0,\ u_1 + u_2 > 2\alpha \Rightarrow u_1 > \alpha \vee u_2 > \alpha$.

$$|G_\alpha| \le \frac{1}{\alpha} \|f - g\|_{L^1} \quad \text{(Chebyshew)}$$

$$|F_\alpha| \le \frac{c_d}{\alpha} \|f - g\|_{L^1} \quad \text{(weak type)}$$

$$|E_\alpha| \le |F_\alpha| + |G_\alpha| \le (\frac{c_d}{\alpha} + \frac{1}{\alpha}) \|f - g\|_{L^1} \le \frac{c_d' \varepsilon}{\alpha}$$

Since $\varepsilon > 0$ was arbitrary $|E_\alpha| = 0$. □

$h \in L^1 \subset L^{1,\infty}$ by Chebyshew: $\infty > \|h\|_{l^1} = \int_{\mathbb{R}^d} |h(y)|\,\mathrm{d}y \ge \int_{h(y) \ge \alpha} |h(y)|\,\mathrm{d}y \ge \alpha |\{|h| > \alpha\}|$.

Would have been enough to replace $L^1(\mathbb{R}^d)$ by L^1_{loc}.

Sets $E \subset \mathbb{R}^d$ measurable, $x \in \mathbb{R}^d$ (not necc. in E) x is a point of Lebesque density of E if

$$\lim_{|B| \to 0, x \in B} \frac{|B \cap E|}{B} = 1$$

Corollary. Let $E \subset \mathbb{R}^d$ be measurable. Then

(i) Almost every $x \in E$ is a point of Lebesque density of E.

(ii) Almost every $x \notin E$ is not a point of Lebesque density.

Functions $f \in L^1_{\text{loc}}(\mathbb{R}^d)$.

$$Leb(f) := \{x \in \mathbb{R}^d : f(x) < \infty \text{ and } \lim_{|B| \to 0, x \in B} \fint_B |f(y) - f(x)| \, \mathrm{d}y = 0\}$$

f continuous at $\bar{x} \Rightarrow \bar{x} \in \text{Leb}(f) \Rightarrow \fint_B f \underset{|B| \to 0, x \in B}{\to} f(\bar{x})$ (all the inverse implications are wrong)

Corollary. $f \in L^1_{\text{loc}}(\mathbb{R}^d) \Rightarrow$ Almost every point belongs to Leb(f).

(By checking the proof again?)

These things also works with other sets that ”shrink regularly to x than balls”. It gets worse however when one takes all parallel rectangles and even worse when arbitrarily oriented rectangles are allowed.

Q.2 Key: bounded varioation (BV) $F : [a, b] \to \mathbb{R}$, $P = \{a = t_0 < t_1 < ... < t_N = b\}$

$$V_F^P = \sum_{j=0}^{N} |F(t_j) - F(t_{j+1})|$$

is the variation of f over P. F is of bounded variation if

$$T_F(a, b) = T_F \sup_P V_F^P < \infty$$

$P \subset \tilde{P}$ partitions $\Rightarrow V_F^P \leq V_P^{\tilde{P}}$

Example. (i) f monotonic (increasing) and bounded, $|F| \leq M \Rightarrow F \in$ BV

$$V_F^P = \sum_{j=1}^{N} |F(t_j)T = Nt_{j-1})| = F(b) - F(a) \leq 2M$$

(ii) F differentiable with F' bounded, $|F'| \leq M$, then by mean value theorem $f \in$ BV. Or F LIpschitz

(iii) F α-Hölder ($\alpha < 1$) $6 \implies$ $F \in$ BV. Take $F : [0, 1] \to \mathbb{R}$, $x \mapsto \mathrm{d}(x, C)^\alpha$, where C is the cantor set. 2^{n-1} intervals of length 3^{-n}

$$\alpha > \frac{\log 2}{\log 3} \Rightarrow \sum_{n=1}^{\infty} 2^{n-1} (3^{-n})^\alpha < \infty$$

- *Total variation* of F on $[a, x]$ (where $a \leq x \leq b$) is

$$T_F(a, x) = \sup \sum_{j=0}^{N} |F(t_j) - F(t_{j-1})|$$

- *Positive variation* of F on $[a, 1]$ is

$$P_F(a, x) = \sup \sum_{(+)} (F(t_j) - F(t_{j-1})) \quad \text{all } j : F(t_j) \geq F(t_{j-1})$$

- *Negative variation* of F on $[a, 1]$ is

$$N_F(a, x) = \sup \sum_{(-)} -(F(t_j) - F(t_{j-1}))$$

Lemma. $f : [a, b] \to \mathbb{R}$. Then

(i) $F(x) = F(a) + P_F(a, x) - N_F(a, x)$

(ii) $T_F(a, x) = PF_(a, x) + N_F(a, x)$

$(\forall x \in [a, b])$

recall from measure theory: $f = f^+ - f^-$, $|f| = f^+ + f^-$

Proof. (i) given $\varepsilon > 0$, $\exists P = \{a = t_0 < t_1 < ... < t_N = x\}$

$$|PVF - \sum_{(+)} (F(t_j) - F(t_{j-1}))| < \varepsilon$$

$$|N_F - \sum_{(-)} -(F(t_j) - F(t_{j-1})))| < \varepsilon$$

Also

$$F(x) - F(a) = \sum_{(+)} F(t_j) - F(t_{j-1}) - \sum_{(-)} -(F(t_j) - F(t_{j-1}))$$

$\square$

Corollary. $F : [a, b] \to \mathbb{R} \in \mathrm{BV}$ iff F is the differenco of two inclasing bounded functions

Theorem. $F : [a, b] \to \mathbb{R} \in \mathrm{BV} \Rightarrow F$ *differentiable a.e.*

Wlog f mononotic increasing, "Wlog" f continuous

Lemma (of the rising sun)**.** $G : \mathbb{R} \to \mathbb{R}$ continuous.

$$E = \{x \in \mathbb{R} : \exists h = h_x > 0 \; G(x + h) > G(x)\}$$

Then

(i) E is open ($E = \bigcup_{n=1}^{\infty} (a_n, b_n)$)

(ii) $g(a_n) = G(b_n)$, provided $b_n - a_n < \infty$.

Proof. Let (a_n, b_n) be a finite interval in the decomposition. $a_k \notin E$ then $g(a_k) \geq G(b_k)$. Assume $G(a_k) > G(b_k)$. $\exists c \in (a_k, b_k)$ $g(c) = \frac{g(a_k)+g(b_k)}{2}$. Choose rightmost such c. $\exists d \in (c, b_k)$ $G(d) > G(c)$. But then by continuity c could not have been chosen rightmost, contradiction. $\square$

Can replace $\mathbb{R}$ by $[a,b]$, but then only get for $a_0 = a$ that $G(a_0) \leq G(b_0)$

Proof. of theorem

$$\begin{aligned}
\Delta_h(F)(x) &= \frac{F(x+h)-F(x)}{h} \\
D^{\pm}(F)(x) &= \limsup_{h\to 0, h><0} \Delta_h(F)(x) \\
D_{\pm}(F)(x) &= \liminf_{h\to 0, h><0} \Delta_h(F)(x)
\end{aligned}$$

Dini numbers. Upshot: They are all the same and finite. $D_- \leq D^-$, $D_+ \leq D^+$ clear. ETS

(i) $D^+(F)(x) < \infty$ (a.e. x)

(ii) $D^+(F)(x) \leq D_-(F)(x)$ (a.e. x)

(ii) is equivalent to $D^-(F)(x) \leq D_+(F)(x)$ by replacing $F(x)$ by $-F(-x)$ somewhere. Then $D^+ \leq D_- \leq D^- \leq D_+ \leq D^+ < \infty$.

(i) relacc: F increasing ,bounded, continuous on $[a,b]$. Fix $\gamma > 0$,

$$E_\gamma := \{x : D^+(F)(x) > \gamma\}$$

- E_γ is measurable
- Apply rising sun to $G(x) = F(x) - \gamma x$

$$E_\gamma \subset E = \{x \in [a,b] : \exists h > 0\ G(x+h) > G(x)\} = \bigcup_{k=1}^{\infty} (a_k, b_k)$$

The condition in the set is equivalent to

$$\begin{aligned}
&\iff \exists h > 0\ F(x+h) - \gamma x - \gamma h > F(x) - \gamma x \\
&\iff \exists h > 0\ \frac{F(x+h)-F(x)}{h} > \gamma \\
&\Leftarrow D^+(F)(x) > \gamma
\end{aligned}$$

$G(a_k) \leq G(b_k) \iff F(a_k) - \gamma a_k \leq F(b_k) - \gamma b_k \iff \gamma(b_k - a_k) \leq F(b_k) = F(a_k)$. Therefore

$$|E_\gamma| \leq |E| \leq \sum_{k=1}^{\infty} (b_k - a_k) \leq \frac{1}{\gamma} \sum_{k=1}^{\infty} (F(b_k) - F(a_k)) = \frac{1}{\gamma}(F(b) - F(a))$$

Take $\gamma \to \infty$, done.

(ii) see Stein-Shakarchi (vol 3)

$\square$

Corollary. F increasing, continuous $\Rightarrow F'$ exists a.e., measurable, nonnegative and

$$\int_a^b F'(x)\,\mathrm{d}x \le F(b) - F(a).$$

Proof. Let

$$G_h(x) = \frac{F(x+\frac{1}{n}) - F(x)}{\frac{1}{h}}$$

By the theorem, $G_h(x) \to F'(x)$ $(h \to 0)$ pointwise a.e. By Fatou

$$\int_{[a,b]} F' \le \liminf_{n\to\infty} \int_a^b G_h(x)\,\mathrm{d}x = \liminf_{n\to\infty} ⨍_b^{b+\frac{1}{n}} F(x)\,\mathrm{d}x - ⨍_a^{a+\frac{1}{n}} F(x)\,\mathrm{d}x$$

$\square$

Cannot do better than $\le$: For the Devil's staircase the left hand side is 0 while the rightn hand side is 1.

Why is the sunrise Lemma a covering Lemma?

$$\left.\begin{array}{r} f_+^* = \sup \frac{1}{h}\int_x^{x+h} |f(y)|\,\mathrm{d}y \\ E_\alpha^+ = \{x \in \mathbb{R} : f_+^*(x) > \alpha\} \end{array}\right\} |E_\alpha^+| = \frac{1}{\alpha}\int_{E_\alpha^+} |f|$$

Why? Let

$$G(x) = \int_0^x |f(y)|\,\mathrm{d}y - \alpha x$$

$$x \in E_\alpha^+ \iff f_+^*(x) > \alpha \iff \exists h > 0\ \frac{1}{h}\int_x^{x+h} |f(y)|\,\mathrm{d}y > \alpha \iff \exists h > 0\ G(x+h) > G(x)$$

$$\{x \in \mathbb{R} : \exists h > 0\ G(x+h) > G(x)\} = \dot{\bigcup_{k\in\mathbb{N}}} (a_k, b_k), \quad G(a_k) = G(b_k)$$

$$|E_\alpha^+| = \sum_k (b_k - a_k) = \frac{1}{\alpha}\sum_k \int_{(a_k,b_k)} |f| = \frac{1}{\alpha}\int_{\bigcup_k (a_k,b_k)} |f| = \frac{1}{\alpha}\int_{|E_\alpha^+|} |f|.$$

Definition. $F : [a,b] \to \mathbb{R}$ is *absolutely continuous* if

$$\forall \varepsilon > 0 \exists \delta > 0 : \sum_{k’1}^{N} (B_k - a_k) < s$$

intervals $(a_k b_k)$ disjoint $(k = 1, ..., N) \Rightarrow$

$$\sum_{k=1}^{N} |F(b_k) - F(a_h)| < \varepsilon$$

Remark. (i) On a bounded Inetrval $I \subset \mathbb{R}$

$$C^1(I) \subset \mathrm{Lip}(I) \subset AC(I) \subset BV(I)$$

So they are diff. a.e.. All the inclusions are strict.

(ii) abs cont $\Rightarrow$ unif. con. $\Rightarrow$ cont.

(iii) $f \in L^1_{\text{loc}}(\mathbb{R})$ $F(x) = \int_0^x f(t)\,\mathrm{d}t$ Then F is absolutely continuous. ($\forall \varepsilon \exists \delta |E| < \delta \Rightarrow \int_E |f| < \varepsilon$)
Upshot: AC functions are the ones which re diff a..e. and vrify FTC.

Theorem. *$F \in AC(a,b) \Rightarrow F'$ exists a.e., $F' = 0$ a.e. $\Rightarrow F$ constant*

- Existence of F' clear $\surd$
- $F' = 0$ a.e. $\Rightarrow F$ constant: refinement of Vitali

Definition. A collection $\mathcal{B} = \{B\}$ of (open) balls ön $\mathbb{R}^d$. is a *Vitali covering* of a set E if

$$\forall x \in E \forall \eta > 0 \exists B \in \mathcal{B} : x \in B, |B| < \eta$$

Lemma. $E \subset \mathbb{R}^d$ meas. $|E| < \infty$, $\mathcal{B}$ Vitali covering of E, $\delta > 0$. Then there exist finitely many disjoint balls $B_1, ..., bvB_N \in \mathcal{B}$

$$\sum_{j=1}^N |B_j| \geq |E| - \delta$$

Recall elementary Vitali: $\mathcal{B} = \{B_1, ..., B_N\}$ finite collection of pen balls in $\mathbb{R}^d \Rightarrow \exists$ disjoint subcollection $B_{i_1}, ..., B_{i_k}$ with

$$|\bigcup_{j=1}^B B_j| \leq 3^d \sum_{j=1}^k |B_{i_j}|$$

Proof of Lemma. wlog $\delta > |E|$. Vitali $\Rightarrow \exists$ disjoint subcollection $B_1, ..., B_N \in \mathcal{B}$

$$\sum_{i=1}^{N_1} |B_i| \geq 3^{-d}\delta$$

Sequence of balls $B_1, ..., B_N$. question: Is $\sum_{j=1} |B_j| \geq |E| - \delta$? Yes: done with $N = N_1$. No: work harder.

$$\sum_{j=1}^{N_1} |B_j| < |E| - \delta$$

$$E_2 = E \bigcup_{j=1}^{N_1} \bar{B}_j$$

$$|E_2| \geq |E| - \sum_{j=1}^{N_1} |B_j| > |E| - (|E| - \delta) = \delta$$

$\mathcal{B}$ Vitali covering $\Rightarrow$ balls in $\mathcal{B}$ disjoint frm $\bigcup_{i=1}^{N_1} \bar{B}_i$ still covers E_2. Vitali $\Rightarrow \exists$ finite disjoint subcollection of these balls $B_{N_1+1}, ..., B_{N_2}$

$$\sum_{N_1 < j < N_2} |B_j| \geq 3^{-d}\delta.$$

After k steps, $B_1, ..., B_{N_1}, ...B_{N_1}, ..., B_{N_k}$ with

$$\sum_{j=1}^{N_k} \geq h3^{-d}\delta \geq |E| - \delta$$

iff $k \geq 3^d \frac{|E|-\delta}{\delta}$, stop. $\square$

need to approximate with compact from inside somewhere and with open from outside somewhere else

Corollary. The balls can be arranged in such a way that

$$E \bigcup_{i=1}^{N} B_i| < 2\delta$$

Proof. Choose open $O \supset E : |O \ E| < \delta$. $\mathcal{B}$ Vitali covering $\Rightarrow$ wlog all balls in $\mathcal{B}$ are contained in O.

$$(E \bigcup_{i=1}^{N} B_i) \cup \bigcup_{i=1}]NB_i \subset O. : |E \bigcup_{i=1}^{N} B_i| \leq |O| - \sum_{i=1}^{N} |B_i| \leq |E| + \delta - (|E| - \delta) = 2\delta$$

□

$F \in AC$ *Back to the real lin:* Goal: $F' = 0$ a.e. $\Rightarrow F$ constant. ETS $F(a) = F(b)$

$$E = \{x \in (a,b) : F'(x) \text{ exists and } = 0\} \quad |E| = b - a\}$$

Fix $\varepsilon > 0$. For $x \in E$, $\lim_{h\to 0} |\frac{F(x+h)-F(x)}{h}| = 0$. $\forall \eta > 0 \exists$ open interval $I = (a_x, b_x) \subset [a,b]$ containing x. $F(b_x) - F(a_x)| \leq \varepsilon(b_x - a_x)$ and $b_x - a_x < \eta$. The collection of these intervals (over all $\eta > 0$) forms a Vitali covering of E. Lemma $\Rightarrow$ Given $\delta > 0$ can select finitely many, disjoint $I_j = (a_j, b_j)_{j=1}^{N}$ such that

$$\sum_{j=1}^{N} |I_j| \geq |E| - \delta = b - a - \delta$$

But

$$\sum_{j=1}^{N} |F(b_j) T F(a_j)| \leq \varepsilon \sum_{j=1}^{N} (b_j - a_j) \leq \varepsilon(b-a)$$

$$[a,b] \bigcup_{j=1}^{N} I_j = \bigcup_{k=1}^{M} [\alpha_k, \beta_k]$$

with total length $\leq \delta$. .:$F \in AC$

$$\sum_{k=1}^{M} |F(\beta_k) - F(\alpha_k)| < \varepsilon$$

$$|F(b) - F(a)| \leq \sum |F(b_j) - F(a_j)| + \sum |F(\beta_k) - F(\alpha_k)| \leq \varepsilon(b-a) + \varepsilon,$$

done. □

Theorem. $F \in AC(a,b)$. *Then*

(i) F' *exists a.e. and is integrable*

(ii)

$$F(x) - F(a) = \int_a^x F'(t)\,\mathrm{d}t \quad (\forall a \leq x \leq b)$$

Conversely, if $f \in L^1(,b)$, *then there exists* $F \in AC(a,b) : F' = f$ *a.e..*

Proof. $\Rightarrow$

(i) seen last lecture.

(ii)

$$G(x) := \int_a^x F'(t)\,\mathrm{d}t$$

$.\!:\! G \in AC$ $.\!:\! F - G \in AC$. Lebesque diff. $\Rightarrow G'(x) = F'(x)$ (a.e. x) $.\!:$ $(F-G)' = 0$ a.e.. Therefore $(F-G)(x) = (F-G)(a)$, $F(x) - G(x) = F(a)$, equivalent to (*)

$\Leftarrow$

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

AC $\surd$. Leb. diff $\Rightarrow F' = f$ a.e. $\square$

Next: Monotone functions which are not nec. continuous. Wlog. F increasing, bounded on $[a,b]$.

$$F(x^-)F \lim_{y \to x, y<x} F(y) \quad F(x^+) := ...$$

$F(x^-) \leq F(x) \leq F(x^+)$, F cont. at x if $F(x^-) = F(x^+)$. Otherwise F has a jump discontinuity at x.

Obs: A (bounded) increasing finction F on $[a,b]$ has at most countable many jumps. There exists an injective map $\mathrm{Disc}(F) \to \mathbb{Q}$

$.\!:$ $\mathrm{Disc}(F) = \{x_n\}_{n=1}^\infty$ $\alpha_n = F(x_n^+) - F(x_n^-)$ = jump of F at x_n. $F(x_n^+) = F(x_n^-) + \alpha_n$ $F(x_n) = F(x_n^-) + \theta_n \alpha_n$, $\theta_n \in [0,1]$. $F(x) = \mu((-\infty, x])$. Corresponds to singular + abs. cont measures.

$$j_n(x) = \begin{cases} 0 & x < x_n \\ \phi_n & x = x_n \\ 1 & x > x_n \end{cases}$$

Jump function associated to F is

$$J_F(x) = \sum_{n=1}^\infty j_n(x)$$

$$\sum_{n=1}^\infty \alpha_n = \sum_{n=1}^\infty F(x_n^+) - F(x_n^-) \leq F(b) - F(a) < \infty$$

because F incr, and F bounded.

Lemma. F increasing, bounded on $[a,b]$, $\mathrm{Disc}(F) = \{x_n\}_{n=1}^\infty$

(i) $J_F(x)$ is discontinuous precisely at $\{x_n\}_{n=1}^\infty$, has a jump at x_n equal to that if F.

(ii) Th function $F - J_F$ is increasing and continuous.

Proof. (i) $x \neq x_n (\forall n) \Rightarrow$ each j_n is continuous at $x \Rightarrow J_F$ is continuous at x because of uniform convergence. $x = x_N (\exists N) \Rightarrow J_F = \sum_{n=1}^N \alpha_n j_n(x) + \sum_{n>N} \alpha_n j_n(x)$. First sum has jump discontinuity x_N of size α_N

(ii) $F - J_F$ is continuous

$$F(x) - J_F(x) \le F(y) T J_F(y) \iff J_F(y) - J_F(y) \le F(y) - F(x)$$

, where

$$J_F(y) = \sum_{x < x_n \le y} \alpha_n = \sum_{x < x_n \le y} F(x_n^+) - F(x_n^-) \le F(y) - F(x)$$

□

Since $F = (F - J_F) + J_F$ ETS J_F is diff a.e.. This was essential step of

$$\mu = \mu_{AC} + \mu_S + \mu_{PP}$$

$z(t) = (x(t), y(t))$. curve γ. $x, y : [a,b] \to \mathbb{R}$ continuous.
γ rectifiable if length

$$L(\gamma) = \sup \sum_{j=1}^{N} |z(t_j) - z(t_{j-1})| < \infty$$

sup over all partitions $P = \{a = t_0 < t_1 <_< t_N = b\}$ of $[a,b]$
When is

$$L(\gamma) = \int_a^b |z'(t)| \, \mathrm{d}t?$$

Lemma. γ is rectifiable iff x, y are of bounded variation (and cont.).

see $F = x + iy$
Assume γ rectifiable, let $L(A,B)$ length of $\gamma(A,B)$, $(a \le A \le B \le b)$

(i) $L(A,B) = T_F(A,B)$ (where $F(t) = z(t)$)

(ii) $L(A,C) + L(C,B) = L(A,B)$ $(A \le C \le B)$

(iii) $A \mapsto L(A,B)$ (fix B) is continuous
$B \mapsto L(A,B)$ (fix A)
seen: $F \in \mathrm{BV}(a,b)$, cont. $\Rightarrow T_F$ cont.

Warning: $[0,1] \ni t \mapsto (F(t), F(t))$, F Cantor. F cont. incr. $F(0) = 0$, $F(1) = 1$, $F' = 0$ a.e.

