

Geometric Aspects of Harmonic Analysis

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Q1

$$\left. \begin{array}{l} f : [a, b] \rightarrow \mathbb{R} \text{ integrable} \\ F(x) = \int_a^x f(t) dt \end{array} \right] \stackrel{?}{\Rightarrow} F \text{ diff. (a.e. } x), F' = f$$

Q2 Conditions of F (on $[a, b]$) s.t.

- $F'(x)$ exists a.e.
- F' integrable
- $\int_a^b F'(x) dx = F(b) - F(a)$

?

Q1 Differentiation of the integral

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t) dt = \frac{1}{|I|} \int_I f = \text{avg}_I f = {}_I f$$

$I = (x, x+h)$, $|I|$ Lebesgue measure of I .

Q1 equivalent to averaging problem: Given $f \in L^1(\mathbb{R}^d)$, is it true, that

$$\lim_{|B| \rightarrow 0, x \in B} \frac{1}{|B|} \int_B f = f(x) \quad (x\text{-a.e.})?$$

$B \subset \mathbb{R}^d$ open ball

Yes, if f continuous $\forall \varepsilon \exists \delta |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon. x \in B$

$$|f(x) - \fint_B f| = \left| \fint_B (f(y) - f(x)) dy \right| < \varepsilon \quad (1)$$

provided B is an open ball of radius $< \frac{\delta}{2}$ containing x

Yes, if f is integrable (not so easy). Hardy, Littlewood (1D, rearrangements; later Wiener for $d > 1$). $f \in L^1(\mathbb{R}^d)$

$$(Mf)(x) := \sup_{x \in B} \frac{1}{|B|} \int_B |f|$$

uncentered HL maximal function

Theorem. Let f be integrable on \mathbb{R}^d . Then

(i) Mf is measurable.

(ii) $(Mf)(x) < \infty$ a.e. x

(iii)

$$|\{x \in \mathbb{R}^d : (Mf)(x) > \alpha\}| < \frac{c}{\alpha} \|f\|_{L^1(\mathbb{R}^d)} \quad (\forall \alpha > 0). \quad (2)$$

$c = c_d = 3^d$, independent of f, α .

$f \neq 0 \in L^1 \Rightarrow Mf(x) \sim |x|^{-d}$ for large radius of x . So then $Mf \notin L^1$.

$$M : \begin{matrix} L^1 \rightarrow L^1 \\ L^1 \rightarrow L^{1,\infty} \end{matrix}$$

Proof. (i) easy $E_\alpha = \{x \in \mathbb{R}^d : (Mf)(x) > \alpha\}$ is open ($\forall \alpha > 0$) (because Mf is lower semicontinuous)

(ii) $|\{x \in \mathbb{R}^d : (Mf)(x) = \infty\}| \subset |\{x \in \mathbb{R}^d : Mf(x) > \alpha\}|$, take $\alpha \rightarrow \infty$.

(iii) follows from an elementary version of *Vitali covering*

□

Lemma. Let $B = \{B_1, B_2, \dots, B_N\}$ be a finite collection of open balls on \mathbb{R}^d . Then there exists a disjoint subcollection $B_{i_1}, B_{i_2}, \dots, B_{i_k}$ of B such that

$$|\bigcup_{j=1}^n B_j| \leq 3^d \sum_{j=1}^k |B_{i_j}|$$

Proof. (i) $B_{i_1} =$ largest ball

(ii) Delete B_{i_1} and its neighbors

(iii) $B_{i_2} =$ largest ball

(iv) repeat...

- Algorithm stops in at most N steps
- output has desired properties:
 - disjointness is clear
 - size $B \cap B' \neq \emptyset$, $r_{B'} \leq r_B$. $B^* =$ ball with the same center as B but 3 times the radius. $\Rightarrow B' \subset B^*$. $|B^*| = 3^d |B|$

□

Back to (iii): Choose $\alpha > 0$, $E_\alpha = \{x \in \mathbb{R}^d : (Mf)(x) > \alpha\}$. For each

$$x \in E_\alpha \exists B = B_x := \frac{1}{|B_x|} \int_{B_x} |f(y)| dy > \alpha$$

equivalent

$$|B_x| < \alpha^{-1} \int_{B_x} |f(y)| dy$$

Fix $K \ll E_\alpha$ compact subset covered by $\bigcup_{x \in K} B_x$, $K \subset \bigcup_{l=1}^N NB_l$

$$|K| \leq \left| \bigcup_{l=1}^N B_l \right| \stackrel{\text{Vitali}}{\leq} 3^d \sum_{j=1}^k |B_{i_j}| \leq \frac{3^d}{\alpha} \sum_{j=1}^k \int_{B_{i_j}} |f(y)| dy = \frac{3^d}{\alpha} \int_{\bigcup_{j=1}^k B_{i_j}} |f(y)| dy \leq \frac{3^d}{\alpha} \|f\|_{L^1(\mathbb{R}^d)}$$

Since K was chosen arbitrary (cpt.), it follows that

$$|E_\alpha| \leq \frac{3^d}{\alpha} \|f\|_{L^1}$$

Can interpolate between weak type L^1 -inequality and $L^\infty \rightarrow L^\infty$ (very easy).

Corollary (Lebesgue differentiation theorem). Let $f \in L^1(\mathbb{R}^d)$. Then

$$\lim_{|B| \rightarrow 0, x \in B} \oint_B f = f(x) \quad \text{x-a.e.} \quad (3)$$

Proof.

$$E_\alpha = \{x \in \mathbb{R}^d : \limsup_{|B| \rightarrow 0, x \in B} \oint_B f - f(x) > 2\alpha\}$$

ETS $|E_\alpha| = 0 \forall \alpha > 0$. Then $E = \bigcup_{n \in \mathbb{N}} E_{\frac{1}{n}} = 0$ and (3) holds on E^c .

Fix $\alpha > 0$, given $\varepsilon > 0$ choose $g \in C_0^\infty(\mathbb{R}^d)$ s.t. $\|f - g\|_{L^1} < \varepsilon$. Already seen

$$\lim_{|B| \rightarrow 0, x \in B} \oint_B g = g(x) \quad \forall x$$

$$\oint_B f - f(x) = \oint_B (f - g) + \oint_B g - g(x) + g(x) - f(x)$$

$$F_\alpha = \{x : M(f - g)(x) > \alpha\}$$

$$G_\alpha = \{x : |f(x) - g(x)| > \alpha\}$$

$E_\alpha \subset F_\alpha \cup G_\alpha$ since $u_1, u_2 > 0$, $u_1 + u_2 > 2\alpha \Rightarrow u_1 > \alpha \vee u_2 > \alpha$.

$$|G_\alpha| \leq \frac{1}{\alpha} \|f - g\|_{L^1} \quad (\text{Chebyshev})$$

$$|F_\alpha| \leq \frac{c_d}{\alpha} \|f - g\|_{L^1} \quad (\text{weak type})$$

$$|E_\alpha| \leq |F_\alpha| + |G_\alpha| \leq \left(\frac{c_d}{\alpha} + \frac{1}{\alpha}\right) \|f - g\|_{L^1} \leq \frac{c'_d \varepsilon}{\alpha}$$

Since $\varepsilon > 0$ was arbitrary $|E_\alpha| = 0$. □

$h \in L^1 \subset L^{1,\infty}$ by Chebyshev: $\infty > \|h\|_{L^1} = \int_{\mathbb{R}^d} |h(y)| dy \geq \int_{h(y) \geq \alpha} |h(y)| dy \geq \alpha |\{h > \alpha\}|$.

Would have been enough to replace $L^1(\mathbb{R}^d)$ by L^1_{loc} .

Sets $E \subset \mathbb{R}^d$ measurable, $x \in \mathbb{R}^d$ (not necc. in E) x is a point of Lebesgue density of E if

$$\lim_{|B| \rightarrow 0, x \in B} \frac{|B \cap E|}{|B|} = 1$$

Corollary. Let $E \subset \mathbb{R}^d$ be measurable. Then

- (i) Almost every $x \in E$ is a point of Lebesgue density of E .
- (ii) Almost every $x \notin E$ is not a point of Lebesgue density.

Functions $f \in L^1_{\text{loc}}(\mathbb{R}^d)$.

$$\text{Leb}(f) := \{x \in \mathbb{R}^d : f(x) < \infty \text{ and } \lim_{|B| \rightarrow 0, x \in B} \int_B |f(y) - f(x)| dy = 0\}$$

f continuous at $\bar{x} \Rightarrow \bar{x} \in \text{Leb}(f) \Rightarrow \int_B f \rightarrow \int_B f(\bar{x})$ (all the inverse implications are wrong)

Corollary. $f \in L^1_{\text{loc}}(\mathbb{R}^d) \Rightarrow$ Almost every point belongs to $\text{Leb}(f)$.

(By checking the proof again?)

These things also works with other sets that "shrink regularly to x than balls". It gets worse however when one takes all parallel rectangles and even worse when arbitrarily oriented rectangles are allowed.

Q.2 Key: bounded variation (BV) $F : [a, b] \rightarrow \mathbb{R}$, $P = \{a = t_0 < t_1 < \dots < t_N = b\}$

$$V_F^P = \sum_{j=0}^N |F(t_j) - F(t_{j+1})|$$

is the variation of f over P . F is of bounded variation if

$$T_F(a, b) = \sup_P V_F^P < \infty$$

$$P \subset \tilde{P} \text{ partitions} \Rightarrow V_F^P \leq V_F^{\tilde{P}}$$

Example. (i) f monotonic (increasing) and bounded, $|f| \leq M \Rightarrow F \in \text{BV}$

$$V_F^P = \sum_{j=1}^N |F(t_j) - F(t_{j-1})| = F(b) - F(a) \leq 2M$$

(ii) F differentiable with F' bounded, $|F'| \leq M$, then by mean value theorem $f \in \text{BV}$. Or F Lipschitz

(iii) F α -Hölder ($\alpha < 1$) $\Rightarrow F \in \text{BV}$. Take $F : [0, 1] \rightarrow \mathbb{R}$, $x \mapsto d(x, C)^\alpha$, where C is the cantor set. 2^{n-1} intervals of length 3^{-n}

$$\alpha > \frac{\log 2}{\log 3} \Rightarrow \sum_{n=1}^{\infty} 2^{n-1} (3^{-n})^\alpha < \infty$$

- *Total variation* of F on $[a, x]$ (where $a \leq x \leq b$) is

$$T_F(a, x) = \sup \sum_{j=0}^N |F(t_j) - F(t_{j-1})|$$

- *Positive variation* of F on $[a, 1]$ is

$$P_F(a, x) = \sup_{(+)} \sum (F(t_j) - F(t_{j-1})) \quad \text{all } j : F(t_j) \geq F(t_{j-1})$$

- *Negative variation* of F on $[a, 1]$ is

$$N_F(a, x) = \sup_{(-)} \sum -(F(t_j) - F(t_{j-1}))$$

Lemma. $f : [a, b] \rightarrow \mathbb{R}$. Then

$$(i) \quad F(x) = F(a) + P_F(a, x) - N_F(a, x)$$

$$(ii) \quad T_F(a, x) = P_F(a, x) + N_F(a, x)$$

($\forall x \in [a, b]$)

recall from measure theory: $f = f^+ - f^-$, $|f| = f^+ + f^-$

Proof. (i) given $\varepsilon > 0$, $\exists P = \{a = t_0 < t_1 < \dots < t_N = x\}$

$$|PVF - \sum_{(+)} (F(t_j) - F(t_{j-1}))| < \varepsilon$$

$$|N_F - \sum_{(-)} -(F(t_j) - F(t_{j-1}))| < \varepsilon$$

Also

$$F(x) - F(a) = \sum_{(+)} F(t_j) - F(t_{j-1}) - \sum_{(-)} -(F(t_j) - F(t_{j-1}))$$

□

Corollary. $F : [a, b] \rightarrow \mathbb{R} \in \text{BV}$ iff F is the difference of two increasing bounded functions

Theorem. $F : [a, b] \rightarrow \mathbb{R} \in \text{BV} \Rightarrow F$ differentiable a.e.

Wlog f monotonically increasing, "Wlog" f continuous

Lemma (of the rising sun). $G : \mathbb{R} \rightarrow \mathbb{R}$ continuous.

$$E = \{x \in \mathbb{R} : \exists h = h_x > 0 \ G(x+h) > G(x)\}$$

Then

$$(i) \quad E \text{ is open } (E = \bigcup_{n=1}^{\infty} (a_n, b_n))$$

$$(ii) \quad g(a_n) = G(b_n), \text{ provided } b_n - a_n < \infty.$$

Proof. Let (a_n, b_n) be a finite interval in the decomposition. $a_k \notin E$ then $g(a_k) \geq G(b_k)$. Assume $G(a_k) > G(b_k)$. $\exists c \in (a_k, b_k)$ $g(c) = \frac{g(a_k) + g(b_k)}{2}$. Choose rightmost such c . $\exists d \in (c, b_k)$ $G(d) > G(c)$. But then by continuity c could not have been chosen rightmost, contradiction. \square

Can replace \mathbb{R} by $[a, b]$, but then only get for $a_0 = a$ that $G(a_0) \leq G(b_0)$

Proof. of theorem

$$\begin{aligned}\Delta_h(F)(x) &= \frac{F(x+h) - F(x)}{h} \\ D^\pm(F)(x) &= \limsup_{h \rightarrow 0, h > < 0} \Delta_h(F)(x) \\ D_\pm(F)(x) &= \liminf_{h \rightarrow 0, h > < 0} \Delta_h(F)(x)\end{aligned}$$

Dini numbers. Upshot: They are all the same and finite. $D_- \leq D^-$, $D_+ \leq D^+$ clear. ETS

(i) $D^+(F)(x) < \infty$ (a.e. x)

(ii) $D^+(F)(x) \leq D_-(F)(x)$ (a.e. x)

(ii) is equivalent to $D^-(F)(x) \leq D_+(F)(x)$ by replacing $F(x)$ by $-F(-x)$ somewhere. Then $D^+ \leq D_- \leq D^- \leq D_+ \leq D^+ < \infty$.

(i) relacc: F increasing, bounded, continuous on $[a, b]$. Fix $\gamma > 0$,

$$E_\gamma := \{x : D^+(F)(x) > \gamma\}$$

- E_γ is measurable
- Apply rising sun to $G(x) = F(x) - \gamma x$

$$E_\gamma \subset E = \{x \in [a, b] : \exists h > 0 \ G(x+h) > G(x)\} = \bigcup_{k=1}^{\infty} (a_k, b_k)$$

The condition in the set is equivalent to

$$\begin{aligned}\Leftrightarrow \exists h > 0 \ F(x+h) - \gamma x - \gamma h > F(x) - \gamma x \\ \Leftrightarrow \exists h > 0 \ \frac{F(x+h) - F(x)}{h} > \gamma \\ \Leftrightarrow D^+(F)(x) > \gamma\end{aligned}$$

$$G(a_k) \leq G(b_k) \Leftrightarrow F(a_k) - \gamma a_k \leq F(b_k) - \gamma b_k \Leftrightarrow \gamma(b_k - a_k) \leq F(b_k) - F(a_k).$$

Therefore

$$|E_\gamma| \leq |E| \leq \sum_{k=1}^{\infty} (b_k - a_k) \leq \frac{1}{\gamma} \sum_{k=1}^{\infty} (F(b_k) - F(a_k)) = \frac{1}{\gamma} (F(b) - F(a))$$

Take $\gamma \rightarrow \infty$, done.

(ii) see Stein-Shakarchi (vol 3)

\square

Corollary. F increasing, continuous $\Rightarrow F'$ exists a.e., measurable, nonnegative and

$$\int_a^b F'(x) dx \leq F(b) - F(a).$$

Proof. Let

$$G_h(x) = \frac{F(x + \frac{1}{n}) - F(x)}{\frac{1}{h}}$$

By the theorem, $G_h(x) \rightarrow F'(x)$ ($h \rightarrow 0$) pointwise a.e. By Fatou

$$\int_{[a,b]} F' \leq \liminf_{n \rightarrow \infty} \int_a^b G_h(x) dx = \liminf_{n \rightarrow \infty} \int_b^{b+\frac{1}{n}} F(x) dx - \int_a^{a+\frac{1}{n}} F(x) dx$$

□

Cannot do better than \leq : For the Devil's staircase the left hand side is 0 while the right hand side is 1.

Why is the sunrise Lemma a covering Lemma?

$$\left. \begin{aligned} f_+^* &= \sup \frac{1}{h} \int_x^{x+h} |f(y)| dy \\ E_\alpha^+ &= \{x \in \mathbb{R} : f_+^*(x) > \alpha\} \end{aligned} \right\} |E_\alpha^+| = \frac{1}{\alpha} \int_{E_\alpha^+} |f|$$

Why? Let

$$G(x) = \int_0^x |f(y)| dy - \alpha x$$

$$x \in E_\alpha^+ \iff f_+^*(x) > \alpha \iff \exists h > 0 \frac{1}{h} \int_x^{x+h} |f(y)| dy > \alpha \iff \exists h > 0 G(x+h) > G(x)$$

$$\{x \in \mathbb{R} : \exists h > 0 G(x+h) > G(x)\} = \bigcup_{k \in \mathbb{N}} (a_k, b_k), \quad G(a_k) = G(b_k)$$

$$|E_\alpha^+| = \sum_k (b_k - a_k) = \frac{1}{\alpha} \sum_k \int_{(a_k, b_k)} |f| = \frac{1}{\alpha} \int_{\bigcup_k (a_k, b_k)} |f| = \frac{1}{\alpha} \int_{|E_\alpha^+|} |f|.$$

Definition. $F : [a, b] \rightarrow \mathbb{R}$ is *absolutely continuous* if

$$\forall \varepsilon > 0 \exists \delta > 0 : \sum_{k=1}^N (B_k - a_k) < \delta \implies \sum_{k=1}^N |F(b_k) - F(a_k)| < \varepsilon$$

intervals (a_k, b_k) disjoint ($k = 1, \dots, N$) \Rightarrow

$$\sum_{k=1}^N |F(b_k) - F(a_k)| < \varepsilon$$

Remark. (i) On a bounded interval $I \subset \mathbb{R}$

$$C^1(I) \subset \text{Lip}(I) \subset AC(I) \subset BV(I)$$

So they are diff. a.e.. All the inclusions are strict.

(ii) abs cont \Rightarrow unif. con. \Rightarrow cont.

(iii) $f \in L^1_{\text{loc}}(\mathbb{R})$ $F(x) = \int_0^x f(t) dt$ Then F is absolutely continuous. $(\forall \varepsilon \exists \delta |E| < \delta \Rightarrow \int_E |f| < \varepsilon)$

Upshot: AC functions are the ones which re diff a.e. and vrfy FTC.

Theorem. $F \in AC(a, b) \Rightarrow F'$ exists a.e., $F' = 0$ a.e. $\Rightarrow F$ constant

- Existence of F' clear \checkmark
- $F' = 0$ a.e. $\Rightarrow F$ constant: refinement of Vitali

Definition. A collection $\mathcal{B} = \{B\}$ of (open) balls on \mathbb{R}^d . is a *Vitali covering* of a set E if

$$\forall x \in E \forall \eta > 0 \exists B \in \mathcal{B} : x \in B, |B| < \eta$$

Lemma. $E \subset \mathbb{R}^d$ meas. $|E| < \infty$, \mathcal{B} Vitali covering of E , $\delta > 0$. Then there exist finitely many disjoint balls $B_1, \dots, B_N \in \mathcal{B}$

$$\sum_{j=1}^N |B_j| \geq |E| - \delta$$

Recall elementary Vitali: $\mathcal{B} = \{B_1, \dots, B_N\}$ finite collection of pen balls in $\mathbb{R}^d \Rightarrow \exists$ disjoint subcollection B_{i_1}, \dots, B_{i_k} with

$$|\bigcup_{j=1}^B B_j| \leq 3^d \sum_{j=1}^k |B_{i_j}|$$

Proof of Lemma. wlog $\delta > |E|$. Vitali $\Rightarrow \exists$ disjoint subcollection $B_1, \dots, B_N \in \mathcal{B}$

$$\sum_{i=1}^{N_1} |B_i| \geq 3^{-d} \delta$$

Sequence of balls B_1, \dots, B_N . question: Is $\sum_{j=1}^{N_1} |B_j| \geq |E| - \delta$? Yes: done with $N = N_1$. No: work harder.

$$\sum_{j=1}^{N_1} |B_j| < |E| - \delta$$

$$E_2 = E \setminus \bigcup_{j=1}^{N_1} B_j$$

$$|E_2| \geq |E| - \sum_{j=1}^{N_1} |B_j| > |E| - (|E| - \delta) = \delta$$

\mathcal{B} Vitali covering \Rightarrow balls in \mathcal{B} disjoint from $\bigcup_{i=1}^{N_1} B_i$ still covers E_2 . Vitali $\Rightarrow \exists$ finite disjoint subcollection of these balls $B_{N_1+1}, \dots, B_{N_2}$

$$\sum_{N_1 < j < N_2} |B_j| \geq 3^{-d} \delta.$$

After k steps, $B_1, \dots, B_{N_1}, \dots, B_{N_k}$ with

$$\sum_{j=1}^{N_k} |B_j| \geq k 3^{-d} \delta \geq |E| - \delta$$

iff $k \geq 3^d \frac{|E| - \delta}{\delta}$, stop. □

need to approximate with compact from inside somewhere and with open from outside somewhere else

Corollary. The balls can be arranged in such a way that

$$|E \cup \bigcup_{i=1}^N B_i| < 2\delta$$

Proof. Choose open $O \supset E$: $|O \setminus E| < \delta$. \mathcal{B} Vitali covering \Rightarrow wlog all balls in \mathcal{B} are contained in O .

$$(E \cup \bigcup_{i=1}^N B_i) \cup \bigcup_{i=1}^N NB_i \subset O. : |E \cup \bigcup_{i=1}^N B_i| \leq |O| - \sum_{i=1}^N |B_i| \leq |E| + \delta - (|E| - \delta) = 2\delta$$

□

$F \in AC$ Back to the real line: Goal: $F' = 0$ a.e. $\Rightarrow F$ constant. ETS $F(a) = F(b)$

$$E = \{x \in (a, b) : F'(x) \text{ exists and } = 0\} \quad |E| = b - a$$

Fix $\varepsilon > 0$. For $x \in E$, $\lim_{h \rightarrow 0} \left| \frac{F(x+h) - F(x)}{h} \right| = 0$. $\forall \eta > 0 \exists$ open interval $I = (a_x, b_x) \subset [a, b]$ containing x . $|F(b_x) - F(a_x)| \leq \varepsilon(b_x - a_x)$ and $b_x - a_x < \eta$. The collection of these intervals (over all $\eta > 0$) forms a Vitali covering of E . Lemma \Rightarrow Given $\delta > 0$ can select finitely many, disjoint $I_j = (a_j, b_j)_{j=1}^N$ such that

$$\sum_{j=1}^N |I_j| \geq |E| - \delta = b - a - \delta$$

But

$$\sum_{j=1}^N |F(b_j) - F(a_j)| \leq \varepsilon \sum_{j=1}^N (b_j - a_j) \leq \varepsilon(b - a)$$

$$[a, b] \supset \bigcup_{j=1}^N I_j = \bigcup_{k=1}^M [\alpha_k, \beta_k]$$

with total length $\leq \delta$. $\therefore F \in AC$

$$\sum_{k=1}^M |F(\beta_k) - F(\alpha_k)| < \varepsilon$$

$$|F(b) - F(a)| \leq \sum |F(b_j) - F(a_j)| + \sum |F(\beta_k) - F(\alpha_k)| \leq \varepsilon(b - a) + \varepsilon,$$

done. □

Theorem. $F \in AC(a, b)$. Then

(i) F' exists a.e. and is integrable

(ii)

$$F(x) - F(a) = \int_a^x F'(t) dt \quad (\forall a \leq x \leq b)$$

Conversely, if $f \in L^1(a, b)$, then there exists $F \in AC(a, b) : F' = f$ a.e..

