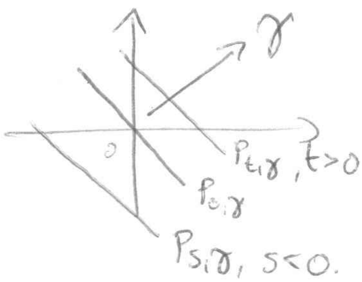
Radon transforms

$\mathcal{R}(f)(t,\gamma) = \int_{P_{t,\gamma}} f.$ where...

$f: \mathbb{R}^d \to \mathbb{R}$, $t \in \mathbb{R}$, $\gamma \in S^{d-1} \subseteq \mathbb{R}^d$

$P_{t,\gamma} = \{x \in \mathbb{R}^d : \underbrace{x \cdot \gamma}_{\text{inner pr.}} = t\}$ hyperplane



$P_{t,\gamma}$ equipped with natural $(d-1)$-dim. Leb. measure, denoted by m_{d-1} (coincides with $(d-1)$-Hausdorff measure)

Remarks (i) $f \in C_0^0(\mathbb{R}^d) \Rightarrow f$ integrable on every $P_{t,\gamma}$
$\Rightarrow R(f)(t,\gamma)$ defined for every (t,γ).
($R(f)$ cont. fct. of (t,γ), cpctly spp. in t

(ii) $f \in L^1(\mathbb{R}^d) \Rightarrow f$ may fail to be measurable/integrable on some $P_{t,\gamma}$ ($\to R(f)(t,\gamma)$ not defined.)

(iii) $f = \chi_E$ ($E \subset \mathbb{R}^d$ mb.) $\Rightarrow R(f)(t,\gamma) = m_{d-1}(E_{t,\gamma})$
$E_{t,\gamma} = E \cap P_{t,\gamma}$ if $E_{t,\gamma}$ measurable.

Look instead at maximal Radon transform:

$$R^*(f)(\gamma) = \sup_{t \in \mathbb{R}} |R(f)(t,\gamma)|.$$

$\to$ Want to study L^p-mapping properties of R in order to study regularity of subsets of $\mathbb{R}^d$.

<u>Thm. 1</u> $f \in C_0^\infty(\mathbb{R}^d)$, $n \geq 3$:

<u>(*)</u> $$\int_{S^{d-1}} \mathcal{R}^*(f)(\gamma)\, d\sigma_\gamma \leq c\left(\|f\|_{L^1(\mathbb{R}^d)} + \|f\|_{L^2(\mathbb{R}^d)} \right).$$

~~(i.e. f needs~~

<u>Rem.</u> i) necessary conditions:

∙) $f \in L^1$: $f(x) = (1+|x|^{d-1})^{-1} \in (L^2 \setminus L^1)(\mathbb{R}^d)$ if $d \geq 3$

f is not integrable in any plane $P_{t,\gamma}$.

($\to f \in L^1$ gives global control.)

∙) $f \in L^2$: $f_\varepsilon(x) = (|x|+\varepsilon)^{-d+\delta}$ if $|x| \leq 1$, $\delta \in (0,1)$ fixed.

Let $\varepsilon \to 0^+$ to see that (*) fails if $\|\cdot\|_{L^2}$ on the RHS is not there

($\to f \in L^2$ gives local control.)

<u>Key</u>: Interplay between Radon and Fourier transform.

$t \mapsto \lambda \in \mathbb{R}$ dual variable.

Fourier transform:

$$\hat{R}(f)(\lambda,\gamma) = \int_{-\infty}^{\infty} R(f)(t,\gamma)\, e^{-2\pi i \lambda t}\, dt$$

<u>Lemma 1</u> $f \in C_0^\infty(\mathbb{R}^d)$, $\gamma \in S^{d-1}$:

$$\hat{R}(f)(\lambda,\gamma) = \hat{f}(\lambda\gamma).$$

<u>Proof</u> $\hat{f}(\lambda\gamma) = \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i x\cdot(\lambda\gamma)}\, dx$

$$= \int_{-\infty}^{\infty} \left(\int_{\mathbb{R}^{d-1}} f(u,t)\, du \right) e^{-2\pi i \lambda t}\, dt. = \int_{-\infty}^{\infty} \left(\int_{P_{t,\gamma}} f \right) e^{-2\pi i \lambda t}\, dt.$$

Choose coordinates
$x = (u,t)$, $t = x\cdot\gamma = x_d \in \mathbb{R}$,
$u = (x_1, \dots, x_{d-1}) \in \mathbb{R}^{d-1}$.