Theorem. $z : [a,b] \to \mathbb{R}^2, t \mapsto (x(t), y(t)) \sim$ *curve* γ. $x, y \in \mathrm{AC}(a,b)$. $\Rightarrow \gamma$ *rectifiable and*

$$L(\gamma) = \int_a^b |z'(t)| \, \mathrm{d}t$$

why? $F : [a,b] \to \mathbb{C}$ is $\mathrm{AC}(a,b)$

$$\Rightarrow T_F(a,b) = \int_a^b |F'(t)| \, \mathrm{d}t.$$

$$\sum_{j=1}^{N} |F(t_j) - F(t_{j-1})| = \sum_{j=1}^{N} \left| \int_{t_{j-1}}^{t_j} F'(t) \, \mathrm{d}t \right| \le \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} |F'(t)| \, \mathrm{d}t = \int_a^b |F'(t)| \, \mathrm{d}t$$

First inequality by FTC. For $\ge$, write $F' = g + h$, g step function, h small in L^1 $G, H =$ def. integrals of g, h. Check $T_F \ge T_G, T_H$, T_H small, $T_G \ge \int_a^b |g(t)| \, \mathrm{d}t$.

Minkowski content of a curve simple, simple closed, quasi-simple curves.

trace of γ: $\Gamma = \{z(t) \in \mathbb{R}^2 : t \in [a,b]\}$.

Given $K \Subset \mathbb{R}^2$ and $\delta > 0$ define

$$K^\delta = \{x \in \mathbb{R}^2 : \mathrm{d}(x,K) < \delta\}$$

where $\mathrm{d}(x,K) = \inf_{k \in K} \mathrm{d}(x,k)$

Definition. The set K has (1D) Minkowski content if

$$\lim_{\delta \to 0} \frac{|K^\delta|}{2\delta}$$

exists (in $\mathbb{R}$), denoted $M(K)$.

Theorem. *Let $\Gamma = \{z(t) : a \le t \le b\}$ be (the trace of) a quasi-simple curve γ. Then Γ has Minkowski content iff γ is rectifiable (in which case $M(\Gamma) = L(\gamma)$).*

Upper Mink: content.

$$\limsup_{\delta \to 0^+} \frac{|K^\delta|}{2\delta} =: M^*(K)$$

lower

$$\liminf_{\delta \to 0^+} \frac{|K^\delta|}{2\delta} =: M_*(K).$$

Proposition. $T = \{z(t) : a \le t \le b\}$ quasi simple. If $M_*(\Gamma) < \infty$, then γ is rectifiable and $L(\gamma) \le M_*(\Gamma)$.

Proposition. $\Gamma = \{z(t) : a \le t \le b\}$ rectifiable γ. Then $M^*(\Gamma) \le L(\gamma)$.

Proof of Prop. 1, for simple curves. Obs: $\Gamma = \{z(t) : a \le t \le b\}$ any curve. $\Delta = |z(b) - z(a)|$. $|\Gamma^\delta| \ge 2\delta\Delta$.

Take any partition P of $[a,b)$. $L_P = \sum_{j=1}^N |z(t_j) - z(t_{j-1})|$. Given $\varepsilon > 0, \exists N$ proper closed subintervals $I_j = [a_j, b_j] \subset (t_{j-1}, t_j)$:

$$\sum_{j=1}^N |z(b_j) - z(a_j)| \le L_P - \varepsilon$$

$I_1, ..., I_N$ disjoint $\Rightarrow \Gamma_1, ..., \Gamma_N$ disjoint because Γ is simple. $\Leftrightarrow \Gamma_1^\delta, ..., \Gamma_N^\delta$ disjoint, provided $\delta > 0$ small enough.

$$\bigcup_{j=1}^N \Gamma_j^\delta \subset \Gamma^\delta$$

$$|\Gamma^\delta| \ge \sum_{j=1}^N |\Gamma_j^\delta| \ge 2\delta \sum_{j=1}^N |z(t_j) - z(t_{j-1})| \ge 2\delta(L_p - \varepsilon)$$

$\square$

Isoperimetric inequality (soft) $\gamma : [a,b] \to \mathbb{R}^2,\ \gamma \in C^1(a,b) : \gamma'(s) \neq 0 \forall s,\ \gamma(a) = \gamma(b)$. Arclength parametrization: $\gamma : [0,L] \to \mathbb{R}^2,\ |\gamma'(s)| = 1 \forall s$.

Theorem. *$\Gamma \subset \mathbb{R}^2$ simple colsed C^2 curve of length L. A area of the region enclosed by Γ.*

$$A = \frac{1}{2}| \int_\Gamma > (x \,\mathrm{d}y - y \,\mathrm{d}x)| = \frac{1}{2}| \int_0^L (x(s)y'(s) - x'(s)y(s)) \,\mathrm{d}s.$$

Then $4\pi A \leq L^2$. Equality iff Γ is a circle.

Proof. wlog (rescale) $L = 2\pi$.: WTS $A \leq \pi$, equality iff Γ is a circle of radius 1.
$\gamma : [0, 2\pi] \to \mathbb{R}^2,\ s \mapsto \gamma(s) = (x(s), y(s))$ arclength par. $x'(s)^2 + y'(s)^2 = 1 \forall s$.:

$$\frac{1}{2\pi} \int_0^{2\pi} (x'(s)^2 + y'(s)^2) \,\mathrm{d}s = 1$$

Γ closed $\Rightarrow x(s), y(s)$ 2η-periodic.

$$x'(s) = \sum_n a_n i n e^{ins}$$

$$y'(s) = \sum_n b_n i n e^{ins}$$

Parseval $\Rightarrow$

$$\sum_{n \in \mathbb{Z}} n^2 (|a_n|^2 + |b_n|^2) = 1$$

$$A = \frac{1}{2} \int_0^{2\pi} (x(s)y'(s) - x'(s)y(s)) \,\mathrm{d}s| = \pi | \sum_{n \in \mathbb{Z}} n(a_n \bar{b}_n - b_n \bar{a}_n)|$$

by bilinear Parseval

$$|a_n \bar{b}_n - b_n \bar{a}_n| \leq 2|a_n||b_n| \leq |a_n|^2 + |b_n|^2$$
$$A \leq \pi \sum_{n \in \mathbb{Z}} n^2 (|a_n|^2 + |b_n|^2) \leq \pi$$

□

Cases of equality: $A = \pi \Rightarrow$

$$x(s) = a_{-1} e^{-is} + a_0 + a_1 e^{is}$$

$$y(s) = b_{-1} e^{-is} + b_0 + b_1 e^{is}$$

x, y real-valued $\Rightarrow a_1 = \bar{a}_{-1},\ b_1 = \bar{b}_{-1}$. (**) $\Rightarrow 2(|a_1|^2 + |b_1|^2) = 1$.
(***) $\Rightarrow$

$$|a_1| = |b_1| = \frac{1}{2}.: \quad a_1 = \frac{1}{2} e^{i\alpha} \quad b_1 = \frac{1}{2} e^{i\beta} \quad (\alpha, \beta \in \mathbb{R})$$

$$1 = 2|a_1 \bar{b}_1 - \bar{a}_1 b_1| = \sin(\alpha - \beta)|.: \quad \alpha - \beta = \frac{k\pi}{2} \quad (\text{odd } k)$$

$$x(s) = a_0 + \cos(s + \alpha)$$

$$y(s) = b_0 \pm \sin(s + \alpha)$$

$\pm$ dep. on parity of $\frac{k-1}{2}$.

Isoperimetric inequality (hard) $\Omega \subset \mathbb{R}^2$ bounded, open, $\partial\Omega = \bar{\Omega} - \Omega =: \Gamma$ rectifiable curve (not nec. simple) with length $l(\Gamma)$.

Theorem.

$$4\pi|\Omega| \leq l(\Omega)^2$$

Proof. inner:

$$\Omega_-^\delta = \{x \in \mathbb{R}^2 : \mathrm{d}(x, \mathbb{R}^2 \ \Omega) \geq \delta\}$$

outer:

$$\Omega_+^\delta = \{x \in \mathbb{R}^2 : \mathrm{d}(x, \bar{\Omega}) < \delta\}$$

$$\Gamma^\delta = \{x : \mathrm{d}(x, \Gamma) < \delta\}$$

$$\Omega_+^\delta = \Omega_-^\delta \dot{\cup} \Gamma >^\delta$$

$A, B \subset \mathbb{R}^d,\ A + B = \{a + b : a \in A, b \in B\}$ Note: $\Omega + B_\delta \subset \Omega_+^\delta,\ \Omega_-^\delta + B_\delta \subset \Omega$.

Brunn-Minkowski: $A, B \subset \mathbb{R}^2$ meas., $A + B$ meas.

$$|A + B|^{\frac{1}{2}} \geq |A|^{\frac{1}{2}} + |B|^{\frac{1}{2}}$$

$$|\Omega_-^\delta| \geq (|\Omega|^{\frac{1}{2}} + |B_\delta|^{\frac{1}{2}})^2 \geq |\Omega| + 2|\Omega|^{\frac{1}{2}} \underbrace{|B_\delta|^{\frac{1}{2}}}_{=(\pi\delta^2)^{\frac{1}{2}}}$$

$$|\Omega| \geq (|\Omega_-^\delta|^{\frac{1}{2}} + |B_\delta|^{\frac{1}{2}})^2 \geq |\Omega_-^\delta| + 2|\Omega_-^\delta|^{\frac{1}{2}}|B_\delta|^{\frac{1}{2}}$$

$$|\Gamma^\delta| \geq |\Omega| + 2|\Omega|^{\frac{1}{2}}\sqrt{\pi} - |\Omega| + |\Omega_-^\delta|^{\frac{1}{2}}\sqrt{\pi}$$

$$\limsup_{\delta \to 0^+} \frac{|\Gamma^\delta|}{2\delta} \geq 2|\Omega|^{\frac{1}{2}}\sqrt{\pi}$$

$$4\pi|\Omega| \leq M^*(\Gamma)^2 \leq l(\Gamma)^2$$

Note, that only in the very last inequality did we use the rectifiability of Γ. □

Brunn-Minkowski ineq. ($\mathbb{R}^d$) $A, b \subset \mathbb{R}^d$ measurable. $A + B = \{a + b : a \in A, , b \in B\}$. $\lambda A = \{\lambda a : a \in A\}$ ($\lambda > 0$).

Q.: Can $|A + B|$ be controlled in terms of $|A|, |B|$? No! There exist sets A, B $|A| = |B| = 0$ with $|A + B| > 0$. Example $[0,1] \times [0,1]$. Another example $A = B = C \subset [0,1]$ Cantor set. Then $A + B = [0,2]$.

Q.: Can $|A + B|^\alpha \geq c_\alpha(|A|^\alpha + |B|^\alpha)$ hold? (for some $\alpha > 0$ with $c_\alpha < \infty$, indep of A, B) Best possible $c_\alpha = 1$.

What about α? Convex sets play a role. $A =$ convex, $B = \lambda A$. $|B| = |\lambda A| = \lambda^d|A|$. $|A + B| = |A + \lambda A| = |(1 + \lambda)A| = (1 + \lambda)^d|A|$ because A is convex.

($\lambda_1 A + \lambda_2 A = (\lambda_1 + \lambda_2)A$ iff A is convex.)

$|A + B|^\alpha \geq |A|^\alpha + |B|^\alpha$ iff $(1 + \lambda)^{d\alpha} \geq 1 + \lambda^{d\alpha} \Rightarrow \alpha \geq \frac{1}{d}$.

$(a + b)^\gamma \geq a^\gamma + b^\gamma \forall a, b \geq 0,\ \gamma \geq 1$.

Candidate inequality:

$$|A + B|^{\frac{1}{d}} \geq |A|^{\frac{1}{d}} + |B|^{\frac{1}{d}}$$

(BM)

A, B measurable $6 \Longrightarrow$ $A + B$ measurable. Take $[0,1] \times$ nonmeasurable.

(i) A, B closed $\Rightarrow A + B$ measurable

(ii) A, B compact $\Rightarrow A + B$ compact

(iii) A, B open $\Rightarrow A + B$ open

Theorem. *(BM) holds if $A, B, A + B$ measurable.*

(i) A, B rectangles with sidelengths $\{a_j\}_{j=1}^{\infty}$, $\{b_j\}_{j=1}^{\infty}$

(ii) A, B unions of fifinely many rectangles with disjoint interiors.

(iii) A, B open sets of finite measure

(iv) A, B compact

(v) $A, B, A + B$ measurable.

Proof. (i) (BM) becomes

$$\prod_{j=1}^{d}(a_j + b_j)^{\frac{1}{d}} \geq \prod_{j=1}^{d} a_j^{\frac{1}{d}} + \prod_{j=1}^{d} b_j^{\frac{1}{d}}$$

$a_j \to \lambda_l a_j$, $b_j \to \lambda_j b_j$. Both sides are multiplied by $(\lambda_1 \lambda_2 ... \lambda_d)^{\frac{1}{d}}$: wglog can assume $a_j + b_j = 1 \forall j$ (Choose $\lambda_j = a_j + b_j$)

AMGM:

$$\prod_{j=1}^{d} a_j^{\frac{1}{d}} \leq \frac{1}{d} \sum_{j_1}^{d} a_j$$

$$\prod_{j=1}^{d} b_j^{\frac{1}{d}} \leq \frac{1}{d} \sum_{j_1}^{d} b_j$$

$$\prod a_j^{\frac{1}{d}} + \prod b_j^{\frac{1}{d}} \leq \frac{1}{d} \sum_{j=1}^{d} (a_j + b_j) = 1$$

(ii) Induction on $n =$ numeber of rectangles in A and B. Choose pair of disjoint rectangles R_1, R_2 in A. Can rotate s.t. R_1 and R_2 are separated by hyperplane $\{x_j = 0\}$. R_1 lies in $A_+ = A \cap \{x_j \geq 0\}$, $A_I = A \cap \{x_j \leq 0\}$.

Rem.: Both A_+, A_- contain at leas one less rectangle than A, $A = A_+ \subset A_-$ and $A_B \cap A_-$ has measure zero.

Now: translate B s.t. B_- and B_+ satisfy

$$\frac{|B_\pm|}{|B|} = \frac{|A_\pm|}{|A|}$$

$(A_+ + B_+) \cup (A_- + B_-) \subset A + B$ Number of rectangles in A_+ and B_+, number of rectangles in A_- and B_- is $< n$.

$$\begin{aligned}
|A + B| &\geq |A_+ + B_-| + |A_- B +_-| \geq (|A_+|^{\frac{1}{d}} + |B_+|^{\frac{1}{d}})^d + (|A_-|^{\frac{1}{d}} + |B_-|^{\frac{1}{d}})^d \\
&= (|A_+|(1 + (\frac{|B_+|}{|A_+|})^{\frac{1}{d}})^d + |A_-|(1 + (\frac{|B_-|}{|A_-|})^{\frac{1}{d}})^d = (|A_+| + |A_-|)(1 + (\frac{|B|}{|A|})^{\frac{1}{d}})^d \\
&= (|A|^{\frac{1}{d}} + |B|^{\frac{1}{d}})^d.
\end{aligned}$$

(iii) Open sets of finite measure A, B. $\forall \varepsilon > 0 \exists A_\varepsilon, B_\varepsilon$ finet unions of parallel rectangles with disjoint interiors. $A_\varepsilon \subset A, B_\alpha \subset B$, $|A| \leq |A_\varepsilon| + \varepsilon$, $|B| \leq |B_\varepsilon| + \varepsilon$.

$|A+B| \geq |A_\varepsilon + B_\varepsilon| \geq (|A_\varepsilon|^{\frac{1}{d}} + |B_\varepsilon|^{\frac{1}{d}})^d \geq ((|A|-\varepsilon)^{\frac{1}{d}} + (|B|-\varepsilon)^{\frac{1}{d}})^d$. Let $\varepsilon \to 0^+$, done.

(iv) A, B compact. Let $A^\varepsilon = \{x : d(x, A) < \varepsilon\}$. $A + B \subset A^\varepsilon + B^\varepsilon \subset (A+B)^{2\varepsilon}$

(v) $A, B, A+B$ measurable: usi inner regularity of Lebesque measure.

□

Remark. A, B open sets of finite positive measure. Equality in (BM) iff A, B convex and similar. $\exists \delta > 0 \exists h \in \mathbb{R}^d : A = \delta B + h$ (A convex iff $\lambda_i A + \lambda_2 A = (\lambda_1 + \lambda_2)A$)

Consequences for isoperimetric inequality $A \subset \mathbb{R}^d$ bounded open with smooth boundary. (∂A, $B \subset \mathbb{R}^d$ ball $|B| = |A|$)

$$|\partial A| = \lim_{\varepsilon \to 0^+} \frac{|A + \varepsilon B| - |A|}{\varepsilon}$$

Isoper ineq.: $|\partial A| \geq |\partial B|$.

Proof.

$$\frac{|A+\varepsilon B| - |A|}{\varepsilon} \geq \frac{(|A|^{\frac{1}{d}} + |\varepsilon B|^{\frac{1}{d}})^d - |A|}{\varepsilon} = \frac{(1+\varepsilon)^d - 1}{\varepsilon}|B| \to d|B| = |\partial B|$$

for $\varepsilon \to 0$. □

Better: $A \subset \mathbb{R}^d$ has finite perimeter ($\iff$ $1_A \in \mathrm{BV}(U)$, $U \subset \mathbb{R}^d$ bdd open)

$$\frac{\mathcal{H}^{d-1}(\partial A)}{|A|^{\frac{d-1}{d}}} \geq \frac{\mathcal{H}^{d-1}(S^{d-1})}{|B^d(0,1)|^{\frac{d-1}{d}}}$$

Hausdorff measure Q: How does a set replicate under scaling? $E \to nE = E_1 \cup ... \cup E_m$ disjoint congruent copies of E. Examples: line $m = n^1$, square $m = n^2$, cube $m = n^3$, Cantor set $3C = C_1 \cup C_2$ $2 = 3^\alpha \iff \alpha = \frac{\log 3}{\log 2}$

$\#(\varepsilon)$ =least # of segments that arise from such poygonal lines. Γ rectifiable iff $\#(\varepsilon) \sim \varepsilon^{-1}$ as $\varepsilon \to 0^+$. If $\#(\varepsilon) \sim \varepsilon^{-\alpha}$ ($\alpha > 1$) In this case, say "Γ has dim α". Snowflake has $\alpha = \frac{\log 4}{\log 3} > 1$.

Upshot: E $\alpha > 1$. $m_\alpha(E)$ = α-dimensional mass of E among sets of "dimension" α.

- $\alpha > \dim(E) \Rightarrow m_\alpha(E) = 0$
- $\alpha < \dim(E) \Rightarrow m_\alpha(E) = \infty$
- $\alpha = \dim(E)$ interesting

R. Gardener Bulletin AMS more about Brunn-Minkowski, geometrically, including more proofs, e.g. with induction of the dimension.

Hausdorff measure $E \subset \mathbb{R}^d$ any subset.

$$m_\alpha^*(E) := \lim_{\delta\to 0^+} \underbrace{\inf\{\sum_k (\operatorname{diam} F_k)^\alpha : E \subset \bigcup_{k=1}^\infty F_k \ \ \operatorname{diam}(F_k) \le \Delta\}}_{H_\alpha^\delta(E)}$$

exterior/outer α-dim Hausdorff measure.

Remark. $H_\alpha^\delta(E) \le H_\alpha^\delta(E) \le m_\alpha^*(E)(\forall\delta > 0)$. $H_\alpha^\delta(E)$ increases when δ ecreases. $\therefore m_\alpha^*(E) = \lim_{\delta\to 0^+} H_\alpha^\delta(E)$ exists

Remark. Coverings must be by sets of arb. small measure. (If we allowed the δ to be arbitrary then two parallel lines would get the same 1d-measure as one of them.)

Remark (Skaling)*.* "The measure of a set should scale like its dimension". E.g.: $\Gamma \subset \mathbb{R}^d$ smoot cureve of length L *sim* $\lambda\Gamma$ has length λL. $Q \subset \mathbb{R}^d$ cube *sum* λQ has measure $\lambda^d|Q|$. $|F|$ scaled by $\lambda \Rightarrow (\operatorname{diam} F)^\alpha$ scaled by λ^α

Properties

(i) $E_1 \subset E_2 \Rightarrow m_\alpha^*(E_1) \le m_\alpha^*(E_2)$

(ii) $\{E_j\} \subset \mathbb{R}^d$ countable family of sets $\Rightarrow m_\alpha^*(\bigcup_{j=1}^\infty E_j) \le \sum_{j=1}^\infty m_\alpha^*(E_j)$

(iii) (Finite additility) $\inf_{x\in E_1, y\in E_1} |x-y| = \operatorname{d}(E_1, E_2) > 0 \Rightarrow m_\alpha^*(E_1 \cup E_2) = m_\alpha^*(E_1) + m_\alpha^*(E_2)$

Proof. ETS $\ge$. Fix $0 < \varepsilon < \operatorname{d}(E_1, E_2)$. Given any cover of $E_1 \cup E_2$ with sets $F_1, F_, \ldots$ of diam $\le \delta < \varepsilon$, let $F_j' = F_j \cap E_1$, $F_j'' = F_j \cap E_2$.

$$\sum(\operatorname{diam}_j F_j')^\alpha + \sum_j (\operatorname{diam} F_j'')^\alpha \le \sum_k \operatorname{diam}(F_k)^\alpha$$

Take inf over all covers, let $\delta \to 0^+$, done. □

m_α^* satisfies all properties of a Caratheodory outer measure $\therefore m_\alpha^*$ is a countabley additive maisure when restricted to Borel sets, call it m_α= α-dim Hausdorff measure.

(iv) $\{E_j\}$ countable family of disjoint Borel sets $\Rightarrow$

$$m_\alpha(\dot{\bigcup}_{j=1}^\infty E_j) = \sum_{j=1}^\infty m_\alpha(E_j)$$

(v) Hausdorff masure is invariant under translation and rotations. It scales like:

$$m_\alpha(\lambda E) = \lambda^\alpha m_\alpha(E)$$

(vi) $m_0(E) = \#E$, $m_1(E) = |E|$ (=1D LEbesgue measure of E), $E \subset \mathbb{R}$ Borel.

(vii) $E \subset \mathbb{R}^d$ Borel, $m_\alpha(E) \simeq |E|$

Proof. (i) Isodiametric inequality: $|E| \le v_d(\frac{\operatorname{diam} E}{2})^d$, v_d volume of the unit ball in $\mathbb{R}^d$. Prove first for sets $E = -E$ and then something hard.

(ii) Covering argument: Given $\varepsilon, \delta > 0$, there exists a covering of E by balls $\{B_j\}$: $\mathrm{diam} B_j < \delta$, $\sum_j |B_j| \le |E| + \varepsilon$

$$H_d^\delta(E) \le \sum_j (\mathrm{diam} B_j)^d = c_d \sum_j |B_j| \le c_d(|E| + \varepsilon),$$

let $\delta, \varepsilon \to 0^+$, get one of the inequalities.

□

(viii) if $m_\alpha^*(E) < \infty$ and $\beta > \alpha$, then $m_\beta^*(E) = 0$. If $m_\alpha^*(E) > 0$ and $\beta < \alpha$, then $m_\beta^(E) = \infty$.