Proof. \Rightarrow

(i) seen last lecture.

(ii)

$$G(x) := \int_a^x F'(t) dt$$

$\therefore G \in AC \therefore F - G \in AC$. Lebesgue diff. $\Rightarrow G'(x) = F'(x)$ (a.e. x) $\therefore (F - G)' = 0$ a.e..
Therefore $(F - G)(x) = (F - G)(a)$, $F(x) - G(x) = F(a)$, equivalent to (*)

\Leftarrow

$$F(x) = \int_a^x f(t) dt$$

AC \checkmark Leb. diff $\Rightarrow F' = f$ a.e. \square

Next: Monotone functions which are not nec. continuous. Wlog. F increasing, bounded on $[a, b]$.

$$F(x^-)F \lim_{y \rightarrow x, y < x} F(y) \quad F(x^+) := \dots$$

$F(x^-) \leq F(x) \leq F(x^+)$, F cont. at x if $F(x^-) = F(x^+)$. Otherwise F has a jump discontinuity at x .

Obs: A (bounded) increasing function F on $[a, b]$ has at most countable many jumps. There exists an injective map $\text{Disc}(F) \rightarrow \mathbb{Q}$

$\therefore \text{Disc}(F) = \{x_n\}_{n=1}^\infty$ $\alpha_n = F(x_n^+) - F(x_n^-) = \text{jump of } F \text{ at } x_n$. $F(x_n^+) = F(x_n^-) + \alpha_n$
 $F(x_n) = F(x_n^-) + \theta_n \alpha_n$, $\theta_n \in [0, 1]$. $F(x) = \mu((-\infty, x])$. Corresponds to singular + abs. cont measures.

$$j_n(x) = \begin{cases} 0 & x < x_n \\ \phi_n & x = x_n \\ 1 & x > x_n \end{cases}$$

Jump function associated to F is

$$J_F(x) = \sum_{n=1}^\infty j_n(x)$$

$$\sum_{n=1}^\infty \alpha_n = \sum_{n=1}^\infty F(x_n^+) - F(x_n^-) \leq F(b) - F(a) < \infty$$

because F incr, and F bounded.

Lemma. F increasing, bounded on $[a, b]$, $\text{Disc}(F) = \{x_n\}_{n=1}^\infty$

(i) $J_F(x)$ is discontinuous precisely at $\{x_n\}_{n=1}^\infty$, has a jump at x_n equal to that of F .

(ii) The function $F - J_F$ is increasing and continuous.

Proof. (i) $x \neq x_n (\forall n) \Rightarrow$ each j_n is continuous at $x \Rightarrow J_F$ is continuous at x because of uniform convergence. $x = x_N (\exists N) \Rightarrow J_F = \sum_{n=1}^N \alpha_n j_n(x) + \sum_{n>N} \alpha_n j_n(x)$. First sum has jump discontinuity x_N of size α_N

(ii) $F - J_F$ is continuous

$$F(x) - J_F(x) \leq F(y) - J_F(y) \iff J_F(y) - J_F(x) \leq F(y) - F(x)$$

, where

$$J_F(y) = \sum_{x < x_n \leq y} \alpha_n = \sum_{x < x_n \leq y} F(x_n^+) - F(x_n^-) \leq F(y) - F(x)$$

□

Since $F = (F - J_F) + J_F$ ETS J_F is diff a.e.. This was essential step of

$$\mu = \mu_{AC} + \mu_S + \mu_{PP}$$

$z(t) = (x(t), y(t))$. curve γ . $x, y : [a, b] \rightarrow \mathbb{R}$ continuous.

γ rectifiable if length

$$L(\gamma) = \sup \sum_{j=1}^N |z(t_j) - z(t_{j-1})| < \infty$$

sup over all partitions $P = \{a = t_0 < t_1 < \dots < t_N = b\}$ of $[a, b]$

When is

$$L(\gamma) = \int_a^b |z'(t)| dt?$$

Lemma. γ is rectifiable iff x, y are of bounded variation (and cont.).

see $F = x + iy$

Assume γ rectifiable, let $L(A, B)$ length of $\gamma(A, B)$, ($a \leq A \leq B \leq b$)

(i) $L(A, B) = T_F(A, B)$ (where $F(t) = z(t)$)

(ii) $L(A, C) + L(C, B) = L(A, B)$ ($A \leq C \leq B$)

(iii) $A \mapsto L(A, B)$ (fix B) is continuous

$B \mapsto L(A, B)$ (fix A)

seen: $F \in \text{BV}(a, b)$, cont. $\Rightarrow T_F$ cont.

Warning: $[0, 1] \ni t \mapsto (F(t), F(t))$, F Cantor. F cont. incr. $F(0) = 0$, $F(1) = 1$, $F' = 0$ a.e.

Theorem. $z : [a, b] \rightarrow \mathbb{R}^2, t \mapsto (x(t), y(t)) \sim \text{curve } \gamma$. $x, y \in \text{AC}(a, b)$. $\Rightarrow \gamma$ rectifiable and

$$L(\gamma) = \int_a^b |z'(t)| dt$$

why? $F : [a, b] \rightarrow \mathbb{C}$ is $\text{AC}(a, b)$

$$\Rightarrow T_F(a, b) = \int_a^b |F'(t)| dt.$$

$$\sum_{j=1}^N |F(t_j) - F(t_{j-1})| = \sum_{j=1}^N \left| \int_{t_{j-1}}^{t_j} F'(t) dt \right| \leq \sum_{j=1}^N \int_{t_{j-1}}^{t_j} |F'(t)| dt = \int_a^b |F'(t)| dt$$

First inequality by FTC. For \geq , write $F' = g + h$, g step function, h small in L^1 $G, H = \text{def.}$ integrals of g, h . Check $T_F \geq T_G, T_H$, T_H small, $T_G \geq \int_a^b |g(t)| dt$.

Minkowski content of a curve simple, simple closed, quasi-simple curves.

trace of γ : $\Gamma = \{z(t) \in \mathbb{R}^2 : t \in [a, b]\}$.

Given $K \in \mathbb{R}^2$ and $\delta > 0$ define

$$K^\delta = \{x \in \mathbb{R}^2 : d(x, K) < \delta\}$$

where $d(x, K) = \inf_{k \in K} d(x, k)$

Definition. The set K has (1D) Minkowski content if

$$\lim_{\delta \rightarrow 0} \frac{|K^\delta|}{2\delta}$$

exists (in \mathbb{R}), denoted $M(K)$.

Theorem. Let $\Gamma = \{z(t) : a \leq t \leq b\}$ be (the trace of) a quasi-simple curve γ . Then Γ has Minkowski content iff γ is rectifiable (in which case $M(\Gamma) = L(\gamma)$).

Upper Mink: content.

$$\limsup_{\delta \rightarrow 0^+} \frac{|K^\delta|}{2\delta} =: M^*(K)$$

lower

$$\liminf_{\delta \rightarrow 0^+} \frac{|K^\delta|}{2\delta} =: M_*(K).$$

Proposition. $\Gamma = \{z(t) : a \leq t \leq b\}$ quasi simple. If $M_*(\Gamma) < \infty$, then γ is rectifiable and $L(\gamma) \leq M_*(\Gamma)$.

Proposition. $\Gamma = \{z(t) : a \leq t \leq b\}$ rectifiable γ . Then $M^*(\Gamma) \leq L(\gamma)$.

Proof of Prop. 1, for simple curves. Obs: $\Gamma = \{z(t) : a \leq t \leq b\}$ any curve. $\Delta = |z(b) - z(a)|$. $|\Gamma^\delta| \geq 2\delta\Delta$.

Take any partition P of $[a, b]$. $L_P = \sum_{j=1}^N |z(t_j) - z(t_{j-1})|$. Given $\varepsilon > 0$, $\exists N$ proper closed subintervals $I_j = [a_j, b_j] \subset (t_{j-1}, t_j)$:

$$\sum_{j=1}^N |z(b_j) - z(a_j)| \leq L_P - \varepsilon$$

I_1, \dots, I_N disjoint $\Rightarrow \Gamma_1, \dots, \Gamma_N$ disjoint because Γ is simple. $\Leftrightarrow \Gamma_1^\delta, \dots, \Gamma_N^\delta$ disjoint, provided $\delta > 0$ small enough.

$$\bigcup_{j=1}^N \Gamma_j^\delta \subset \Gamma^\delta$$

$$|\Gamma^\delta| \geq \sum_{j=1}^N |\Gamma_j^\delta| \geq 2\delta \sum_{j=1}^N |z(t_j) - z(t_{j-1})| \geq 2\delta(L_P - \varepsilon)$$

□

Isoperimetric inequality (soft) $\gamma : [a, b] \rightarrow \mathbb{R}^2$, $\gamma \in C^1(a, b) : \gamma'(s) \neq 0 \forall s$, $\gamma(a) = \gamma(b)$.
 Arclength parametrization: $\gamma : [0, L] \rightarrow \mathbb{R}^2$, $|\gamma'(s)| = 1 \forall s$.

Theorem. $\Gamma \subset \mathbb{R}^2$ simple closed C^2 curve of length L . Area of the region enclosed by Γ .

$$A = \frac{1}{2} \left| \int_{\Gamma} (x dy - y dx) \right| = \frac{1}{2} \left| \int_0^L (x(s)y'(s) - x'(s)y(s)) ds \right|$$

Then $4\pi A \leq L^2$. Equality iff Γ is a circle.

Proof. wlog (rescale) $L = 2\pi$: WTS $A \leq \pi$, equality iff Γ is a circle of radius 1.

$\gamma : [0, 2\pi] \rightarrow \mathbb{R}^2$, $s \mapsto \gamma(s) = (x(s), y(s))$ arclength par. $x'(s)^2 + y'(s)^2 = 1 \forall s$:

$$\frac{1}{2\pi} \int_0^{2\pi} (x'(s)^2 + y'(s)^2) ds = 1$$

Γ closed $\Rightarrow x(s), y(s)$ 2π -periodic.

$$x'(s) = \sum_n a_n i n e^{i n s}$$

$$y'(s) = \sum_n b_n i n e^{i n s}$$

Parseval \Rightarrow

$$\sum_{n \in \mathbb{Z}} n^2 (|a_n|^2 + |b_n|^2) = 1$$

$$A = \frac{1}{2} \int_0^{2\pi} (x(s)y'(s) - x'(s)y(s)) ds = \pi \left| \sum_{n \in \mathbb{Z}} n (a_n \bar{b}_n - b_n \bar{a}_n) \right|$$

by bilinear Parseval

$$|a_n \bar{b}_n - b_n \bar{a}_n| \leq 2|a_n||b_n| \leq |a_n|^2 + |b_n|^2$$

$$A \leq \pi \sum_{n \in \mathbb{Z}} n^2 (|a_n|^2 + |b_n|^2) \leq \pi$$

□

Cases of equality: $A = \pi \Rightarrow$

$$x(s) = a_{-1}e^{-is} + a_0 + a_1e^{is}$$

$$y(s) = b_{-1}e^{-is} + b_0 + b_1e^{is}$$

x, y real-valued $\Rightarrow a_1 = \bar{a}_{-1}$, $b_1 = \bar{b}_{-1}$. (**) $\Rightarrow 2(|a_1|^2 + |b_1|^2) = 1$.

(***) \Rightarrow

$$|a_1| = |b_1| = \frac{1}{2} : a_1 = \frac{1}{2}e^{i\alpha} \quad b_1 = \frac{1}{2}e^{i\beta} \quad (\alpha, \beta \in \mathbb{R})$$

$$1 = 2|a_1 \bar{b}_1 - \bar{a}_1 b_1| = \sin(\alpha - \beta) : \alpha - \beta = \frac{k\pi}{2} \quad (\text{odd } k)$$

$$x(s) = a_0 + \cos(s + \alpha)$$

$$y(s) = b_0 \pm \sin(s + \alpha)$$

\pm dep. on parity of $\frac{k-1}{2}$.

Isoperimetric inequality (hard) $\Omega \subset \mathbb{R}^2$ bounded, open, $\partial\Omega = \bar{\Omega} - \Omega =: \Gamma$ rectifiable curve (not nec. simple) with length $l(\Gamma)$.

Theorem.

$$4\pi|\Omega| \leq l(\Omega)^2$$

Proof. inner:

$$\Omega_-^\delta = \{x \in \mathbb{R}^2 : d(x, \mathbb{R}^2 \setminus \Omega) \geq \delta\}$$

outer:

$$\Omega_+^\delta = \{x \in \mathbb{R}^2 : d(x, \bar{\Omega}) < \delta\}$$

$$\Gamma^\delta = \{x : d(x, \Gamma) < \delta\}$$

$$\Omega_+^\delta = \Omega_-^\delta \dot{\cup} \Gamma^\delta > \delta$$

$A, B \subset \mathbb{R}^d$, $A + B = \{a + b : a \in A, b \in B\}$ Note: $\Omega + B_\delta \subset \Omega_+^\delta$, $\Omega_-^\delta + B_\delta \subset \Omega$.

Brunn-Minkowski: $A, B \subset \mathbb{R}^2$ meas., $A + B$ meas.

$$|A + B|^{\frac{1}{2}} \geq |A|^{\frac{1}{2}} + |B|^{\frac{1}{2}}$$

$$|\Omega_-^\delta| \geq (|\Omega|^{\frac{1}{2}} + |B_\delta|^{\frac{1}{2}})^2 \geq |\Omega| + 2|\Omega|^{\frac{1}{2}} \underbrace{|B_\delta|^{\frac{1}{2}}}_{=(\pi\delta^2)^{\frac{1}{2}}}$$

$$|\Omega| \geq (|\Omega_-^\delta|^{\frac{1}{2}} + |B_\delta|^{\frac{1}{2}})^2 \geq |\Omega_-^\delta| + 2|\Omega_-^\delta|^{\frac{1}{2}}|B_\delta|^{\frac{1}{2}}$$

$$|\Gamma^\delta| \geq |\Omega| + 2|\Omega|^{\frac{1}{2}}\sqrt{\pi} - |\Omega_-^\delta| + |\Omega_-^\delta|^{\frac{1}{2}}\sqrt{\pi}$$

$$\limsup_{\delta \rightarrow 0^+} \frac{|\Gamma^\delta|}{2\delta} \geq 2|\Omega|^{\frac{1}{2}}\sqrt{\pi}$$

$$4\pi|\Omega| \leq M^*(\Gamma)^2 \leq l(\Gamma)^2$$

Note, that only in the very last inequality did we use the rectifiability of Γ . □

Brunn-Minkowski ineq. (\mathbb{R}^d) $A, B \subset \mathbb{R}^d$ measurable. $A + B = \{a + b : a \in A, b \in B\}$. $\lambda A = \{\lambda a : a \in A\}$ ($\lambda > 0$).

Q.: Can $|A + B|$ be controlled in terms of $|A|, |B|$? No! There exist sets A, B $|A| = |B| = 0$ with $|A + B| > 0$. Example $[0, 1] \times [0, 1]$. Another example $A = B = C \subset [0, 1]$ Cantor set. Then $A + B = [0, 2]$.

Q.: Can $|A + B|^\alpha \geq c_\alpha(|A|^\alpha + |B|^\alpha)$ hold? (for some $\alpha > 0$ with $c_\alpha < \infty$, indep of A, B) Best possible $c_\alpha = 1$.

What about α ? Convex sets play a role. A = convex, $B = \lambda A$. $|B| = |\lambda A| = \lambda^d |A|$. $|A + B| = |A + \lambda A| = |(1 + \lambda)A| = (1 + \lambda)^d |A|$ because A is convex.

($\lambda_1 A + \lambda_2 A = (\lambda_1 + \lambda_2)A$ iff A is convex.)

$|A + B|^\alpha \geq |A|^\alpha + |B|^\alpha$ iff $(1 + \lambda)^{d\alpha} \geq 1 + \lambda^{d\alpha} \Rightarrow \alpha \geq \frac{1}{d}$.

$(a + b)^\gamma \geq a^\gamma + b^\gamma \forall a, b \geq 0, \gamma \geq 1$.

Candidate inequality:

$$|A + B|^{\frac{1}{d}} \geq |A|^{\frac{1}{d}} + |B|^{\frac{1}{d}}$$

(BM)

A, B measurable $\Rightarrow A + B$ measurable. Take $[0, 1] \times \text{nonmeasurable}$.

(i) A, B closed $\Rightarrow A + B$ measurable

(ii) A, B compact $\Rightarrow A + B$ compact

(iii) A, B open $\Rightarrow A + B$ open

Theorem. (BM) holds if $A, B, A + B$ measurable.

(i) A, B rectangles with sidelengths $\{a_j\}_{j=1}^\infty, \{b_j\}_{j=1}^\infty$

(ii) A, B unions of finitely many rectangles with disjoint interiors.

(iii) A, B open sets of finite measure

(iv) A, B compact

(v) $A, B, A + B$ measurable.

Proof. (i) (BM) becomes

$$\prod_{j=1}^d (a_j + b_j)^{\frac{1}{d}} \geq \prod_{j=1}^d a_j^{\frac{1}{d}} + \prod_{j=1}^d b_j^{\frac{1}{d}}$$

$a_j \rightarrow \lambda_1 a_j, b_j \rightarrow \lambda_1 b_j$. Both sides are multiplied by $(\lambda_1 \lambda_2 \dots \lambda_d)^{\frac{1}{d}}$: wlog can assume $a_j + b_j = 1 \forall j$ (Choose $\lambda_j = a_j + b_j$)

AMGM:

$$\prod_{j=1}^d a_j^{\frac{1}{d}} \leq \frac{1}{d} \sum_{j=1}^d a_j$$

$$\prod_{j=1}^d b_j^{\frac{1}{d}} \leq \frac{1}{d} \sum_{j=1}^d b_j$$

$$\prod_{j=1}^d a_j^{\frac{1}{d}} + \prod_{j=1}^d b_j^{\frac{1}{d}} \leq \frac{1}{d} \sum_{j=1}^d (a_j + b_j) = 1$$

(ii) Induction on n = number of rectangles in A and B . Choose pair of disjoint rectangles R_1, R_2 in A . Can rotate s.t. R_1 and R_2 are separated by hyperplane $\{x_j = 0\}$. R_1 lies in $A_+ = A \cap \{x_j \geq 0\}$, $A_- = A \cap \{x_j \leq 0\}$.

Rem.: Both A_+, A_- contain at least one less rectangle than A , $A = A_+ \cup A_-$ and $A_B \cap A_-$ has measure zero.

Now: translate B s.t. B_- and B_+ satisfy

$$\frac{|B_\pm|}{|B|} = \frac{|A_\pm|}{|A|}$$

$(A_+ + B_+) \cup (A_- + B_-) \subset A + B$ Number of rectangles in A_+ and B_+ , number of rectangles in A_- and B_- is $< n$.

$$\begin{aligned} |A + B| &\geq |A_+ + B_+| + |A_- + B_-| \geq (|A_+|^{\frac{1}{d}} + |B_+|^{\frac{1}{d}})^d + (|A_-|^{\frac{1}{d}} + |B_-|^{\frac{1}{d}})^d \\ &= (|A_+|(1 + (\frac{|B_+|}{|A_+|})^{\frac{1}{d}}))^d + |A_-|(1 + (\frac{|B_-|}{|A_-|})^{\frac{1}{d}})^d = (|A_+| + |A_-|)(1 + (\frac{|B|}{|A|})^{\frac{1}{d}})^d \\ &= (|A|^{\frac{1}{d}} + |B|^{\frac{1}{d}})^d. \end{aligned}$$

(iii) Open sets of finite measure A, B . $\forall \varepsilon > 0 \exists A_\varepsilon, B_\varepsilon$ finet unions of parallel rectangles with disjoint interiors. $A_\varepsilon \subset A, B_\varepsilon \subset B, |A| \leq |A_\varepsilon| + \varepsilon, |B| \leq |B_\varepsilon| + \varepsilon$.

$$|A + B| \geq |A_\varepsilon + B_\varepsilon| \geq (|A_\varepsilon|^{\frac{1}{d}} + |B_\varepsilon|^{\frac{1}{d}})^d \geq ((|A| - \varepsilon)^{\frac{1}{d}} + (|B| - \varepsilon)^{\frac{1}{d}})^d. \text{ Let } \varepsilon \rightarrow 0^+, \text{ done.}$$

(iv) A, B compact. Let $A^\varepsilon = \{x : d(x, A) < \varepsilon\}$. $A + B \subset A^\varepsilon + B^\varepsilon \subset (A + B)^{2\varepsilon}$

(v) $A, B, A + B$ measurable: use inner regularity of Lebesgue measure.

□

Remark. A, B open sets of finite positive measure. Equality in (BM) iff A, B convex and similar. $\exists \delta > 0 \exists h \in \mathbb{R}^d : A = \delta B + h$ (A convex iff $\lambda_1 A + \lambda_2 A = (\lambda_1 + \lambda_2)A$)

Consequences for isoperimetric inequality $A \subset \mathbb{R}^d$ bounded open with smooth boundary. $(\partial A, B \subset \mathbb{R}^d \text{ ball } |B| = |A|)$

$$|\partial A| = \lim_{\varepsilon \rightarrow 0^+} \frac{|A + \varepsilon B| - |A|}{\varepsilon}$$

Isoper ineq.: $|\partial A| \geq |\partial B|$.

Proof.

$$\frac{|A + \varepsilon B| - |A|}{\varepsilon} \geq \frac{(|A|^{\frac{1}{d}} + |\varepsilon B|^{\frac{1}{d}})^d - |A|}{\varepsilon} = \frac{(1 + \varepsilon)^d - 1}{\varepsilon} |B| \rightarrow d|B| = |\partial B|$$

for $\varepsilon \rightarrow 0$.