$\square$

Lemma 2 $f \in C_0^0(\mathbb{R}^d)$:

$$\int_{S^{d-1}} \left(\int_{-\infty}^{\infty} |\hat{R}(f)(\lambda,\gamma)|^2 |\lambda|^{d-1} d\lambda \right) d\sigma_\gamma = 2 \int_{\mathbb{R}^d} |f(x)|^2 dx$$

Proof $2 \int_{\mathbb{R}^d} |f(x)|^2 dx = 2 \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 d\xi$

$$= 2 \int_{S^{d-1}} \left(\int_0^{\infty} |\hat{f}(\lambda\gamma)|^2 \lambda^{d-1} d\lambda \right) d\sigma_\gamma.$$

$$= 2 \int_{S^{d-1}} \left(\int_{-\infty}^{0} |\hat{f}(\lambda\gamma)|^2 |\lambda|^{d-1} d\lambda \right) d\sigma_\gamma$$

done by Lemma 2. $\square$

Lemma 3 Suppose $F(t) = \int_{-\infty}^{\infty} \hat{F}(\lambda) e^{2\pi i \lambda t} d\lambda$

where ~~f~~ $\sup_{\lambda \in \mathbb{R}} |\hat{F}(\lambda)| \leq A$ and $\int_{-\infty}^{\infty} |\hat{F}(\lambda)|^2 |\lambda|^{d-1} d\lambda \leq B^2$.

Then $\sup_{t \in \mathbb{R}} |F(t)| \leq c(A+B)$ (This is essential in the proof of Thm. 1)

Moreover, if $0 < \alpha < \frac{1}{2}$, then $|F(t_1) - F(t_2)| \leq c_\alpha |t_1 - t_2|^\alpha \cdot (A+B)$

($\to$ α-Hölder)

Proof.

$$F(t) = \underbrace{\int_{|\lambda|<1} \hat{F}(\lambda) e^{2\pi i \lambda t} d\lambda}_{\leq CA} + \underbrace{\int_{|\lambda|>1} \hat{F}(\lambda) e^{2\pi i \lambda t} d\lambda}_{\leq \int_{|\lambda|>1} \hat{F}(\lambda) d\lambda}$$

$$\overset{CS}{\leq} \underbrace{\left(\int_{|\lambda|>1} |\hat{F}(\lambda)|^2 |\lambda|^{d-1} d\lambda \right)^{1/2}}_{\leq B} \cdot \underbrace{\left(\int_{|\lambda|>1} |\lambda|^{-d+1} d\lambda \right)^{1/2}}_{<\infty \text{ if } -d+1<-1 \iff d \geq 3.}$$ ✓

$\Rightarrow$ first estimate.

$$|F(t_1)-F(t_2)| = \int_{-\infty}^{\infty} \hat{F}(\lambda)\underbrace{\left(e^{2\pi i\lambda t_1}-e^{2\pi i\lambda t_2}\right)}_{|\cdot|\le C_\alpha |t_1-t_2|^\alpha |\lambda|^\alpha \text{ if } \alpha\in(0,1]} d\lambda$$

$(x \to e^{ix}$ Lipschitz$)$

$$= \int_{|\lambda|\le 1} \Big\} \le C_\alpha A\, |t_1-t_2|^\alpha$$

$$+ \int_{|\lambda|>1} \Big\} \le |t_1-t_2|^\alpha \underbrace{\int_{|\lambda|>1} |\hat{F}(\lambda)|\,|\lambda|^\alpha d\lambda}$$

$$\le \left(\int |\hat{F}(\lambda)|^2 |\lambda|^{d-1} d\lambda\right)^{1/2} \cdot \Big(\underbrace{\int_{|\lambda|>1} |\lambda|^{-d+1+2\alpha} d\lambda}_{<\infty \text{ if } \alpha<\frac{1}{2} \text{ for } d\le 3.}\Big)$$

$\square$

Proof of Thm. For each $\gamma \in S^{d-1}$, let $F(t)=R(f)(t,\gamma)$

$\Rightarrow \sup_{t\in\mathbb{R}} |F(t)| = R^*(f)(\gamma)$.