Proof. $\mathrm{diam} F < \delta, \beta > \alpha \Rightarrow (\mathrm{diam} F)^\beta = (\mathrm{diam} F)^{\beta-\alpha}(\mathrm{diam} F)^\alpha < \delta^{\beta-\alpha}(\mathrm{diam} F)^\alpha$ □

Consequence: Given $E \subset \mathbb{R}^d$ Borel, $\exists!\, \alpha$ such that

$$m_\beta(E) = \begin{cases} \infty & \beta < \alpha \\ 0 \beta > \alpha \end{cases}$$

$$\alpha = \sup\{\beta : m_\beta(E) = \infty\} = \inf\{\beta : m_\beta(E) = 0\} := \text{Hausdorff dimension of } E = \dim E$$

At the critical value $\alpha = \dim E$ $0 \le m_\alpha(E) \le \infty$. If E is bounded and the enequalities are strict, we say that E has strict Hausdorff dimension α.

Theorem. *The Cantor set $C \subset [0,1)$ has strict Hausdorff dimensios $\frac{\log 2}{\log 3}$.*

ETS: $0 < m_\alpha(C) \le 1$

Proof. $m_\alpha(C) \le 1$: $C = \bigcap C_k$ where each C_k is a finite unionof 2^k inetrvals of length 3^{-k}.. Given $\delta > 0$ coose k large enough tuch tht $3^{-k} < \delta$. C_k covers C and Consists of 2^k intervals of diameter $3^{-k} < \delta$. $H_\alpha^\delta(C) \le 2^k(3^{-k})^\alpha = 1$, let $\delta \to 0^+$, done.

$m_\alpha(C) > 0$:

Lemma. $E \Subset d$ compact, $f : E \to \mathbb{R}$ γ-Hölder,

$$|f(x) - f(y)| \le m|x-y|^\gamma \quad (\forall x, y \in E) \quad 0 < \gamma \le 1$$

Then

(i) $m_\beta(f(E)) \le M^\beta m_\alpha(E)$ if $\beta = \frac{\alpha}{\gamma}$.

(ii) $\dim f(E) \le \frac{1}{\gamma} \dim(E)$

Proof. $\{F_k\}$ countable family of sets that overs E $\therefore \{f(F_k \cap E)\}$ covers $f(E)$. $\mathrm{diam} f(F_k \cap E) \le M(\mathrm{diam} F_k)^\gamma$.

$$\sum_k (\mathrm{diam} f(E \aleph F_k))^{\frac{\alpha}{\gamma}} \le M^{\frac{\alpha}{\gamma}} \sum_k (\mathrm{diam} F_k)^\alpha,$$

done. and 1 implies 2. □

Lemma. The Cantor-Lebesgue function $F : C \to [0,1]$ is $\gamma = \frac{\log 2}{\log 3}$-Hölder.

Proof. Goal: $|F(x) - F(y)| \leq c|x-y|^\gamma \ \forall x, y \in C$.

F_n increases at most 2^{-n} on an interval of length 3^{-n}. $\therefore$ slope $\leq (\frac{3}{2})^n$ $\therefore |F_n(x) - F_n(y)| \leq (\frac{3}{2})^n |x-y|$. $|F_n(x) - F(x)| \leq 2^{-n}$. Given x, y chose n: $3^n|x-y| \sim 1$, $3^\gamma = 2$.

$$|F(x)-F(y)| \leq |F_n(x)-F_n(y)|+|F_n(x)-F(x)|+|F_n(y)-F(y)| \leq (\frac{3}{2})^n|x-y|+2\cdot 2^{-n} \leq c2^{-n} = c(3^{-n})^\gamma \leq c'|x-y|^\gamma$$

□

Apply LEmma 1 with $E = C,\ f = F,\ \gamma = \frac{\log 2}{\log 3} \Rightarrow 1 = m_1([0,1]) \leq M m_\alpha(C)$, $\dim C = \frac{\log 2}{\log 3}$ □

Rectifiable curves

Theorem. $\gamma : [a,b] \to \mathbb{R}^d$ *continuous and simple. Then* γ *is rectifiable iff* $\Gamma = \{\gamma(t) : a \leq t \leq b\}$ *has strict Hausdorff dimension equal to 1.* $m_1(\Gamma) = l(\gamma)$.

Proof. $\Rightarrow$: Let γ be rectifiable of length L. Consider acrlength parametrization $\tilde{\gamma}$. $\Gamma = \{\tilde{\gamma}(s) : 0 \leq s \leq L\}$.

$$|\tilde{\gamma}(s_1) - \tilde{\gamma}(s_2)| \leq |s_1 - s_2|$$

By Lemma 1 (i) $m_1(\Gamma) \leq L$. Wh $m_1(\Gamma) \geq L$?

$$\Gamma_j = \{\gamma(t) : t_j \leq t \leq t_{j+1}\}$$

$$\Gamma = \bigcup_{j=1}^{N-1} \Gamma_j \quad m_1(\Gamma) = \sum_{j=0}^{N-1} m_1(\Gamma_j)$$

Claim: $m_1(\Gamma_j) \geq l_j := |\gamma(t_j) - \gamma(t_{j+1})|$

Proof. $\pi : \mathbb{R}^2 \infty \mathbb{R}$ $(x,y) \mapsto x$ Lipschitz, $\pi(\Gamma_j) \subset [0, l_j]$ Lemma 1 (i) implies the claim. □

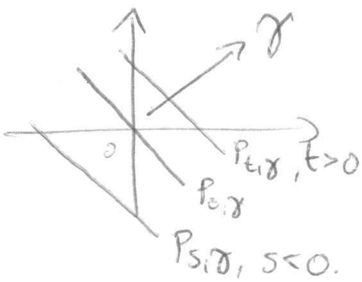
$\therefore m_1(\Gamma) \geq \sum m_1(\Gamma_j) \geq \sum l_j,\ L := \sup_p \sum l_j \therefore m_1(\Gamma) \geq L$, done. □

Radon transforms

$$\mathcal{R}(f)(t,\gamma) = \int_{P_{t,\gamma}} f. \quad \text{where} \dots$$

$f: \mathbb{R}^d \to \mathbb{R}$, $t \in \mathbb{R}$, $\gamma \in S^{d-1} \subseteq \mathbb{R}^d$

$P_{t,\gamma} = \{x \in \mathbb{R}^d : \underbrace{x \cdot \gamma}_{\text{inner pr.}} = t\}$ hyperplane



$P_{t,\gamma}$ equipped with natural $(d-1)$-dim. Leb. measure, (coincides with $(d-1)$-Hausdorff measure) denoted by m_{d-1}

Remarks (i) $f \in C_0^0(\mathbb{R}^d) \Rightarrow f$ integrable on every $P_{t,\gamma}$
$\Rightarrow R(f)(t,\gamma)$ defined for every (t,γ).
($R(f)$ cont. fct. of (t,γ), cpctly spp. in t

(ii) $f \in L^1(\mathbb{R}^d) \Rightarrow f$ may fail to be measurable/integrable on some $P_{t,\gamma}$ ($\to R(f)(t,\gamma)$ not defined.)

(iii) $f = \chi_E$ ($E \subset \mathbb{R}^d$ mb.) $\Rightarrow R(f)(t,\gamma) = m_{d-1}(E_{t,\gamma})$
$E_{t,\gamma} = E \cap P_{t,\gamma}$ if $E_{t,\gamma}$ measurable.

Look instead at maximal Radon transform:

$$R^*(f)(\gamma) = \sup_{t \in \mathbb{R}} |R(f)(t,\gamma)|.$$

$\to$ Want to study L^p-mapping properties of R in order to study regularity of subsets of $\mathbb{R}^d$.

Thm. 1 $f \in C_0^\circ(\mathbb{R}^d)$, $n \geq 3$:

$$(*) \quad \int_{S^{d-1}} \mathcal{R}^*(f)(\gamma)\, d\sigma_\gamma \leq c\left(\|f\|_{L^1(\mathbb{R}^d)} + \|f\|_{L^2(\mathbb{R}^d)}\right).$$

~~(i.e. f needs~~

Rem. i) necessary conditions:

$\cdot)$ $f \in L^1$: $f(x) = (1+|x|^{d-1})^{-1} \in (L^2 \setminus L^1)(\mathbb{R}^d)$ if $d \geq 3$

f is not integrable in any plane $P_{t,\gamma}$.

($\to f \in L^1$ gives global control.)

$\cdot)$ $f \in L^2$: $f_\varepsilon(x) = (|x|+\varepsilon)^{-d+\delta}$ if $|x| \leq 1$, $\delta \in (0,1)$ fixed.

Let $\varepsilon \to 0^+$ to see that $(*)$ fails if $\|\cdot\|_{L^2}$ on the RHS is not there

($\to f \in L^2$ gives local control.)

Key: Interplay between Radon and Fourier transform.

$t \mapsto \lambda \in \mathbb{R}$ dual variable.

Fourier transform:

$$\hat{R}(f)(\lambda,\gamma) = \int_{-\infty}^{\infty} R(f)(t,\gamma)\, e^{-2\pi i \lambda t}\, dt$$

Lemma 1 $f \in C_0^\circ(\mathbb{R}^d)$, $\gamma \in S^{d-1}$:

$$\hat{R}(f)(\lambda,\gamma) = \hat{f}(\lambda\gamma).$$

Proof $\hat{f}(\lambda\gamma) = \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i x \cdot (\lambda\gamma)}\, dx$

$$= \int_{-\infty}^{\infty} \left(\int_{\mathbb{R}^{d-1}} f(u,t)\, du\right) e^{-2\pi i \lambda t}\, dt. = \int_{-\infty}^{\infty} \left(\int_{P_{t,\gamma}} f\right) e^{-2\pi i \lambda t}\, dt.$$

Choose coordinates
$x = (u,t)$, $t = x\cdot\gamma = x_d \in \mathbb{R}$,
$u = (x_1, \dots, x_{d-1}) \in \mathbb{R}^{d-1}$.

$\square$

(2

<u>Lemma 2</u> $f \in C_0^0(\mathbb{R}^d)$:

$$\int_{S^{d-1}} \left(\int_{-\infty}^{\infty} |\hat{R}(f)(\lambda, \gamma)|^2 |\lambda|^{d-1} d\lambda \right) d\sigma_\gamma = 2 \int_{\mathbb{R}^d} |f(x)|^2 dx$$

<u>Proof</u> $2 \int_{\mathbb{R}^d} |f(x)|^2 dx = 2 \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 d\xi$

$= 2 \int_{S^{d-1}} \left(\int_0^\infty |\hat{f}(\lambda\gamma)|^2 \lambda^{d-1} d\lambda \right) d\sigma_\gamma.$

$= 2 \int_{S^{d-1}} \left(\int_{-\infty}^0 |\hat{f}(\lambda\gamma)|^2 |\lambda|^{d-1} d\lambda \right) d\sigma_\gamma$

done by Lemma 2. $\square$

<u>Lemma 3</u> Suppose $F(t) = \int_{-\infty}^{\infty} \hat{F}(\lambda) e^{2\pi i \lambda t} d\lambda$

where $\sup_{\lambda \in \mathbb{R}} |\hat{F}(\lambda)| \leq A$ and $\int_{-\infty}^{\infty} |\hat{F}(\lambda)|^2 |\lambda|^{d-1} d\lambda \leq B^2$.

Then $\sup_{t \in \mathbb{R}} |F(t)| \leq c(A+B)$ (This is essential in the proof of Thm. 1)

Moreover, if $0 < \alpha < \frac{1}{2}$, then $|F(t_1) - F(t_2)| \leq C_\alpha |t_1 - t_2|^\alpha \cdot (A+B)$

($\rightarrow$ α-Hölder)

<u>Proof.</u>

$$F(t) = \underbrace{\int_{|\lambda|<1} \hat{F}(\lambda) e^{2\pi i \lambda t} d\lambda}_{\leq CA} + \underbrace{\int_{|\lambda|>1} \hat{F}(\lambda) e^{2\pi i \lambda t} d\lambda}_{\leq \int_{|\lambda|>1} \hat{F}(\lambda) d\lambda}$$

$$\overset{CS}{\leq} \underbrace{\left(\int_{|\lambda|>1} |\hat{F}(\lambda)|^2 |\lambda|^{d-1} d\lambda \right)^{1/2}}_{\leq B} \cdot \underbrace{\left(\int_{|\lambda|>1} |\lambda|^{-d+1} d\lambda \right)^{1/2}}_{< \infty \text{ if } -d+1 < -1 \iff d \geq 3. \checkmark}$$

$\Rightarrow$ first estimate.

$$|F(t_1)-F(t_2)| = \int_{-\infty}^{\infty} \hat{F}(\lambda) \underbrace{(e^{2\pi i \lambda t_1} - e^{2\pi i \lambda t_2})}_{|\cdot| \le C_\alpha |t_1-t_2|^\alpha |\lambda|^\alpha \text{ if } \alpha\in(0,1]} d\lambda$$

$(x \to e^{ix}$ Lipschitz$)$

$$= \int_{|\lambda|\le 1} \} \le C_\alpha A |t_1-t_2|^\alpha$$

$$+ \int_{|\lambda|>1} \} \le |t_1-t_2|^\alpha \underbrace{\int_{|\lambda|>1} |\hat{F}(\lambda)| |\lambda|^\alpha d\lambda}_{\le \left(\int |\hat{F}(\lambda)|^2 |\lambda|^{d-1} d\lambda\right)^{1/2} \cdot \left(\int_{|\lambda|>1} |\lambda|^{-d+1+2\alpha} d\lambda\right)}$$

$$\underbrace{\int_{|\lambda|>1} |\lambda|^{-d+1+2\alpha} d\lambda}_{<\infty \text{ if } \alpha < \frac{1}{2} \text{ for } d\le 3.}$$

$\square$

Proof of Thm. For each $\gamma \in S^{d-1}$, let $F(t) = R(f)(t,\gamma)$

$\Rightarrow \sup_{t\in\mathbb{R}} |F(t)| = R^*(f)(\gamma)$.

Let $A(\gamma) = \sup_{\lambda\in\mathbb{R}} |\hat{F}(\lambda)|$, $B^2(\gamma) = \int_{-\infty}^{\infty} |\hat{F}(\lambda)|^2 |\lambda|^{d-1} d\lambda$

Lemma 3 $\overset{\text{check}}{\underset{\text{assumptions!}}{\Rightarrow}}$ $\sup_{t\in\mathbb{R}} |F(t)| \le C(A(\gamma)+B(\gamma))$

Lemma 1 $\Rightarrow$ $\hat{F}(\lambda) = \hat{R}(f)(\lambda,\gamma) = \hat{f}(\lambda\gamma) \Rightarrow A(\gamma) \le \|f\|_{L^1(\mathbb{R}^d)}$

Lemma 2 $\Rightarrow$ $\int_{S^{d-1}} B^2(\gamma)\, d\sigma_\gamma = 2\|f\|^2_{L^2(\mathbb{R}^d)}$.

We have $\sup_{t\in\mathbb{R}} |F(t)|^2 \le C'(A^2(\gamma)+B^2(\gamma))$

Integrate both sides:

$$\int_{S^{d-1}} R^*(f)(\gamma)^2 d\sigma_\gamma \lesssim \underbrace{\underbrace{\int A^2(\gamma)\, d\sigma_\gamma}_{\lesssim \|f\|_1^2} + \underbrace{\int B^2(\gamma)\, d\sigma_\gamma}_{\simeq \|f\|^2_{L^2}}}_{\lesssim (\|f\|_{L^1} + \|f\|_{L^2})^2}$$

Use Hölder, because on cpt. space:

$\|R^*(f)\|^2_{L^1} \lesssim \int_{S^{d-1}} R^*(f)(\gamma)^2 d\sigma_\gamma$.

$\square$

Regularity of sets when $d \geq 3$.

$E \subset \mathbb{R}^d$ meas. $E_{t,\gamma} = E \cap P_{t,\gamma}$ (t varies, γ fixed)

Fubini $\Rightarrow$ $E_{t,\gamma}$ is m_{d-1}-measurable for a.e. t

$t \mapsto m_{d-1}(E_{t,\gamma})$ is a measurable fctn of t.

Thm. 2 $E \subset \mathbb{R}^d$ ($d \geq 3$) of finite measure.

Then for a.e. $\gamma \in S^{d-1}$:

(i) $E_{t,\gamma}$ is m_{d-1}-measurable for every t.

(ii) $t \mapsto m_{d-1}(E_{t,\gamma})$ is a cont. fct. of t.

Moreover, this fctn is α-Hölder $\forall \alpha \in (0, \frac{1}{2})$.

Cor. $d \geq 3$, $E \subset \mathbb{R}^d$ of Lebesgue measure zero.

Then, for a.e. $\gamma \in S^{d-1}$, the slice $E_{t,\gamma}$ has zero measure for every $t \in \mathbb{R}$.

Prop. $d \geq 3$, $f \in (L^1 \cap L^2)(\mathbb{R}^d)$. Then for a.e. $\gamma \in S^{d-1}$:

i) f is meas. and int. on the plane for every $t \in \mathbb{R}$.

ii) $t \mapsto \mathcal{R}(f)(t,\gamma)$ is cont. and α-Hölder if $\alpha < \frac{1}{2}$.

Moreover, estimate $(*)$ from Thm. 1 holds for f.

Rem. Prop. implies Thm. 2 by taking char. fctn of E.

$\mathcal{R}(\chi_E)(t,\gamma) = m_{d-1}(E_{t,\gamma})$.

We skip the proof of Prop. (follows from Thm. 1 using some delicate measure theory.)

What about $d = 2$?

Given $f \in L^1(\mathbb{R}^2)$, define

$$\mathcal{R}_\delta(f)(t,\gamma) = \frac{1}{2\delta} \int_{t-\delta}^{t-\delta} \mathcal{R}(f)(s,\gamma)\, ds. \quad \text{(averaged version of } \mathcal{R}\text{.)}$$

(integration over thickened line (= hyperplane).)

$$= \frac{1}{2\delta} \int_{\{t-\delta \leq x\cdot\gamma \leq t+\delta\}} f(x)\, dx.$$

<u>Thm. 3</u> $f \in C_0^\circ(\mathbb{R}^2)$, $0 < \delta \leq 1/2$.

$$\int_{S^1} R_\delta^*(f)(\gamma)\, d\sigma_\gamma \lesssim \left(\log \frac{1}{\delta}\right)^{1/2} \cdot \left(\|f\|_{L^1(\mathbb{R}^2)} + \|f\|_{L^2(\mathbb{R}^2)}\right)$$

Theorem. *$f \in C_0^0(\mathbb{R}^2)$, $0 < \delta \le \frac{1}{2}$. Then*

$$\int_{S^1} R_\delta^*(f)(\gamma)\,\mathrm{d}\sigma_\gamma \lesssim (\log\frac{1}{\delta})^{\frac{1}{2}}(\|f\|_{L^1(\mathbb{R}^2)} + \|f\|_{L^2(\mathbb{R}^2)})$$

Proof of Theorem. Modified version of lemma 3: Setting

$$F_\delta(t) = \int_{-\infty}^{\infty} \hat{F}(\lambda)(\frac{e^{2\pi i(t+\delta)\lambda} - e^{2\pi i(t-\delta)\lambda}}{2\pi i\lambda(2\delta)})\,\mathrm{d}\lambda$$

Suppose $\sup_{\lambda\in\mathbb{R}} |\hat{F}(\lambda)| \le A$ and $\int_{-\infty}^{\infty} |\hat{F}(\lambda)|^2|\lambda|^{d-1}\,\mathrm{d}\lambda \le B^2$.

Claim:

$$\sup_t |F_\delta(t)| \lesssim (\log\frac{1}{\delta})^{\frac{1}{2}}(A+B)$$

$$F_\delta(t) = \int_{-\infty}\infty = \int_{|\lambda|\le 1} + \int_{|\lambda|>1} \le cA + \int_{1<|\lambda|\le\frac{1}{\delta}} |\hat{F}(\lambda)|\,\mathrm{d}\lambda + \frac{c}{\delta}\int_{|\lambda|>\frac{1}{\delta}} |\hat{F}(\lambda)||\lambda|^{-1}\,\mathrm{d}\lambda = I + II$$

CS:

$$I \lesssim (\int_{\mathbb{R}} |\hat{F}(\lambda)|^2|\lambda|\,\mathrm{d}\lambda)^{\frac{1}{2}}(\int_{1<|\lambda|\le\frac{1}{\delta}} |\lambda|^{-1}\,\mathrm{d}\lambda)^{\frac{1}{2}} \le B(\log\frac{1}{\delta})^{\frac{1}{2}}$$

$$II \lesssim \frac{c}{\delta}(\int_{\mathbb{R}^2} |\hat{F}(\lambda)|^2|\lambda|\,\mathrm{d}\lambda)^{\frac{1}{2}}(\int_{|\lambda|>\frac{1}{\delta}} |\lambda|^{-3}\,\mathrm{d}\lambda)^{\frac{1}{2}} \lesssim B$$

□

Theorem. *There exists a subset $K \subset \mathbb{R}^2$ such that*

(i) K is compact

(ii) K has Lebesgue measure zero

(iii) K contains a translate of every unit line segment

Theorem. *Suppose F is any set that satisfies conditions (i) and (iii) from Theorem 1. Then F has Hausdorff dimension 2.*

Proof of Theorem 2. Let F be a Kakeya set. Fix $0 < \alpha < 2$. Let $F \subset \bigcup_{i=1}^\infty B_i$ be a covering with balls B_i of diameter $\le \delta$. It is enough to show

$$\sum(\operatorname{diam} B_i)^\alpha \ge c_\alpha > 0$$

for $\alpha < 2$.

Case 1: Assume $\operatorname{diam} B_1 = \delta \le \frac{1}{2}$ and let $N < \infty$ be the number of balls in the covering. WTS $N\delta^\alpha \ge c_\alpha$. $B_i^* =$ double of B_i. $F^* = \bigcup_i B_i^*$. $|F^*| \le \sum |B_i^*| = cN\delta^2$. F Kakeya $\Rightarrow \forall\gamma \in S^1 \exists s_\gamma \perp \gamma$ unit lime segment: $s_\gamma \subset F$. $s_\gamma^\delta \subset F^*$. $\therefore R_\delta^*(\chi_{F^*})(\gamma) \ge 1$ $(\forall\gamma \in S^1)$. Take $f = \chi_{F^*}$ in (*). Since $L^2 \subset L^1$,

$$\|\chi_{F^*}\|_{L^1} \lesssim \|\chi_{F^*}\|_{L^2} = |F^*|^{\frac{1}{2}} \lesssim N^{\frac{1}{2}}\delta.$$

(*) $\Rightarrow 0 < c \le (\log\frac{1}{\delta})^{\frac{1}{2}} N^{\frac{1}{2}}\delta$. This implies $N\delta^\alpha \ge c_\alpha > 0$.