□

Better: $A \subset \mathbb{R}^d$ has finite perimeter ($\Leftrightarrow 1_A \in \text{BV}(U), U \subset \mathbb{R}^d$ bdd open)

$$\frac{\mathcal{H}^{d-1}(\partial A)}{|A|^{\frac{d-1}{d}}} \geq \frac{\mathcal{H}^{d-1}(S^{d-1})}{|B^d(0, 1)|^{\frac{d-1}{d}}}$$

Hausdorff measure Q: How does a set replicate under scaling? $E \rightarrow nE = E_1 \cup \dots \cup E_m$ disjoint congruent copies of E . Examples: line $m = n^1$, square $m = n^2$, cube $m = n^3$, Cantor set $3C = C_1 \cup C_2$ $2 = 3^\alpha \Leftrightarrow \alpha = \frac{\log 2}{\log 3}$

$\#(\varepsilon)$ = least $\#$ of segments that arise from such polygonal lines. Γ rectifiable iff $\#(\varepsilon) \sim \varepsilon^{-1}$ as $\varepsilon \rightarrow 0^+$. If $\#(\varepsilon) \sim \varepsilon^{-\alpha}$ ($\alpha > 1$) In this case, say " Γ has dim α ". Snowflake has $\alpha = \frac{\log 4}{\log 3} > 1$.

Upshot: E $\alpha > 1$. $m_\alpha(E) = \alpha$ -dimensional mass of E among sets of "dimension" α .

- $\alpha > \dim(E) \Rightarrow m_\alpha(E) = 0$
- $\alpha < \dim(E) \Rightarrow m_\alpha(E) = \infty$
- $\alpha = \dim(E)$ interesting

R. Gardner Bulletin AMS more about Brunn-Minkowski, geometrically, including more proofs, e.g. with induction of the dimension.

Hausdorff measure $E \subset \mathbb{R}^d$ any subset.

$$m_\alpha^*(E) := \lim_{\delta \rightarrow 0^+} \inf \left\{ \underbrace{\sum_k (\text{diam} F_k)^\alpha : E \subset \bigcup_{k=1}^{\infty} F_k, \text{diam}(F_k) \leq \delta}_{H_\alpha^\delta(E)} \right\}$$

exterior/outer α -dim Hausdorff measure.

Remark. $H_\alpha^\delta(E) \leq H_\alpha^\delta(E) \leq m_\alpha^*(E) (\forall \delta > 0)$. $H_\alpha^\delta(E)$ increases when δ decreases. $\therefore m_\alpha^*(E) = \lim_{\delta \rightarrow 0^+} H_\alpha^\delta(E)$ exists

Remark. Coverings must be by sets of arb. small measure. (If we allowed the δ to be arbitrary then two parallel lines would get the same 1d-measure as one of them.)

Remark (Scaling). "The measure of a set should scale like its dimension". E.g.: $\Gamma \subset \mathbb{R}^d$ smooth curve of length L *sim* $\lambda\Gamma$ has length λL . $Q \subset \mathbb{R}^d$ cube *sum* λQ has measure $\lambda^d |Q|$. $|F|$ scaled by $\lambda \Rightarrow (\text{diam} F)^\alpha$ scaled by λ^α

Properties

- (i) $E_1 \subset E_2 \Rightarrow m_\alpha^*(E_1) \leq m_\alpha^*(E_2)$
- (ii) $\{E_j\} \subset \mathbb{R}^d$ countable family of sets $\Rightarrow m_\alpha^*(\bigcup_{j=1}^{\infty} E_j) \leq \sum_{j=1}^{\infty} m_\alpha^*(E_j)$
- (iii) (Finite additivity) $\inf_{x \in E_1, y \in E_2} |x - y| = d(E_1, E_2) > 0 \Rightarrow m_\alpha^*(E_1 \cup E_2) = m_\alpha^*(E_1) + m_\alpha^*(E_2)$

Proof. ETS \geq . Fix $0 < \varepsilon < d(E_1, E_2)$. Given any cover of $E_1 \cup E_2$ with sets F_1, F_2, \dots of $\text{diam} \leq \delta < \varepsilon$, let $F'_j = F_j \cap E_1$, $F''_j = F_j \cap E_2$.

$$\sum (\text{diam}_j F'_j)^\alpha + \sum (\text{diam}_j F''_j)^\alpha \leq \sum_k (\text{diam}(F_k))^\alpha$$

Take inf over all covers, let $\delta \rightarrow 0^+$, done. □

m_α^* satisfies all properties of a Caratheodory outer measure $\therefore m_\alpha^*$ is a countably additive measure when restricted to Borel sets, call it $m_\alpha = \alpha$ -dim Hausdorff measure.

- (iv) $\{E_j\}$ countable family of disjoint Borel sets \Rightarrow

$$m_\alpha(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} m_\alpha(E_j)$$

- (v) Hausdorff measure is invariant under translation and rotations. It scales like:

$$m_\alpha(\lambda E) = \lambda^\alpha m_\alpha(E)$$

- (vi) $m_0(E) = \#E$, $m_1(E) = |E|$ (=1D Lebesgue measure of E), $E \subset \mathbb{R}$ Borel.

- (vii) $E \subset \mathbb{R}^d$ Borel, $m_\alpha(E) \simeq |E|$

Proof. (i) Isodiametric inequality: $|E| \leq v_d \left(\frac{\text{diam} E}{2}\right)^d$, v_d volume of the unit ball in \mathbb{R}^d .
Prove first for sets $E = -E$ and then something hard.

- (ii) Covering argument: Given $\varepsilon, \delta > 0$, there exists a covering of E by balls $\{B_j\}$:
 $\text{diam} B_j < \delta$, $\sum_j |B_j| \leq |E| + \varepsilon$

$$H_d^\delta(E) \leq \sum_j (\text{diam} B_j)^d = c_d \sum_j |B_j| \leq c_d(|E| + \varepsilon),$$

let $\delta, \varepsilon \rightarrow 0^+$, get one of the inequalities.

□

- (viii) if $m_\alpha^*(E) < \infty$ and $\beta > \alpha$, then $m_\beta^*(E) = 0$. If $m_\alpha^*(E) > 0$ and $\beta < \alpha$, then $m_\beta^*(E) = \infty$.

Proof. $\text{diam} F < \delta, \beta > \alpha \Rightarrow (\text{diam} F)^\beta = (\text{diam} F)^{\beta-\alpha} (\text{diam} F)^\alpha < \delta^{\beta-\alpha} (\text{diam} F)^\alpha$ □

Consequence: Given $E \subset \mathbb{R}^d$ Borel, $\exists! \alpha$ such that

$$m_\beta(E) = \begin{cases} \infty & \beta < \alpha \\ 0 & \beta > \alpha \end{cases}$$

$$\alpha = \sup\{\beta : m_\beta(E) = \infty\} = \inf\{\beta : m_\beta(E) = 0\} := \text{Hausdorff dimension of } E = \dim E$$

At the critical value $\alpha = \dim E$ $0 \leq m_\alpha(E) \leq \infty$. If E is bounded and the inequalities are strict, we say that E has strict Hausdorff dimension α .

Theorem. The Cantor set $C \subset [0, 1]$ has strict Hausdorff dimension $\frac{\log 2}{\log 3}$.

ETS: $0 < m_\alpha(C) \leq 1$

Proof. $m_\alpha(C) \leq 1$: $C = \bigcap C_k$ where each C_k is a finite union of 2^k intervals of length 3^{-k} . Given $\delta > 0$ choose k large enough such that $3^{-k} < \delta$. C_k covers C and consists of 2^k intervals of diameter $3^{-k} < \delta$. $H_\alpha^\delta(C) \leq 2^k (3^{-k})^\alpha = 1$, let $\delta \rightarrow 0^+$, done.

$$m_\alpha(C) > 0:$$

Lemma. $E \subseteq \mathbb{R}^d$ compact, $f : E \rightarrow \mathbb{R}$ γ -Hölder,

$$|f(x) - f(y)| \leq m|x - y|^\gamma \quad (\forall x, y \in E) \quad 0 < \gamma \leq 1$$

Then

$$(i) \quad m_\beta(f(E)) \leq M^\beta m_\alpha(E) \text{ if } \beta = \frac{\alpha}{\gamma}.$$

$$(ii) \quad \dim f(E) \leq \frac{1}{\gamma} \dim(E)$$

Proof. $\{F_k\}$ countable family of sets that covers E : $\{f(F_k \cap E)\}$ covers $f(E)$. $\text{diam} f(F_k \cap E) \leq M(\text{diam} F_k)^\gamma$.

$$\sum_k (\text{diam} f(F_k \cap E))^{\frac{\alpha}{\gamma}} \leq M^{\frac{\alpha}{\gamma}} \sum_k (\text{diam} F_k)^\alpha,$$

done. and 1 implies 2. □

Lemma. The Cantor-Lebesgue function $F : C \rightarrow [0, 1]$ is $\gamma = \frac{\log 2}{\log 3}$ -Hölder.

Proof. Goal: $|F(x) - F(y)| \leq c|x - y|^\gamma \forall x, y \in C$.

F_n increases at most 2^{-n} on an interval of length 3^{-n} . \therefore slope $\leq (\frac{3}{2})^n \therefore |F_n(x) - F_n(y)| \leq (\frac{3}{2})^n |x - y|$. $|F_n(x) - F(x)| \leq 2^{-n}$. Given x, y chose n : $3^n |x - y| \sim 1$, $3^\gamma = 2$.

$$|F(x) - F(y)| \leq |F_n(x) - F_n(y)| + |F_n(x) - F(x)| + |F_n(y) - F(y)| \leq (\frac{3}{2})^n |x - y| + 2 \cdot 2^{-n} \leq c 2^{-n} = c(3^{-n})^\gamma \leq c' |x - y|^\gamma$$

□

Apply Lemma 1 with $E = C$, $f = F$, $\gamma = \frac{\log 2}{\log 3} \Rightarrow 1 = m_1([0, 1]) \leq M m_\alpha(C)$, $\dim C = \frac{\log 2}{\log 3}$

□

Rectifiable curves

Theorem. $\gamma : [a, b] \rightarrow \mathbb{R}^d$ continuous and simple. Then γ is rectifiable iff $\Gamma = \{\gamma(t) : a \leq t \leq b\}$ has strict Hausdorff dimension equal to 1. $m_1(\Gamma) = l(\gamma)$.

Proof. \Rightarrow : Let γ be rectifiable of length L . Consider arclength parametrization $\tilde{\gamma}$. $\Gamma = \{\tilde{\gamma}(s) : 0 \leq s \leq L\}$.

$$|\tilde{\gamma}(s_1) - \tilde{\gamma}(s_2)| \leq |s_1 - s_2|$$

By Lemma 1 (i) $m_1(\Gamma) \leq L$. Wh $m_1(\Gamma) \geq L$?

$$\Gamma_j = \{\gamma(t) : t_j \leq t \leq t_{j+1}\}$$

$$\Gamma = \bigcup_{j=1}^{N-1} \Gamma_j \quad m_1(\Gamma) = \sum_{j=0}^{N-1} m_1(\Gamma_j)$$

Claim: $m_1(\Gamma_j) \geq l_j := |\gamma(t_j) - \gamma(t_{j+1})|$

Proof. $\pi : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2$ $(x, y) \mapsto x$ Lipschitz, $\pi(\Gamma_j) \subset [0, l_j]$ Lemma 1 (i) implies the claim.

□

$$\therefore m_1(\Gamma) \geq \sum m_1(\Gamma_j) \geq \sum l_j, \quad L := \sup_p \sum l_j : m_1(\Gamma) \geq L, \text{ done.}$$

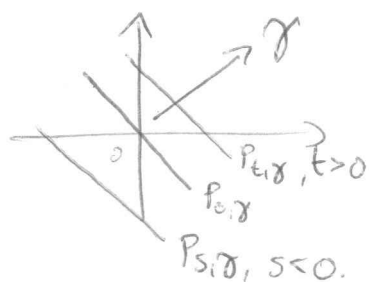
□

Radon transforms

$$R(f)(t, \gamma) = \int_{P_{t, \gamma}} f. \quad \text{where ...}$$

$$f: \mathbb{R}^d \rightarrow \mathbb{R}, \quad t \in \mathbb{R}, \quad \gamma \in S^{d-1} \subseteq \mathbb{R}^d$$

$$P_{t, \gamma} = \{x \in \mathbb{R}^d : \underbrace{x \cdot \gamma}_{\text{inner pr.}} = t\} \quad \text{hyperplane}$$



$P_{t, \gamma}$ equipped with natural $(d-1)$ -dim. Leb. measure, denoted by m_{d-1} (coincides with $(d-1)$ -Hausdorff measure)

Remarks (i) $f \in C_0^\infty(\mathbb{R}^d) \Rightarrow f$ integrable on every $P_{t, \gamma} \Rightarrow R(f)(t, \gamma)$ defined for every (t, γ) .
($R(f)$ cont. fct. of (t, γ) ,
cpctly supp. in t)

(ii) $f \in L^1(\mathbb{R}^d) \Rightarrow f$ may fail to be measurable/integrable on some $P_{t, \gamma}$ ($\rightarrow R(f)(t, \gamma)$ not defined.)

(iii) $f = \chi_E$ ($E \subset \mathbb{R}^d$ mb.) $\Rightarrow R(f)(t, \gamma) = m_{d-1}(E_{t, \gamma})$ if $E_{t, \gamma}$ measurable.
 $E_{t, \gamma} = E \cap P_{t, \gamma}$

Look instead at maximal Radon transform:

$$R^*(f)(\gamma) = \sup_{t \in \mathbb{R}} |R(f)(t, \gamma)|.$$

\rightarrow Want to study L^p -mapping properties of R in order to study regularity of subsets of \mathbb{R}^d .

Thm. 1 $f \in C^0(\mathbb{R}^d)$, $n \geq 3$:

(*) $\int_{S^{d-1}} R^*(f)(\gamma) d\sigma_\gamma \leq C(\|f\|_{L^1(\mathbb{R}^d)} + \|f\|_{L^2(\mathbb{R}^d)}).$

~~the f needs~~

Rem. i) necessary conditions:

•) $f \in L^1$: $f(x) = (1+|x|^{d-1})^{-1} \in (L^2 \setminus L^1)(\mathbb{R}^d)$ if $d \geq 3$
 f is not integrable in any plane $P_{t,\gamma}$.
($\rightarrow f \in L^1$ gives global control.)

•) $f \in L^2$: $f_\varepsilon(x) = (|x|+\varepsilon)^{-d+\delta}$ if $|x| \leq 1$, $\delta \in (0,1)$ fixed.
Let $\varepsilon \rightarrow 0^+$ to see that (*) fails if $\|\cdot\|_{L^2}$
on the RHS is not there
($\rightarrow f \in L^2$ gives local control.)

Key: Interplay between Radon and Fourier transform.
 $t \mapsto \lambda \in \mathbb{R}$ dual variable.

Fourier transform:

$$\hat{R}(f)(\lambda, \gamma) = \int_{-\infty}^{\infty} R(f)(t, \gamma) e^{-2\pi i \lambda t} dt$$

Lemma 1 $f \in C^0(\mathbb{R}^d)$, $\gamma \in S^{d-1}$:

$$\hat{R}(f)(\lambda, \gamma) = \hat{f}(\lambda \gamma).$$

Proof $\hat{f}(\lambda \gamma) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot (\lambda \gamma)} dx$

$$= \int_{-\infty}^{\infty} \left(\int_{\mathbb{R}^{d-1}} f(u, t) du \right) e^{-2\pi i \lambda t} dt = \int_{-\infty}^{\infty} \left(\int_{P_{t,\gamma}} f \right) e^{-2\pi i \lambda t} dt.$$

Choose coordinates

$$x = (u, t), \quad t = x \cdot \gamma = x_d \in \mathbb{R},$$

$$u = (x_1, \dots, x_{d-1}) \in \mathbb{R}^{d-1}.$$

□

Lemma 2 $f \in C_0^\infty(\mathbb{R}^d)$: (2)

$$\int_{S^{d-1}} \left(\int_{-\infty}^{\infty} |\hat{F}(f)(\lambda, y)|^2 |\lambda|^{d-1} d\lambda \right) d\sigma_y = 2 \int_{\mathbb{R}^d} |f(x)|^2 dx$$

Proof $2 \int_{\mathbb{R}^d} |f(x)|^2 dx = 2 \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 d\xi$
 $= 2 \int_{S^{d-1}} \left(\int_0^\infty |\hat{f}(\lambda y)|^2 \lambda^{d-1} d\lambda \right) d\sigma_y$
 $= 2 \int_{S^{d-1}} \left(\int_{-\infty}^\infty |\hat{f}(\lambda y)|^2 |\lambda|^{d-1} d\lambda \right) d\sigma_y$
 done by Lemma 2. \square

Lemma 3 Suppose $F(t) = \int_{-\infty}^{\infty} \hat{F}(\lambda) e^{2\pi i \lambda t} d\lambda$
 where $\sup_{\lambda \in \mathbb{R}} |\hat{F}(\lambda)| \leq A$ and $\int_{-\infty}^{\infty} |\hat{F}(\lambda)|^2 |\lambda|^{d-1} d\lambda \leq B^2$.

Then $\sup_{t \in \mathbb{R}} |F(t)| \leq C(A+B)$ (This is essential in the proof of Thm. 1)

Moreover, if $0 < \alpha < \frac{1}{2}$, then $|F(t_1) - F(t_2)| \leq C_\alpha |t_1 - t_2|^\alpha \cdot (A+B)$

($\rightarrow \alpha$ -Hölder)

Proof.

$$F(t) = \underbrace{\int_{|\lambda| \leq 1} \hat{F}(\lambda) e^{2\pi i \lambda t} d\lambda}_{\leq CA} + \underbrace{\int_{|\lambda| > 1} \hat{F}(\lambda) e^{2\pi i \lambda t} d\lambda}_{\leq \int_{|\lambda| > 1} \hat{F}(\lambda) d\lambda}$$

$$\stackrel{CS}{\leq} \left(\int_{|\lambda| > 1} |\hat{F}(\lambda)|^2 |\lambda|^{d-1} d\lambda \right)^{1/2} \cdot \underbrace{\left(\int_{|\lambda| > 1} |\lambda|^{-d+1} d\lambda \right)^{1/2}}_{< \infty \text{ if } -d+1 < -1 \Rightarrow d \geq 3. \checkmark} \leq B$$

\Rightarrow first estimate.

$$\begin{aligned}
|F(t_1) - F(t_2)| &= \int_{-\infty}^{\infty} \hat{F}(\lambda) \underbrace{(e^{2\pi i \lambda t_1} - e^{2\pi i \lambda t_2})}_{1 - |e^{2\pi i \lambda(t_1 - t_2)}|} d\lambda \\
&\leq \int_{|\lambda| \leq 1} \hat{F}(\lambda) d\lambda + \int_{|\lambda| > 1} |\hat{F}(\lambda)| |\lambda|^{-\alpha} d\lambda \\
&\leq C_\alpha A |t_1 - t_2|^\alpha \quad (\text{Lipschitz}) \\
&\quad + \int_{|\lambda| > 1} |\hat{F}(\lambda)| |\lambda|^{-\alpha} d\lambda \\
&\leq \left(\int_{|\lambda| > 1} |\hat{F}(\lambda)|^2 |\lambda|^{-d+1} d\lambda \right)^{1/2} \\
&\quad \cdot \left(\int_{|\lambda| > 1} |\lambda|^{-d+1+2\alpha} d\lambda \right)^{1/2} \\
&\leq \infty \text{ if } \alpha < \frac{1}{2} \text{ for } d \leq 3.
\end{aligned}$$

□

Proof of Thm. For each $\gamma \in S^{d-1}$, let $F(t) = R(f)(t, \gamma)$

$$\Rightarrow \sup_{t \in \mathbb{R}} |F(t)| = R^*(f)(\gamma).$$

$$\text{Let } A(\gamma) = \sup_{\lambda \in \mathbb{R}} |\hat{F}(\lambda)|, \quad B^2(\gamma) = \int_{-\infty}^{\infty} |\hat{F}(\lambda)|^2 |\lambda|^{d-1} d\lambda$$

$$\text{Lemma 3 } \stackrel{\text{check assumptions!}}{\Rightarrow} \sup_{t \in \mathbb{R}} |F(t)| \leq C(A(\gamma) + B(\gamma))$$

$$\text{Lemma 1 } \Rightarrow \hat{F}(\lambda) = \hat{R}(f)(\lambda, \gamma) = \hat{f}(\lambda \gamma) \Rightarrow A(\gamma) \leq \|f\|_{L^1(\mathbb{R}^d)}$$

$$\text{Lemma 2 } \Rightarrow \int_{S^{d-1}} B^2(\gamma) d\sigma_\gamma = 2 \|f\|_{L^2(\mathbb{R}^d)}^2.$$

$$\text{We have } \sup_{t \in \mathbb{R}} |F(t)|^2 \leq C^2 (A^2(\gamma) + B^2(\gamma))$$

Integrate both sides:

$$\int_{S^{d-1}} R^*(f)(\gamma)^2 d\sigma_\gamma \lesssim \underbrace{\int_{S^{d-1}} A^2(\gamma) d\sigma_\gamma}_{\approx \|f\|_1^2} + \underbrace{\int_{S^{d-1}} B^2(\gamma) d\sigma_\gamma}_{\approx \|f\|_{L^2}^2}$$

Use Hölder, because on cpt. space:

$$\lesssim (\|f\|_{L^1}^2 + \|f\|_{L^2}^2)^2$$

$$\|R^*(f)\|_{L^1}^2 \lesssim \int_{S^{d-1}} R^*(f)(\gamma)^2 d\sigma_\gamma.$$

□

Regularity of sets when $d \geq 3$.

(3)

$E \subset \mathbb{R}^d$ meas. $E_{t,\gamma} = E \cap P_{t,\gamma}$ (t varies, γ fixed)

Fubini $\Rightarrow E_{t,\gamma}$ is m_{d-1} -measurable for a.e. t

$t \mapsto m_{d-1}(E_{t,\gamma})$ is a measurable fctn of t .

Thm. 2 $E \subset \mathbb{R}^d$ ($d \geq 3$) of finite measure.

Then for a.e. $\gamma \in S^{d-1}$:

(i) $E_{t,\gamma}$ is m_{d-1} -measurable for every t .

(ii) $t \mapsto m_{d-1}(E_{t,\gamma})$ is a cont. fct. of t .

Moreover, this fctn is α -Hölder $\forall \alpha \in (0, \frac{1}{2})$.

Cor. $d \geq 3$, $E \subset \mathbb{R}^d$ of Lebesgue measure zero.