Let $A(\gamma)=\sup_{\lambda\in\mathbb{R}} |\hat{F}(\lambda)|$, $B^2(\gamma)=\int_{-\infty}^{\infty} |\hat{F}(\lambda)|^2 |\lambda|^{d-1} d\lambda$

Lemma 3 $\underset{\text{assumptions!}}{\overset{\text{check}}{\Rightarrow}}$ $\sup_{t\in\mathbb{R}} |F(t)| \le C(A(\gamma)+B(\gamma))$

Lemma 1 $\Rightarrow$ $\hat{F}(\lambda)=\hat{R}(f)(\lambda,\gamma)=\hat{f}(\lambda\gamma) \Rightarrow A(\gamma)\le \|f\|_{L^1(\mathbb{R}^d)}$

Lemma 2 $\Rightarrow$ $\int_{S^{d-1}} B^2(\gamma)\, d\sigma_\gamma = 2\|f\|^2_{L^2(\mathbb{R}^d)}$.

We have $\sup_{t\in\mathbb{R}} |F(t)|^2 \le C'(A^2(\gamma)+B^2(\gamma))$

Integrate both sides:

$$\int_{S^{d-1}} R^*(f)(\gamma)^2 d\sigma_\gamma \lesssim \underbrace{\underbrace{\int A^2(\gamma)\, d\sigma_\gamma}_{\lesssim \|f\|_1^2} + \underbrace{\int B^2(\gamma)\, d\sigma_\gamma}_{\simeq \|f\|^2_{L^2}}}_{\lesssim (\|f\|_{L^1}+\|f\|_{L^2})^2}$$

Use Hölder, because on cpt. space:

$\|R^*(f)\|^2_{L^1} \lesssim \int_{S^{d-1}} R^*(f)(\gamma)^2 d\sigma_\gamma$.

$\square$

Regularity of sets when $d \geq 3$.

$E \subset \mathbb{R}^d$ meas. $E_{t,\gamma} = E \cap P_{t,\gamma}$ (t varies, γ fixed)

Fubini $\Rightarrow$ $E_{t,\gamma}$ is m_{d-1}-measurable for a.e. t

$t \mapsto m_{d-1}(E_{t,\gamma})$ is a measurable fctn of t.

<u>Thm. 2</u> $E \subset \mathbb{R}^d$ ($d \geq 3$) of finite measure.

Then for a.e. $\gamma \in S^{d-1}$:

(i) $E_{t,\gamma}$ is m_{d-1}-measurable for <u>every</u> t.

(ii) $t \mapsto m_{d-1}(E_{t,\gamma})$ is a <u>cont.</u> fct. of t.

Moreover, this fctn is α-Hölder $\forall \alpha \in (0, \frac{1}{2})$.

<u>Cor.</u> $d \geq 3$, $E \subset \mathbb{R}^d$ of Lebesgue measure zero.

Then, for a.e. $\gamma \in S^{d-1}$, the slice $E_{t,\gamma}$ has zero measure for <u>every</u> $t \in \mathbb{R}$.

<u>Prop.</u> $d \geq 3$, $f \in (L^1 \cap L^2)(\mathbb{R}^d)$. Then for a.e. $\gamma \in S^{d-1}$:

<u>i)</u> f is meas. and int. on the plane for every $t \in \mathbb{R}$.

<u>ii)</u> $t \mapsto \mathcal{R}(f)(t,\gamma)$ is cont. and α-Hölder if $\alpha < \frac{1}{2}$.

Moreover, estimate $(*)$ from Thm. 1 holds for f.

<u>Rem.</u> Prop. implies Thm. 2 by taking char. fctn of E.

$\mathcal{R}(\chi_E)(t,\gamma) = m_{d-1}(E_{t,\gamma})$.

We skip the proof of Prop. (follows from Thm. 1 using some delicate measure theory.)

What about $d=2$?

Given $f \in L^1(\mathbb{R}^2)$, define

$$\mathcal{R}_\delta(f)(t,\gamma) = \frac{1}{2\delta} \int_{t-\delta}^{t+\delta} \mathcal{R}(f)(s,\gamma)\, ds. \quad \text{(averaged version of } \mathcal{R}.)$$

(integration over thickened Line (= hyperplane).)

$$= \frac{1}{2\delta} \int_{\{t-\delta \leq x\cdot\gamma \leq t+\delta\}} f(x)\, dx.$$

Thm. 3 $f \in C_0^\circ(\mathbb{R}^2)$, $0 < \delta \leq 1/2$.

$$\int_{S^1} R_\delta^*(f)(\gamma)\, d\sigma_\gamma \lesssim \left(\log \frac{1}{\delta}\right)^{1/2} \cdot \left(\|f\|_{L^1(\mathbb{R}^2)} + \|f\|_{L^2(\mathbb{R}^2)} \right)$$