Case 2: General case. $F \subset \bigcup_{i=1}^{\infty} B_i$ with each ball B_i of diameter ≤ 1. For each $k \in \mathbb{N}$, let N_k be the number of balls in $\{B_i\}$ with diameter $B_k \sim 2^{-k}$, i.e. $\in [2^{-k-1}, 2^{-k}]$. WTS

$$\sum_{k=0}^{\infty} N_k 2^{-k\alpha} \geq c_\alpha > 0.$$

ETS $\exists k' : N_{k'} 2^{-k'\alpha} \geq c_\alpha$.

$$F_k = F \cap (\bigcup_{\mathrm{diam} B_i \sim 2^{-k}} B_i)$$

$$F_k^* = \bigcup_{\mathrm{diam} B_v \sim 2^{-k}} B_i^*$$

$$|F_k^*| \leq c N_k 2^{-2k} \quad \forall k$$

F Kakeya $\Rightarrow \forall \gamma \in S^2 \exists s_\gamma \perp \gamma : s_\gamma \subset F$ (in particular $m_1(s_\gamma \cap F) = 1$).

Key: For some k, a large proportion of s_γ belongs to F_k. Pick $\{a_k\}_{k=0}^{\infty}$ such that $0 \leq a_k < 1$, $\sum a_k = 1$, (a_k) dos not nend to 0 too quickly, e.g. $a_k = c_\varepsilon 2^{-k\varepsilon}$ (for sufficiently small ε.

Claim:

$$\exists k : m_1(s_\gamma \cap F_k) \geq a_k.$$

Otherwise $m_1(s_\gamma \cap F) \leq \sum_k m_1(s_\gamma \cap F_k) < \sum a_k = 1$, contradicts (**)

For this value of k,

$$R^*_{2^{-k}}(\chi_{F_k^*})(\gamma) \geq a_k.$$

Since this choice of k depends on γ, let

$$E_k = \{\gamma \in S^1 : R^*_{2^{-k}}(\chi_{F_k^*})(\gamma) \geq a_k\}.$$

$S^1 = \bigcup_{k_1}^{\infty} E_k$. Therefore $\exists k' : |E_{k'}| \geq 2\pi a_{k'}$.

$$2\pi a_{k'}^2 = 2\pi a_{k'} a_{k'} \leq \int_{E_{k'}} a_{k'} \, \mathrm{d}\sigma \leq_{S^1} R^*_{2^{-k'}}(\chi_{F^*_{k'}})(\gamma) \, \mathrm{d}\sigma_\gamma$$

$$2^{-2k^\varepsilon} \sim a_{k'}^2 \leq c(\log 2^{k'})^{\frac{1}{2}} |F^*_{k'}|^{\frac{1}{2}} \leq c(\log 2^{k'})^{\frac{1}{2}} N_{k'}^{\frac{1}{2}} 2^{-k'}$$

$\Rightarrow N_{k'} 2^{-\alpha k'} \geq c_\alpha$, provided $4\varepsilon < 2 - \alpha$. □

Construction of a Kakeya set I (Stein-Shakarchi, III)

Thinner Cantor set, always taking away the half.

Take two of them, E_0, E_1, where E_1 has twice the length. Put E_0 on $y = 1$ and E_1 on $y = 0$. Let F be the union of all line segments that join a point in E_0 with one in E_1.

Construction of an ε-Kakeya set (Stein)

Theorem. *Given $\varepsilon > 0$, $\exists N = N_\varepsilon$ and 2^N rectangles $R_1, ..., R_{2^N}$ with side lengths 1×2^{-N} such that*

(i)

$$|\bigcup_{j=1}^{2^N} R_j| < \varepsilon$$

(ii) the reaches $\tilde{R}_j$ are mutually disjoint, i.e.

$$|\bigcup_{j=1}^{2^N} \tilde{R}_j| = 1$$

Proof. Fix $\alpha \in (\frac{1}{2}, 1)$. Symmetric triangle ABC with M opposite C. Push the right part into the left part and call the resulting body $\Phi(T)$. It consists of heart $\Phi_h(T)$ and arms $\Phi_a(T)$. Then

$$|\Phi_h(T)| = \alpha^2 |T|$$

$$|\Phi_a(T)| = 2(1-\alpha)^2 |T|$$

Conclusion

$$|\Phi(T)| = (\alpha^2 + 2(1-\alpha)^2)|T|$$

n-fold iteration (Peron trees): Split not into two but 2^n parts and do everything pairwise. Key: right side of $\Phi_h(A_0A_2C)$ // left side of $\Phi_n(A_2A_4C)$ // CA_2

Then look at heart/arms again.

$$|\text{arms of } \Psi_1(ABC)| \leq 2(1-\alpha)^2 |T|.$$

$$|\text{heart of } \Psi_1(ABC)| = \alpha^2 |T|$$

$$\therefore |\Psi_1(ABC)| = (\alpha^2 + 2(1-\alpha)^2)|T|.$$

Iterate: Carry out this process on the heart of $\Psi_1(ABC)$ with n replaced by $n-1$, given are the union of 2^{n-1} triangles.

Then retranslate all 2^n original triangles to obtain figure $\Psi_2(ABC)$.

$$|\text{heart of } \Psi_2(ABC) = \alpha^2 \alpha^2 |T|$$

$$|\text{additional arms of } \Psi_2(ABC)| \leq 2(1-\alpha)^2 \alpha^2 |T|$$

$$\begin{aligned}
|\Psi_n(ABC)| &\leq (\alpha^{2n} + 2(1-\alpha)^2 + 2(1-\alpha)^2\alpha^2 + \ldots + 2(1-\alpha)^2\alpha^{2n-2}) \\
&\leq \alpha^{2n} + 2(1-\alpha)^2 \underbrace{\sum_{n=0}^{\infty} \alpha^{2n}}_{=\frac{1}{1-\alpha^2}} \\
&\leq \alpha^{2n} + 2(1-\alpha)
\end{aligned}$$

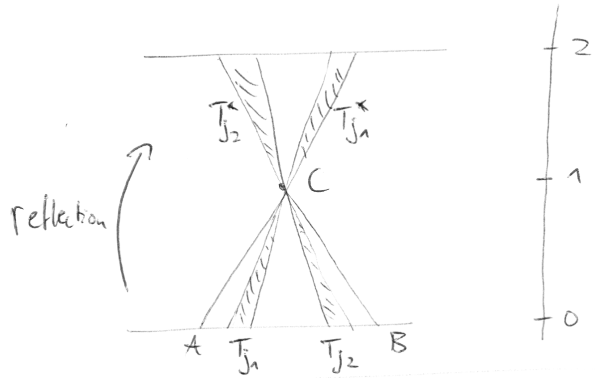


Figure 1: Obtaining mutually disjoint reaches by reflecting in C.

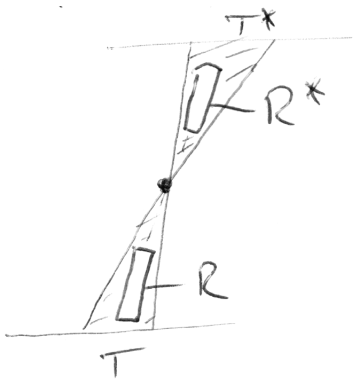


Figure 2: Go from triangles to rectangles by placing rectangles into the triangles with half the length.

$\square$

Application Maximal functions and counterexamples. Q: Given a collection $\mathcal{C} = \{C\}$ of sets, for which class of functions do we have

$$\lim_{\operatorname{diam}(C)\to 0,\ c\in\mathcal{C}} \frac{1}{|C|}\int_C f(x-y)\,\mathrm{d}y = f(x) \quad x-\text{a.e.?}$$

Seen:

$$(M_{\mathcal{C}}f)(x) = \sup_{c\in\mathcal{C}} \frac{1}{|C|}\int_C |f(x)-y)|\,\mathrm{d}y$$

$\mathcal{C} = \{\text{balls}\}$, weak-type (1,1) inequality for $M_{\mathcal{C}} \Rightarrow$ a.e. convergence of averages. A converse also holds!

$\{\mathrm{d}\mu_j\}_{j=1}^{\infty}$ collection of finite, nonnegative measures on $\mathbb{R}^d$: $\operatorname{supp}(\mu_j) \subset K \Subset \mathbb{R}^d$. Define the maximal operator

$$(Mf)(x) = \sup_j |f * \mu_j|(x).$$

Proposition. $1 \le p < \infty$. Assume for each $f \in L^p(\mathbb{R}^d)$ that $(Mf)(x) < \infty$ for some set of x having positive measure. Then $f \mapsto Mf$ is of weak-type (p,p), i.e.

$$\exists A < \infty : |\{x \in \mathbb{R}^d : (Mf)(x) > \alpha\}| \le \frac{A}{\alpha^p}\|f\|_{L^p} \quad (\forall \alpha > 0)$$

Lemma. $\{E_j\}$ collection of subsets of a fixed compact set:

$$\sum_{j=1}^{\infty} |E_j| = \infty.$$

Then there exists a sequence of translates $F_j = E_j + x_j$:

$$\limsup F_j = \bigcap_{k=1}^{\infty}\Big(\bigcup_{j=k}^{\infty} F_j\Big) = \mathbb{R}^n \quad \text{(a.e.)}$$

The above set equals $\{x \in \mathbb{R}^d : x \in F_j \text{ infinitely often}\}$.

$$\liminf F_j = \bigcup_{k=1}^{\infty}(\bigcup_{j=k}^{\infty} F_j)$$

is a subset.

Proof of Lemma. $Q \subset \mathbb{R}^d$ unit cube. $A_1, A_2 \subset Q$. Then $\exists h \in \mathbb{R}^d : |A_1 \cap (A_2 - h)| \geq 2^{-d}|A_1||A_2|$. Why?

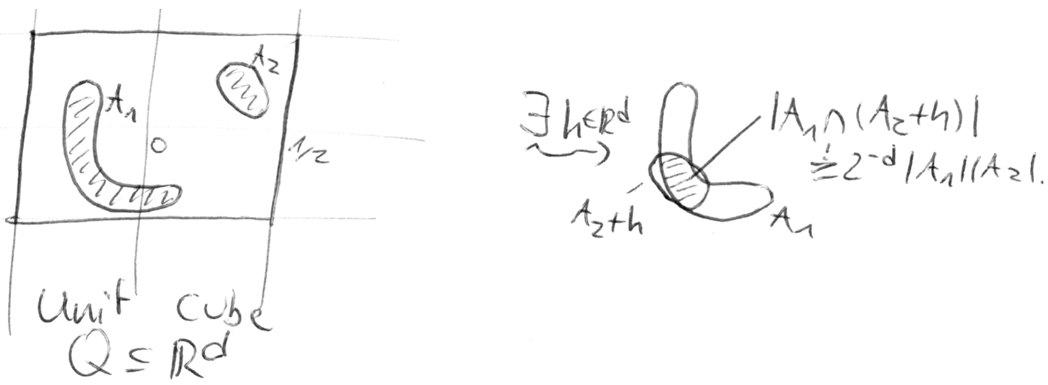


Figure 3: Translation of subsets A_1 and A_2 of a unit cube Q.

$$\eta(x) = \int_{\mathbb{R}^d} \chi_{A_1}(y)\chi_{A_2}(x+y)\,\mathrm{d}y \sim \chi_{A_1} * \chi_{A_1}(x)$$

$$\int_{\mathbb{R}^d} |A_1||A_1|$$

$$\operatorname{supp}(\eta) \subset Q^*$$

$|Q^*| = 2^d$.

$$\exists h \in Q^* : \eta(h) \geq \operatorname{avg}_{Q^*}(\eta) = \frac{1}{|Q^*|}\int_{\mathbb{R}^d} \eta = \frac{|A_1||A_2|}{2^d}$$

Wlog $\operatorname{supp}(E_j) \subset Q$.

Step 2: There exist translates $F_j = E_j + x_j$ that cover Q at least once.

$$Q \subset \bigcup_j F_j$$

Why? $F_1 = E_1$. Suppose (inductively) that $F_1, ..., F_{j-1}$ have been constructed. Let $A_1 = Q \cap (F_1 \cup ... \cup F_{j-1})^{\complement}$ and $A_2 = E_j$. Step 1 $\Rightarrow$ $\exists h : |A_1 \cap (A_2 - h)| \geq 2^{-d}|A_1||A_2|$. Set $F_j = A_2 - h = E_j - h$. Let $p_j = |Q \cap (F_1 \cup ... \cup F_j)|$. Then

$$\begin{aligned} p_j &= p_{j-1} + |\underbrace{Q \cap (F_1 \cup ... \cup F_{j-1})^{\complement}}_{A_1} \cap \underbrace{F_j}_{A_2 - h}| = p_{j-1} + |A_1 \cap (A_2 - h)| \\ &\geq p_{j-1} + 2^{-d}|A_1||E_j| = p_{j-1} + 2^{-d}(1 - p_{j-1})|E_j| \end{aligned}$$

$$\therefore p_j - p_{j-1} \geq 2^{-d}(1 - p_{j-1})|E_j|$$

$$\sum_{j=2}^{\infty}(p_j - p_{j-1}) = \lim_{j\to\infty} p_j - p_1 \therefore \lim_j p_j = 1$$

Step 3: Decompose (twice) $\{E_j\}$ into a countable infinite number of subcollections so that on each subcollection the sum of the measures diverges. $\square$

Proof of Proposition. Take a ball B such that $B \supset Q + K$. $\operatorname{supp}(F) \subset Q \Rightarrow \operatorname{supp}(F * \mu_j) \subset\Rightarrow \operatorname{supp}(Mf) \subset B$. Key: Estimate (*) (the violation of the weak type estimate) holds if $\operatorname{supp}(f) \subset Q$. For each $k, \exists \alpha_k > 0 \exists g_k \subset L^p : \operatorname{supp}(g_k) \subset Q$ such that

$$|\{x \in B : Mg_k(x) > \alpha_k\}| \geq \frac{2^k}{\alpha_k^p}\|g_k\|_{L^p}^p$$

Replace g_k by $\tilde{g}_k = \frac{k}{\alpha_k} g_k$.

$$\frac{2^k}{k^p} \leq \frac{|\{x \in B : M\tilde{g}_k(x) > k\}|}{\|\tilde{g}_k\|_{L^p}^p} \to \infty \quad \text{as} k \to \infty$$

$\therefore$ There exists a sequence $\{f_k\} \subset L^p$ and a sequence of constants $R_k \to \infty$ such that with $E_k = \{x \in B : Mf_k(x) > R_k\}$ we get

$$\sum_k |E_k| = \infty \qquad \sum_k \|f_k\|_{L^p}^p < \infty.$$

Remark. $d\mu_j \geq 0$ wlog $f_k \geq 0$.

By the lemma $\exists \{x_k\}$ such that $F_k = E_k + x_k$ satisfy $\limsup F_k = \mathbb{R}^d$ (a.e). Let

$$\tilde{f}_k(x) = f_k(x + x_k), \qquad F(x) = \sup_k \tilde{f}_k(x)$$

Then

$$M(F) = \sup_j |F * \mu_j| = \sup_j |(\sup_k \tilde{f}_k) * \mu_j| \geq \sup_k \sup_j |\tilde{f}_k * \mu_j| = \sup_k M(\tilde{f}_k)$$

Also $M(\tilde{f}_k) > R_k$ on $F_k \therefore M(F) = \infty$ a.e. Check $f \in L^p$:

$$|F|^p = |\sup_k \tilde{f}_k|^p \leq \sum_p |\tilde{f}_k|^p$$

$$\|F\|_{L^p}^p \leq \sum_k \|f_k u|_{L^p}^p < \infty$$

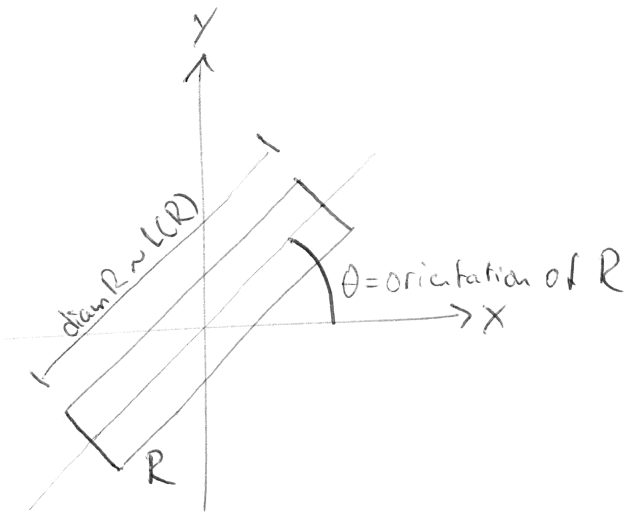
Full conclusion

$$f = \sum f\chi_{Q_j} =: \sum f_j$$
$$M(f) \leq \sum_j M(f_j)$$

$\square$

Example. Rectangles with arbitrary orientation

$$\mathcal{C} = \mathcal{R} = \{\textit{all} \text{ rectangles in } \mathbb{R}^2 \text{ centered at } 0\}$$



Corollary. Given $1 \leq p < \infty, \exists f \in L^p(\mathbb{R}^n)$ such that

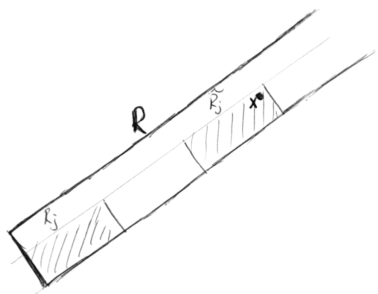
$$\limsup_{\operatorname{diam}(R)\to 0 R\in\mathcal{R}} \frac{1}{|R|}\int_R f(x-y)\,\mathrm{d}y = \infty \quad (x - \text{a.e.})$$

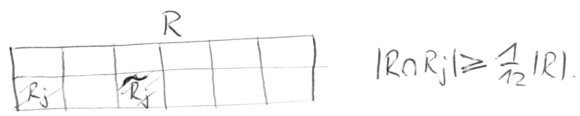
Idea: Use the ε-Kakeya set to show that M is not weak (p,p)

$$(Mf)(x) = \sup_{\operatorname{diam}(R)<8} \frac{1}{|R|}\left|\int_R f(x-y)\,\mathrm{d}y\right|$$

Let $E = \bigcup_{j=1}^{2^N} R_j$ as before. $\|\chi_E\|_{L^p}^p = |E| < \varepsilon$. If $x \in \tilde{R}_j$, then $\exists$ rectangle R such that

- R is centered at x
- $\operatorname{diam}(R) \leq 8$
- $|R \cap R_j| \geq \frac{1}{12}|R|$





$y \in x - E = -(E-x),\ y - x \in -E,\ x - y \in E$

$$M(\chi_E)(x) \geq \fint_{R-x} \chi_E(x-y)\,\mathrm{d}y = \frac{|(R-x)\cap(E-x)|}{|R|} \geq \frac{1}{12}$$

Conclusion: $M\chi_E \geq \frac{1}{12}$ on the set $\bigcup_{j-1}^{2^N} \tilde{R}_j$ (of measure 1)

$$\forall A > 0 \ \exists \text{ set } E : |\{x \in \mathbb{R}^d : M\chi_E > \alpha\}| \leq A\alpha^{-p}\|\chi_E\|_{L]p}^p$$

does not hold! $\therefore M$ is not of weak typ (p,p).

Note, that this is not the complete proof. Therefore still have to replace 8 by δ.

Bochner-Ries summability Q: In which way does Fourier inversion hold? (In $L^p(\mathbb{R}^d)$)

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi)e^{2\pi i x\xi}\,\mathrm{d}\xi = \lim_{R\to\infty} \underbrace{\int_{|\xi|<R} \hat{f}(\xi)(1-\frac{|\xi|^2}{R^2})^\delta e^{2\pi i x\xi}\,\mathrm{d}\xi}_{f * k_\delta} ?$$

for suitable $\delta \geq 0$.

$$\delta > \frac{d-1}{2} \Rightarrow k_\delta \in L^1(\mathbb{R}^d)$$

$$\delta \leq \frac{d-1}{2}$$

$$\delta = 0 \text{ today}$$

$d = 1$: The second equality holds in L^p-norm $(1 < p < \infty)$ $\Longleftarrow$ L^p boundedness of Hilbert transform

It also holds a.e. ($p = 2$ Carleson '66 $1 < p < \infty$ Hunt '68)

$d \geq 2$ Let us only consider norm convergence

$$f \mapsto (S^\delta f)(x) = \int_{|\xi|\leq 1} \hat{f}(\xi)(1-|\xi|^2)^\delta e^{2\pi i x\xi}\,\mathrm{d}\xi$$

$$Sf(x) = \int_{|\xi|\leq 1} \hat{f}(\xi)e^{2\pi i x\xi}\,\mathrm{d}\xi$$

$$\widehat{Sf} = 1_{|\xi|\leq 1}\hat{f}$$

$(\widehat{Hf} = i\pi \ \mathrm{sgn}\xi\hat{f})$

$$\|Sf\|_{L^2(\mathbb{R}^d)} = \|\widehat{Sf}\|_{L^2} = \|1_{|\xi|\leq 1}\hat{f}\|_{L^2} \leq \|\hat{f}\|_{L^2} = \|f\|_{L^2}$$

He said something about every bounded operator can be written like this or so? Fourier multiplier S is bounded iff multiplier function is bounded.

Theorem (C.Fefferman '71 – Annals of mathematics "The multiplier problem for the ball")**.** *Suppose $q \geq 2$ and $p \neq 2$. Then the operator S (initially defined on $L^p \cap L^2$) is not extendable to a bounded operator from $L^p(\mathbb{R}^d)$ to itself.*

Proof. Let $B \subset \mathbb{R}^d$ ball, let S_B be the multiplier orator associated to B:

$$\widehat{S_B f} = 1_B \hat{f}.$$

Given $u \in S^{d-1} \subset \mathbb{R}^d$, let S^n be the multiplier operator associated to the half-space with normal u:

$$\widehat{S^u f} = 1_{\{\xi u > 0\}} \hat{f} \qquad (S^u f)(x) = \int_{\xi u > 0} \hat{f}(\xi) e^{2\pi i x \xi} \,\mathrm{d}\xi$$

Upshot: L^p-bound for S implies an L^p vector-valued inequality for S_Bs and S_us.

Lemma (Y. Méyer)**.** Suppose

$$\|Sf\|_{L^p} \le A_p \|f\|_{L^p}$$

$f \in (L^2 \cap L^p)(\mathbb{R}^d)$ holds for some $p \in [1, \infty]$. Suppose $f_1, ..., f_M \in L^2 \cap L^p,\ u_1, ..., u_M \in S^{d-1} \subset \mathbb{R}^d$. Then

$$\|(\sum_{j=1}^{M} ||S^{u_j}(f_j)|^2)^{\frac{1}{2}}\|_{L^p(\mathbb{R}^d)} \le A_p \|(\sum_{j=1}^{M} |f_j|^2)^{\frac{1}{2}}\|_{L^p(\mathbb{R}^d)} \tag{4}$$

where A_p is the same constant as above.