Then, for a.e. $\gamma \in S^{d-1}$, the slice $E_{t,\gamma}$ has zero measure for every $t \in \mathbb{R}$.

Prop. $d \geq 3$, $f \in (L^1 \cap L^2)(\mathbb{R}^d)$. Then for a.e.

$\gamma \in S^{d-1}$:

i) f is meas. and int. on the plane for every $t \in \mathbb{R}$.

ii) $t \mapsto \mathcal{R}(f)(t,\gamma)$ is cont. and α -Hölder if $\alpha < \frac{1}{2}$.

Moreover, estimate (*) from Thm. 1 holds for f .

Rem. Prop. implies Thm. 2 by taking char. fctn of E .

$$\mathcal{R}(\chi_E)(t,\gamma) = m_{d-1}(E_{t,\gamma}).$$

We skip the proof of Prop. (follows from Thm. 1 using some delicate measure theory)

What about $d=2$?

Given $f \in L^1(\mathbb{R}^2)$, define

$$\mathcal{R}_\delta(f)(t,\gamma) = \frac{1}{2\delta} \int_{t-\delta}^{t+\delta} \mathcal{R}(f)(s,\gamma) ds. \quad (\text{averaged version of } \mathcal{R})$$

(integration over thickened line = hypplane.)

$$= \frac{1}{2\delta} \int_{\{t-\delta \leq x \cdot \gamma \leq t+\delta\}} f(x) dx.$$

Thm. 3 $f \in C_0^\infty(\mathbb{R}^2)$, $0 < \delta \leq 1/2$.

$$\int_{S^1} R_\delta^*(f)(\gamma) d\sigma_\gamma \lesssim (\log \frac{1}{\delta})^{1/2} \cdot (\|f\|_{L^1(\mathbb{R}^2)} + \|f\|_{L^2(\mathbb{R}^2)})$$

Theorem. $f \in C_0^0(\mathbb{R}^2)$, $0 < \delta \leq \frac{1}{2}$. Then

$$\int_{S^1} R_\delta^*(f)(\gamma) d\sigma_\gamma \lesssim (\log \frac{1}{\delta})^{\frac{1}{2}} (\|f\|_{L^1(\mathbb{R}^2)} + \|f\|_{L^2(\mathbb{R}^2)})$$

Proof of Theorem. Modified version of lemma 3: Setting

$$F_\delta(t) = \int_{-\infty}^{\infty} \hat{F}(\lambda) \left(\frac{e^{2\pi i(t+\delta)\lambda} - e^{2\pi i(t-\delta)\lambda}}{2\pi i\lambda(2\delta)} \right) d\lambda$$

Suppose $\sup_{\lambda \in \mathbb{R}} |\hat{F}(\lambda)| \leq A$ and $\int_{-\infty}^{\infty} |\hat{F}(\lambda)|^2 |\lambda|^{d-1} d\lambda \leq B^2$.

Claim:

$$\sup_t |F_\delta(t)| \lesssim (\log \frac{1}{\delta})^{\frac{1}{2}} (A + B)$$

$$F_\delta(t) = \int_{-\infty}^{\infty} \hat{F}(\lambda) \left(\frac{e^{2\pi i(t+\delta)\lambda} - e^{2\pi i(t-\delta)\lambda}}{2\pi i\lambda(2\delta)} \right) d\lambda = \int_{|\lambda| \leq 1} \hat{F}(\lambda) d\lambda + \int_{1 < |\lambda| \leq \frac{1}{\delta}} \hat{F}(\lambda) d\lambda + \frac{c}{\delta} \int_{|\lambda| > \frac{1}{\delta}} |\hat{F}(\lambda)| |\lambda|^{-1} d\lambda = I + II$$

CS:

$$I \lesssim \left(\int_{\mathbb{R}} |\hat{F}(\lambda)|^2 |\lambda| d\lambda \right)^{\frac{1}{2}} \left(\int_{1 < |\lambda| \leq \frac{1}{\delta}} |\lambda|^{-1} d\lambda \right)^{\frac{1}{2}} \leq B (\log \frac{1}{\delta})^{\frac{1}{2}}$$

$$II \lesssim \frac{c}{\delta} \left(\int_{\mathbb{R}^2} |\hat{F}(\lambda)|^2 |\lambda| d\lambda \right)^{\frac{1}{2}} \left(\int_{|\lambda| > \frac{1}{\delta}} |\lambda|^{-3} d\lambda \right)^{\frac{1}{2}} \lesssim B$$

□

Theorem. There exists a subset $K \subset \mathbb{R}^2$ such that

- (i) K is compact
- (ii) K has Lebesgue measure zero
- (iii) K contains a translate of every unit line segment

Theorem. Suppose F is any set that satisfies conditions (i) and (iii) from Theorem 1. Then F has Hausdorff dimension 2.

Proof of Theorem 2. Let F be a Kakeya set. Fix $0 < \alpha < 2$. Let $F \subset \bigcup_{i=1}^{\infty} B_i$ be a covering with balls B_i of diameter $\leq \delta$. It is enough to show

$$\sum (\text{diam} B_i)^\alpha \geq c_\alpha > 0$$

for $\alpha < 2$.

Case 1: Assume $\text{diam} B_1 = \delta \leq \frac{1}{2}$ and let $N < \infty$ be the number of balls in the covering. WTS $N\delta^\alpha \geq c_\alpha$. B_i^* = double of B_i . $F^* = \bigcup_i B_i^*$. $|F^*| \leq \sum |B_i^*| = cN\delta^2$. F Kakeya $\Rightarrow \forall \gamma \in S^1 \exists s_\gamma \perp \gamma$ unit line segment: $s_\gamma \subset F$. $s_\gamma^\delta \subset F^*$. $\therefore R_\delta^*(\chi_{F^*})(\gamma) \geq 1$ ($\forall \gamma \in S^1$). Take $f = \chi_{F^*}$ in (*). Since $L^2 \subset L^1$,

$$\|\chi_{F^*}\|_{L^1} \lesssim \|\chi_{F^*}\|_{L^2} = |F^*|^{\frac{1}{2}} \lesssim N^{\frac{1}{2}} \delta.$$

(*) $\Rightarrow 0 < c \leq (\log \frac{1}{\delta})^{\frac{1}{2}} N^{\frac{1}{2}} \delta$. This implies $N\delta^\alpha \geq c_\alpha > 0$.

Case 2: General case. $F \subset \bigcup_{i=1}^{\infty} B_i$ with each ball B_i of diameter ≤ 1 . For each $k \in \mathbb{N}$, let N_k be the number of balls in $\{B_i\}$ with diameter $B_k \sim 2^{-k}$, i.e. $\in [2^{-k-1}, 2^{-k}]$. WTS

$$\sum_{k=0}^{\infty} N_k 2^{-k\alpha} \geq c_{\alpha} > 0.$$

ETS $\exists k' : N_{k'} 2^{-k'\alpha} \geq c_{\alpha}$.

$$F_k = F \cap \left(\bigcup_{\text{diam } B_i \sim 2^{-k}} B_i \right)$$

$$F_k^* = \bigcup_{\text{diam } B_v \sim 2^{-k}} B_v^*$$

$$|F_k^*| \leq c N_k 2^{-2k} \quad \forall k$$

F Keakeya $\Rightarrow \forall \gamma \in S^2 \exists s_{\gamma} \perp \gamma : s_{\gamma} \subset F$ (in particular $m_1(s_{\gamma} \cap F) = 1$).

Key: For some k , a large proportion of s_{γ} belongs to F_k . Pick $\{a_k\}_{k=0}^{\infty}$ such that $0 \leq a_k < 1$, $\sum a_k = 1$, (a_k) does not tend to 0 too quickly, e.g. $a_k = c_{\varepsilon} 2^{-k\varepsilon}$ (for sufficiently small ε).

Claim:

$$\exists k : m_1(s_{\gamma} \cap F_k) \geq a_k.$$

Otherwise $m_1(s_{\gamma} \cap F) \leq \sum_k m_1(s_{\gamma} \cap F_k) < \sum a_k = 1$, contradicts (**)

For this value of k ,

$$R_{2^{-k}}^*(\chi_{F_k^*})(\gamma) \geq a_k.$$

Since this choice of k depends on γ , let

$$E_k = \{\gamma \in S^1 : R_{2^{-k}}^*(\chi_{F_k^*})(\gamma) \geq a_k\}.$$

$S^1 = \bigcup_{k=1}^{\infty} E_k$. Therefore $\exists k' : |E_{k'}| \geq 2\pi a_{k'}$.

$$2\pi a_{k'}^2 = 2\pi a_{k'} a_{k'} \leq \int_{E_{k'}} a_{k'} d\sigma \leq_{S^1} \int_{E_{k'}} R_{2^{-k'}}^*(\chi_{F_{k'}^*})(\gamma) d\sigma_{\gamma}$$

$$2^{-2k\varepsilon} \sim a_{k'}^2 \leq c(\log 2^{k'})^{\frac{1}{2}} |F_{k'}^*|^{\frac{1}{2}} \leq c(\log 2^{k'})^{\frac{1}{2}} N_{k'}^{\frac{1}{2}} 2^{-k'}$$

$\Rightarrow N_{k'} 2^{-\alpha k'} \geq c_{\alpha}$, provided $4\varepsilon < 2 - \alpha$. □

Construction of a Keakeya set I (Stein-Shakarchi, III)

Thinner Cantor set, always taking away the half.

Take two of them, E_0, E_1 , where E_1 has twice the length. Put E_0 on $y = 1$ and E_1 on $y = 0$. Let F be the union of all line segments that join a point in E_0 with one in E_1 .

Construction of an ε -Keakeya set (Stein)

Theorem. Given $\varepsilon > 0$, $\exists N = N_{\varepsilon}$ and 2^N rectangles R_1, \dots, R_{2^N} with side lengths 1×2^{-N} such that

(i)

$$\left| \bigcup_{j=1}^{2^N} R_j \right| < \varepsilon$$

(ii) the reaches \tilde{R}_j are mutually disjoint, i.e.

$$|\bigcup_{j=1}^{2^N} \tilde{R}_j| = 1$$

Proof. Fix $\alpha \in (\frac{1}{2}, 1)$. Symmetric triangle ABC with M opposite C . Push the right part into the left part and call the resulting body $\Phi(T)$. It consists of heart $\Phi_h(T)$ and arms $\Phi_a(T)$. Then

$$|\Phi_h(T)| = \alpha^2 |T|$$

$$|\Phi_a(T)| = 2(1 - \alpha)^2 |T|$$

Conclusion

$$|\Phi(T)| = (\alpha^2 + 2(1 - \alpha)^2) |T|$$

n -fold iteration (Peron trees): Split not into two but 2^n parts and do everything pairwise. Key: right side of $\Phi_h(A_0 A_2 C)$ // left side of $\Phi_n(A_2 A_4 C)$ // CA_2

Then look at heart/arms again.

$$|\text{arms of } \Psi_1(ABC)| \leq 2(1 - \alpha)^2 |T|.$$

$$|\text{heart of } \Psi_1(ABC)| = \alpha^2 |T|$$

$$\therefore |\Psi_1(ABC)| = (\alpha^2 + 2(1 - \alpha)^2) |T|.$$

Iterate: Carry out this process on the heart of $\Psi_1(ABC)$ with n replaced by $n - 1$, given are the union of 2^{n-1} triangles.

Then retranslate all 2^n original triangles to obtain figure $\Psi_2(ABC)$.

$$|\text{heart of } \Psi_2(ABC)| = \alpha^2 \alpha^2 |T|$$

$$|\text{additional arms of } \Psi_2(ABC)| \leq 2(1 - \alpha)^2 \alpha^2 |T|$$

$$\begin{aligned} |\Psi_n(ABC)| &\leq (\alpha^{2n} + 2(1 - \alpha)^2 + 2(1 - \alpha)^2 \alpha^2 + \dots + 2(1 - \alpha)^2 \alpha^{2n-2}) \\ &\leq \alpha^{2n} + 2(1 - \alpha)^2 \underbrace{\sum_{n=0}^{\infty} \alpha^{2n}}_{= \frac{1}{1 - \alpha^2}} \\ &\leq \alpha^{2n} + 2(1 - \alpha) \end{aligned}$$

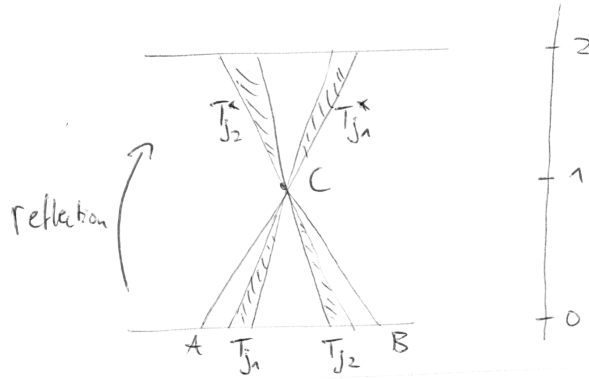


Figure 1: Obtaining mutually disjoint reaches by reflecting in C .

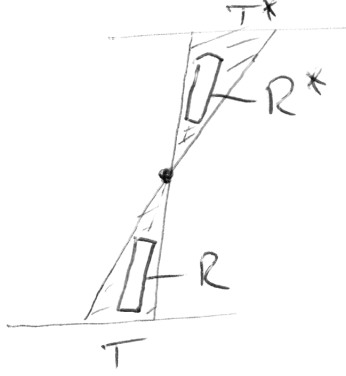


Figure 2: Go from triangles to rectangles by placing rectangles into the triangles with half the length.

□

Application Maximal functions and counterexamples. Q: Given a collection $\mathcal{C} = \{C\}$ of sets, for which class of functions do we have

$$\lim_{\text{diam}(C) \rightarrow 0, c \in \mathcal{C}} \frac{1}{|C|} \int_C f(x-y) dy = f(x) \quad x - \text{a.e.}?$$

Seen:

$$(M_{\mathcal{C}} f)(x) = \sup_{c \in \mathcal{C}} \frac{1}{|C|} \int_C |f(x-y)| dy$$

$\mathcal{C} = \{\text{balls}\}$, weak-type (1,1) inequality for $M_{\mathcal{C}} \Rightarrow$ a.e. convergence of averages. A converse also holds!

$\{\mu_j\}_{j=1}^{\infty}$ collection of finite, nonnegative measures on $\mathbb{R}^d : \text{supp}(\mu_j) \subset K \Subset \mathbb{R}^d$. Define the maximal operator

$$(Mf)(x) = \sup_j |f * \mu_j|(x).$$

Proposition. $1 \leq p < \infty$. Assume for each $f \in L^p(\mathbb{R}^d)$ that $(Mf)(x) < \infty$ for some set of x having positive measure. Then $f \mapsto Mf$ is of weak-type (p, p) , i.e.

$$\exists A < \infty : |\{x \in \mathbb{R}^d : (Mf)(x) > \alpha\}| \leq \frac{A}{\alpha^p} \|f\|_{L^p}^p \quad (\forall \alpha > 0)$$

Lemma. $\{E_j\}$ collection of subsets of a fixed compact set:

$$\sum_{j=1}^{\infty} |E_j| = \infty.$$

Then there exists a sequence of translates $F_j = E_j + x_j$:

$$\limsup F_j = \bigcap_{k=1}^{\infty} \left(\bigcup_{j=k}^{\infty} F_j \right) = \mathbb{R}^n \quad (\text{a.e.})$$

The above set equals $\{x \in \mathbb{R}^d : x \in F_j \text{ infinitely often}\}$.

$$\liminf F_j = \bigcup_{k=1}^{\infty} \left(\bigcap_{j=k}^{\infty} F_j \right)$$

is a subset.

Proof of Lemma. $Q \subset \mathbb{R}^d$ unit cube. $A_1, A_2 \subset Q$. Then $\exists h \in \mathbb{R}^d : |A_1 \cap (A_2 - h)| \geq 2^{-d} |A_1| |A_2|$. Why?

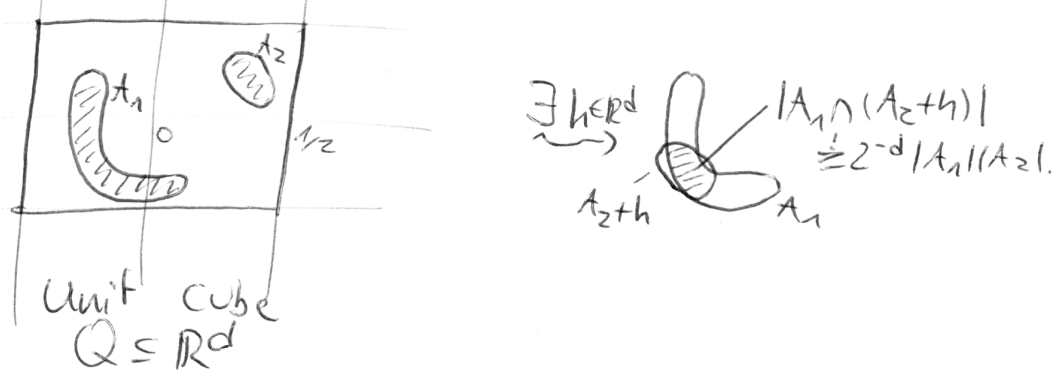


Figure 3: Translation of subsets A_1 and A_2 of a unit cube Q .

$$\eta(x) = \int_{\mathbb{R}^d} \chi_{A_1}(y) \chi_{A_2}(x+y) dy \sim \chi_{A_1} * \chi_{A_2}(x)$$

$$\int_{\mathbb{R}^d} |A_1| |A_2|$$

$$\text{supp}(\eta) \subset Q^*$$

$$|Q^*| = 2^d.$$

$$\exists h \in Q^* : \eta(h) \geq \text{avg}_{Q^*}(\eta) = \frac{1}{|Q^*|} \int_{\mathbb{R}^d} \eta = \frac{|A_1| |A_2|}{2^d}$$

Wlog $\text{supp}(\eta) \subset Q$.

Step 2: There exist translates $F_j = E_j + x_j$ that cover Q at least once.

$$Q \subset \bigcup_j F_j$$

Why? $F_1 = E_1$. Suppose (inductively) that F_1, \dots, F_{j-1} have been constructed. Let $A_1 = Q \cap (F_1 \cup \dots \cup F_{j-1})^c$ and $A_2 = E_j$. Step 1 $\Rightarrow \exists h : |A_1 \cap (A_2 - h)| \geq 2^{-d} |A_1| |A_2|$. Set $F_j = A_2 - h = E_j - h$. Let $p_j = |Q \cap (F_1 \cup \dots \cup F_j)|$. Then

$$p_j = p_{j-1} + \underbrace{|Q \cap (F_1 \cup \dots \cup F_{j-1})^c|}_{A_1} \underbrace{|E_j|}_{A_2 - h} = p_{j-1} + |A_1 \cap (A_2 - h)|$$

$$\geq p_{j-1} + 2^{-d} |A_1| |E_j| = p_{j-1} + 2^{-d} (1 - p_{j-1}) |E_j|$$

$$\therefore p_j - p_{j-1} \geq 2^{-d} (1 - p_{j-1}) |E_j|$$

$$\sum_{j=2}^{\infty} (p_j - p_{j-1}) = \lim_{j \rightarrow \infty} p_j - p_1 = \lim_j p_j = 1$$

Step 3: Decompose (twice) $\{E_j\}$ into a countable infinite number of subcollections so that on each subcollection the sum of the measures diverges. \square

Proof of Proposition. Take a ball B such that $B \supset Q + K$. $\text{supp}(F) \subset Q \Rightarrow \text{supp}(F * \mu_j) \subset \text{supp}(Mf) \subset B$. Key: Estimate (*) (the violation of the weak type estimate) holds if $\text{supp}(f) \subset Q$. For each $k, \exists \alpha_k > 0 \exists g_k \in L^p : \text{supp}(g_k) \subset Q$ such that

$$|\{x \in B : M g_k(x) > \alpha_k\}| \geq \frac{2^k}{\alpha_k^p} \|g_k\|_{L^p}^p$$

Replace g_k by $\tilde{g}_k = \frac{k}{\alpha_k} g_k$.

$$\frac{2^k}{k^p} \leq \frac{|\{x \in B : M \tilde{g}_k(x) > k\}|}{\|\tilde{g}_k\|_{L^p}^p} \rightarrow \infty \quad \text{as } k \rightarrow \infty$$

\therefore There exists a sequence $\{f_k\} \subset L^p$ and a sequence of constants $R_k \rightarrow \infty$ such that with $E_k = \{x \in B : M f_k(x) > R_k\}$ we get

$$\sum_k |E_k| = \infty \quad \sum_k \|f_k\|_{L^p}^p < \infty.$$

Remark. $d\mu_j \geq 0$ wlog $f_k \geq 0$.