Proof of Lemma. Step 1: $B = B_R =$ ball of radius R centered at 0. Then

$$\|S_B(f)\|_{L^p} \le A_p \|f\|_{L^p} \quad (g \in L^2 \cap L^p) \tag{5}$$

Why? Scaling:

$$\delta_R(g)(x) = g(\frac{x}{R})$$

Check $\delta_{R^{-1}} \circ S \circ \delta_R = S_{B_R}$ since $S = S_{B_1}$. $\hat{\delta}_\rho(g)(\xi) = R^d \hat{g}(R\xi)$.

Step 2: M balls. $(p < \infty)$ $f = (f_1, ..., f_M)$ given M-tuple of functions. $T(f) := (Tf_1, ..., Tf_M)$. Given a unit vector $\omega = (\omega_1, ..., \omega_M) \in \mathbb{C}^M$, let

$$S_\omega(f) = \sum_{j=1}^{M} \bar{\omega}_j S_B(f_j) = S_B(\sum_j \bar{\omega}_j f_j) = S_B(f_\omega) \qquad f_\omega = \sum_{j=1}^{M} \bar{\omega}_j f_j$$

$$(5) \Rightarrow \int_{\mathbb{R}^d} |S_\omega f(x)|^p \,\mathrm{d}x \le A_p^k \int_{\mathbb{R}^d} |f_\omega(x)|^p \,\mathrm{d}x \tag{6}$$

$(S_\omega f)(x) = ?$. $x, y \in \mathbb{C}^M,\ \langle x, y\rangle = \sum_{i=1}^{M} x_j \bar{y}_j$

$$\begin{aligned} S_\omega(f)(x) = S_B(f_\omega)(x) = S_B(\sum_{j=1}^{M} \bar{\omega}_j f_j)(x) = \sum_{j=1}^{M} \bar{\omega}_j S_B(f_j)(x)| = |\langle S_B(f)(x), \omega\rangle| \\ = |S_B(f)(x)||\langle \frac{S_B(f)(x)}{|S_B(f)(x)|}, \omega, \rangle| = (\sum_{j=1}^{M} |S_B(f_j)(x)|^2)^{\frac{1}{2}} |\varphi(\omega, S_B(f)(x))| \end{aligned}$$

Integrate both sides of (6) with respect to ω (before integrating in x).

$$\begin{aligned} \text{LHS} &= \int_{\mathbb{R}^d} (\int_{|\omega|=1} |S_\omega(f)(x)|^p \,\mathrm{d}\omega) \,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} (\sum_{j=1}^{M} |S_B(f_j)(x)|^2)^{\frac{p}{2}} \underbrace{(\int_{|\omega|=1} |\Phi(\omega, S_B(f)(x)|^p \,\mathrm{d}\omega) \,\mathrm{d}x))}_{\gamma_p} \,\mathrm{d}x \end{aligned}$$

$$0 \neq \gamma_p = \int_{|\omega|=1} |\Phi(\omega, 1)|^p \,\mathrm{d}\omega$$

For fixed $\nu \in S^{d-1}$ $\int_{S^{d-1}} |\langle \omega, \nu \rangle|^p \,\mathrm{d}\sigma_\omega = \omega_{d-2} \int_{-1}^{1} (1-t^2)^{\frac{d-3}{2}} \,\mathrm{d}t$.

Similarly,

$$\text{RHS} = \int_{\mathbb{R}^d} (\sum_{j=1}^{M} |f_j(x)^2)^{\frac{p}{2}} \,\mathrm{d}x \gamma_p$$

$$(6) \Rightarrow \|(\sum_{j=1}^{M} |S_B(f_j)|^2)^{\frac{1}{2}}\|_{L^p(\mathbb{R}^d)} \leq A_p \|(\sum_{j=1}^{M} |f_j|^2)^{\frac{1}{2}}\|_{L^p(\mathbb{R}^d)}$$

Step 3: From balls to half-spaces

$B_R^u =$ ball of radius R centered at Ru. Upshot: $B_R^u \to \{\xi u > 0\}$ as $R \to \infty$.

Translation

$$(T_y f)(x) = f(x-y)$$

$$\widehat{T_y f}(\xi) = e^{i\xi y} \hat{f}(\xi)$$

$$S_{B_R^u}(f)(x) = e^{2\pi i u R x} S_{B_R}(f e^{-2\pi i u R x})$$

(4) implies

$$\|(\sum |S_{B_R^{u_j}}(f_j)|^2)^{\frac{1}{2}}\|_{L^p} \leq A_p \|(\sum |f_j|^2)^{\frac{1}{2}}\|_{L^p}$$

Let $R \to \infty$ to finish:

$$\therefore S_{B_R^{u_j}}(f_j) \to S^{u_j}(f_j) \quad R \to \infty (\text{ in } L^2)$$

$\therefore$ there exists an almost everywhere converging subsequence, done. $\square$

$\widehat{Sf} = 1_{B(0,1)} \hat{f}$. S is not bounded in $L^p(\mathbb{R}^d)$ unless $d = 1$ or $p = 2$. Focus on multiplier operator for the half-space (S^u), $d = 1$.

$$(S^+ f)(x) = \int_0^\infty \hat{f}(\xi) e^{2\pi i x \xi} \,\mathrm{d}\xi \quad (f \in L^2)$$

$$|(S^+ f)(x)| \geq \frac{c}{|x|} \quad \text{if } |x| \geq \frac{1}{2} \tag{7}$$

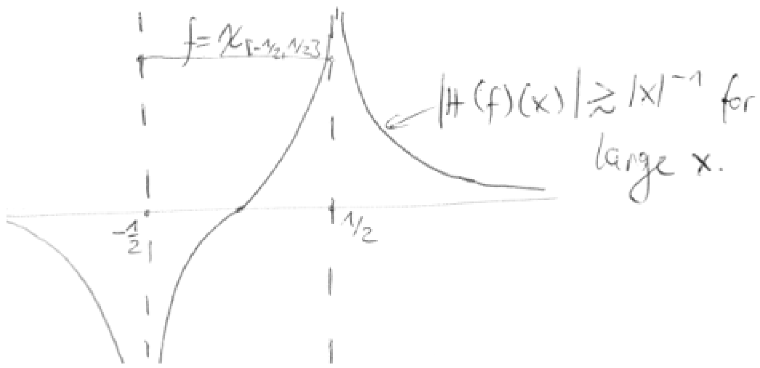
Proof.

$$(S^+ f)(x) = \lim_{\varepsilon \to 0} \int_0^\infty \hat{f}(\xi) e^{2\pi i (x + i\varepsilon)\xi} \,\mathrm{d}\xi \quad \text{in } L^2$$

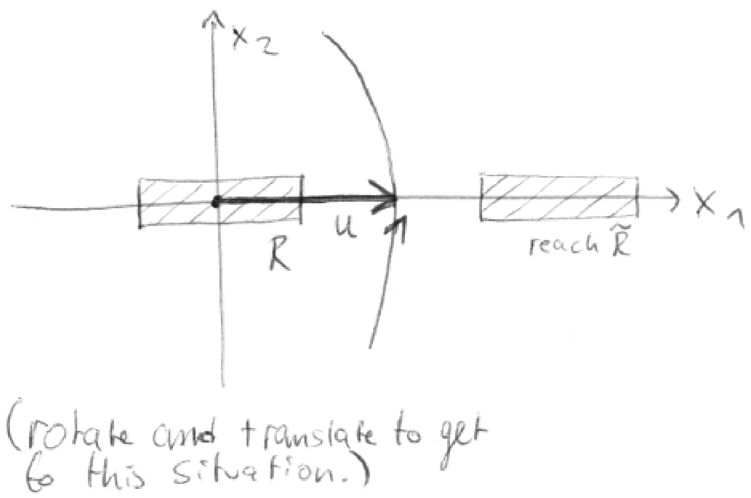
$$\int_{-\infty}^{\infty} (\int_0^\infty e^{-2\pi i y \xi} e^{2\pi i (x+i\varepsilon)\xi} \,\mathrm{d}\xi) f(y) \,\mathrm{d}y = \frac{1}{2\pi i} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{y - x - i\varepsilon} \,\mathrm{d}y$$

This has absolute value $\lesssim c|x|$ if $|x| \geq \frac{1}{2}$.

Alternative proof: $|H(f)(x)| \geq |x|^{-1}$ for large x, $S^+ = \frac{1}{2}(I + iH)$.



$\square$



For $R = (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2^{N+1}}, \frac{1}{2^{N+1}})$

$$1_R = 1_{(-\frac{1}{2},\frac{1}{2})} \otimes 1_{(-2^{-(N+1)},2^{-(N+1)})}$$

If u points in the direction of x_1 then

$$(S^u 1_R)(x_1, x_2) = (S^+ 1_{(-\frac{1}{2},\frac{1}{2})})(x_1) 1_{(-2^{-(N+1)},2^{-(N+1)})}(x_2)$$

$$(7) \Rightarrow |S^u(1_R)| \geq c' 1_{\tilde{R}}$$

Similarly for any 1×2^{-N} rectangle R_j. $u_j \in S^1$ in the positive direction of the longest side of R_j. Rotate and translate, then we get

$$|S^{u_j}(1_{R_j})| \geq c' 1_{\tilde{R}_j} \tag{8}$$

Take $R_1, ..., R_{2^N}$ to be the collection given by ε-Kakeya construction, plug that into result from lemma, get a contradiction.

Key: $p < 2$ and $d = 2$.

Lemma with $f_j = 1_{R_j}$ and $M = 2^N$

$$c' \leq \|(\sum_{j=1}^{2^N} |S^{u_j}(1_{R_j})|^2)^{\frac{1}{2}}\|_{L^p(\mathbb{R}^2)} \qquad (8),\ |\bigcup \tilde{R}_j| = 1$$

$$\leq A_p \|(\sum_{j=1}^{2^N} |1_{R_j}|^2)^{\frac{1}{2}}\|_{L^p(\mathbb{R}^2)} = A_p \underbrace{(\int_E (\sum_{j=1}^{2^N} |1_{R_j}|^2)^{\frac{p}{2}} \,\mathrm{d}x)^{\frac{1}{p}}}_{I}$$

$$I \leq |E|^{\frac{1}{q}} (\int (\sum |1_{R_j}|^2)^{\frac{p}{2}\frac{2}{p}} \,\mathrm{d}x)^{\frac{1}{p}\frac{p}{2}} \qquad \text{Hölder}$$

$$= |E|^{\frac{1}{q}} \sum_{j=1}^{2^N} |R_j| = |E|^{\frac{1}{q}}$$

$$E = \bigcup_{j=1}^{2^N} R_j, \quad |E| < \varepsilon, \quad \frac{1}{q} + \frac{1}{p/2} = 1, \quad \frac{1}{q} = 1 - \frac{p}{2}, \quad \frac{1}{pq} = \frac{1}{p}(1 - \frac{p}{2}) > 0$$

In the end we get

$$c' \leq A_p \varepsilon^{\frac{1}{pq}}.$$

Let $\varepsilon \to 0^+$ to finish.

$d > 2$:

$$f_j(\underbrace{x}_{\in \mathbb{R}^d}) = f_j(x_1, x_2, \underbrace{x'}_{\in \mathbb{R}^{d-2}}) = 1_{R_j}(x_1, x_2) f(x') \quad f \in S(\mathbb{R}^{d-2})$$

$p > 2$:

$$\langle Sf, g\rangle = \langle \widehat{Sf}, \hat{g}\rangle = \langle 1_B \hat{f}, \hat{g}\rangle = \langle \hat{f}, 1_B \hat{g}\rangle = \langle f, Sg\rangle \quad S = S^*$$

□

Oscillatory integrals in harmonic analysis Stein (VIII,IX), Stein-Shakarchi (Chapter 8), Sogge

- averaging operators
- restriction theory
- Bochner-Riesz summability

Motivation (in $\mathbb{R}^3$)

$$(Af)(x) = \fint_{\mathbb{S}^2} f(x-y) \,\mathrm{d}\sigma_y = \frac{1}{4\pi} f * \sigma(x)$$

σ surface measure in $\mathbb{S}^2$.

Smoothing properties:

$$\|\frac{\partial}{\partial x_j} A(f)\|_{L^2(\mathbb{R}^3)} \lesssim \|f\|_{L^2(\mathbb{R}^3)} \quad (j = 1, 2, 3) \tag{9}$$

$f \in L^2 \Rightarrow$

$$\|f * \sigma\|_{L^2} \leq \|f\|_{L^2} \underbrace{|\sigma|(\mathbb{R}^3)}_{<\infty}$$

(use Minkowski integral inequality) $\therefore f * \sigma \in L^2$ if $f \in L^2$.

Idea $\widehat{(f * \sigma)} = \hat{f}\hat{\sigma}$

$$(\widehat{\frac{\partial}{\partial x_j} A(f)})(\xi) = \xi_j \hat{f}(\xi)\hat{\sigma}(\xi),$$

so we should study $\hat{\sigma}$.

without loss of generality $\xi = (0, 0, |\xi|)$ because the integral is spherically symmetric.

$$\hat{\sigma}(\xi) = \int_{\mathbb{S}^2} e^{-2\pi i \omega \xi} \,\mathrm{d}\sigma_\omega = 2\pi \int_0^\pi e^{-2\pi i |\xi| \cos\theta} \sin\theta \,\mathrm{d} = 2\pi \int_{-1}^1 e^{2\pi i |\xi| t} \,\mathrm{d}t = \frac{2\sin(2\pi|\xi|)}{|\xi|}$$

$$t = \cos\theta \quad \mathrm{d}t = \sin\theta \,\mathrm{d}\theta$$

$$\therefore |\hat{\sigma}(\xi)| \lesssim (1 + |\xi|)^{-1}$$

(9) follows from this and Plancherel.

This also generalizes to higher dimensions. There we get a factor $(1-t^2)^{\frac{d-3}{2}}$ in the last integral.

Oscillatory integrals

$$I(\lambda) = \int_{\mathbb{R}^d} e^{i\lambda\phi(x)} \psi(x) \,\mathrm{d}x$$

where $\lambda \in \mathbb{R}$ is the oscillatory parameter, $\phi \in C^\infty(\mathbb{R}^d, \mathbb{R})$ the phase and $\psi(x) \in C$ the amplitude.

Q.: How does $I(\lambda)$ behave for large $|\lambda|$? General principle: Main contribution comes from the critical points of the phase, $x_0 : \ \nabla\phi(x_0) = 0$.

Principle of non-stationary phase $\phi \in C^\infty, \ \psi \in C_0^\infty : |\nabla\phi(x)| > 0 \ (\forall x \in \operatorname{supp}\psi)$. Then for any $N \in \mathbb{N}$

$$|I(\lambda)| \leq c_N |\lambda|^{-N}$$

Proof. involves integration by parts □

$d = 1$:

$$I_1(\lambda) = \int_a^b e^{i\lambda\phi(x)} \,\mathrm{d}x$$

$$0 < a < b < \infty \quad \psi(x) = \chi_{[a,b]}(x) \quad \text{which is rough!}$$

This means we will not get such a fast decay

Lemma (van der Corput (I))**.** $\phi \in C^2, \ \phi'$ monotonic, $|\phi'(x)| \geq 1 \ (\forall x \in [a, b])$. Then

$$|I_1(\lambda)| \leq \frac{3}{|\lambda|} \quad (\forall \lambda > 0)$$

Remark. (i) 3 is neither important nor sharp; independence of a, b, ϕ is the key!

(ii) Order of decrease in λ is sharp ($\phi(x) = x \therefore I_1(\lambda) = \frac{e^{i\lambda b} - e^{i\lambda a}}{i\lambda}$)

(iii) monotonicity of ϕ' is essential

Proof. Integrate by parts(...) □

What if critical points are present? ($d = 1$)

$x_0 : \phi'(x_0) = 0$ (critical point) and $\phi''(x_0) \neq 0$ (non degenerate), e.g. $\phi(x) = x^2,\ x_0 = 0$. In this case

$$\int_{\mathbb{R}} e^{i\lambda x^2}\psi(x)\,\mathrm{d}x = c_0\lambda^{\frac{-1}{2}} + \mathcal{O}(|\lambda|^{-\frac{3}{2}}) = \sum_{k=0}^{N} a_x \lambda^{-\frac{1}{2}-k} + \mathcal{O}(|\lambda|^{-\frac{3}{2}-N}) \quad (\forall N,\ \lambda \to \infty)$$

Lemma (van der Corput (II))**.** $\phi \in C^2[a,b],\ |\phi''(x)| \geq 1\ (\forall x \in [a,b])$. Then

$$|I_1(\lambda)| \leq \frac{8}{\lambda^{\frac{1}{2}}} \quad (\forall \lambda > 0)$$

Remark. More generally: $\mathcal{O}(|\lambda|^{\frac{1}{k}})$ if $|\phi^{(k)}| \geq 1$.

Proof. Integration by parts not needed. Instead split up region in small area around critical point with properly chosen size, and rest, and then use results from above. □

Corollary. Same assumptions as van der Corput (II). $\psi \in C^1[a,b]$.

$$|\int e^{i\lambda\phi(x)}\psi(x)\,\mathrm{d}x| \leq c_\psi \lambda^{-\frac{1}{2}}$$

Application: Asymptotics of Bessel functions

$$J_m(r) = \frac{1}{2\pi}\int_0^{2\pi} e^{ir\sin x}e^{-imx}\,\mathrm{d}x \quad \phi(x) = \sin x, \quad \psi(x) = e^{-imx} \quad (m \in \mathbb{Z})$$

Corollary.

$$|J_m(r)| \leq cr^{-\frac{1}{2}} \quad r \to \infty$$

Recall: Averaging operator in $\mathbb{R}^d$ ($d > 1$) is

$$(Af)(x) = (f * \sigma)(x) \quad \sigma \text{ surface measure on } \mathbb{S}^{d-1}$$

Theorem. *$f \mapsto A(f)$ is bounded from $L^2(\mathbb{R}^d)$ to $L^2_k(\mathbb{R}^d)$ with $= \frac{d-1}{2}$.*

Proof.

$$\hat{\sigma}(\xi) = 2\pi|\xi|^{-\frac{d}{2}+1} \underbrace{J_{\frac{d}{2}-1}(2\pi|\xi|)}_{=\mathcal{O}(|\xi|^{-\frac{1}{2}}),\ |\xi|\to\infty}$$

$$\therefore |\hat{\sigma}(\xi)| = \mathcal{O}(|\xi|^{-\frac{d-1}{2}}) \quad |\xi| \to \infty$$

□

"What is van der Corput's lemma in higher dimension?" (Carbery-Wright, 2000)

$$I(\lambda) = \int_{\mathbb{R}^d} e^{i\lambda\phi(x)}\psi(x)\,\mathrm{d}x$$

(ϕ smooth, ψ smooth, compactly supported)

nondegeneracy hypothesis

$$\det(\nabla^2\phi)(x) \neq 0 \quad \forall x \in \mathrm{supp}(\psi)$$

Theorem. *Under above assumptions*

$$|I(\lambda)| = \mathcal{O}(|\lambda|^{-\frac{1}{2}}) \quad \lambda \to \infty$$

Remark. (i) Decay rate is sharp

(ii) Proof uses TT^* method: $|I(\lambda)|^2 = I(\lambda)\overline{I(\lambda)}$

(iii) variant: $\mathrm{rk}(\nabla^2\phi) \geq m$ for some $0 < m \leq d$ on $\mathrm{supp}(\psi)$. Then

$$|I(\lambda)| = \mathcal{O}(|\lambda|^{-\frac{m}{2}})$$

Application: Fourier transform of surface-carried measures

Recall: $(\mathbb{S}^{d-1}, \sigma)$

$$|\widehat{\sigma}(\xi)| \lesssim (1+|\xi|)^{-\frac{d-1}{2}}$$

(not a Bessel coincidence)

(local) C^∞-hypersurface M. After translation and rotation $x_0 = 0$, $T_{x_0}M = \{x_d = 0\}$. M can be represented as

$$M = \{(x', x_d) \in B \subset \mathbb{R}^d : x_d = \varphi(x')\}$$

Can arrange $\varphi(0) = 0 = (\nabla_{x'}\varphi)(x')|_{x'=0}$.

$$\varphi(x') = \frac{1}{2}\sum_{k,j=1}^{d-1} \underbrace{\frac{\partial^2\varphi}{\partial x_k \partial x_j}}_{(a_{jk})} x_k x_j + \mathcal{O}(|x'|^3) = \frac{1}{2}\sum_{j=0}^{d-1} k_j x_j^2 + \mathcal{O}(|x'|^3)$$

(a_{jk}) $(d-1)\times(d-1)$ $\mathbb{R}$-valued symmetric matrix $\therefore$ diagonizable. k_j principal curvatures of M at x_0. $k := \prod_{j=1}^{d-1} k_j$ is the Gaussian curvature of M at x_0 ($k = \det(\nabla^2\varphi)$)

E.g.

(i) $\mathbb{S}^{d-1} \subset \mathbb{R}^d$. $k_j = 1$ $(\forall j) \therefore k = 1$

(ii) $\{x_3 = \underbrace{x_1^2 - x_2^2}_{\varphi(x_1,x_2)}\} \subset \mathbb{R}^3$, $\frac{1}{2}\nabla^2\varphi(x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(iii) $\{x_1^2 = |x'|^2 : x \neq 0\}$, $x' \in \mathbb{R}^{d-1}$. $d-2$ identical nonvanishing principal curvatures x_d^{-2} $+1$ vanishing principal curvature.

surface measure σ

$$\int_M f \,\mathrm{d}\sigma = \int_{\mathbb{R}^{d-1}} f(x', \varphi(x')) \underbrace{\sqrt{1 + |\nabla_{x'}\varphi(x')|^2}\,\mathrm{d}x'}_{\mathrm{d}\sigma \text{ in our coordinate sys.}}$$

$$\mathrm{d}\mu = \psi\,\mathrm{d}\sigma, \quad \psi \in C_0^\infty(M, \sigma)$$

is a surface carried measure.

$$\widehat{\mu}(\xi) = \int_M e^{-2\pi i x\xi}\,\mathrm{d}\mu(x) = \int_M e^{-2\pi i x\xi}\psi(x)\,\mathrm{d}\sigma_x$$

is bounded on $\mathbb{R}^d$ because $|\mu|(\mathbb{R}^d) < \infty$.

Theorem. *Hypersurface $M \subset \mathbb{R}^d$ with nonvanishing Gaussion curvature at each point of* supp(ψ). *Then*

$$|\hat{\mu}(\xi)| = \mathcal{O}(|\xi|^{-\frac{d-1}{2}}), \quad |\xi| \to \infty$$

Corollary. If M has at last m non vanishing principal curvatures (at each point of supp(ψ)), then

$$|\hat{\mu}(\xi))| = \mathcal{O}(|\xi|^{-\frac{m}{2}}), \quad |\xi| \to \infty$$

Last time: oscillatory integrals and averaging operators

$$(Af)(x) = (F * \sigma)(x) = \fint_{\mathbb{S}^{d-1}} f(x-y)\,\mathrm{d}\sigma_y \quad (d > 1)$$

Smoothin property: $f \mapsto A(f)$ is bounded from $L^2(\mathbb{R}^d)$ to $L^2_k(\mathbb{R}^d)$ with $k = \frac{d-1}{2}$. Here we used

$$|\hat{\sigma}(\xi)| \lesssim |\xi|^{-\frac{d-1}{2}}.$$

A few weeks ago:

$$R(f)(t.\gamma) = \int_{P_{t,\gamma}} f$$

where $P_{t,\gamma} = \{x \in \mathbb{R}^d : x\gamma = t\}$.

$$R^*(f)(\gamma) = \sup_{t \in \mathbb{R}} |R(f)(t,\gamma)|$$

if $d \geq 3$ then

$$\int_{\mathbb{S}^{d-1}} R^*(f)(\gamma)\,\mathrm{d}\sigma_\gamma \lesssim \|f\|_{L^2} + \|f\|_{L^2}$$

This estimate was based on

$$\int_{\mathbb{S}^{d-1}} \int_{-\infty}^{\infty} |\hat{R}(f)(\lambda,\gamma)|^2 |\lambda|^{d-1}\,\mathrm{d}\lambda\,\mathrm{d}\sigma_\gamma = 2\int_{\mathbb{R}^d} |f(x)|^2\,\mathrm{d}x$$

due to $\hat{R}(f)(\lambda,\gamma) = \hat{f}(\lambda\gamma)$. Consider $d = 3$ then this becomes

$$\int_{\mathbb{S}^2} \int_{\mathbb{R}} |\frac{\mathrm{d}}{\mathrm{d}t} R(f)(t,\gamma)|^2\,\mathrm{d}t\,\mathrm{d}\sigma_\gamma = 8\pi^2 \int_{\mathbb{R}^3} |f(x)|^2\,\mathrm{d}x \tag{10}$$

by Plancherel $t \leftrightarrow \lambda$. Note, that something like this also holds for higher dimensions.

Now consider the following "linearized" version of the Radon transform:

$$R_B(f) = \int_{\mathbb{R}^{d-1}} f(y', x_d - B(x',y'))\,\mathrm{d}y' = \int_{M_x} f$$

where $x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$, $y = (y', y_d)$ and $B : \mathbb{R}^{d-1} \times \mathbb{R}^{d-1} \to \mathbb{R}$ is a nondegenerate bilinear form, and

$$M_x = \{(y', y_d) \mid y_d = x_d - B(x',y')\}.$$

E.g. $B(x',y') = \langle x', y'\rangle$ (usual inner product on $\mathbb{R}^d$). $y_d = x_d - \langle x', y'\rangle \iff \langle x', y'\rangle + y_d = x_d \iff \langle (x',1), (y', y_d)\rangle = x_d$. The map

$$\begin{aligned} \mathbb{R}^d &\to \{\text{affine hyperplanes on } \mathbb{R}^d\} \\ x &\mapsto M_x \end{aligned}$$

is injective and surjective onto {hyperplanes not orthogonal to $M_0 = \{x_d = 0\}$}. The excerpted collection of hyperplanes is lower dimensional, so we can think of R_B as a substitute for R.

An analogue of (10) is

$$\int_{\mathbb{R}^d} |\frac{\partial}{\partial x_3} R_B(f)(x)|^2 \, \mathrm{d}x = c_B \int_{\mathbb{R}^3} |f(x)|^2 \, \mathrm{d}x$$

for $f \in C_0^0(\mathbb{R})$

Proof.

$$\int_{\mathbb{R}^3} |\frac{\partial}{\partial x_3} R_B(f)(x)|^2 \, \mathrm{d}x = \int_{\mathbb{R}} \int_{\mathbb{R}^2} | \underbrace{(\frac{\partial}{\partial x_3} R_B(f))^\wedge(x', \xi_3)}_{=2\pi\xi_3 \int_{\mathbb{R}^2} e^{-2\pi i \xi_3 B(x',y')} \hat{f}(y',\xi_3)\,\mathrm{d}y'} |^2 \, \mathrm{d}x' \, \mathrm{d}\xi_3 \qquad x = (x', x_3)$$

The last equality follows from

$$\begin{aligned}
\hat{R}_B(f)(x', \xi_3) &= \int_{\mathbb{R}} e^{-2\pi i \xi_3 x_3} R_B(f)(x', x_3) \, \mathrm{d}x_3 \\
&= \int e^{-2\pi i \xi_3 x_3} \int_{\mathbb{R}^2} f(y', \underbrace{x_3 - B(x', y')}_{y_3}) \, \mathrm{d}y' \, \mathrm{d}x_3 \\
&= \int \int_{\mathbb{R}^{2+1}} e^{-2\pi i \xi_3 (y_3 + B(x', y'))} f(y' y_3) \, \mathrm{d}y' \, \mathrm{d}y_3 \\
&= \int_{\mathbb{R}^2} e^{-2\pi i \xi_3 B(x', y')} \underbrace{(\int_{\mathbb{R}} e^{-2\pi i \xi_3 y_3} f(y', y_3) \, \mathrm{d}y_3)}_{\hat{f}(y', \xi_3)} \mathrm{d}y'
\end{aligned}$$

Since B is nondegenetare $\exists C : \mathbb{R}^2 \to \mathbb{R}^2$ linear, invertible such that $B(x', y') = \langle C(x'), y' \rangle$. Change variables $\xi_3 C(x') = u \in \mathbb{R}^2$ is well defined since C is invertible. $\therefore \xi_3 B(x', y') = \langle \xi_3 C(x'), y' \rangle = \langle u, y' \rangle \therefore \xi_3^2 |\det C| \, \mathrm{d}x' = \mathrm{d}u$. Then the first integral becomes

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} | \int_{\mathbb{R}^2} e^{-2\pi i u y'} \hat{f}(y', \xi_3) \, \mathrm{d}y' |^2 \frac{\mathrm{d}u}{|\det C|} \, \mathrm{d}\xi_3 \simeq \int \int |\hat{f}(y', \xi_3)|^2 \, \mathrm{d}y' \, \mathrm{d}\xi_3 \simeq \int_{\mathbb{R}^3} |f(y)|^2 \, \mathrm{d}y$$

by 2D-Plancherel $y' \leftrightarrow u$ and 1D-Plancherel $\xi_3 \leftrightarrow y_3$. □

Rotational curvature Both the averaging operator A and the Radon transform R_B are of the form $f \mapsto \int_{M_x} f(y) \, \mathrm{d}_x(y)$, where for each $x \in \mathbb{R}^d$ we have a manifold M_x (depending smoothly on x) over which we integrate.

$$\begin{aligned}
A: &\quad M_x = x + M_0 \quad M_0 \text{ curved} \\
R_B: &\quad M_x = \{y = (y', y_d) \mid y' \in \mathbb{R}^{d-1}, \; y_d = x_d - B(x', y')\} \quad \text{flat but } M_x \text{ rotates as } x \text{ varies.}
\end{aligned}$$

Start with a smooth "double defining" function $\rho = \rho(x, y)$ given in a ball in $\mathbb{R}^d \times \mathbb{R}^d$. Its rotational matrix is

$$M = M(\rho) = \begin{pmatrix} \rho & \frac{\partial \rho}{\partial y_1} & \cdots & \frac{\partial \rho}{\partial y_d} \\ \frac{\partial \rho}{\partial x_1} & & & \\ \vdots & (\frac{\partial^2 \rho}{\partial x_j \partial y_k})_{j,k=1}^d & & \\ \frac{\partial \rho}{\partial x_d} & & & \end{pmatrix}$$

containing the mixed Hessian. The rotational curvature of ρ is

$$\operatorname{rotcurv}(\rho) := \det(M(\rho))$$

We want $\rho = 0 \Rightarrow \operatorname{rotcurv}(\rho) \neq 0$. $M_x = \{y : \rho(x,y) = 0\}$. The fact that $\nabla_y \rho \neq 0$ if $\rho = 0$ implies that M_x is a smooth surface (or something like that).

Examples/Properties:

(i) Translatior invariant case: $\rho(x,y) = \rho(x-y)$. $M_x = M_0 + x$ and $\operatorname{rotcurv}(\rho) \neq 0$ iff M_0 has nonvanishing Gaussian curvature.

(ii) Case of R_B: $\rho(x,y) = y_d - x_d + B(x', y')$. $\operatorname{rotcurv}(\rho) \neq 0$ iff B is nondegenerate.

(iii) $\tilde{\rho}(x,y) = a(x,y)\rho(x,y)$ with $a(x,y) \neq 0$. Then $\tilde{\rho}$ is another defining function for $\{M_x\}$, and $\operatorname{rotcurv}(\tilde{\rho}) = a^{d+1} \operatorname{rotcurv}(\rho)$

(iv) $x \mapsto \psi_1(x)$, $y \mapsto \psi_2(y)$ local diffeomorphisms of $\mathbb{R}^d$. For $\tilde{\rho}(x,y) = \rho(\psi_1(x), \psi_2(y))$ then $\operatorname{rotcurv}(\tilde{\rho}) = J_1(x)J_2(y) \operatorname{rotcurv}(\rho)$ with $J_k = \det \operatorname{jac}(\psi_k)$, $k = 1, 2$

Define the general averaging operator A by

$$A(f)(x) = \int_{M_x} f(y)\psi_0(x,y) \,\mathrm{d}\sigma_x(y)$$

initially for $f \in C_0^0(\mathbb{R}^d)$. $M_x = \{y \mid \rho(x,y) = 0\}$ with induced Lebesgue measure $\mathrm{d}\sigma_x$. ρ is a double defining function with $\operatorname{rotcurv}(\rho) \neq 0$. $\psi_0 \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$.

Theorem. *The operator A extends to a bounded linear map from $L^2(\mathbb{R}^d)$ to $L^2_k(\mathbb{R}^d)$ where $k = \frac{d-1}{2}$.*

Proof. Step 1: Oscillatory integral operators (FIOs)

Step 2: L^2 estimate via dyadic decomposition of "almost-orthogonal" parts.

Step 1: Define

$$T_\lambda(f)(x) = \int_{\mathbb{R}^d} e^{i\lambda\varphi(x,y)}\psi(x,y)f(y) \,\mathrm{d}y$$

where $\varphi \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and $\psi \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ with

$$\det(\nabla^2_{x,y}\varphi) = \det\left(\frac{\partial^2\varphi}{\partial x_k \partial y_j}\right)_{k,j=1}^d \neq 0 \quad \text{on } \operatorname{supp}(\psi)$$

last week:

$$I(\lambda) = \int_{\mathbb{R}^d} e^{i\lambda\varphi(y)}\psi(y) \,\mathrm{d}y \Rightarrow |I(\lambda)| \lesssim |\lambda|^{-\frac{d}{2}}$$

if $\det \nabla^2\varphi \neq 0$ on $\operatorname{supp}(\psi)$. We used $|I(\lambda)|^2 = I(\lambda)\overline{I(\lambda)}$ where then appeared the term $\varphi(u + y) - \varphi(y)$.

Proposition. Under the above assumptions,

$$\|T_\lambda\|_{L^2 \to L^2} \leq c\lambda^{-\frac{d}{2}} \quad \forall \lambda > 0$$

Proof. Similar to its scalar version, omitted. □

Consequence: For the corresponding oscillatory integral operator involving ρ

$$S_\lambda(f)(x) = \int_{\mathbb{R}\times\mathbb{R}^d} e^{i\lambda y_0 \rho(x,y)} \psi(x, y_0, y) f(y) \,\mathrm{d}y_0 \,\mathrm{d}y$$

with $(y_0, y) \in \mathbb{R} \times \mathbb{R}^d$ and $\psi \in C_0^\infty$ is supported away from $y_0 = 0$.

Corollary. If $\rho = 0 \Rightarrow$ rotcurv$(\rho) \neq 0$ then

$$\|S_\lambda\|_{L^2\to L^2} \le c\lambda^{-\frac{d+1}{2}}$$

Proof of Corollary. $\bar{x} = (x_0, x), \bar{y} = (y_0, y) \in \mathbb{R}\times\mathbb{R}^d$. Set $\varphi(\bar{x}, \bar{y}) = x_0 y_0 \rho(x, y)$ then $\det(\nabla^2_{x,y}\varphi) = (x_0 y_0)^{d+1}$ rotcurv(ρ). Define

$$F_\lambda(x_0, x) = F_\lambda(\bar{x}) = \int_{\mathbb{R}^{d+1}} e^{i\lambda\varphi(\bar{x},\bar{y})} \psi_1(x_0, x, y_0, y) f(y) \,\mathrm{d}y_0 \,\mathrm{d}y$$

with $\psi_1(1, \lambda, y_0, y) = \psi(x, y_0, y)$ (from S_λ). Then $S_\lambda(f)(x) = F_\lambda(1, x)$.

Observation: If $I \subset \mathbb{R}$ interval of length 1, $g \in C^1(I),\ x_0 \in I$ then

$$|g(u_0)|^2 \le 2\Big(\int_I |g(u)|^2 \,\mathrm{d}u + \int_I |g'(u)|^2 \,\mathrm{d}u\Big)$$

Apply this observation with $I = [1, 2],\ u_0 = 1,\ g(u) = F_\lambda(u, x)$ to get:

$$\int_{\mathbb{R}^d} |S_\lambda(f)(x)|^2 \,\mathrm{d}x \le 2\Big(\int |F_\lambda(x_0, x)|^2 \,\mathrm{d}x + \int \Big|\frac{\partial}{\partial x_0} F_\lambda(x_0, x)\Big|^2 \,\mathrm{d}x_0 \,\mathrm{d}x\Big)$$

The first integral is $\lesssim \lambda^{-(d+1)} \|f\|^2_{L^2}$ by Proposition with $\mathbb{R}^{d+1}$ instead of $\mathbb{R}^d$).

$$\frac{\partial}{\partial x_0}(e^{i\lambda x_0 y_0 \rho(x,y)}) = \frac{\partial}{\partial y_0}(e^{i\lambda x_0 y_0 \rho(x_0,y_0)}) \frac{y_0}{x_0}$$

we integrate by parts somewhere and use our form of φ. Therefore the second summand also satisfies the desired estimate. $\square$

Step 2: Dyadic decomposition of A.

Co-area formula (see Evans-Cariery (?), Stein-Shakarchi IV, Exercise 8 ,Ch. 8): Fix $h \in S(\mathbb{R})$ sucht that $\int h = 1,\ M = \{x \in \mathbb{R}^d \mid \rho(x) = 0\}$. Then

$$\int_M f \frac{\mathrm{d}\sigma}{|\nabla\rho|} = \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon} \int_{\mathbb{R}^d} h\Big(\frac{\rho(x)}{\varepsilon}\Big) f(x) \,\mathrm{d}x.$$

E.g. $\rho(x) = |x| - 1 \therefore M = \mathbb{S}^{d-1},\ \frac{\nabla}{\rho}(x) = x|x|, \therefore |\nabla\rho| = 1$

$$\int_{\mathbb{S}^{d-1}} f \,\mathrm{d}\varepsilon = \lim_{\varepsilon\to *^+} \frac{1}{\varepsilon} \int_{\mathbb{R}^d} h\Big(\frac{|x| - 1}{\varepsilon}\Big) F(x) \,\mathrm{d}x$$

where $f = F|_{\mathbb{S}^{d-1}}$.

$$A(f)(x) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{\mathbb{R}^d} h(\rho(x, y)\varepsilon) \psi(x, y) f(y) \,\mathrm{d}y$$

where $\psi(x,y) = \psi_0(x,y)|\nabla_y \rho| \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$. Let $\gamma \in C_0^\infty(\mathbb{R})$ such that $\gamma = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$ and 0 on $[-1,1]^\complement$. Let $h = \widehat{\gamma}$. Then

$$h(\rho) = \int_{\mathbb{R}} e^{2\pi i \xi u} \gamma(u) \, \mathrm{d}u.$$

Then

$$\int h = \int \widehat{\gamma} = \gamma(0) = 1$$

and

$$\int_{\mathbb{R}} e^{2\pi i u \rho} \gamma(\varepsilon u) \, \mathrm{d}u = (\delta_\varepsilon \gamma)^\vee(\rho) = \varepsilon^{-1} \gamma^\vee(\varepsilon^{-1}\rho) = \varepsilon^{-1} h(\varepsilon^{-1}\rho)$$

where $\delta_\varepsilon \gamma)(u) = \gamma(\varepsilon u)$. Choose $\varepsilon = 2^{-r}, \ r \in \mathbb{N}$. Note

$$\gamma(2^{-r}u) = \gamma(u) + \sum_{k=1}^{r} (\gamma(\frac{u}{2^k} - \gamma(\frac{u}{2^{k-1}}))$$

Let $r \to \infty$ to get

$$1 = \gamma(u) + \sum_{k=1}^{\infty} \eta(\frac{u}{2^k})$$

where $\eta(\cdot) = \gamma(\cdot) - \gamma(2\cdot)$ because γ is continuous. Then $\eta \in C_0^\infty(\mathbb{R}), \ \mathrm{supp}(\eta) \subset \{\frac{1}{4} \leq |u| \leq 1\}$. Whenever f is continuous we get by Fourier inversion that

$$\begin{aligned}
A(f)(x) &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\mathbb{R}^d} \int_{\mathbb{R}} e^{2\pi i u \frac{\rho(x,y)}{\varepsilon}} \gamma(u) \psi(x,y) f(y) \, \mathrm{d}u \, \mathrm{d}y \\
&= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \int_{\mathbb{R}} e^{2\pi i u \rho(x,y) \gamma(\varepsilon u)} \psi(x,y) f(y) \, \mathrm{d}u \, \mathrm{d}y \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}} e^{2\pi i u \rho(x,y)} \gamma(u) \psi(x,y) f(y) \, \mathrm{d}u \, \mathrm{d}y \\
&= \sum_{k=1}^{\infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}} e^{2\pi i u \rho(x,y)} \eta(\frac{u}{2^k}) \psi(x,y) f(y) \, \mathrm{d}u \, \mathrm{d}y
\end{aligned}$$

since $\gamma(\varepsilon u) \to \gamma(0) = 1 \ \varepsilon \to 0$. Call the summands $A_k(f)(x)$. Properties of A_k:

(i) $f \in L^2(\mathbb{R}^d) \Rightarrow$

$$A_k(f) \in C_0^\infty(\mathbb{R}^d)$$

(ii)

$$\|A_k(f)\|_{L^2} \leq c 2^{-k(\frac{d-1}{2})} \|f\|_{L^2}.$$

Recall $\|S_\lambda\|_{L^2 \to L^2} \lesssim \lambda^{-\frac{d+1}{2}i}$ and change variables in the definition of $A_k(f)$.

(iii) $\exists m : |j-k| \geq m \ \forall N$

$$\|(A_k^* A_j)(f)\|_{L^1} \lesssim_N 2^{-N \max(k,j)} \|f\|_{L^2}.$$

Similarly for $A_k A_j^*$. For the proof, invoke nonstationary phase. Also, recall $|I(\lambda)|^2 = I(\lambda)\overline{I(\lambda)}$.

(iv) $A_k^{(\alpha)} = (\frac{\partial}{\partial x})^\alpha A_k$. Then

$$\|A_k^{(\alpha)}\|_{L^2 \to L^2} \lesssim 2^{k|\alpha|} 2^{-k(\frac{d-1}{2})}$$

and

$$\|A_k^{(\alpha)} (A_j^{(\alpha)})^*\|_{L^2 \to L^2} \lesssim_{\alpha, N} 2^{-N \max(k,j)}.$$

Step 3: Almost-orthogonality

Assume that $\{T_k\}_{k=1}^{\infty}$ is a sequence of bounded operators on $L^2(\mathbb{R}^d)$ and that $\{a(k)\}_{k\in\mathbb{Z}}$ are positive constants with

$$A = \sum_{k\in\mathbb{Z}} a(k) < \infty.$$

Lemma (Cotlar-Knapp-Stein)**.** Assume for $\|T_k T_j^*\|_{L^2\to L^2}$ that $\|T_k^* T_j\| \leq a(k-j)^2$. Then, for every r,

$$\|\sum_{k=0}^{r} T_k\| \leq A.$$

Note, that the bound A is independent of r.

Write $T = \sum_{k=0}^{r} T_k$. Recall $\|T\|^2 = \|T^*T\|$ since $\|AB\| \leq \|A\|\|B\|$ and $\|Tx\|^2 = \langle Tx, Tx\rangle = \langle x, T^*Tx\rangle \leq \|T^*T\|\|x\|^2$ and plug in an extremizing sequence of $\|Tx\|$ for x. Then $\|T\|^4 = (\|T\|^2)^2 = \|T^*T\|^2 = \|(T^*T)^2\|$ since T^*T is self adjoint. By induction we get

$$\|T\|^{2n} = \|(T^*T)^n\|.$$

$$(T^*T)^n = \sum_{i_1,i_2,\ldots,i_{2n}} (T_{i_1}T_{i_2}^*\ldots T_{i_{2n-1}}T_{i_{2n}}^*)$$

(i)

$$\|(T_{i_1}T_{i_2}^*)\ldots(T_{i_{2n-1}}T_{2_{2n}}^*)\| \leq a(i_1-i_2)^2 a(i_3-i_4)^2\ldots a(2_{2n-1})^2$$

(ii)

$$\|T_{i_1}(T_{i_2}T_{i_3})\ldots(T_{i_{2n-2}}T_{i_{2n-1}})T_{i_{2n}}\| \leq Aa(i_2-i_3)^2 a(i_4-i_5)^2\ldots a(i_{2n-2}-i_{2n-1})^2 A$$

Take geometric mean of (i) and (ii) and get

$$\|T_{i_1}T_{i_2}\ldots T_{i_{2n-1}}T_{i_{2n}}^*\| \leq Aa(i_1-a_2)a(i_2-i_3)\ldots a(i_{2n-1}-i_{2n})$$

Now sum the whole thing in $i_1, i_2, \ldots, i_{2n-1}$. Then sum by sum, each of the factors turns into an A. In the end the sum in i_{2n} gives a factor $r+1$. So,

$$\sum_{i_1,i_2,\ldots i_{2n}} \|T_{i_1}T_{i_2}^*\ldots T_{i_{2n-1}}T_{i_{2n}}^*\| \leq A^{2n}(r+1)$$

$$\therefore \|T\| \leq A(1+r)^{\frac{1}{n}} \to A \quad n\to\infty$$

(This is called 'Tensor power trick')

Putting everything together: Case 1: d odd ($\therefore \frac{d-1}{2} \in \mathbb{Z}$). ETS $\forall |\alpha| \leq \frac{d-1}{2} \quad \forall f \in L^2(\mathbb{R}^d)$

- $\partial_x^\alpha A(f)$ exists (in the sense of distributions) and is an L^2 function
- $f \mapsto \partial_x^\alpha A(f)$ is bounded on L^2.