By the lemma $\exists \{x_k\}$ such that $F_k = E_k + x_k$ satisfy $\limsup F_k = \mathbb{R}^d$ (a.e). Let

$$\tilde{f}_k(x) = f_k(x + x_k), \quad F(x) = \sup_k \tilde{f}_k(x)$$

Then

$$M(F) = \sup_j |F * \mu_j| = \sup_j |(\sup_k \tilde{f}_k) * \mu_j| \geq \sup_k \sup_j |\tilde{f}_k * \mu_j| = \sup_k M(\tilde{f}_k)$$

Also $M(\tilde{f}_k) > R_k$ on F_k . $M(F) = \infty$ a.e. Check $f \in L^p$:

$$|F|^p = |\sup_k \tilde{f}_k|^p \leq \sum_p |\tilde{f}_k|^p$$

$$\|F\|_{L^p}^p \leq \sum_k \|f_k\|_{L^p}^p < \infty$$

Full conclusion

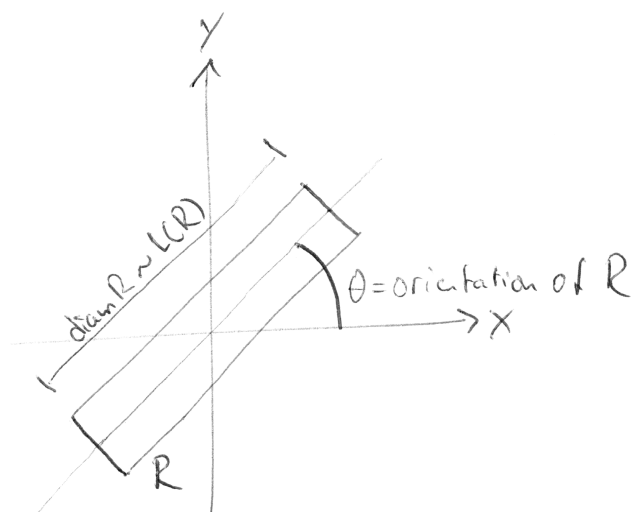
$$f = \sum f \chi_{Q_j} =: \sum f_j$$

$$M(f) \leq \sum_j M(f_j)$$

\square

Example. Rectangles with arbitrary orientation

$$\mathcal{C} = \mathcal{R} = \{\text{all rectangles in } \mathbb{R}^2 \text{ centered at } 0\}$$



Corollary. Given $1 \leq p < \infty$, $\exists f \in L^p(\mathbb{R}^n)$ such that

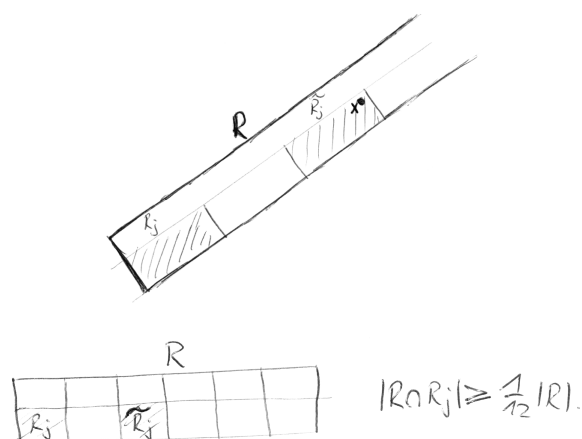
$$\limsup_{\text{diam}(R) \rightarrow 0, R \in \mathcal{R}} \frac{1}{|R|} \int_R f(x-y) dy = \infty \quad (x - \text{a.e.})$$

Idea: Use the ε -Kakeya set to show that M is not weak (p, p)

$$(Mf)(x) = \sup_{\text{diam}(R) < 8} \frac{1}{|R|} \left| \int_R f(x-y) dy \right|$$

Let $E = \bigcup_{j=1}^{2^N} R_j$ as before. $\|\chi_E\|_{L^p}^p = |E| < \varepsilon$. If $x \in \tilde{R}_j$, then \exists rectangle R such that

- R is centered at x
- $\text{diam}(R) \leq 8$
- $|R \cap R_j| \geq \frac{1}{12}|R|$



$$y \in x - E = -(E - x), \quad y - x \in -E, \quad x - y \in E$$

$$M(\chi_E)(x) \geq \int_{R-x} \chi_E(x-y) dy = \frac{|(R-x) \cap (E-x)|}{|R|} \geq \frac{1}{12}$$

Conclusion: $M\chi_E \geq \frac{1}{12}$ on the set $\bigcup_{j=1}^{2^N} \tilde{R}_j$ (of measure 1)

$$\forall A > 0 \exists \text{ set } E : |\{x \in \mathbb{R}^d : M\chi_E > A\}| \leq A\alpha^{-p} \|\chi_E\|_{L^p}^p$$

does not hold! $\therefore M$ is not of weak typ (p, p) .

Note, that this is not the complete proof. Therefore still have to replace 8 by δ .

Bochner-Ries summability Q: In which way does Fourier inversion hold? (In $L^p(\mathbb{R}^d)$)

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i x \xi} d\xi = \lim_{R \rightarrow \infty} \underbrace{\int_{|\xi| < R} \hat{f}(\xi) \left(1 - \frac{|\xi|^2}{R^2}\right)^\delta e^{2\pi i x \xi} d\xi}_{f * k_\delta} ?$$

for suitable $\delta \geq 0$.

$$\delta > \frac{d-1}{2} \Rightarrow k_\delta \in L^1(\mathbb{R}^d)$$

$$\delta \leq \frac{d-1}{2}$$

$$\delta = 0 \text{ today}$$

$d = 1$: The second equality holds in L^p -norm ($1 < p < \infty$) \iff L^p boundedness of Hilbert transform

It also holds a.e. ($p = 2$ Carleson '66 $1 < p < \infty$ Hunt '68)

$d \geq 2$ Let us only consider norm convergence

$$f \mapsto (S^\delta f)(x) = \int_{|\xi| \leq 1} \hat{f}(\xi) (1 - |\xi|^2)^\delta e^{2\pi i x \xi} d\xi$$

$$Sf(x) = \int_{|\xi| \leq 1} \hat{f}(\xi) e^{2\pi i x \xi} d\xi$$

$$\widehat{Sf} = 1_{|\xi| \leq 1} \hat{f}$$

$$(\widehat{Hf} = i\pi \operatorname{sgn} \xi \hat{f})$$

$$\|Sf\|_{L^2(\mathbb{R}^d)} = \|\widehat{Sf}\|_{L^2} = \|1_{|\xi| \leq 1} \hat{f}\|_{L^2} \leq \|\hat{f}\|_{L^2} = \|f\|_{L^2}$$

He said something about every bounded operator can be written like this or so? Fourier multiplier S is bounded iff multiplier function is bounded.

Theorem (C.Fefferman '71 – Annals of mathematics "The multiplier problem for the ball").
Suppose $q \geq 2$ and $p \neq 2$. Then the operator S (initially defined on $L^p \cap L^2$) is not extendable to a bounded operator from $L^p(\mathbb{R}^d)$ to itself.

Proof. Let $B \subset \mathbb{R}^d$ ball, let S_B be the multiplier operator associated to B :

$$\widehat{S_B f} = 1_B \hat{f}.$$

Given $u \in S^{d-1} \subset \mathbb{R}^d$, let S^u be the multiplier operator associated to the half-space with normal u :

$$\widehat{S^u f} = 1_{\{\xi u > 0\}} \hat{f} \quad (S^u f)(x) = \int_{\xi u > 0} \hat{f}(\xi) e^{2\pi i x \xi} d\xi$$

Upshot: L^p -bound for S implies an L^p vector-valued inequality for S_B s and S_u s.

Lemma (Y. Méyer). Suppose

$$\|Sf\|_{L^p} \leq A_p \|f\|_{L^p}$$

$f \in (L^2 \cap L^p)(\mathbb{R}^d)$ holds for some $p \in [1, \infty]$. Suppose $f_1, \dots, f_M \in L^2 \cap L^p$, $u_1, \dots, u_M \in S^{d-1} \subset \mathbb{R}^d$. Then

$$\left\| \left(\sum_{j=1}^M |S^{u_j}(f_j)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^d)} \leq A_p \left\| \left(\sum_{j=1}^M |f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^d)} \quad (4)$$

where A_p is the same constant as above.

Proof of Lemma. Step 1: $B = B_R =$ ball of radius R centered at 0. Then

$$\|S_B(f)\|_{L^p} \leq A_p \|f\|_{L^p} \quad (g \in L^2 \cap L^p) \quad (5)$$

Why? Scaling:

$$\delta_R(g)(x) = g\left(\frac{x}{R}\right)$$

Check $\delta_{R^{-1}} \circ S \circ \delta_R = S_{B_R}$ since $S = S_{B_1}$. $\hat{\delta}_\rho(g)(\xi) = R^d \hat{g}(R\xi)$.

Step 2: M balls. ($p < \infty$) $f = (f_1, \dots, f_M)$ given M -tuple of functions. $T(f) := (Tf_1, \dots, Tf_M)$. Given a unit vector $\omega = (\omega_1, \dots, \omega_M) \in \mathbb{C}^M$, let

$$S_\omega(f) = \sum_{j=1}^M \bar{\omega}_j S_B(f_j) = S_B\left(\sum_j \bar{\omega}_j f_j\right) = S_B(f_\omega) \quad f_\omega = \sum_{j=1}^M \bar{\omega}_j f_j$$

$$(5) \Rightarrow \int_{\mathbb{R}^d} |S_\omega f(x)|^p dx \leq A_p^k \int_{\mathbb{R}^d} |f_\omega(x)|^p dx \quad (6)$$

$$(S_\omega f)(x) = ?. \quad x, y \in \mathbb{C}^M, \quad \langle x, y \rangle = \sum_{i=1}^M x_i \bar{y}_i$$

$$\begin{aligned} S_\omega(f)(x) &= S_B(f_\omega)(x) = S_B\left(\sum_{j=1}^M \bar{\omega}_j f_j\right)(x) = \sum_{j=1}^M \bar{\omega}_j S_B(f_j)(x) = |\langle S_B(f)(x), \omega \rangle| \\ &= |S_B(f)(x)| \left| \left\langle \frac{S_B(f)(x)}{|S_B(f)(x)|}, \omega \right\rangle \right| = \left(\sum_{j=1}^M |S_B(f_j)(x)|^2 \right)^{\frac{1}{2}} |\varphi(\omega, S_B(f)(x))| \end{aligned}$$

Integrate both sides of (6) with respect to ω (before integrating in x).

$$\begin{aligned} \text{LHS} &= \int_{\mathbb{R}^d} \left(\int_{|\omega|=1} |S_\omega(f)(x)|^p d\omega \right) dx \\ &= \int_{\mathbb{R}^d} \left(\sum_{j=1}^M |S_B(f_j)(x)|^2 \right)^{\frac{p}{2}} \underbrace{\left(\int_{|\omega|=1} |\Phi(\omega, S_B(f)(x))|^p d\omega \right)}_{\gamma_p} dx \end{aligned}$$

$$0 \neq \gamma_p = \int_{|\omega|=1} |\Phi(\omega, 1)|^p d\omega$$

For fixed $\nu \in S^{d-1}$ $\int_{S^{d-1}} |\langle \omega, \nu \rangle|^p d\sigma_\omega = \omega_{d-2} \int_{-1}^1 (1-t^2)^{\frac{d-3}{2}} dt$.

Similarly,

$$\text{RHS} = \int_{\mathbb{R}^d} \left(\sum_{j=1}^M |f_j(x)|^2 \right)^{\frac{p}{2}} dx \gamma_p$$

$$(6) \Rightarrow \left\| \left(\sum_{j=1}^M |S_B(f_j)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^d)} \leq A_p \left\| \left(\sum_{j=1}^M |f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^d)}$$

Step 3: From balls to half-spaces

B_R^u = ball of radius R centered at Ru . Upshot: $B_R^u \rightarrow \{\xi u > 0\}$ as $R \rightarrow \infty$.

Translation

$$(T_y f)(x) = f(x - y)$$

$$\widehat{T_y f}(\xi) = e^{i\xi y} \hat{f}(\xi)$$

$$S_{B_R^u}(f)(x) = e^{2\pi i u R x} S_{B_R}(f e^{-2\pi i u R x})$$

(4) implies

$$\left\| \left(\sum |S_{B_R^{u_j}}(f_j)|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \leq A_p \left\| \left(\sum |f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^p}$$

Let $R \rightarrow \infty$ to finish:

$$\therefore S_{B_R^{u_j}}(f_j) \rightarrow S^{u_j}(f_j) \quad R \rightarrow \infty \text{ (in } L^2)$$

\therefore there exists an almost everywhere converging subsequence, done. \square

$\widehat{Sf} = 1_{B(0,1)} \hat{f}$. S is not bounded in $L^p(\mathbb{R}^d)$ unless $d = 1$ or $p = 2$. Focus on multiplier operator for the half-space (S^u) , $d = 1$.

$$(S^+ f)(x) = \int_0^\infty \hat{f}(\xi) e^{2\pi i x \xi} d\xi \quad (f \in L^2)$$

$$|(S^+ f)(x)| \geq \frac{c}{|x|} \quad \text{if } |x| \geq \frac{1}{2} \tag{7}$$

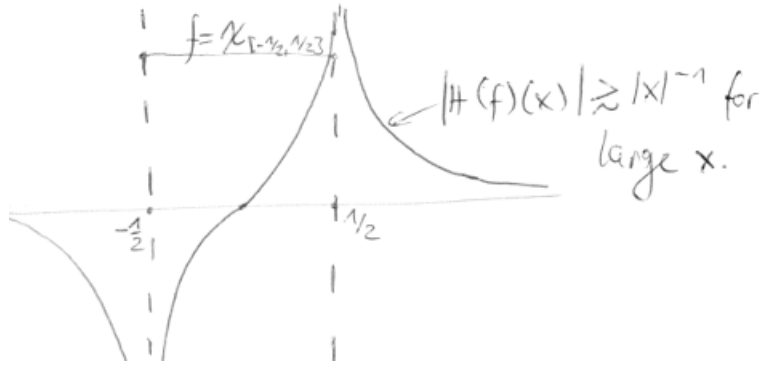
Proof.

$$(S^+ f)(x) = \lim_{\varepsilon \rightarrow 0} \int_0^\infty \hat{f}(\xi) e^{2\pi i (x+i\varepsilon)\xi} d\xi \quad \text{in } L^2$$

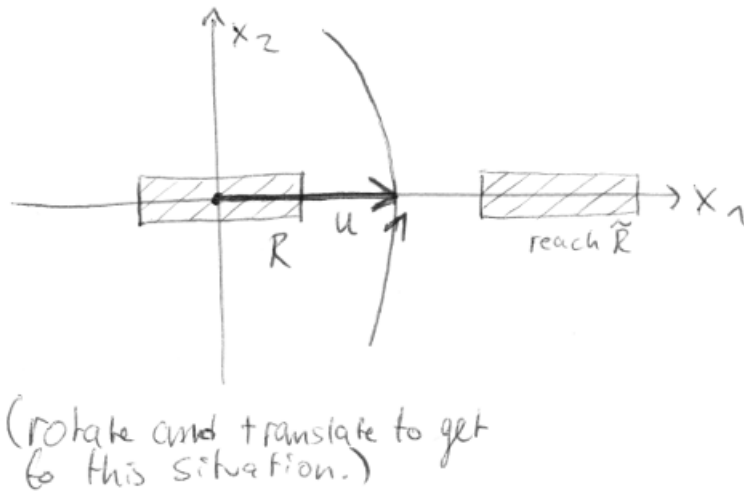
$$\int_{-\infty}^\infty \left(\int_0^\infty e^{-2\pi i y \xi} e^{2\pi i (x+i\varepsilon)\xi} d\xi \right) f(y) dy = \frac{1}{2\pi i} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{y - x - i\varepsilon} dy$$

This has absolute value $\lesssim c|x|$ if $|x| \geq \frac{1}{2}$.

Alternative proof: $|H(f)(x)| \geq |x|^{-1}$ for large x , $S^+ = \frac{1}{2}(I + iH)$.



□



For $R = (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2^{N+1}}, \frac{1}{2^{N+1}})$

$$1_R = 1_{(-\frac{1}{2}, \frac{1}{2})} \otimes 1_{(-2^{-(N+1)}, 2^{-(N+1)})}$$

If u points in the direction of x_1 then

$$(S^u 1_R)(x_1, x_2) = (S^+ 1_{(-\frac{1}{2}, \frac{1}{2})})(x_1) 1_{(-2^{-(N+1)}, 2^{-(N+1)})}(x_2)$$

$$(7) \Rightarrow |S^u(1_R)| \geq c' 1_{\tilde{R}}$$

Similarly for any 1×2^{-N} rectangle R_j . $u_j \in S^1$ in the positive direction of the longest side of R_j . Rotate and translate, then we get

$$|S^{u_j}(1_{R_j})| \geq c' 1_{\tilde{R}_j} \quad (8)$$

Take R_1, \dots, R_{2^N} to be the collection given by ε -Kakeya construction, plug that into result from lemma, get a contradiction.

Key: $p < 2$ and $d = 2$.

Lemma with $f_j = 1_{R_j}$ and $M = 2^N$

$$\begin{aligned} c' &\leq \left\| \left(\sum_{j=1}^{2^N} |S^{u_j}(1_{R_j})|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)} \quad (8), \quad \left| \bigcup \tilde{R}_j \right| = 1 \\ &\leq A_p \left\| \left(\sum_{j=1}^{2^N} |1_{R_j}|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)} = A_p \underbrace{\left(\int_E \left(\sum_{j=1}^{2^N} |1_{R_j}|^2 \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}}}_I \end{aligned}$$

$$\begin{aligned} I &\leq |E|^{\frac{1}{q}} \left(\int \left(\sum |1_{R_j}|^2 \right)^{\frac{p}{2} \frac{2}{p}} dx \right)^{\frac{1}{p} \frac{p}{2}} \quad \text{Hölder} \\ &= |E|^{\frac{1}{q}} \sum_{j=1}^{2^N} |R_j| = |E|^{\frac{1}{q}} \end{aligned}$$

$$E = \bigcup_{j=1}^{2^N} R_j, \quad |E| < \varepsilon, \quad \frac{1}{q} + \frac{1}{p/2} = 1, \quad \frac{1}{q} = 1 - \frac{p}{2}, \quad \frac{1}{pq} = \frac{1}{p} \left(1 - \frac{p}{2}\right) > 0$$

In the end we get

$$c' \leq A_p \varepsilon^{\frac{1}{pq}}.$$

Let $\varepsilon \rightarrow 0^+$ to finish.

$d > 2$:

$$f_j(\underset{\in \mathbb{R}^d}{x}) = f_j(x_1, x_2, \underset{\in \mathbb{R}^{d-2}}{x'}) = 1_{R_j}(x_1, x_2) f(x') \quad f \in S(\mathbb{R}^{d-2})$$

$p > 2$:

$$\langle Sf, g \rangle = \langle \widehat{Sf}, \hat{g} \rangle = \langle 1_B \hat{f}, \hat{g} \rangle = \langle \hat{f}, 1_B \hat{g} \rangle = \langle f, Sg \rangle \quad S = S^*$$

□

Oscillatory integrals in harmonic analysis Stein (VIII, IX), Stein-Shakarchi (Chapter 8), Sogge

- averaging operators
- restriction theory
- Bochner-Riesz summability

Motivation (in \mathbb{R}^3)

$$(Af)(x) = \int_{\mathbb{S}^2} f(x-y) d\sigma_y = \frac{1}{4\pi} f * \sigma(x)$$

σ surface measure in \mathbb{S}^2 .

Smoothing properties:

$$\left\| \frac{\partial}{\partial x_j} A(f) \right\|_{L^2(\mathbb{R}^3)} \lesssim \|f\|_{L^2(\mathbb{R}^3)} \quad (j = 1, 2, 3) \quad (9)$$

$f \in L^2 \Rightarrow$

$$\|f * \sigma\|_{L^2} \leq \|f\|_{L^2} \underbrace{|\sigma|(\mathbb{R}^3)}_{< \infty}$$

(use Minkowski integral inequality) $\therefore f * \sigma \in L^2$ if $f \in L^2$.

Idea $\widehat{(f * \sigma)} = \widehat{f} \widehat{\sigma}$

$$\widehat{\left(\frac{\partial}{\partial x_j} A(f)\right)}(\xi) = \xi_j \widehat{f}(\xi) \widehat{\sigma}(\xi),$$

so we should study $\widehat{\sigma}$.

without loss of generality $\xi = (0, 0, |\xi|)$ because the integral is spherically symmetric.

$$\widehat{\sigma}(\xi) = \int_{\mathbb{S}^2} e^{-2\pi i \omega \xi} d\sigma_\omega = 2\pi \int_0^\pi e^{-2\pi i |\xi| \cos \theta} \sin \theta d\theta = 2\pi \int_{-1}^1 e^{2\pi i |\xi| t} dt = \frac{2 \sin(2\pi |\xi|)}{|\xi|}$$

$$t = \cos \theta \quad dt = -\sin \theta d\theta$$

$$\therefore |\widehat{\sigma}(\xi)| \lesssim (1 + |\xi|)^{-1}$$

(9) follows from this and Plancherel.

This also generalizes to higher dimensions. There we get a factor $(1-t^2)^{\frac{d-3}{2}}$ in the last integral.

Oscillatory integrals

$$I(\lambda) = \int_{\mathbb{R}^d} e^{i\lambda \phi(x)} \psi(x) dx$$

where $\lambda \in \mathbb{R}$ is the oscillatory parameter, $\phi \in C^\infty(\mathbb{R}^d, \mathbb{R})$ the phase and $\psi(x) \in C$ the amplitude.

Q.: How does $I(\lambda)$ behave for large $|\lambda|$? General principle: Main contribution comes from the critical points of the phase, $x_0 : \nabla \phi(x_0) = 0$.

Principle of non-stationary phase $\phi \in C^\infty, \psi \in C_0^\infty : |\nabla \phi(x)| > 0 (\forall x \in \text{supp } \psi)$. Then for any $N \in \mathbb{N}$

$$|I(\lambda)| \leq c_N |\lambda|^{-N}$$

Proof. involves integration by parts □

$d = 1$:

$$I_1(\lambda) = \int_a^b e^{i\lambda \phi(x)} dx$$

$$0 < a < b < \infty \quad \psi(x) = \chi_{[a,b]}(x) \quad \text{which is rough!}$$

This means we will not get such a fast decay

Lemma (van der Corput (I)). $\phi \in C^2, \phi'$ monotonic, $|\phi'(x)| \geq 1 (\forall x \in [a, b])$. Then

$$|I_1(\lambda)| \leq \frac{3}{|\lambda|} \quad (\forall \lambda > 0)$$

Remark. (i) 3 is neither important nor sharp; independence of a, b, ϕ is the key!

(ii) Order of decrease in λ is sharp ($\phi(x) = x : I_1(\lambda) = \frac{e^{i\lambda b} - e^{i\lambda a}}{i\lambda}$)

(iii) monotonicity of ϕ' is essential

Proof. Integrate by parts(...) □

What if critical points are present? ($d = 1$)
 $x_0 : \phi'(x_0) = 0$ (critical point) and $\phi''(x_0) \neq 0$ (non degenerate), e.g. $\phi(x) = x^2$, $x_0 = 0$. In this case

$$\int_{\mathbb{R}} e^{i\lambda x^2} \psi(x) dx = c_0 \lambda^{-\frac{1}{2}} + \mathcal{O}(|\lambda|^{-\frac{3}{2}}) = \sum_{k=0}^N a_k \lambda^{-\frac{1}{2}-k} + \mathcal{O}(|\lambda|^{-\frac{3}{2}-N}) \quad (\forall N, \lambda \rightarrow \infty)$$

Lemma (van der Corput (II)). $\phi \in C^2[a, b]$, $|\phi''(x)| \geq 1$ ($\forall x \in [a, b]$). Then

$$|I_1(\lambda)| \leq \frac{8}{\lambda^{\frac{1}{2}}} \quad (\forall \lambda > 0)$$

Remark. More generally: $\mathcal{O}(|\lambda|^{\frac{1}{k}})$ if $|\phi^{(k)}| \geq 1$.

Proof. Integration by parts not needed. Instead split up region in small area around critical point with properly chosen size, and rest, and then use results from above. \square

Corollary. Same assumptions as van der Corput (II). $\psi \in C^1[a, b]$.

$$|\int e^{i\lambda \phi(x)} \psi(x) dx| \leq c_{\psi} \lambda^{-\frac{1}{2}}$$

Application: Asymptotics of Bessel functions

$$J_m(r) = \frac{1}{2\pi} \int_0^{2\pi} e^{ir \sin x} e^{-imx} dx \quad \phi(x) = \sin x, \quad \psi(x) = e^{-imx} \quad (m \in \mathbb{Z})$$

Corollary.

$$|J_m(r)| \leq cr^{-\frac{1}{2}} \quad r \rightarrow \infty$$

Recall: Averaging operator in \mathbb{R}^d ($d > 1$) is

$$(Af)(x) = (f * \sigma)(x) \quad \sigma \text{ surface measure on } \mathbb{S}^{d-1}$$

Theorem. $f \mapsto A(f)$ is bounded from $L^2(\mathbb{R}^d)$ to $L_k^2(\mathbb{R}^d)$ with $= \frac{d-1}{2}$.

Proof.

$$\begin{aligned} \hat{\sigma}(\xi) &= 2\pi |\xi|^{-\frac{d}{2}+1} \underbrace{J_{\frac{d}{2}-1}(2\pi|\xi|)}_{=\mathcal{O}(|\xi|^{-\frac{1}{2}}), |\xi| \rightarrow \infty} \\ \therefore |\hat{\sigma}(\xi)| &= \mathcal{O}(|\xi|^{-\frac{d-1}{2}}) \quad |\xi| \rightarrow \infty \end{aligned}$$

\square

"What is van der Corput's lemma in higher dimension?" (Carbery-Wright, 2000)

$$I(\lambda) = \int_{\mathbb{R}^d} e^{i\lambda \phi(x)} \psi(x) dx$$

(ϕ smooth, ψ smooth, compactly supported)
 nondegeneracy hypothesis

$$\det(\nabla^2 \phi)(x) \neq 0 \quad \forall x \in \text{supp}(\psi)$$

Theorem. Under above assumptions

$$|I(\lambda)| = \mathcal{O}(|\lambda|^{-\frac{1}{2}}) \quad \lambda \rightarrow \infty$$

Remark. (i) Decay rate is sharp

(ii) Proof uses TT^* method: $|I(\lambda)|^2 = I(\lambda)\overline{I(\lambda)}$

(iii) variant: $\text{rk}(\nabla^2\phi) \geq m$ for some $0 < m \leq d$ on $\text{supp}(\psi)$. Then

$$|I(\lambda)| = \mathcal{O}(|\lambda|^{-\frac{m}{2}})$$

Application: Fourier transform of surface-carried measures

Recall: $(\mathbb{S}^{d-1}, \sigma)$

$$|\hat{\sigma}(\xi)| \lesssim (1 + |\xi|)^{-\frac{d-1}{2}}$$

(not a Bessel coincidence)

(local) C^∞ -hypersurface M . After translation and rotation $x_0 = 0$, $T_{x_0}M = \{x_d = 0\}$. M can be represented as

$$M = \{(x', x_d) \in B \subset \mathbb{R}^d : x_d = \varphi(x')\}$$

Can arrange $\varphi(0) = 0 = (\nabla_{x'}\varphi)(x')|_{x'=0}$.

$$\varphi(x') = \frac{1}{2} \sum_{k,j=1}^{d-1} \underbrace{\frac{\partial^2 \varphi}{\partial x_k \partial x_j}}_{(a_{jk})} x_k x_j + \mathcal{O}(|x'|^3) = \frac{1}{2} \sum_{j=0}^{d-1} k_j x_j^2 + \mathcal{O}(|x'|^3)$$

(a_{jk}) $(d-1) \times (d-1)$ \mathbb{R} -valued symmetric matrix \therefore diagonalizable. k_j principal curvatures of M at x_0 . $k := \prod_{j=1}^{d-1} k_j$ is the Gaussian curvature of M at x_0 ($k = \det(\nabla^2\varphi)$)

E.g.

(i) $\mathbb{S}^{d-1} \subset \mathbb{R}^d$. $k_j = 1$ ($\forall j$) $\therefore k = 1$

(ii) $\{x_3 = \underbrace{x_1^2 - x_2^2}_{\varphi(x_1, x_2)}\} \subset \mathbb{R}^3$, $\frac{1}{2}\nabla^2\varphi(x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(iii) $\{x_1^2 = |x'|^2 : x \neq 0\}$, $x' \in \mathbb{R}^{d-1}$. $d-2$ identical nonvanishing principal curvatures $x_d^{-2} + 1$ vanishing principal curvature.

surface measure σ

$$\int_M f d\sigma = \int_{\mathbb{R}^{d-1}} f(x', \varphi(x')) \underbrace{\sqrt{1 + |\nabla_{x'}\varphi(x')|^2}}_{d\sigma \text{ in our coordinate sys.}} dx'$$

$$d\mu = \psi d\sigma, \quad \psi \in C_0^\infty(M, \sigma)$$

is a surface carried measure.

$$\hat{\mu}(\xi) = \int_M e^{-2\pi i x \xi} d\mu(x) = \int_M e^{-2\pi i x \xi} \psi(x) d\sigma_x$$

is bounded on \mathbb{R}^d because $|\mu|(\mathbb{R}^d) < \infty$.

Theorem. Hypersurface $M \subset \mathbb{R}^d$ with nonvanishing Gaussian curvature at each point of $\text{supp}(\psi)$. Then

$$|\hat{\mu}(\xi)| = \mathcal{O}(|\xi|^{-\frac{d-1}{2}}), \quad |\xi| \rightarrow \infty$$

Corollary. If M has at last m non vanishing principal curvatures (at each point of $\text{supp}(\psi)$), then

$$|\hat{\mu}(\xi)| = \mathcal{O}(|\xi|^{-\frac{m}{2}}), \quad |\xi| \rightarrow \infty$$

Last time: oscillatory integrals and averaging operators

$$(Af)(x) = (F * \sigma)(x) = \int_{\mathbb{S}^{d-1}} f(x-y) d\sigma_y \quad (d > 1)$$

Smoothin property: $f \mapsto A(f)$ is bounded from $L^2(\mathbb{R}^d)$ to $L_k^2(\mathbb{R}^d)$ with $k = \frac{d-1}{2}$. Here we used

$$|\hat{\sigma}(\xi)| \lesssim |\xi|^{-\frac{d-1}{2}}.$$

A few weeks ago:

$$R(f)(t, \gamma) = \int_{P_{t, \gamma}} f$$

where $P_{t, \gamma} = \{x \in \mathbb{R}^d : x\gamma = t\}$.

$$R^*(f)(\gamma) = \sup_{t \in \mathbb{R}} |R(f)(t, \gamma)|$$

if $d \geq 3$ then

$$\int_{\mathbb{S}^{d-1}} R^*(f)(\gamma) d\sigma_\gamma \lesssim \|f\|_{L^2} + \|f\|_{L^2}$$

This estimate was based on

$$\int_{\mathbb{S}^{d-1}} \int_{-\infty}^{\infty} |\hat{R}(f)(\lambda, \gamma)|^2 |\lambda|^{d-1} d\lambda d\sigma_\gamma = 2 \int_{\mathbb{R}^d} |f(x)|^2 dx$$

due to $\hat{R}(f)(\lambda, \gamma) = \hat{f}(\lambda\gamma)$. Consider $d = 3$ then this becomes

$$\int_{\mathbb{S}^2} \int_{\mathbb{R}} \left| \frac{d}{dt} R(f)(t, \gamma) \right|^2 dt d\sigma_\gamma = 8\pi^2 \int_{\mathbb{R}^3} |f(x)|^2 dx \quad (10)$$

by Plancherel $t \leftrightarrow \lambda$. Note, that something like this also holds for higher dimensions.

Now consider the following "linearized" version of the Radon transform:

$$R_B(f) = \int_{\mathbb{R}^{d-1}} f(y', x_d - B(x', y')) dy' = \int_{M_x} f$$

where $x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$, $y = (y', y_d)$ and $B : \mathbb{R}^{d-1} \times \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ is a nondegenerate bilinear form, and

$$M_x = \{(y', y_d) \mid y_d = x_d - B(x', y')\}.$$

E.g. $B(x', y') = \langle x', y' \rangle$ (usual inner product on \mathbb{R}^d). $y_d = x_d - \langle x', y' \rangle \iff \langle x', y' \rangle + y_d = x_d \iff \langle (x', 1), (y', y_d) \rangle = x_d$. The map

$$\begin{aligned} \mathbb{R}^d &\rightarrow \{\text{affine hyperplanes on } \mathbb{R}^d\} \\ x &\mapsto M_x \end{aligned}$$

is injective and surjective onto {hyperplanes not orthogonal to $M_0 = \{x_d = 0\}$ }. The excerpted collection of hyperplanes is lower dimensional, so we can think of R_B as a substitute for R .

An analogue of (10) is

$$\int_{\mathbb{R}^d} \left| \frac{\partial}{\partial x_3} R_B(f)(x) \right|^2 dx = c_B \int_{\mathbb{R}^3} |f(x)|^2 dx$$

for $f \in C_0^0(\mathbb{R})$

Proof.

$$\begin{aligned} \int_{\mathbb{R}^3} \left| \frac{\partial}{\partial x_3} R_B(f)(x) \right|^2 dx &= \int_{\mathbb{R}} \int_{\mathbb{R}^2} \left| \underbrace{\left(\frac{\partial}{\partial x_3} R_B(f) \right)^\wedge(x', \xi_3)}_{=2\pi\xi_3 \int_{\mathbb{R}^2} e^{-2\pi i \xi_3 B(x', y')} \hat{f}(y', \xi_3) dy'} \right|^2 dx' d\xi_3 \quad x = (x', x_3) \end{aligned}$$

The last equality follows from

$$\begin{aligned} \hat{R}_B(f)(x', \xi_3) &= \int_{\mathbb{R}} e^{-2\pi i \xi_3 x_3} R_B(f)(x', x_3) dx_3 \\ &= \int e^{-2\pi i \xi_3 x_3} \int_{\mathbb{R}^2} f(y', \underbrace{x_3 - B(x', y')}_{y_3}) dy' dx_3 \\ &= \int \int_{\mathbb{R}^{2+1}} e^{-2\pi i \xi_3 (y_3 + B(x', y'))} f(y', y_3) dy' dy_3 \\ &= \int_{\mathbb{R}^2} e^{-2\pi i \xi_3 B(x', y')} \underbrace{\left(\int_{\mathbb{R}} e^{-2\pi i \xi_3 y_3} f(y', y_3) dy_3 \right)}_{\hat{f}(y', \xi_3)} dy' \end{aligned}$$

Since B is nondegenetare $\exists C : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ linear, invertible such that $B(x', y') = \langle C(x'), y' \rangle$. Changle variables $\xi_3 C(x') = u \in \mathbb{R}^2$ is well defined since C is invertible. $\therefore \xi_3 B(x', y') = \langle \xi_3 C(x'), y' \rangle = \langle u, y' \rangle \therefore \xi_3^2 |\det C| dx' = du$. Then the first integral becomes

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} e^{-2\pi i u y'} \hat{f}(y', \xi_3) dy' \right|^2 \frac{du}{|\det C|} d\xi_3 \simeq \int \int |\hat{f}(y', \xi_3)|^2 dy' d\xi_3 \simeq \int_{\mathbb{R}^3} |f(y)|^2 dy$$

by 2D-Plancherel $y' \leftrightarrow u$ and 1D-Plancherel $\xi_3 \leftrightarrow y_3$. \square

Rotational curvature Both the averaging operator A and the Radon transform R_B are of the form $f \mapsto \int_{M_x} f(y) d_x(y)$, where for each $x \in \mathbb{R}^d$ we have a manifold M_x (depending smoothly on x) over which we integrate.

$A : M_x = x + M_0 \quad M_0$ curved

$R_B : M_x = \{y = (y', y_d) \mid y' \in \mathbb{R}^{d-1}, y_d = x_d - B(x', y')\} \quad \text{flat but } M_x \text{ rotates as } x \text{ varies.}$

Start with a smooth "double defining" function $\rho = \rho(x, y)$ given in a ball in $\mathbb{R}^d \times \mathbb{R}^d$. Its rotational matrix is

$$M = M(\rho) = \begin{pmatrix} \rho & \frac{\partial \rho}{\partial y_1} & \cdots & \frac{\partial \rho}{\partial y_d} \\ \frac{\partial \rho}{\partial x_1} & & & \\ \vdots & \left(\frac{\partial^2 \rho}{\partial x_j \partial y_k} \right)_{j,k=1}^d & & \\ \frac{\partial \rho}{\partial x_d} & & & \end{pmatrix}$$

containing the mixed Hessian. The rotational curvature of ρ is

$$\text{rotcurv}(\rho) := \det(M(\rho))$$

We want $\rho = 0 \Rightarrow \text{rotcurv}(\rho) \neq 0$. $M_x = \{y : \rho(x, y) = 0\}$. The fact that $\nabla_y \rho \neq 0$ if $\rho = 0$ implies that M_x is a smooth surface (or something like that).

Examples/Properties:

- (i) Translational invariant case: $\rho(x, y) = \rho(x - y)$. $M_x = M_0 + x$ and $\text{rotcurv}(\rho) \neq 0$ iff M_0 has nonvanishing Gaussian curvature.
- (ii) Case of R_B : $\rho(x, y) = y_d - x_d + B(x', y')$. $\text{rotcurv}(\rho) \neq 0$ iff B is nondegenerate.
- (iii) $\tilde{\rho}(x, y) = a(x, y)\rho(x, y)$ with $a(x, y) \neq 0$. Then $\tilde{\rho}$ is another defining function for $\{M_x\}$, and $\text{rotcurv}(\tilde{\rho}) = a^{d+1} \text{rotcurv}(\rho)$
- (iv) $x \mapsto \psi_1(x)$, $y \mapsto \psi_2(y)$ local diffeomorphisms of \mathbb{R}^d . For $\tilde{\rho}(x, y) = \rho(\psi_1(x), \psi_2(y))$ then $\text{rotcurv}(\tilde{\rho}) = J_1(x)J_2(y) \text{rotcurv}(\rho)$ with $J_k = \det \text{jac}(\psi_k)$, $k = 1, 2$

Define the general averaging operator A by

$$A(f)(x) = \int_{M_x} f(y) \psi_0(x, y) d\sigma_x(y)$$

initially for $f \in C_0^0(\mathbb{R}^d)$. $M_x = \{y \mid \rho(x, y) = 0\}$ with induced Lebesgue measure $d\sigma_x$. ρ is a double defining function with $\text{rotcurv}(\rho) \neq 0$. $\psi_0 \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$.

Theorem. The operator A extends to a bounded linear map from $L^2(\mathbb{R}^d)$ to $L_k^2(\mathbb{R}^d)$ where $k = \frac{d-1}{2}$.

Proof. Step 1: Oscillatory integral operators (FIOs)

Step 2: L^2 estimate via dyadic decomposition of "almost-orthogonal" parts.

Step 1: Define

$$T_\lambda(f)(x) = \int_{\mathbb{R}^d} e^{i\lambda\varphi(x, y)} \psi(x, y) f(y) dy$$

where $\varphi \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and $\psi \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ with

$$\det(\nabla_{x, y}^2 \varphi) = \det\left(\frac{\partial^2 \varphi}{\partial x_k \partial y_j}\right)_{k, j=1}^d \neq 0 \quad \text{on } \text{supp}(\psi)$$

last week:

$$I(\lambda) = \int_{\mathbb{R}^d} e^{i\lambda\varphi(y)} \psi(y) dy \Rightarrow |I(\lambda)| \lesssim |\lambda|^{-\frac{d}{2}}$$

if $\det \nabla^2 \varphi \neq 0$ on $\text{supp}(\psi)$. We used $|I(\lambda)|^2 = I(\lambda) \overline{I(\lambda)}$ where then appeared the term $\varphi(u + y) - \varphi(y)$.

Proposition. Under the above assumptions,

$$\|T_\lambda\|_{L^2 \rightarrow L^2} \leq c\lambda^{-\frac{d}{2}} \quad \forall \lambda > 0$$

Proof. Similar to its scalar version, omitted. □

Consequence: For the corresponding oscillatory integral operator involving ρ

$$S_\lambda(f)(x) = \int_{\mathbb{R} \times \mathbb{R}^d} e^{i\lambda y_0 \rho(x,y)} \psi(x, y_0, y) f(y) dy_0 dy$$

with $(y_0, y) \in \mathbb{R} \times \mathbb{R}^d$ and $\psi \in C_0^\infty$ is supported away from $y_0 = 0$.

Corollary. If $\rho = 0 \Rightarrow \text{rotcurv}(\rho) \neq 0$ then

$$\|S_\lambda\|_{L^2 \rightarrow L^2} \leq c\lambda^{-\frac{d+1}{2}}$$

Proof of Corollary. $\bar{x} = (x_0, x), \bar{y} = (y_0, y) \in \mathbb{R} \times \mathbb{R}^d$. Set $\varphi(\bar{x}, \bar{y}) = x_0 y_0 \rho(x, y)$ then $\det(\nabla_{x,y}^2 \varphi) = (x_0 y_0)^{d+1} \text{rotcurv}(\rho)$. Define

$$F_\lambda(x_0, x) = F_\lambda(\bar{x}) = \int_{\mathbb{R}^{d+1}} e^{i\lambda \varphi(\bar{x}, \bar{y})} \psi_1(x_0, x, y_0, y) f(y) dy_0 dy$$

with $\psi_1(1, \lambda, y_0, y) = \psi(x, y_0, y)$ (from S_λ). Then $S_\lambda(f)(x) = F_\lambda(1, x)$.

Observation: If $I \subset \mathbb{R}$ interval of length 1, $g \in C^1(I)$, $x_0 \in I$ then

$$|g(u_0)|^2 \leq 2 \left(\int_I |g(u)|^2 du + \int_I |g'(u)|^2 du \right)$$

Apply this observation with $I = [1, 2]$, $u_0 = 1$, $g(u) = F_\lambda(u, x)$ to get:

$$\int_{\mathbb{R}^d} |S_\lambda(f)(x)|^2 dx \leq 2 \left(\int |F_\lambda(x_0, x)|^2 dx + \int \left| \frac{\partial}{\partial x_0} F_\lambda(x_0, x) \right|^2 dx_0 dx \right)$$

The first integral is $\lesssim \lambda^{-(d+1)} \|f\|_{L^2}^2$ by Proposition with \mathbb{R}^{d+1} instead of \mathbb{R}^d .

$$\frac{\partial}{\partial x_0} (e^{i\lambda x_0 y_0 \rho(x,y)}) = \frac{\partial}{\partial y_0} (e^{i\lambda x_0 y_0 \rho(x,y)}) \frac{y_0}{x_0}$$

we integrate by parts somewhere and use our form of φ . Therefore the second summand also satisfies the desired estimate. \square

Step 2: Dyadic decomposition of A .

Co-area formula (see Evans-Cariery (?), Stein-Shakarchi IV, Exercice 8, Ch. 8): Fix $h \in S(\mathbb{R})$ such that $\int h = 1$, $M = \{x \in \mathbb{R}^d \mid \rho(x) = 0\}$. Then

$$\int_M f \frac{d\sigma}{|\nabla \rho|} = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{\mathbb{R}^d} h\left(\frac{\rho(x)}{\varepsilon}\right) f(x) dx.$$

E.g. $\rho(x) = |x| - 1$, $M = \mathbb{S}^{d-1}$, $\frac{\nabla}{\rho}(x) = x|x|, \therefore |\nabla \rho| = 1$

$$\int_{\mathbb{S}^{d-1}} f d\varepsilon = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{\mathbb{R}^d} h\left(\frac{|x| - 1}{\varepsilon}\right) F(x) dx$$

where $f = F|_{\mathbb{S}^{d-1}}$.

$$A(f)(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\mathbb{R}^d} h(\rho(x, y)\varepsilon) \psi(x, y) f(y) dy$$

where $\psi(x, y) = \psi_0(x, y)|\nabla_y \rho| \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$. Let $\gamma \in C_0^\infty(\mathbb{R})$ such that $\gamma = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$ and 0 on $[-1, 1]^c$. Let $h = \hat{\gamma}$. Then

$$h(\rho) = \int_{\mathbb{R}} e^{2\pi i \xi u} \gamma(u) du.$$

Then

$$\int h = \int \hat{\gamma} = \gamma(0) = 1$$

and

$$\int_{\mathbb{R}} e^{2\pi i u \rho} \gamma(\varepsilon u) du = (\delta_\varepsilon \gamma)^\vee(\rho) = \varepsilon^{-1} \gamma^\vee(\varepsilon^{-1} \rho) = \varepsilon^{-1} h(\varepsilon^{-1} \rho)$$

where $\delta_\varepsilon \gamma(u) = \gamma(\varepsilon u)$. Choose $\varepsilon = 2^{-r}$, $r \in \mathbb{N}$. Note

$$\gamma(2^{-r} u) = \gamma(u) + \sum_{k=1}^r (\gamma(\frac{u}{2^k}) - \gamma(\frac{u}{2^{k-1}}))$$

Let $r \rightarrow \infty$ to get

$$1 = \gamma(u) + \sum_{k=1}^{\infty} \eta(\frac{u}{2^k})$$

where $\eta(\cdot) = \gamma(\cdot) - \gamma(2\cdot)$ because γ is continuous. Then $\eta \in C_0^\infty(\mathbb{R})$, $\text{supp}(\eta) \subset \{\frac{1}{4} \leq |u| \leq 1\}$. Whenever f is continuous we get by Fourier inversion that

$$\begin{aligned} A(f)(x) &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\mathbb{R}^d} \int_{\mathbb{R}} e^{2\pi i u \frac{\rho(x, y)}{\varepsilon}} \gamma(u) \psi(x, y) f(y) du dy \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^d} \int_{\mathbb{R}} e^{2\pi i u \rho(x, y) \gamma(\varepsilon u)} \psi(x, y) f(y) du dy \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}} e^{2\pi i u \rho(x, y) \gamma(u)} \psi(x, y) f(y) du dy \\ &= \sum_{k=1}^{\infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}} e^{2\pi i u \rho(x, y) \eta(\frac{u}{2^k})} \psi(x, y) f(y) du dy \end{aligned}$$

since $\gamma(\varepsilon u) \rightarrow \gamma(0) = 1$ $\varepsilon \rightarrow 0$. Call the summands $A_k(f)(x)$. Properties of A_k :

(i) $f \in L^2(\mathbb{R}^d) \Rightarrow$

$$A_k(f) \in C_0^\infty(\mathbb{R}^d)$$

(ii)

$$\|A_k(f)\|_{L^2} \leq c 2^{-k(\frac{d-1}{2})} \|f\|_{L^2}.$$

Recall $\|S_\lambda\|_{L^2 \rightarrow L^2} \lesssim \lambda^{-\frac{d+1}{2}i}$ and change variables in the definition of $A_k(f)$.