For each r, set

$$\partial_x^\alpha \sum_{k=0}^{r} =: \sum_{k=0}^{r} T_k \quad (T_k = A_k^{(\alpha)})$$

The estimates 1 and 2 imply that the hypotheses of CKS are satisfied with $a(k) = c_n 2^{-|k|N}$ ($\forall N, \therefore$ can choose $N=1$).

$$\therefore \|\partial_x^\alpha \sum_{k=0}^{r} A_k(f)\|_{L^2} \leq A\|f\|_{L^2}$$

where $A = \sum_{k\in\mathbb{Z}}$, provided $|\alpha| \leq \frac{d-1}{2}$.

(i) with $\alpha = 0$ we get

$$\lim_{r\to\infty} \sum_{k=0}^{r} A_k(f) = A(f)$$

in L^2 and therefore in the weak sense.

$$\lim_{r\to\infty} \partial_x^\alpha \sum_{k=1}^{r} A_k(f) = \partial_x^\alpha A(f)$$

in the weak sense and therefore in L^2.

Conclusion

$$\|\partial_x^\alpha A(f)\|_{L^2} \le A\|f\|_{L^2}$$

whenever $f \in C_0^0(\mathbb{R}^d),\ |\alpha| \le \frac{d-1}{2}$ (and d is odd) $\square$

today:

$$(A_t f)(x) = \int_{|y|=1} f(x-ty)\,\mathrm{d}\sigma(y) = (f * \sigma_t)(x)$$

which is the average of f over the sphere of radius t contered at x,

$$\int_{|x|=t} g(x)\,\mathrm{d}\sigma_t(x) := \int_{|x|=1} g(tx)\,\mathrm{d}\sigma(x).$$

This definition works fine provided g is continuous.

Question: We would like to know for general f whether $(A_t f)(x) \to f(x)$ (x-a.e.) as $t \to 0$. Recall that we already did this for balls instead of spheres.

Is $t \mapsto (A_t f)(x)$ continuous (for any x)? Is $\sup_{t_1<t<t_2} |A_t f(x)|$ measurable in x? This actually has to be discussed before we can think about the first question.

A priori estimate for the spherical maximal averages:

Theorem. *Let $f \in C_0^0(\mathbb{R}^d),\ d \ge 4$. Then*

$$\|\sup_{t>0} |(A_t f)(x)|\|_{L_x^2(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)}.$$

Note that this can be improved up to $d \ge 2$, but fails for $d = 1$.

Key: Let $f \in C_0^1(\mathbb{R}^d)$. Then

$$(Sf)(x) = (\int_0^\infty |\frac{\partial A_t f}{\partial t}(x)|^2 t\,\mathrm{d}t)^{\frac{1}{2}}.$$

This also holds for C_0^0 by density.

Lemma.

$$\sup_{t>0} |(A_t f)(x)| \le (Mf)(x) + c(Sf)(x)$$

where Mf is the standard Hardy-Littlewood maximal function over centered balls.

Proof fo Lemma 1.

$$(A_t f)(x) = t^{-d} \int_0^t \frac{\partial}{\partial s}[s^d (A_s f)(x)]\,\mathrm{d}s = I + II$$

using that the integrand equals

$$ds^{d-1}A_s f)(x) + s^d \frac{\partial (A_s)f)}{\partial s}(x)$$

$$I = dt^{-d}\int_0^t s^{d-1}(\int_{|y|=1} f(x-sy)\,\mathrm{d}\sigma(y))\,\mathrm{d}s = dt^{-d}\int_{B(x,t)=\{y:|x-y|\le t\}} f(y)\,\mathrm{d}y = \frac{1}{\frac{t^d}{d}}\int_{B(x,t)} f \le \sup_{t>0} \fint_{B(x,t)} f = (Mf)(x)$$

due to our choice of normalization $|B(x,t)| = \int_0^t \underbrace{\omega_{d-1}}_{1} s^{d-1}\,\mathrm{d}s = \frac{t^d}{d}$.

$$II = t^{-d}\int_0^t s^{d-\frac{1}{2}}(\frac{\partial A_s f)}{\partial s}(x) s^{\frac{1}{2}})\,\mathrm{d}s \le (\int_0^\infty |\frac{\partial (A_s f)(x)}{\partial s}|^2 s\,\mathrm{d}s)^{\frac{1}{2}} \cdot t^{-d}(\int_0^t s^{2d-1}\,\mathrm{d}s)^{\frac{1}{2}} \lesssim_d (Sf)(x)$$

□

Lemma. If $d \ge 4$ then

$$\|Sf\|_{L^2(\mathbb{R}^d)} \le A\|f\|_{L^2(\mathbb{R}^d)}$$

Proof. $A_s f = f * \sigma_s$ ∴

$$(\widehat{A_s f})(\xi) = \hat{f}(\xi)\hat{\sigma}_s(\xi) = \hat{f}(\xi)\hat{\sigma}(s\xi)$$

since

$$\hat{\sigma}_s(\xi) = \int_{|x|=1} e^{2\pi i x\xi}\,\mathrm{d}\sigma_s(x) = \int_{|x|=1} e^{2\pi i s x\xi}\,\mathrm{d}\sigma(x) = \hat{\sigma}(s\xi).$$

It follows that

$$\frac{\partial(\hat{A}_s f)}{\partial s}(\xi) = \frac{\mu(s\xi)}{s}\hat{f}(\xi)$$

where

$$\mu(\xi) = \sum_{j=1}^d \xi_j \frac{\partial\hat{\sigma}}{\partial \xi_j}(\xi) = \langle \xi, \nabla\hat{\sigma}(\xi)\rangle$$

because we differentiate with respect to a variable independent from the Fourier transform.

Key:

$$|\mu(\xi)| \le A\min\{|\xi|, |\xi|^{-\frac{d-3}{2}}\}$$

Proof.

$$\frac{\partial\hat{\sigma}}{\partial\xi_j}(\xi) = 2\pi\int_{|x|=1} x_j e^{2\pi i x\xi}\,\mathrm{d}\sigma(x)$$

Since $x_j \in [-1,1]$ we get

$$|\frac{\partial\hat{\sigma}}{\partial\xi_j}(\xi)| \le A(1+|\xi|)^{-\frac{d-1}{2}}$$

The statement follows from this and Cauchy-Schwarz applied to the definition of $\mu(\xi)$. □

By Plancherel we get

$$\int_{\mathbb{R}^d} |\frac{\partial(A_s f)}{\partial s}(x)|^2\,\mathrm{d}x = \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \frac{|(s\xi)|^2}{s^2}\,\mathrm{d}\xi$$

Now we multiply by s and integrate with respect to s and get

$$\int_{\mathbb{R}^d} |Sf(x)|^2 \,\mathrm{d}x = \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \underbrace{(\int_0^\infty \frac{|\mu(s\xi)|^2}{s} \,\mathrm{d}s)}_{*} \,\mathrm{d}\xi$$

It is enough to show, that $*$ is bounded with respect to ξ.

$$\int_0^\infty \frac{|\mu(s\xi)|^2}{s} \,\mathrm{d}s = \int_0^{\frac{1}{|\xi|}} + \int_{\frac{1}{|\xi|}}^\infty \leq A(|\xi|^2 \int_0^{\frac{1}{|\xi|}} s^2 \frac{\mathrm{d}s}{s} + |\xi|^{-(d-3)} \int_{\frac{1}{|\xi|}}^\infty s^{-(d-3)} \frac{\mathrm{d}s}{s}) \leq C <$$

where C is independent of ξ. The whole thing is true iff $-(d-3)-1 < -1,\ d \leq 4$, since this takes care of the second summand and the first summand is no problem. □

Now the theorem follows from the Lemmas and the L^2 boundedness of the maximal function.

Now look at d_1. Then the theorem fails because

$$A_t f(x) = \frac{1}{2}(f(x+t) + f(x-t))$$

and we can take as a counteraxample a function nonnegative which blows up close to 0 but is in L^p for every $p < \infty$ such that $\sup_{t>0}(A_t f)(x) = \infty$ everywhere.

He said something with the wave equation and its smoothing properties for $1 \leq d \leq 3$.

$d \geq 2$: The maximal operator $f \mapsto \sup_{t>0}(A_t(f)|$ is bounded on $L^p(\mathbb{R}^d)$ if $p > \frac{d}{d-1}$.

- For $d \geq 3$, this is in Stein's Chapter XI paragraph 3. Ideas: rotational curvature, FIOs, dyadic decomposition, almost orthogonality
- For $d = 2$, this is a theorem of Bourgan (1986) with alternative proofs by Sogge (1991): Cinematic curvature. Mockenhaupt-Seeger-Sogge (1993): Local smoothing for wave equation. $d = 2$ is also in Stein's Chapter XI paragraph 4D
- No such result holds in $L^p(\mathbb{R}^d)$ if $p \leq \frac{d}{d-1}$: Let

$$f(y) = \frac{|y|^{1-d}}{\log \frac{1}{|y|}} 1_{\{|y| \leq \frac{1}{2}\}}(y)$$

Then $f \in L^p$ if $p \leq \frac{d}{d-1}$: First, forget about the log for a moment. It is only there to take care of the endpoint.

$$\|f\|^p_{L^p(\mathbb{R}^d)} \simeq \int_0^{\frac{1}{2}} \frac{r^{(1-d)p}}{(\log \frac{1}{r})^p} rwd - 1 \,\mathrm{d}r = \int_0^{\frac{1}{2}} r^{(1-d)(p-1)} \,\mathrm{d}r$$

$$(1-d)(p-1) > -1 \quad (d-1)(p-1) < 1 \quad p < \frac{1}{d-1} + 1 = \frac{d}{d-1}$$

For any x, the quantity $(A_t f)(x)$ is unbounded when $f \sim |x|$:

$$\therefore \sup_t (A_t f)(x) = \infty \quad \text{everywhere}$$

Next: Averages with respect to a fixed curve:

$$(Mf)(x) = \sup_{h>0} \frac{1}{2h} | \int_{-h}^h f(x - (t, t^2)) \,\mathrm{d}t|$$

Maximal operator along a parabola $f : \mathbb{R}^2 \to \mathbb{C}$

$$\tilde{A} f(x) = \frac{1}{2t} \int_{-t}^{t} f(x - \gamma(s)) \,\mathrm{d}s$$

Maximal operator $\sup_{t>0} \tilde{A}_t f(x)$.

$2^k < t \leq 2^{k+1}$.

$$\frac{1}{2t} \int_{-t}^{t} f(x - \gamma(s)) \,\mathrm{d}s \leq \frac{1}{22^k} \int_{-2^{k+1}}^{2^{k+1}} |f(x - \gamma(s))| \,\mathrm{d}s \leq 4 \underbrace{\frac{1}{2^{k-1}} \int |f(x - \gamma(s)| \eta(2^{-k-1} s) \,\mathrm{d}s}_{A_{-k-1} f(x)}$$

where $\eta : \mathbb{R} \to \mathbb{R}_+$ smooth such that

$$\eta(s) = \begin{cases} 1 & |s| \leq 1 \\ 0 & |s| \geq 2 \end{cases}$$

Then the maximal operator we consider equal to

$$\sup_{k \in \mathbb{Z}} |A_k f(x)|$$

up to a factor (right?)

Now we consider $\gamma(s) = (s, s^2)$. Then

$$A_j f(x) = 2^j \int f(x_1 - s, x_2 - s^2) \eta(2^j s) \,\mathrm{d}s = 2^{j+k} \int f(x_1 - 2^k \tilde{s}, x_2 - 2^{2k} \tilde{s}^2) \eta(2^{jk} \tilde{s}) \,\mathrm{d}\tilde{s} = 2^{j+k} \int f(2^h (2^{-k} x_1 - \tilde{s}), 2^{2k} (2^{-2k} x$$

where $s = 2^{-k} \tilde{s}$ and

$$P_k f(x) = f(2^k x_1, 2^{2k} x_2)$$

$$\Rightarrow M = P_{-k} M P_k$$

Compare A_j with

$$B_j f(x) = 2^{2j} \int f(x - y) \psi(2^j y_1, 2^{2j} y_2) \,\mathrm{d}y$$

where ψ is compactly supported and smooth and

$$\int_{\mathbb{R}^2} \psi = \int_{\mathbb{R}} \eta$$

$$\psi_j(x) = 2^{3j} \psi(2^j x, 2^{2j} x_2)$$

is supported in a ball of radius 2^{-j} with respect to the metric

$$\rho(x, y) = \max(|x_1 - y_2|, |x_2 - y_2|^{\frac{1}{2}})$$

$$B_j f(x) \lesssim M_\rho f(x)$$

where M is the maximal function with respect to $(\mathbb{R}^2, \rho, \text{Leb. measure})$, doubling metric measure space. M_ρ is L^p-bounded for some reason. Therefore

$$f \mapsto \sup_j |B_j|$$

is bounded on $L^p(\mathbb{R}^2)$, $1 < p \leq \infty$. Need to estimate

$$\sup_j |(A_j - B_j)f(x)| \leq (\sum_j |(A_j - B_j)f(x)|^2)^{\frac{1}{2}}$$

$$\int f \,\mathrm{d}\mu_j = 2^j \int f(s, s^2)\eta(2^j s)\,\mathrm{d}s$$

$$A_j f = \mu_j * f, \quad B_j f = \psi_j * f$$

Define

$$\sigma_j := \mu_j - \psi_j$$

Goal: Show that

$$\|(\sum_j |\sigma_j * f|^2)^{\frac{1}{2}}\|_p \lesssim \|f\|_p$$

By interpolation it suffices to show this for $1 < p \leq 2$. $p = 2$: Let $f \in S$. Then by Plancherel

$$\|(\sum_j |\sigma_j * f|^2)^{\frac{1}{2}}\|_2^2 = \int \sum_j \sigma_j * f|^2 = \sum_j \int |\sigma_j * f|^2 = \sum_j \int |\hat{\sigma}_j|^2 |\hat{f}|^2 \leq \int |\hat{f}(\xi)|^2 \sum_j |\hat{\sigma}_j(\xi)|^2 \lesssim \int |\hat{f}|^2 = \|f\|_2^2$$

if we show that $\sum_j |\hat{\sigma}_j(\xi)|^2$ is bounded by a constant. Since the parabola has nonvanishing curvature we have

$$|\hat{\mu}_0(\xi)| \lesssim |\xi|^{-\frac{1}{2}}$$

Since ψ_0 is smooth we have for all N

$$|\hat{\psi}_0(\xi)| \lesssim |\xi|^{-N}.$$

That implies

$$|\hat{\sigma}_0(\xi)| = |\hat{\mu}_0(\xi) - \hat{\psi}_0(\xi)| \lesssim |\xi|^{-\frac{1}{2}}.$$

Also,

$$\int_{\mathbb{R}^2} \mathrm{d}(\mu_0 - \psi_0) = 0 \Rightarrow \hat{\sigma}_0(0) = 0$$

and since σ_0 has compact support and

$$\int_{\mathbb{R}^2} \mathrm{d}|\sigma_0| < \infty$$

$\hat{\sigma}_0(\xi)$ is differentiable in zero and therefore

$$|\hat{\sigma}_0(\xi)| \lesssim |\xi|.$$

Now

$$\hat{\mu}_j(\xi) = 2^j \int e^{-2\pi i(\xi_1 s + \xi_2 s^2)}\eta(2^2 s)\,\mathrm{d}s = \int e^{-2\pi i(2^{-j}\xi\tilde{s} + 2^{-2j}\xi_2\tilde{s}^2)}\eta(\tilde{s})\,\mathrm{d}\tilde{s} = \hat{\mu}_0(2^{-j}\xi_1, 2^{-2j}\xi_2)$$

Also,

$$|\hat{\sigma}_0(\xi)| \leq \min(\rho(\xi), \rho(\xi)^{-\frac{1}{2}})$$

$$|\hat{\sigma}_j(\xi)| = |\hat{\sigma}_0(2^{-j}\xi_2, 2^{-2j}\xi_2)| \lesssim \min(2^{-j}\rho(\xi), (2^{-j}\rho(\xi))^{-\frac{1}{2}})$$

Now we sum

$$\sum_j |\sigma_j(\xi)|^2 \lesssim \sum_{j \in \mathbb{Z}} \min((2^{-j}\rho(\xi))^2, (2^{-j}\rho(\xi))^{-1}) \lesssim 1$$

$1 < p < 2$.

Lemma. Let $1 < q, p_0 < \infty$ with $\frac{1}{1q} = |\frac{1}{2} - \frac{1}{p_0}|$. Assume

$$\sigma^* f := \sup_k |\sigma_k| * |f|$$

is bounded on L^q. Let $(g_k) \in L^{p_0}(l^2)$. Then

$$\|(\sum_k |\sigma_k * g_k|^2)^{\frac{1}{2}}\|_{p_0} \lesssim \|(\sum_k |g_k|^2)^{\frac{1}{2}}\|_{p_0}$$

Something with an operator

Remark. For fixed q there are two p_0s. If $p_0 < 2$ then

$$\frac{1}{2q} = \frac{1}{p_0} - \frac{1}{2} \Rightarrow p_0 = \frac{1}{\frac{1}{2q} + \frac{1}{2}} = \frac{2q}{1+q} < q$$

If $p_0 > 2$, then

$$\frac{1}{2q} = \frac{1}{2} - \frac{1}{p_0} \Rightarrow \frac{1}{p_0} = \frac{1}{2} - \frac{1}{2q} \Rightarrow \frac{2}{p_0} = 1 - \frac{1}{q} \Rightarrow q = (\frac{p_0}{2})'$$

Proof. Result for $p_0 < 2$ follows from $p_0 > 2$ by duality because for $L^{p_0}(l^2) \to L^{p_0}(l^2)$ the adjoint maps on $(L^{p_0}(l^2))' = L^{p_0'}(l^2)$. For $g \in L^{p_0}(l^2),\ f \in L^{p_0'}(l^2)$ we use that

$$\langle (Tg)_k, f\rangle = \int \sum_k (Tg)_k f_k = \int (\sigma_k * g) f_k = \int g_k(\tilde{\sigma}_k * f_k)$$

where $\tilde{\sigma}_k$ is σ_k reflected at 0.

$p_0 > 2$.

$$\|(\sum_k |\sigma_k * g_k|^2)^{\frac{1}{2}}\|_{p_0}^2 = \|\sum_k |\sigma_k * g_k|^2\|_{\frac{p_0}{2}} = \int f \sum_k |\sigma_k * g_k|^2$$

for some f with $1 = \|f\|_{(\frac{p_0}{2})'} = \|f\|_q$.

$$(|\sigma_k| * |g_k(x)|)^2 = (\int |g_k(x-y)| \,\mathrm{d}|\sigma_k|)^2 \le (\int |g_k(x-y)|^2 \,\mathrm{d}|\sigma_k|)\underbrace{(\int \mathrm{d}|\sigma_k|)}_{\lesssim 1}$$

Now the whole sum becomes

$$\begin{aligned} &\lesssim \int |f| \sum_k |\sigma_k| * |g_k|^2 = \int \sum_k (|\tilde{\sigma}_k| * |f|)|g_k|^2 \le \int (\tilde{\sigma}^* f) \sum_k |g_k|^2 \\ &\le \underbrace{\|\sigma^* f\|_q}_{\lesssim 1} \|\sum_k |g_k|^2\|_{\frac{p_0}{2}} \lesssim \|(\sum_k |g_k|^2)^{\frac{1}{2}}\|_{p_0}^2 \end{aligned}$$

$\square$

Theorem. *Let $1 < q < \infty$, $\frac{2q}{1+q} < p \le 2$. Assume σ^* is bounded on $L^q(\mathbb{R}^2)$, and σ_j are measures with*

$$\hat{\sigma}_j(\xi) \lesssim \min(2^j\rho(\xi), (2^j\rho(\xi))^{-1})^{\varepsilon}$$

for some $\varepsilon > 0$ and

$$\int \mathrm{d}|\hat{\sigma}_j| \lesssim 1.$$

Then

$$\|(\sum_j |\sigma_j * f|^2)^{\frac{1}{2}}\|_p \lesssim \|f\|_p.$$

By construction qs and ranges for p iteratively we can cover the whole interval.

This theorem can actually be applied to different situations, e.g. dyadic spheres. One might not get optimal results though. Something with maximal functions for general spheres or so...

Let S_k be Littlewood-Paley projections, i.e.

$$\widehat{S_k f} = \hat{\varphi}_k f$$

where φ_k is a smooth and bounded function with

$$\operatorname{supp} \hat{\varphi}_k \subset \{\xi : 2^{-k-1} \leq \rho(\xi) \leq 2^{-k+1}\}$$

and

$$\sum_k \hat{\phi}_k(\xi) = 1$$

for $\xi \neq 0$, so that

$$f = \sum_k S_k f$$

for $f \in L^p,\ 1 < p < \infty$. Then (or: and?) we have for the square function

$$\|(\sum_k |S_k f|^2)^{\frac{1}{2}}\|_p \lesssim \|f\|_p$$

for $1 < p < \infty$.