(iii) $\exists m : |j - k| \geq m \forall N$

$$\|(A_k^* A_j)(f)\|_{L^1} \lesssim_N 2^{-N \max(k, j)} \|f\|_{L^2}.$$

Similarly for $A_k A_j^*$. For the proof, invoke nonstationary phase. Also, recall $|I(\lambda)|^2 = I(\lambda) \overline{I(\lambda)}$.

(iv) $A_k^{(\alpha)} = (\frac{\partial}{\partial x})^\alpha A_k$. Then

$$\|A_k^{(\alpha)}\|_{L^2 \rightarrow L^2} \lesssim 2^{k|\alpha|} 2^{-k(\frac{d-1}{2})}$$

and

$$\|A_k^{(\alpha)} (A_j^{(\alpha)})^*\|_{L^2 \rightarrow L^2} \lesssim_{\alpha, N} 2^{-N \max(k, j)}.$$

Step 3: Almost-orthogonality

Assume that $\{T_k\}_{k=1}^\infty$ is a sequence of bounded operators on $L^2(\mathbb{R}^d)$ and that $\{a(k)\}_{k \in \mathbb{Z}}$ are positive constants with

$$A = \sum_{k \in \mathbb{Z}} a(k) < \infty.$$

Lemma (Cotlar-Knapp-Stein). Assume for $\|T_k T_j^*\|_{L^2 \rightarrow L^2}$ that $\|T_k^* T_j\| \leq a(k-j)^2$. Then, for every r ,

$$\left\| \sum_{k=0}^r T_k \right\| \leq A.$$

Note, that the bound A is independent of r .

Write $T = \sum_{k=0}^r T_k$. Recall $\|T\|^2 = \|T^* T\|$ since $\|AB\| \leq \|A\| \|B\|$ and $\|Tx\|^2 = \langle Tx, Tx \rangle = \langle x, T^* T x \rangle \leq \|T^* T\| \|x\|^2$ and plug in an extremizing sequence of $\|Tx\|$ for x . Then $\|T\|^4 = (\|T\|^2)^2 = \|T^* T\|^2 = \|(T^* T)^2\|$ since $T^* T$ is self adjoint. By induction we get

$$\|T\|^{2n} = \|(T^* T)^n\|.$$

$$(T^* T)^n = \sum_{i_1, i_2, \dots, i_{2n}} (T_{i_1} T_{i_2}^* \dots T_{i_{2n-1}} T_{i_{2n}}^*)$$

(i)

$$\|(T_{i_1} T_{i_2}^*) \dots (T_{i_{2n-1}} T_{i_{2n}}^*)\| \leq a(i_1 - i_2)^2 a(i_3 - i_4)^2 \dots a(i_{2n-1} - i_{2n})^2$$

(ii)

$$\|T_{i_1} (T_{i_2} T_{i_3}^*) \dots (T_{i_{2n-2}} T_{i_{2n-1}}^*) T_{i_{2n}}\| \leq A a(i_2 - i_3)^2 a(i_4 - i_5)^2 \dots a(i_{2n-2} - i_{2n-1})^2 A$$

Take geometric mean of (i) and (ii) and get

$$\|T_{i_1} T_{i_2} \dots T_{i_{2n-1}} T_{i_{2n}}^*\| \leq A a(i_1 - i_2) a(i_2 - i_3) \dots a(i_{2n-1} - i_{2n})$$

Now sum the whole thing in $i_1, i_2, \dots, i_{2n-1}$. Then sum by sum, each of the factors turns into an A . In the end the sum in i_{2n} gives a factor $r+1$. So,

$$\sum_{i_1, i_2, \dots, i_{2n}} \|T_{i_1} T_{i_2}^* \dots T_{i_{2n-1}} T_{i_{2n}}^*\| \leq A^{2n} (r+1)$$

$$\therefore \|T\| \leq A(1+r)^{\frac{1}{n}} \rightarrow A \quad n \rightarrow \infty$$

(This is called 'Tensor power trick')

Putting everything together: Case 1: d odd ($\therefore \frac{d-1}{2} \in \mathbb{Z}$). ETS $\forall |\alpha| \leq \frac{d-1}{2} \quad \forall f \in L^2(\mathbb{R}^d)$

- $\partial_x^\alpha A(f)$ exists (in the sense of distributions) and is an L^2 function
- $f \mapsto \partial_x^\alpha A(f)$ is bounded on L^2 .

For each r , set

$$\partial_x^\alpha \sum_{k=0}^r =: \sum_{k=0}^r T_k \quad (T_k = A_k^{(\alpha)})$$

The estimates 1 and 2 imply that the hypotheses of CKS are satisfied with $a(k) = c_n 2^{-|k|N}$ ($\forall N, \therefore$ can choose $N = 1$).

$$\therefore \|\partial_x^\alpha \sum_{k=0}^r A_k(f)\|_{L^2} \leq A \|f\|_{L^2}$$

where $A = \sum_{k \in \mathbb{Z}} a(k)$, provided $|\alpha| \leq \frac{d-1}{2}$.

(i) with $\alpha = 0$ we get

$$\lim_{r \rightarrow \infty} \sum_{k=0}^r A_k(f) = A(f)$$

in L^2 and therefore in the weak sense.

$$\lim_{r \rightarrow \infty} \partial_x^\alpha \sum_{k=1}^r A_k(f) = \partial_x^\alpha A(f)$$

in the weak sense and therefore in L^2 .

Conclusion

$$\|\partial_x^\alpha A(f)\|_{L^2} \leq A\|f\|_{L^2}$$

whenever $f \in C_0^0(\mathbb{R}^d)$, $|\alpha| \leq \frac{d-1}{2}$ (and d is odd)

□

today:

$$(A_t f)(x) = \int_{|y|=1} f(x - ty) d\sigma(y) = (f * \sigma_t)(x)$$

which is the average of f over the sphere of radius t centered at x ,

$$\int_{|x|=t} g(x) d\sigma_t(x) := \int_{|x|=1} g(tx) d\sigma(x).$$

This definition works fine provided g is continuous.

Question: We would like to know for general f whether $(A_t f)(x) \rightarrow f(x)$ (x -a.e.) as $t \rightarrow 0$. Recall that we already did this for balls instead of spheres.

Is $t \mapsto (A_t f)(x)$ continuous (for any x)? Is $\sup_{t_1 < t < t_2} |A_t f(x)|$ measurable in x ? This actually has to be discussed before we can think about the first question.

A priori estimate for the spherical maximal averages:

Theorem. Let $f \in C_0^0(\mathbb{R}^d)$, $d \geq 4$. Then

$$\left\| \sup_{t>0} |(A_t f)(x)| \right\|_{L_x^2(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)}.$$

Note that this can be improved up to $d \geq 2$, but fails for $d = 1$.

Key: Let $f \in C_0^1(\mathbb{R}^d)$. Then

$$(Sf)(x) = \left(\int_0^\infty \left| \frac{\partial A_t f}{\partial t}(x) \right|^2 t dt \right)^{\frac{1}{2}}.$$

This also holds for C_0^0 by density.

Lemma.

$$\sup_{t>0} |(A_t f)(x)| \leq (Mf)(x) + c(Sf)(x)$$

where Mf is the standard Hardy-Littlewood maximal function over centered balls.

Proof fo Lemma 1.

$$(A_t f)(x) = t^{-d} \int_0^t \frac{\partial}{\partial s} [s^d (A_s f)(x)] ds = I + II$$

using that the integrand equals

$$ds^{d-1}A_sf)(x) + s^d \frac{\partial(A_sf)}{\partial s}(x)$$

$$I = dt^{-d} \int_0^t s^{d-1} \left(\int_{|y|=1} f(x-sy) d\sigma(y) \right) ds = dt^{-d} \int_{B(x,t)=\{y:|x-y|\leq t\}} f(y) dy = \frac{1}{\frac{t^d}{d}} \int_{B(x,t)} f \leq \sup_{t>0} \int_{B(x,t)} f = (Mf)(x)$$

due to our choice of normalization $|B(x,t)| = \int_0^t \underbrace{\omega_{d-1}}_1 s^{d-1} ds = \frac{t^d}{d}$.

$$II = t^{-d} \int_0^t s^{d-\frac{1}{2}} \left(\frac{\partial A_sf}{\partial s}(x) s^{\frac{1}{2}} \right) ds \leq \left(\int_0^\infty \left| \frac{\partial(A_sf)(x)}{\partial s} \right|^2 s ds \right)^{\frac{1}{2}} \cdot t^{-d} \left(\int_0^t s^{2d-1} ds \right)^{\frac{1}{2}} \lesssim_d (Sf)(x)$$

□

Lemma. If $d \geq 4$ then

$$\|Sf\|_{L^2(\mathbb{R}^d)} \leq A\|f\|_{L^2(\mathbb{R}^d)}$$

Proof. $A_sf = f * \sigma_s$.

$$(\hat{A}_sf)(\xi) = \hat{f}(\xi) \hat{\sigma}_s(\xi) = \hat{f}(\xi) \hat{\sigma}(s\xi)$$

since

$$\hat{\sigma}_s(\xi) = \int_{|x|=1} e^{2\pi i x \xi} d\sigma_s(x) = \int_{|x|=1} e^{2\pi i s x \xi} d\sigma(x) = \hat{\sigma}(s\xi).$$

It follows that

$$\frac{\partial(\hat{A}_sf)}{\partial s}(\xi) = \frac{\mu(s\xi)}{s} \hat{f}(\xi)$$

where

$$\mu(\xi) = \sum_{j=1}^d \xi_j \frac{\partial \hat{\sigma}}{\partial \xi_j}(\xi) = \langle \xi, \nabla \hat{\sigma}(\xi) \rangle$$

because we differentiate with respect to a variable independent from the Fourier transform.

Key:

$$|\mu(\xi)| \leq A \min\{|\xi|, |\xi|^{-\frac{d-3}{2}}\}$$

Proof.

$$\frac{\partial \hat{\sigma}}{\partial \xi_j}(\xi) = 2\pi \int_{|x|=1} x_j e^{2\pi i x \xi} d\sigma(x)$$

Since $x_j \in [-1, 1]$ we get

$$\left| \frac{\partial \hat{\sigma}}{\partial \xi_j}(\xi) \right| \leq A(1 + |\xi|)^{-\frac{d-1}{2}}$$

The statement follows from this and Cauchy-Schwarz applied to the definition of $\mu(\xi)$. □

By Plancherel we get

$$\int_{\mathbb{R}^d} \left| \frac{\partial(A_sf)}{\partial s}(x) \right|^2 dx = \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \frac{|(s\xi)|^2}{s^2} d\xi$$

Now we multiply by s and integrate with respect to s and get

$$\int_{\mathbb{R}^d} |Sf(x)|^2 dx = \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \underbrace{\left(\int_0^\infty \frac{|\mu(s\xi)|^2}{s} ds \right)}_* d\xi$$

It is enough to show, that $*$ is bounded with respect to ξ .

$$\int_0^\infty \frac{|\mu(s\xi)|^2}{s} ds = \int_0^{\frac{1}{|\xi|}} + \int_{\frac{1}{|\xi|}}^\infty \leq A(|\xi|^2 \int_0^{\frac{1}{|\xi|}} s^2 \frac{ds}{s} + |\xi|^{-(d-3)} \int_{\frac{1}{|\xi|}}^\infty s^{-(d-3)} \frac{ds}{s}) \leq C <$$

where C is independent of ξ . The whole thing is true iff $-(d-3) - 1 < -1$, $d \leq 4$, since this takes care of the second summand and the first summand is no problem. \square

Now the theorem follows from the Lemmas and the L^2 boundedness of the maximal function. Now look at d_1 . Then the theorem fails because

$$A_t f(x) = \frac{1}{2}(f(x+t) + f(x-t))$$

and we can take as a counterexample a function nonnegative which blows up close to 0 but is in L^p for every $p < \infty$ such that $\sup_{t>0} (A_t f)(x) = \infty$ everywhere.

He said something with the wave equation and its smoothing properties for $1 \leq d \leq 3$.

$d \geq 2$: The maximal operator $f \mapsto \sup_{t>0} (A_t(f))$ is bounded on $L^p(\mathbb{R}^d)$ if $p > \frac{d}{d-1}$.

- For $d \geq 3$, this is in Stein's Chapter XI paragraph 3. Ideas: rotational curvature, FIOs, dyadic decomposition, almost orthogonality
- For $d = 2$, this is a theorem of Bourgain (1986) with alternative proofs by Sogge (1991): Cinematic curvature. Mockenhaupt-Seeger-Sogge (1993): Local smoothing for wave equation. $d = 2$ is also in Stein's Chapter XI paragraph 4D
- No such result holds in $L^p(\mathbb{R}^d)$ if $p \leq \frac{d}{d-1}$: Let

$$f(y) = \frac{|y|^{1-d}}{\log \frac{1}{|y|}} 1_{\{|y| \leq \frac{1}{2}\}}(y)$$

Then $f \in L^p$ if $p \leq \frac{d}{d-1}$: First, forget about the log for a moment. It is only there to take care of the endpoint.

$$\|f\|_{L^p(\mathbb{R}^d)}^p \simeq \int_0^{\frac{1}{2}} \frac{r^{(1-d)p}}{(\log \frac{1}{r})^p} r^{d-1} dr = \int_0^{\frac{1}{2}} r^{(1-d)(p-1)} dr$$

$$(1-d)(p-1) > -1 \quad (d-1)(p-1) < 1 \quad p < \frac{1}{d-1} + 1 = \frac{d}{d-1}$$

For any x , the quantity $(A_t f)(x)$ is unbounded when $f \sim |x|$:

$$\therefore \sup_t (A_t f)(x) = \infty \quad \text{everywhere}$$

Next: Averages with respect to a fixed curve:

$$(Mf)(x) = \sup_{h>0} \frac{1}{2h} \left| \int_{-h}^h f(x - (t, t^2)) dt \right|$$

Maximal operator along a parabola $f : \mathbb{R}^2 \rightarrow \mathbb{C}$

$$\tilde{A}f(x) = \frac{1}{2t} \int_{-t}^t f(x - \gamma(s)) \, ds$$

Maximal operator $\sup_{t>0} \tilde{A}_t f(x)$.
 $2^k < t \leq 2^{k+1}$.

$$\frac{1}{2t} \int_{-t}^t f(x - \gamma(s)) \, ds \leq \frac{1}{22^k} \int_{-2^{k+1}}^{2^{k+1}} |f(x - \gamma(s))| \, ds \leq 4 \underbrace{\frac{1}{2^{k-1}} \int |f(x - \gamma(s))| \eta(2^{-k-1}s) \, ds}_{A_{-k-1}f(x)}$$

where $\eta : \mathbb{R} \rightarrow \mathbb{R}_+$ smooth such that

$$\eta(s) = \begin{cases} 1 & |s| \leq 1 \\ 0 & |s| \geq 2 \end{cases}$$

Then the maximal operator we consider equal to

$$\sup_{k \in \mathbb{Z}} |A_k f(x)|$$

up to a factor (right?)

Now we consider $\gamma(s) = (s, s^2)$. Then

$$A_j f(x) = 2^j \int f(x_1 - s, x_2 - s^2) \eta(2^j s) \, ds = 2^{j+k} \int f(x_1 - 2^k \tilde{s}, x_2 - 2^{2k} \tilde{s}^2) \eta(2^{jk} \tilde{s}) \, d\tilde{s} = 2^{j+k} \int f(2^h(2^{-k}x_1 - \tilde{s}), 2^{2k}(2^{-2k}x_2 - \tilde{s}^2)) \eta(2^{jk} \tilde{s}) \, d\tilde{s}$$

where $s = 2^{-k} \tilde{s}$ and

$$\begin{aligned} P_k f(x) &= f(2^k x_1, 2^{2k} x_2) \\ \Rightarrow M &= P_{-k} M P_k \end{aligned}$$

Compare A_j with

$$B_j f(x) = 2^{2j} \int f(x - y) \psi(2^j y_1, 2^{2j} y_2) \, dy$$

where ψ is compactly supported and smooth and

$$\int_{\mathbb{R}^2} \psi = \int_{\mathbb{R}} \eta$$

$$\psi_j(x) = 2^{3j} \psi(2^j x, 2^{2j} x_2)$$

is supported in a ball of radius 2^{-j} with respect to the metric

$$\rho(x, y) = \max(|x_1 - y_1|, |x_2 - y_2|^{\frac{1}{2}})$$

$$B_j f(x) \lesssim M_\rho f(x)$$

where M is the maximal function with respect to $(\mathbb{R}^2, \rho, \text{Leb. measure})$, doubling metric measure space. M_ρ is L^p -bounded for some reason. Therefore

$$f \mapsto \sup_j |B_j f|$$

is bounded on $L^p(\mathbb{R}^2)$, $1 < p \leq \infty$. Need to estimate

$$\sup_j |(A_j - B_j)f(x)| \leq \left(\sum_j |(A_j - B_j)f(x)|^2 \right)^{\frac{1}{2}}$$

$$\int f d\mu_j = 2^j \int f(s, s^2) \eta(2^j s) ds$$

$$A_j f = \mu_j * f, \quad B_j f = \psi_j * f$$

Define

$$\sigma_j := \mu_j - \psi_j$$

Goal: Show that

$$\left\| \left(\sum_j |\sigma_j * f|^2 \right)^{\frac{1}{2}} \right\|_p \lesssim \|f\|_p$$

By interpolation it suffices to show this for $1 < p \leq 2$. $p = 2$: Let $f \in S$. Then by Plancherel

$$\left\| \left(\sum_j |\sigma_j * f|^2 \right)^{\frac{1}{2}} \right\|_2^2 = \int \sum_j |\sigma_j * f|^2 = \sum_j \int |\sigma_j * f|^2 = \sum_j \int |\hat{\sigma}_j|^2 |\hat{f}|^2 \leq \int |\hat{f}(\xi)|^2 \sum_j |\hat{\sigma}_j(\xi)|^2 \lesssim \int |\hat{f}|^2 = \|f\|_2^2$$

if we show that $\sum_j |\hat{\sigma}_j(\xi)|^2$ is bounded by a constant. Since the parabola has nonvanishing curvature we have

$$|\hat{\mu}_0(\xi)| \lesssim |\xi|^{-\frac{1}{2}}$$

Since ψ_0 is smooth we have for all N

$$|\hat{\psi}_0(\xi)| \lesssim |\xi|^{-N}.$$

That implies

$$|\hat{\sigma}_0(\xi)| = |\hat{\mu}_0(\xi) - \hat{\psi}_0(\xi)| \lesssim |\xi|^{-\frac{1}{2}}.$$

Also,

$$\int_{\mathbb{R}^2} d(\mu_0 - \psi_0) = 0 \Rightarrow \hat{\sigma}_0(0) = 0$$

and since σ_0 has compact support and

$$\int_{\mathbb{R}^2} d|\sigma_0| < \infty$$

$\hat{\sigma}_0(\xi)$ is differentiable in zero and therefore

$$|\hat{\sigma}_0(\xi)| \lesssim |\xi|.$$

Now

$$\hat{\mu}_j(\xi) = 2^j \int e^{-2\pi i(\xi_1 s + \xi_2 s^2)} \eta(2^j s) ds = \int e^{-2\pi i(2^{-j}\xi \tilde{s} + 2^{-2j}\xi_2 \tilde{s}^2)} \eta(\tilde{s}) d\tilde{s} = \hat{\mu}_0(2^{-j}\xi_1, 2^{-2j}\xi_2)$$

Also,

$$|\hat{\sigma}_0(\xi)| \leq \min(\rho(\xi), \rho(\xi)^{-\frac{1}{2}})$$

$$|\hat{\sigma}_j(\xi)| = |\hat{\sigma}_0(2^{-j}\xi_2, 2^{-2j}\xi_2)| \lesssim \min(2^{-j}\rho(\xi), (2^{-j}\rho(\xi))^{-\frac{1}{2}})$$

Now we sum

$$\sum_j |\sigma_j(\xi)|^2 \lesssim \sum_{j \in \mathbb{Z}} \min((2^{-j}\rho(\xi))^2, (2^{-j}\rho(\xi))^{-1}) \lesssim 1$$

$$1 < p < 2.$$

Lemma. Let $1 < q, p_0 < \infty$ with $\frac{1}{1q} = |\frac{1}{2} - \frac{1}{p_0}|$. Assume

$$\sigma^* f := \sup_k |\sigma_k| * |f|$$

is bounded on L^q . Let $(g_k) \in L^{p_0}(l^2)$. Then

$$\|(\sum_k |\sigma_k * g_k|^2)^{\frac{1}{2}}\|_{p_0} \lesssim \|(\sum_k |g_k|^2)^{\frac{1}{2}}\|_{p_0}$$

Something with an operator

Remark. For fixed q there are two p_0 s. If $p_0 < 2$ then

$$\frac{1}{2q} = \frac{1}{p_0} - \frac{1}{2} \Rightarrow p_0 = \frac{1}{\frac{1}{2q} + \frac{1}{2}} = \frac{2q}{1+q} < q$$

If $p_0 > 2$, then

$$\frac{1}{2q} = \frac{1}{2} - \frac{1}{p_0} \Rightarrow \frac{1}{p_0} = \frac{1}{2} - \frac{1}{2q} \Rightarrow \frac{2}{p_0} = 1 - \frac{1}{q} \Rightarrow q = (\frac{p_0}{2})'$$

Proof. Result for $p_0 < 2$ follows from $p_0 > 2$ by duality because for $L^{p_0}(l^2) \rightarrow L^{p_0}(l^2)$ the adjoint maps on $(L^{p_0}(l^2))' = L^{p_0'}(l^2)$. For $g \in L^{p_0}(l^2)$, $f \in L^{p_0'}(l^2)$ we use that

$$\langle (Tg)_k, f \rangle = \int \sum_k (Tg)_k f_k = \int (\sigma_k * g) f_k = \int g_k (\tilde{\sigma}_k * f_k)$$

where $\tilde{\sigma}_k$ is σ_k reflected at 0.