Proof of Theorem. By the lemma

$$\|(\sum_j |\sigma_j * \sum_k S_{j+k} f|^2)^{\frac{1}{2}}\|_p \leq \sum_k \|(\sum_j |\sigma_j * S_{j+k} f|^2)^{\frac{1}{2}}\|_p$$

Now the summands are

$$\lesssim \|(\sum_j |S_{j+k} f|^2)^{\frac{1}{2}}\|_p \lesssim \|f\|_p$$

where $p = \frac{2q}{1+q}$. But this is not enough. So we do

$$\|(\sum_j |\sigma_j * S_{j+k} f|^2)^{\frac{1}{2}}\|_2^2 = \sum_j \|\sigma_j * S_{j+k} f\|_2^2 = \sum_j \|\hat{\sigma}_j \hat{\varphi}_{j+k} \hat{f}\|_2^2 \lesssim 2^{-2\varepsilon|k|} \sum_j \|\hat{\varphi}_{j+k} \hat{f}\|_2^2 \lesssim 2^{-2\varepsilon|k|} \|\hat{f}\|_2^2$$

since $\hat{\varphi}_{j+k}$ is supported where $\rho(\xi) \sim 2^{-j-k}$ and $\hat{\sigma}_j \lesssim 2^{-\varepsilon|k|}$ on $\operatorname{supp} \hat{\varphi}_{j+k}$. Now we use Marcinkiewic interpolation:

$$\frac{1-\theta}{\frac{2q}{1+q}} + \frac{\theta}{2} = \frac{1}{p_\theta}$$

$$\Rightarrow I_k \lesssim 1^{1-\theta} (2^{-2\varepsilon|k|})^\theta \|f\|_{p_\theta}$$

which is summable in k. For the missing part see Duoandikoetxea and Rubio de Francia 1986 $\square$

Fourier restriction theory Given $f : \mathbb{S}^{d-1} \to \mathbb{C}$, consider the Fourier transform $\widehat{F\sigma}(\xi) = \int_{\mathbb{S}^{d-1}} f(x)e^{-2\pi i x\xi}\, \mathrm{d}\sigma(x)$ for $\xi \in \mathbb{R}^d$ (tempered distribution which turns out to be a function)

- f smooth $\Rightarrow \widehat{f\sigma}$ decays at ∞, e.g.

  $$|\widehat{f\sigma}(\xi)| \le c\|f\|_{C^2}(1+|\xi|)^{-\frac{d-1}{2}}$$

  via stationary phase

- f bounded then no pointwise decay holds in general. E.g.

  $$f_k(x) = e^{2\pi i k x}$$

  Consider $\xi = k$ then $|\widehat{f_k\sigma}(\xi)| = \sigma(\mathbb{S}^{d-1}) \simeq 1$, no decay here. Now take

  $$f = \sum_{j\ge 0} \frac{f_{k_j}}{j^2}$$

  where $|k_j| \to \infty$ sufficiently fast, e.g. $k_j = j!$. Easy to check: f is continuous but $|\widehat{f\sigma}(\xi)| \le c(1+|\xi|)^{-\varepsilon}$ does not hold for any $c < \infty,\ \varepsilon > 0$. (uniformly in ξ).

- Problem of distinguished origin disappears if we take L^q norms (There was some discussion on the wording going on which I did not understand)

Restriction conjecture (Stein): Prove, that if $f \in L^\infty(\mathbb{S}^{d-1})$, then

$$\|\widehat{f\sigma}\|_{L^q(\mathbb{R}^d)} \le c_{q,d}\|f\|_{L^\infty(\mathbb{S}^{d-1},\sigma)}$$

for very $q > \frac{2d}{d-1}$.

- Range of exponents would be sharp: take $f \equiv 1 \in L^\infty(\mathbb{S}^{d-1})$

  $$\|\hat\sigma\|^q_{L^q(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\hat\sigma(\xi)|^q \,\mathrm{d}\xi \lesssim \int_{\mathbb{R}^d} (1+|\xi|)r^{-\frac{d-1}{2}q}\,\mathrm{d}\xi \simeq_d \int_0^\infty (1+r)^{-\frac{d-1}{2}q} r^{d-1}\,\mathrm{d}r$$

  The last expression is $< \infty$ iff $-\frac{d-1}{2}q + (d-1) < -1$ iff $q > \frac{2d}{d-1}$.

Corresponding problem for L^2 densities was solved by Tomas-Stein inequality ($\sim$ 1975). If $f \in L^2(\mathbb{S}^{d-1})$, then

$$\|\widehat{f\sigma}\|_{L^q(\mathbb{R}^d)} \lesssim_{q,d} \|f\|_{L^2(\mathbb{S}^{d-1})}$$

if $q \ge \frac{2d+2}{d-1}$, and this range is the best possible. Note, that this was already proven last term and can be found in Stein/Shakarchi

- Assumptions are of the form $q > q_0$ or $g \ge q_0$. Why?

  $$\|\widehat{f\sigma}\|_{L^\infty(\mathbb{R}^d)} = \sup_{\xi\in\mathbb{R}^d} |\int_{\mathbb{S}^{d-1}} f(x)e^{-2\eta i x\xi}\,\mathrm{d}\sigma_x| \le \int_{\mathbb{S}^{d-1}} |f(x)|\,\mathrm{d}\sigma_x = \|f\|_{L^1(\mathbb{S}^{d-1})} \lesssim \|f\|_{L^2(\mathbb{S}^{d-1})}$$

  + Riesz-Thorin because the sphere is compact

- $q \geq \frac{2d+2}{d-1}$ is the best possible for L^2 densities: Knapp counterexample.

$$C_\delta = \{x \in \mathbb{S}^{d-1} : 1 - xe_d \leq \delta^2\}$$

where $e_d = (0, ..., 0, 1)$. Since $|x - e_d|^2 = 2(1 - xe_d)$,

$$|x - e_d| \leq c\delta \Rightarrow x \in C_\delta \Rightarrow |x - e_d| \leq Cd$$

for appropriate constants $0 < c < C < \infty$. Let $f = 1_{C_\delta}$. Right hand side is easy:

$$\|f\|_{L^2(\mathbb{S}^{d-1})} = |C_\delta|^{\frac{1}{2}} \simeq_d \delta q^{\frac{d-1}{2}}$$

Left hand is trickier. Note: the support of $f\sigma$ is contained in a cylindrical box B_δ centered at e_d with length $\sim \delta^2$ in the $_d$ direction and $\sim \delta$ in the $(d-1)$ orthogonal direction.

$$\|\widehat{f\sigma}\|_{L^q(\mathbb{R}^d)}$$

Uncertainty principle: If a function is supported it a box, then its fourier transform is more or less constant on the dual box, which is the same but with inverse lengths. Idea: look at $\widehat{f\sigma}$ on the dual box B^*_δ (centered at 0), u.e. suppose $|\xi_d| \leq c_1^{-1}\delta^{-1}$ and $|\xi_j| \leq c_1^{-1}\delta^{-1}$ if $j < d$, c_1 is a large constant to be chosen. If $\xi \in B^*_\delta$, then

$$|\widehat{f\sigma}(\xi)| = |\int_{C_\delta} e^{2\pi i x\xi} \, \mathrm{d}\sigma_x| = \int e^{2\pi i(x-e_d)\xi} \, \mathrm{d}\sigma_x| \geq \int_{C_\delta} \cos(2\pi(x - e_d)\xi) \, \mathrm{d}\sigma_x|$$

Conditions on ξ imply that

$$2\pi(x - e_d)\xi| \leq \frac{\pi}{3}$$

if C_1 large enough. Therefore

$$|\widehat{f\sigma}(\xi)| \geq \frac{1}{2}|C_d| \simeq \delta^{d-1}$$

How large is B^*_δ?

$$|B^*_\delta| \simeq \delta^{-2}(\delta^{-1})^{d-1} = \delta^{-(d+1)}$$

Conclusion:

$$\|\widehat{f\sigma}\|_{L^1(\mathbb{R}^d)} \gtrsim (\int_{B^*_\delta} |\widehat{f\sigma}(\xi)|^q \, \mathrm{d}\xi)^{\frac{1}{q}} \gtrsim \delta wd - 1\delta^{-\frac{d+1}{q}}$$

$$\delta^{d-1}\delta^{-\frac{d+1}{q}} \leq \|\widehat{f\sigma}\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{S}^{d-1})} \simeq \delta^{\frac{d-1}{2}}$$

$$\therefore d - 1 - \frac{d+1}{q} \geq \frac{d-1}{2} \Leftrightarrow \frac{d-1}{2} \geq \frac{d+1}{q} \Leftrightarrow q \geq 2d + 2d - 1$$

So, if you violate both of the two obstructions, then the inequality should hold (which obstructions? Maybe ∞-bound on the domain and on the range?))

Technical tool: convolution of Schwartz function with a (compactly supported) measure. $\phi \in S(\mathbb{R}^d),\ \mu \in M(\mathbb{R}^d)$ then

$$(\varphi * \mu)(x) = \int \varphi(x - y) \, \mathrm{d}\mu(y)$$

Notation $\check{\mu} = \hat{\mu}(-\cdot)$

Lemma. (i)

$$\widehat{\check{\phi}\mu} = \phi * \hat{\mu}$$

(ii)

$$\widehat{\phi\mu} = \widehat{\varphi} * \hat{\mu}$$

Proof of (b), (a) is similar. Enough to show $\forall \psi \in S(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \widehat{\phi\mu}\psi \,\mathrm{d}x = \int_{\mathbb{R}^d} (\widehat{\varphi} * \hat{\mu})\psi \,\mathrm{d}x$$

$$\int_{\mathbb{R}^d} \widehat{\phi\mu}\psi \,\mathrm{d}x = \int \widehat{\psi}\varphi \,\mathrm{d}\mu = \int \widehat{\check{\phi} * \psi} \,\mathrm{d}\mu = \int_{\mathbb{R}^d} (\check{\varphi} * \psi)\hat{\mu} \,\mathrm{d}x = \int_{\mathbb{R}^d} (\hat{\phi} * \hat{\mu})\psi \,\mathrm{d}x$$

due to duality, Fourier inversion on S, duality and definition of $*$ + Fubini. ☐

Lemma. $f, g \in S, \mu \in M(\mathbb{R}^d)$. Then

$$\int \hat{f}\bar{\hat{g}} \,\mathrm{d}\mu = \int_{\mathbb{R}^d} (\hat{\mu} * \bar{g}) f \,\mathrm{d}x$$

Proof.

$$\int \hat{f}\bar{\hat{g}} \,\mathrm{d}\mu = \int_{\mathbb{R}^d} f\widehat{\bar{\hat{g}}\mu} \,\mathrm{d}x = \int_{\mathbb{R}^d} f(\bar{g} * \hat{\mu}) \,\mathrm{d}x$$

by duality and lemma 1 ☐

Lemma. μ finite positive measure. Then the following are equivalent

(i) $\|\widehat{f\mu}\|_{L^q} \le c\|f\|_{L^2(\mu)} \quad \forall f \in L^2(\mu)$

(ii) $\|\hat{g}\|_{L^2(\mu)} \le c\|g\|_{L^{q'}} \quad \forall g \in S$

(iii) $\|f * \hat{\mu}\|_{L^q} \le c^2\|f\|_{L^{q'}} \quad \forall f \in S$

Note, that

- $T : L^2 \to L^q,\ f \mapsto \widehat{f\mu}$ (extension) iff
- $T^* : L^{q'} \to L^2,\ g \mapsto \hat{g}|_{\operatorname{supp}\mu}$ (restriction) iff
- $TT^* : L^{q'} \to L^q,\ f \mapsto f * \hat{\mu}$

Again, there were some words about the expression 'extension', which I did not understand.

Proof of Tomas-Stein, up to endpoint. Will show

$$q > \frac{2d+2}{d-1} \Rightarrow \|f * \hat{\mu}\|_{L^q(\mathbb{R}^d)} \lesssim_{q,d} \|f\|_{L^{q'}(\mathbb{R}^d)}$$

Relevant properties of σ:

(i) $|\hat{\sigma}(\xi)| \lesssim (1 + |\xi|)^{-\frac{d-1}{2}}$

(ii) $\sigma(D(x,r)) \simeq r^{d-1}$

Note, that every measure satisfying this will have a Tomas-Stein, because these are the only properties we need.

Let $\phi \in C_0^\infty(\mathbb{R}^d)$, $\operatorname{supp}(\Phi) \subset \{x : \frac{1}{4} \le |x| \le 1\}$, $\sum_{j \ge 0} \varphi(\frac{x}{2^j}) = 1$ if $|x| \ge 1$.

Cut up $\hat{\sigma}$ as follows:

$$\hat{\sigma} = k_{-\infty} + \sum_{j=0}^{\infty} k_j$$

with

$$k_j(x) = \phi(\frac{x}{2^j})\hat{\sigma}(x)$$

and

$$k_{-\infty}(x) = (1 - \sum_{j \ge 0} \phi(\frac{x}{2^j}))\hat{\sigma}(x)$$

Easy: $k_{-\infty} \in C_0^\infty$:

$$\|f * k_{-\infty}\|_{L^q} \lesssim \|f\|_{L^p} \quad \forall p \le q$$

Since $q > 2$ we can take $p = q'$ (Why does it hold for q', then? And don't we want to show it only for q' anyways?).

Trickier: $(k_j)_{j=0}^\infty$. Upshot: Estimate the convolution with k_j $L^1 \to L^\infty$, $L^2 \to L^2$.

First $L^1 \to L^\infty$.

$$\|f * k_j\|_{L^\infty} \le \|k_j\|_{L^\infty} \|f\|_{L^1}$$

where

$$\|k_j\|_{L^\infty} \sim 2^{-j\frac{d-1}{2}}$$

as a consequence of property (i).

$L^2 \to L^2$

$$\|f * k_j\|_{L^2} = \|\hat{f}\hat{k}_j\|_{L^2} \le \|\hat{k}_j\|_{L^\infty} \|f\|_{L^2}$$

Claim :$\|\hat{k}_j\|_{L^\infty} \sim 2^j$ (Next lecture)

Interpolate (Riesz-Thorin)

$$\|f * k_j\|_{L^q} \lesssim 2^{-j\frac{d-1}{2}(1-\theta)} 2^{j\theta} \|f\|_{L^{q'}}$$

$$\frac{1}{q} = \frac{1-\theta}{\infty} + \frac{\theta}{2} \therefore q = \frac{2}{\theta}$$

which is equivalent to

$$\|f * k_j\|_{L^q} \lesssim 2^{j(\frac{d+1}{q} - \frac{d-1}{2})} \|f\|_{L^{q'}}$$

for any $q \in [2, \infty]$. The exponent is less than 0 iff $q > \frac{2d+2}{d-1}$ □

Fourier restriction theory 2 $f \in L^1(\mathbb{R}^d)$ implies $\hat{f}$ is uniformly continuous (so, $\hat{f}$ can be restricted to any set)

$f \in L^2(\mathbb{R}^d)$ iff $\hat{f} \in L^2$, (therefore $\hat{f}$ cannot be restricted to a set of Lebesgue measure 0)

Question 1 What happens for intermediate $p \in (1, 2)$?

Let $M \subset \mathbb{R}^d$ be smooth compact hypersurface equipped with $\mathrm{d}\mu = \psi \,\mathrm{d}\sigma$, $\psi \in C_0^\infty$ and $\mathrm{d}\sigma$ surface measure on M. Given $1 < p < 2$, for which exponents q does

$$(\int_M |\hat{f}(\xi)|^q \,\mathrm{d}\mu_\xi)^{\frac{1}{q}} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$$

hold? A complete answer for $q = 2$ is given by Tomas-Stein: M compact hypersurface whose Gauss curvature does note vanish on $\operatorname{supp}(M)$. Then the restriction inequality holds provided $q = 2$ and $1 \le p \le \frac{2d+2}{d+3}$. Note, that this is the dual version.

Question 2 What happens for $q < 2$? Dimensional analysis implies

$$1 \le p < \frac{2d}{d+1}$$

and Knapp-Type examples

$$q \le (\frac{d-1}{d+1})p'$$

The restriction conjecture states that these conditions are also sufficient.

End of proof of Tomas-Stein. Back to $(\mathbb{S}^{d-1}, \sigma)$. Strategy:

$$\|f * \hat{\sigma}\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^{q'}(\mathbb{R}^d)}$$

if $q > \frac{2d+2}{d-1}$.

Why is

$$\|\hat{k}_j\|_{L^\infty} \lesssim 2^j$$

$k_j = \phi_{2^{-j}}\hat{\sigma}$ implies $\hat{k}_j = \psi^{2^{-j}} * \sigma$ where $\psi^\varepsilon(x) = \varepsilon^{-d}\psi(\varepsilon^{-1}x)$ and $\psi = \hat{\phi} \in S$. Now

$$|\hat{k}_j(\xi)| \lesssim_N 2^{jd} \int_{\mathbb{S}^{d-1}} (1 + 2^j|\xi - \eta|)^{-N} \, \mathrm{d}\sigma(\eta) \qquad \forall N \in \mathbb{N}$$

Now let $D(x,r) = B(x,r) \cap \mathbb{S}^{d-1}$. Then $\sigma(D(x,r)) \lesssim r^{d-1}$. (Is this right?) So

$$\begin{aligned} |\hat{k}_j(\xi)| \lesssim 2^{jd} &\int_{D(\xi,2^{-j})} (1 + 2^j|\xi - \eta|)^{-N} \, \mathrm{d}\sigma(\eta) \\ &+ \sum_{k \ge 0} \int_{D(\xi,2^{k+1-j}) \setminus D(\xi,2^{k-j})} (1 + 2^j|\xi - \eta|)^{-N} \, \mathrm{d}\sigma(\eta) \end{aligned}$$

The first one will dominate because of the rapid decay of ψ.

$$\lesssim 2^{jd}[\underbrace{\sigma(D(\xi, 2^{-j}))}_{\sim 2^{-j(d-1)}} + \sum_{k \ge 0} 2^{-Nk} \underbrace{\sigma(D(\xi, 2^{k+1-j}) \setminus D(\xi, 2^{k-j}))}_{\sim 2^{(d-1)(k-j)}}] \lesssim 2^j$$

just choose $N = d$ such that the sum becomes a geometric series. □

Remark. The $L^2 \to L^2$ bound in the previous argument was based only on dimensionality considerations. Therefore there should be an L^2 bound for $\widehat{f\nu}$ valid under very general conditions.

Theorem. *Let ν be a positive finite measure where*

$$\nu(D(x,r)) \le cr^\alpha$$

Then

$$\|\widehat{f\nu}\|_{L^2(D(0,R))} \le cR^{\frac{d-\alpha}{2}} \|f\|_{L^2(\mathrm{d}\nu)}$$

The proof relies on Schur's test: (x, μ), (Y, ν) measure spaces, $K(x,y)$ measurable on $X \times Y$. If for all y respectively x

$$\int_X |K(x,y)| \, \mathrm{d}\mu(x) \le A, \qquad \int_Y |K(x,y)| \, \mathrm{d}\nu(y) \le B,$$

then for

$$(T_K f)(x) = \int_Y K(x,y) f(y) \, \mathrm{d}\nu(y)$$

we have

$$\|T_K\|_{L^2 \to L^2} \le \sqrt{AB}.$$

Proof of the theorem. Let $\phi \in S(\mathbb{R}^d)$ be radial such that $\phi \geq 1$ on the unit disc $\hat{\phi}$ has compact support. $\phi_\varepsilon(x) := \phi(\varepsilon x)$. Then

$$\|\widehat{f\nu}\|_{L^2(D(0,R))} \leq \|\phi_{R^{-1}}(x)\widehat{f\nu}(-x)\|_{L^2_X(\mathbb{R}^d)} = \|\widehat{\phi_{R^{-1}}} * (f\nu)\|_{L^2(\mathbb{R}^d)} = \|\int_Y R^d\hat{\phi}(R(x-y))f(y)\,\mathrm{d}\nu(y)\|_{L^2}$$

We have the estimates

$$\int_{\mathbb{R}^d} R^d\hat{\phi}(R(x-y))|\,\mathrm{d}x = \|\phi\|_{L^1} < \infty$$

by change of variables.

$$\int_Y R^d|\hat{\phi}(R(x-y))|\,\mathrm{d}\nu(y) \lesssim R^{d-\alpha}$$

by the hypothesis of ν and compact support of $\hat{\phi}$. Now apply Schur. ☐

Proof of the restriction conjecture in $d = 2$ (Zygmund '70, Fefferman '72)

Theorem. *Let $\gamma : I \to \mathbb{R}^2$ be a smooth curve with $\gamma' \neq 0$ and $\gamma'' \neq 0$ on some finite interval I. Let $4 < q \leq \infty$ and $3p' \leq q$. Then $\forall \varphi \in L^p(I)$*

$$\|\int_I \varphi(t)e^{i\gamma(t)\xi}\,\mathrm{d}t\|_{L^q_\xi(\mathbb{R}^2)} \lesssim_{p,q} \|\varphi\|_{L^p(I)}$$

Note, that the unit circle is a special case.

Proof. Step 1: 'Even integer' trick.

$$\|\int_I \varphi(t)e^{i\gamma(t)\xi}\,\mathrm{d}t\|^2_{L^q_\xi(\mathbb{R}^2)} = \int_I\int_I \varphi(t)\overline{\varphi(s)}e^{i(\gamma(t)-\gamma(s))\xi}\,\mathrm{d}t\,\mathrm{d}s\|_{L^{\frac{q}{2}}(\mathbb{R}^2)}$$

Step 2: Change variables $I \times I \to U \subset \mathbb{R}^2$, $(t,s) \mapsto \gamma(t) - \gamma(s) = x$. Choose I small enough such that the changle of variables is invertible with Jacobian $J = \det(\frac{\partial(t,s)}{\partial x})$ satisfying $|J| \simeq |t-s|^{-1}$. Let $\gamma(t) = (x_1(t), x_2(t))$. Then

$$|\frac{\partial x}{\partial(t,s)}| = |\begin{pmatrix} -x_1'(s) & -x_2'(s) \\ x_1'(t) & x_2'(t) \end{pmatrix}| = |x_1(s)x_2'(t) - x_2'(s)x_1'(t)| \simeq |t-s|$$

since in case of γ being the unit circle parametrized by arclength, this is equal to

$$|\cos s \sin t - \sin s \cos t| = |\sin(s-t)| \sim |s-t|.$$

Lets assume γ is already parametrized by arclength. To compare that situation with the unit circle, define

$$\theta(s) = \int_0^s k(t)\,\mathrm{d}t,$$

where k is the (name?) curvature of γ. Then

$$\gamma'(s) = (\cos\theta(s), \sin\theta(s))$$

and due to

$$(\min k)|s-t| \lesssim \|\theta(s) - \theta(t)| \leq \|k\|_{L^\infty}|s-t|$$

we can apply the above estimate.

Step 3: Hausdorff-Young

The integral now becomes

$$\| \int_U e^{ix\xi} F(x) \, \mathrm{d}x \|_{L^{\frac{q}{2}}(\mathbb{R}^2)} \leq \|F\|_{L^r(\mathbb{R}^2)}$$

where

$$F(x) := \varphi(t)\overline{\varphi(s)}|J|$$

provided $\frac{q}{2} = r' \geq 2$.

Step 4: Fractional integration (Hölder-Littlewood-Sobolev)

We treat the following as an integral of $|\varphi|^r$ times a second function.

$$\|F\|_{L^r(\mathbb{R}^2)} \simeq (\int_I \int_I \frac{|\varphi(t)|^r |\varphi(s)|^r}{|t-s|^{r-1}} \, \mathrm{d}t \, \mathrm{d}s)^{\frac{1}{r}} \leq c\|\varphi\|^2_{L^p(I)}$$

provided $1 < r < 2$ and $1 + \frac{1}{(p/r)'} \geq r - 1 + \frac{1}{p/r}$. The first one is fine by assumption. The second is equivalent to $3p' \leq q$. $\square$