$p_0 > 2$.

$$\|(\sum_k |\sigma_k * g_k|^2)^{\frac{1}{2}}\|_{p_0}^2 = \|\sum_k |\sigma_k * g_k|^2\|_{\frac{p_0}{2}} = \int f \sum_k |\sigma_k * g_k|^2$$

for some f with $1 = \|f\|_{(\frac{p_0}{2})'} = \|f\|_q$.

$$(|\sigma_k| * |g_k(x)|)^2 = (\int |g_k(x-y)| d|\sigma_k|)^2 \leq (\int |g_k(x-y)|^2 d|\sigma_k|) (\underbrace{\int d|\sigma_k|}_{\lesssim 1})$$

Now the whole sum becomes

$$\begin{aligned} &\lesssim \int |f| \sum_k |\sigma_k| * |g_k|^2 = \int \sum_k (|\tilde{\sigma}_k| * |f|) |g_k|^2 \leq \int (\tilde{\sigma}^* f) \sum_k |g_k|^2 \\ &\leq \underbrace{\|\sigma^* f\|_q}_{\lesssim 1} \|\sum_k |g_k|^2\|_{\frac{p_0}{2}} \lesssim \|(\sum_k |g_k|^2)^{\frac{1}{2}}\|_{p_0}^2 \end{aligned}$$

□

Theorem. Let $1 < q < \infty$, $\frac{2q}{1+q} < p \leq 2$. Assume σ^* is bounded on $L^q(\mathbb{R}^2)$, and σ_j are measures with

$$\hat{\sigma}_j(\xi) \lesssim \min(2^j \rho(\xi), (2^j \rho(\xi))^{-1})^\varepsilon$$

for some $\varepsilon > 0$ and

$$\int d|\hat{\sigma}_j| \lesssim 1.$$

Then

$$\|(\sum_j |\sigma_j * f|^2)^{\frac{1}{2}}\|_p \lesssim \|f\|_p.$$

By construction q_s and ranges for p iteratively we can cover the whole interval.

This theorem can actually be applied to different situations, e.g. dyadic spheres. One might not get optimal results though. Something with maximal functions for general spheres or so...

Let S_k be Littlewood-Paley projections, i.e.

$$\hat{S}_k f = \hat{\varphi}_k f$$

where φ_k is a smooth and bounded function with

$$\text{supp } \hat{\varphi}_k \subset \{\xi : 2^{-k-1} \leq \rho(\xi) \leq 2^{-k+1}\}$$

and

$$\sum_k \hat{\phi}_k(\xi) = 1$$

for $\xi \neq 0$, so that

$$f = \sum_k S_k f$$

for $f \in L^p$, $1 < p < \infty$. Then (or: and?) we have for the square function

$$\|(\sum_k |S_k f|^2)^{\frac{1}{2}}\|_p \lesssim \|f\|_p$$

for $1 < p < \infty$.

Proof of Theorem. By the lemma

$$\|(\sum_j |\sigma_j * \sum_k S_{j+k} f|^2)^{\frac{1}{2}}\|_p \leq \sum_k \|(\sum_j |\sigma_j * S_{j+k} f|^2)^{\frac{1}{2}}\|_p$$

Now the summands are

$$\lesssim \|(\sum_j |S_{j+k} f|^2)^{\frac{1}{2}}\|_p \lesssim \|f\|_p$$

where $p = \frac{2q}{1+q}$. But this is not enough. So we do

$$\|(\sum_j |\sigma_j * S_{j+k} f|^2)^{\frac{1}{2}}\|_2^2 = \sum_j \|\sigma_j * S_{j+k} f\|_2^2 = \sum_j \|\hat{\sigma}_j \hat{\varphi}_{j+k} \hat{f}\|_2^2 \lesssim 2^{-2\varepsilon|k|} \sum_j \|\hat{\varphi}_{j+k} \hat{f}\|_2^2 \lesssim 2^{-2\varepsilon|k|} \|\hat{f}\|_2^2$$

since $\hat{\varphi}_{j+k}$ is supported where $\rho(\xi) \sim 2^{-j-k}$ and $\hat{\sigma}_j \lesssim 2^{-\varepsilon|k|}$ on $\text{supp } \hat{\varphi}_{j+k}$. Now we use Marcinkiewicz interpolation:

$$\begin{aligned} \frac{1-\theta}{\frac{2q}{1+q}} + \frac{\theta}{2} &= \frac{1}{p_\theta} \\ \Rightarrow I_k &\lesssim 1^{1-\theta} (2^{-2\varepsilon|k|})^\theta \|f\|_{p_\theta} \end{aligned}$$

which is summable in k . For the missing part see Duoandikoetxea and Rubio de Francia 1986 \square

Fourier restriction theory Given $f : \mathbb{S}^{d-1} \rightarrow \mathbb{C}$, consider the Fourier transform $\widehat{f\sigma}(\xi) = \int_{\mathbb{S}^{d-1}} f(x) e^{-2\pi i x \xi} d\sigma(x)$ for $\xi \in \mathbb{R}^d$ (tempered distribution which turns out to be a function)

- f smooth $\Rightarrow \widehat{f\sigma}$ decays at ∞ , e.g.

$$|\widehat{f\sigma}(\xi)| \leq c \|f\|_{C^2} (1 + |\xi|)^{-\frac{d-1}{2}}$$

via stationary phase

- f bounded then no pointwise decay holds in general. E.g.

$$f_k(x) = e^{2\pi i k x}$$

Consider $\xi = k$ then $|\widehat{f_k\sigma}(\xi)| = \sigma(\mathbb{S}^{d-1}) \simeq 1$, no decay here. Now take

$$f = \sum_{j \geq 0} \frac{f_{k_j}}{j^2}$$

where $|k_j| \rightarrow \infty$ sufficiently fast, e.g. $k_j = j!$. Easy to check: f is continuous but $|\widehat{f\sigma}(\xi)| \leq c(1 + |\xi|)^{-\varepsilon}$ does not hold for any $c < \infty$, $\varepsilon > 0$. (uniformly in ξ).

- Problem of distinguished origin disappears if we take L^q norms (There was some discussion on the wording going on which I did not understand)

Restriction conjecture (Stein): Prove, that if $f \in L^\infty(\mathbb{S}^{d-1})$, then

$$\|\widehat{f\sigma}\|_{L^q(\mathbb{R}^d)} \leq c_{q,d} \|f\|_{L^\infty(\mathbb{S}^{d-1}, \sigma)}$$

for very $q > \frac{2d}{d-1}$.

- Range of exponents would be sharp: take $f \equiv 1 \in L^\infty(\mathbb{S}^{d-1})$

$$\|\widehat{\sigma}\|_{L^q(\mathbb{R}^d)}^q = \int_{\mathbb{R}^d} |\widehat{\sigma}(\xi)|^q d\xi \lesssim \int_{\mathbb{R}^d} (1 + |\xi|)^{-\frac{d-1}{2}q} d\xi \simeq_d \int_0^\infty (1 + r)^{-\frac{d-1}{2}q} r^{d-1} dr$$

The last expression is $< \infty$ iff $-\frac{d-1}{2}q + (d-1) < -1$ iff $q > \frac{2d}{d-1}$.

Corresponding problem for L^2 densities was solved by Tomas-Stein inequality (~ 1975). If $f \in L^2(\mathbb{S}^{d-1})$, then

$$\|\widehat{f\sigma}\|_{L^q(\mathbb{R}^d)} \lesssim_{q,d} \|f\|_{L^2(\mathbb{S}^{d-1})}$$

if $q \geq \frac{2d+2}{d-1}$, and this range is the best possible. Note, that this was already proven last term and can be found in Stein/Shakarchi

- Assumptions are of the form $q > q_0$ or $g \geq q_0$. Why?

$$\|\widehat{f\sigma}\|_{L^\infty(\mathbb{R}^d)} = \sup_{\xi \in \mathbb{R}^d} \left| \int_{\mathbb{S}^{d-1}} f(x) e^{-2\pi i x \xi} d\sigma_x \right| \leq \int_{\mathbb{S}^{d-1}} |f(x)| d\sigma_x = \|f\|_{L^1(\mathbb{S}^{d-1})} \lesssim \|f\|_{L^2(\mathbb{S}^{d-1})}$$

+ Riesz-Thorin because the sphere is compact

- $q \geq \frac{2d+2}{d-1}$ is the best possible for L^2 densities: Knapp counterexample.

$$C_\delta = \{x \in \mathbb{S}^{d-1} : 1 - xe_d \leq \delta^2\}$$

where $e_d = (0, \dots, 0, 1)$. Since $|x - e_d|^2 = 2(1 - xe_d)$,

$$|x - e_d| \leq c\delta \Rightarrow x \in C_\delta \Rightarrow |x - e_d| \leq Cd$$

for appropriate constants $0 < c < C < \infty$. Let $f = 1_{C_\delta}$. Right hand side is easy:

$$\|f\|_{L^2(\mathbb{S}^{d-1})} = |C_\delta|^{\frac{1}{2}} \simeq_d \delta q^{\frac{d-1}{2}}$$

Left hand is trickier. Note: the support of $f\sigma$ is contained in a cylindrical box B_δ centered at e_d with length $\sim \delta^2$ in the e_d direction and $\sim \delta$ in the $(d-1)$ orthogonal direction.

$$\|\widehat{f\sigma}\|_{L^q(\mathbb{R}^d)}$$

Uncertainty principle: If a function is supported in a box, then its fourier transform is more or less constant on the dual box, which is the same but with inverse lengths. Idea: look at $\widehat{f\sigma}$ on the dual box B_δ^* (centered at 0), u.e. suppose $|\xi_d| \leq c_1^{-1}\delta^{-1}$ and $|\xi_j| \leq c_1^{-1}\delta^{-1}$ if $j < d$, c_1 is a large constant to be chosen. If $\xi \in B_\delta^*$, then

$$|\widehat{f\sigma}(\xi)| = \left| \int_{C_\delta} e^{2\pi i x \xi} d\sigma_x \right| = \left| \int e^{2\pi i (x - e_d) \xi} d\sigma_x \right| \geq \left| \int_{C_\delta} \cos(2\pi (x - e_d) \xi) d\sigma_x \right|$$

Conditions on ξ imply that

$$2\pi(x - e_d)\xi \leq \frac{\pi}{3}$$

if C_1 large enough. Therefore

$$|\widehat{f\sigma}(\xi)| \geq \frac{1}{2}|C_d| \simeq \delta^{d-1}$$

How large is B_δ^* ?

$$|B_\delta^*| \simeq \delta^{-2}(\delta^{-1})^{d-1} = \delta^{-(d+1)}$$

Conclusion:

$$\|\widehat{f\sigma}\|_{L^1(\mathbb{R}^d)} \gtrsim \left(\int_{B_\delta^*} |\widehat{f\sigma}(\xi)|^q d\xi \right)^{\frac{1}{q}} \gtrsim \delta^{wd} - 1 \delta^{-\frac{d+1}{q}}$$

$$\delta^{d-1} \delta^{-\frac{d+1}{q}} \leq \|\widehat{f\sigma}\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{S}^{d-1})} \simeq \delta^{\frac{d-1}{2}}$$

$$\therefore d-1 - \frac{d+1}{q} \geq \frac{d-1}{2} \Leftrightarrow \frac{d-1}{2} \geq \frac{d+1}{q} \Leftrightarrow q \geq 2d+2d-1$$

So, if you violate both of the two obstructions, then the inequality should hold (which obstructions? Maybe ∞ -bound on the domain and on the range?)

Technical tool: convolution of Schwartz function with a (compactly supported) measure. $\phi \in S(\mathbb{R}^d)$, $\mu \in M(\mathbb{R}^d)$ then

$$(\phi * \mu)(x) = \int \phi(x - y) d\mu(y)$$

Notation $\check{\mu} = \hat{\mu}(\cdot)$

Lemma. (i)

$$\widehat{\check{\phi}\mu} = \phi * \hat{\mu}$$

(ii)

$$\widehat{\phi\mu} = \hat{\varphi} * \hat{\mu}$$

Proof of (b), (a) is similar. Enough to show $\forall \psi \in S(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \widehat{\phi\mu}\psi \, dx = \int_{\mathbb{R}^d} (\hat{\varphi} * \hat{\mu})\psi \, dx$$

$$\int_{\mathbb{R}^d} \widehat{\phi\mu}\psi \, dx = \int \widehat{\psi\varphi} \, d\mu = \int \widehat{\check{\phi} * \psi} \, d\mu = \int_{\mathbb{R}^d} (\check{\varphi} * \psi)\hat{\mu} \, dx = \int_{\mathbb{R}^d} (\hat{\phi} * \hat{\mu})\psi \, dx$$

due to duality, Fourier inversion on S , duality and definition of $*$ + Fubini. \square

Lemma. $f, g \in S, \mu \in M(\mathbb{R}^d)$. Then

$$\int \widehat{f\hat{g}} \, d\mu = \int_{\mathbb{R}^d} (\hat{\mu} * \bar{g})f \, dx$$

Proof.

$$\int \widehat{f\hat{g}} \, d\mu = \int_{\mathbb{R}^d} \widehat{f\hat{g}\mu} \, dx = \int_{\mathbb{R}^d} f(\bar{g} * \hat{\mu}) \, dx$$

by duality and lemma 1 \square

Lemma. μ finite positive measure. Then the following are equivalent

- (i) $\|\widehat{f\mu}\|_{L^q} \leq c\|f\|_{L^2(\mu)} \quad \forall f \in L^2(\mu)$
- (ii) $\|\hat{g}\|_{L^2(\mu)} \leq c\|g\|_{L^{q'}} \quad \forall g \in S$
- (iii) $\|f * \hat{\mu}\|_{L^q} \leq c^2\|f\|_{L^{q'}} \quad \forall f \in S$

Note, that

- $T : L^2 \rightarrow L^q, f \mapsto \widehat{f\mu}$ (extension) iff
- $T^* : L^{q'} \rightarrow L^2, g \mapsto \hat{g}|_{\text{supp } \mu}$ (restriction) iff
- $TT^* : L^{q'} \rightarrow L^q, f \mapsto f * \hat{\mu}$

Again, there were some words about the expression 'extension', which I did not understand.

Proof of Tomas-Stein, up to endpoint. Will show

$$q > \frac{2d+2}{d-1} \Rightarrow \|f * \hat{\mu}\|_{L^q(\mathbb{R}^d)} \lesssim_{q,d} \|f\|_{L^{q'}(\mathbb{R}^d)}$$

Relevant properties of σ :

- (i) $|\hat{\sigma}(\xi)| \lesssim (1 + |\xi|)^{-\frac{d-1}{2}}$
- (ii) $\sigma(D(x, r)) \simeq r^{d-1}$

Note, that every measure satisfying this will have a Tomas-Stein, because these are the only properties we need.

Let $\phi \in C_0^\infty(\mathbb{R}^d)$, $\text{supp}(\Phi) \subset \{x : \frac{1}{4} \leq |x| \leq 1\}$, $\sum_{j \geq 0} \varphi(\frac{x}{2^j}) = 1$ if $|x| \geq 1$.

Cut up $\hat{\sigma}$ as follows:

$$\hat{\sigma} = k_{-\infty} + \sum_{j=0}^{\infty} k_j$$

with

$$k_j(x) = \phi(\frac{x}{2^j}) \hat{\sigma}(x)$$

and

$$k_{-\infty}(x) = (1 - \sum_{j \geq 0} \phi(\frac{x}{2^j})) \hat{\sigma}(x)$$

Easy: $k_{-\infty} \in C_0^\infty$:

$$\|f * k_{-\infty}\|_{L^q} \lesssim \|f\|_{L^p} \quad \forall p \leq q$$

Since $q > 2$ we can take $p = q'$ (Why does it hold for q' , then? And don't we want to show it only for q' anyways?).

Trickier: $(k_j)_{j=0}^\infty$. Upshot: Estimate the convolution with k_j $L^1 \rightarrow L^\infty$, $L^2 \rightarrow L^2$.

First $L^1 \rightarrow L^\infty$.

$$\|f * k_j\|_{L^\infty} \leq \|k_j\|_{L^\infty} \|f\|_{L^1}$$

where

$$\|k_j\|_{L^\infty} \sim 2^{-j \frac{d-1}{2}}$$

as a consequence of property (i).

$$L^2 \rightarrow L^2$$

$$\|f * k_j\|_{L^2} = \|\widehat{f} \widehat{k}_j\|_{L^2} \leq \|\widehat{k}_j\|_{L^\infty} \|f\|_{L^2}$$

Claim: $\|\widehat{k}_j\|_{L^\infty} \sim 2^j$ (Next lecture)

Interpolate (Riesz-Thorin)

$$\|f * k_j\|_{L^q} \lesssim 2^{-j \frac{d-1}{2} (1-\theta)} 2^{j\theta} \|f\|_{L^{q'}}$$

$$\frac{1}{q} = \frac{1-\theta}{\infty} + \frac{\theta}{2} \therefore q = \frac{2}{\theta}$$

which is equivalent to

$$\|f * k_j\|_{L^q} \lesssim 2^{j(\frac{d+1}{q} - \frac{d-1}{2})} \|f\|_{L^{q'}}$$

for any $q \in [2, \infty]$. The exponent is less than 0 iff $q > \frac{2d+2}{d-1}$

□

Fourier restriction theory 2 $f \in L^1(\mathbb{R}^d)$ implies \widehat{f} is uniformly continuous (so, \widehat{f} can be restricted to any set)

$f \in L^2(\mathbb{R}^d)$ iff $\widehat{f} \in L^2$, (therefore \widehat{f} cannot be restricted to a set of Lebesgue measure 0)

Question 1 What happens for intermediate $p \in (1, 2)$?

Let $M \subset \mathbb{R}^d$ be smooth compact hypersurface equipped with $d\mu = \psi d\sigma$, $\psi \in C_0^\infty$ and $d\sigma$ surface measure on M . Given $1 < p < 2$, for which exponents q does

$$\left(\int_M |\widehat{f}(\xi)|^q d\mu_\xi \right)^{\frac{1}{q}} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$$

hold? A complete answer for $q = 2$ is given by Tomas-Stein: M compact hypersurface whose Gauss curvature does not vanish on $\text{supp}(M)$. Then the restriction inequality holds provided $q = 2$ and $1 \leq p \leq \frac{2d+2}{d+3}$. Note, that this is the dual version.

Question 2 What happens for $q < 2$? Dimensional analysis implies

$$1 \leq p < \frac{2d}{d+1}$$

and Knapp-Type examples

$$q \leq \left(\frac{d-1}{d+1}\right)p'$$

The restriction conjecture states that these conditions are also sufficient.

End of proof of Tomas-Stein. Back to $(\mathbb{S}^{d-1}, \sigma)$. Strategy:

$$\|f * \hat{\sigma}\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^{q'}(\mathbb{R}^d)}$$

if $q > \frac{2d+2}{d-1}$.

Why is

$$\|\hat{k}_j\|_{L^\infty} \lesssim 2^j$$

$k_j = \phi_{2^{-j}} \hat{\sigma}$ implies $\hat{k}_j = \psi^{2^{-j}} * \sigma$ where $\psi^\varepsilon(x) = \varepsilon^{-d} \psi(\varepsilon^{-1}x)$ and $\psi = \hat{\phi} \in S$. Now

$$|\hat{k}_j(\xi)| \lesssim_N 2^{jd} \int_{\mathbb{S}^{d-1}} (1 + 2^j |\xi - \eta|)^{-N} d\sigma(\eta) \quad \forall N \in \mathbb{N}$$

Now let $D(x, r) = B(x, r) \cap \mathbb{S}^{d-1}$. Then $\sigma(D(x, r)) \lesssim r^{d-1}$. (Is this right?) So

$$\begin{aligned} |\hat{k}_j(\xi)| &\lesssim 2^{jd} \int_{D(\xi, 2^{-j})} (1 + 2^j |\xi - \eta|)^{-N} d\sigma(\eta) \\ &\quad + \sum_{k \geq 0} \int_{D(\xi, 2^{k+1-j}) \setminus D(\xi, 2^{k-j})} (1 + 2^j |\xi - \eta|)^{-N} d\sigma(\eta) \end{aligned}$$

The first one will dominate because of the rapid decay of ψ .

$$\lesssim 2^{jd} [\underbrace{\sigma(D(\xi, 2^j))}_{\sim 2^{-j(d-1)}} + \sum_{k \geq 0} 2^{-Nk} \underbrace{\sigma(D(\xi, 2^{k+1-j}) \setminus D(\xi, 2^{k-j}))}_{\sim 2^{(d-1)(k-j)}}] \lesssim 2^j$$

just choose $N = d$ such that the sum becomes a geometric series. \square

Remark. The $L^2 \rightarrow L^2$ bound in the previous argument was based only on dimensionality considerations. Therefore there should be an L^2 bound for $\widehat{f}\nu$ valid under very general conditions.

Theorem. Let ν be a positive finite measure where

$$\nu(D(x, r)) \leq cr^\alpha$$

Then

$$\|\widehat{f}\nu\|_{L^2(D(0, R))} \leq cR^{\frac{d-\alpha}{2}} \|f\|_{L^2(d\nu)}$$

The proof relies on Schur's test: (x, μ) , (Y, ν) measure spaces, $K(x, y)$ measurable on $X \times Y$. If for all y respectively x

$$\int_X |K(x, y)| d\mu(x) \leq A, \quad \int_Y |K(x, y)| d\nu(y) \leq B,$$

then for

$$(T_K f)(x) = \int_Y K(x, y) f(y) d\nu(y)$$

we have

$$\|T_K\|_{L^2 \rightarrow L^2} \leq \sqrt{AB}.$$

Proof of the theorem. Let $\phi \in S(\mathbb{R}^d)$ be radial such that $\phi \geq 1$ on the unit disc $\widehat{\phi}$ has compact support. $\phi_\varepsilon(x) := \phi(\varepsilon x)$. Then

$$\|\widehat{f\nu}\|_{L^2(D(0,R))} \leq \|\phi_{R^{-1}}(x)\widehat{f\nu}(-x)\|_{L^2_X(\mathbb{R}^d)} = \|\widehat{\phi_{R^{-1}} * (f\nu)}\|_{L^2(\mathbb{R}^d)} = \left\| \int_Y R^d \widehat{\phi}(R(x-y)) f(y) d\nu(y) \right\|_{L^2}$$

We have the estimates

$$\int_{\mathbb{R}^d} R^d |\widehat{\phi}(R(x-y))| dx = \|\phi\|_{L^1} < \infty$$

by change of variables.

$$\int_Y R^d |\widehat{\phi}(R(x-y))| d\nu(y) \lesssim R^{d-\alpha}$$

by the hypothesis of ν and compact support of $\widehat{\phi}$. Now apply Schur. \square

Proof of the restriction conjecture in $d = 2$ (Zygmund '70, Fefferman '72)

Theorem. Let $\gamma : I \rightarrow \mathbb{R}^2$ be a smooth curve with $\gamma' \neq 0$ and $\gamma'' \neq 0$ on some finite interval I . Let $4 < q \leq \infty$ and $3p' \leq q$. Then $\forall \varphi \in L^p(I)$

$$\left\| \int_I \varphi(t) e^{i\gamma(t)\xi} dt \right\|_{L^q_\xi(\mathbb{R}^2)} \lesssim_{p,q} \|\varphi\|_{L^p(I)}$$

Note, that the unit circle is a special case.

Proof. Step 1: 'Even integer' trick.

$$\left\| \int_I \varphi(t) e^{i\gamma(t)\xi} dt \right\|_{L^q_\xi(\mathbb{R}^2)}^2 = \left\| \int_I \int_I \varphi(t) \overline{\varphi(s)} e^{i(\gamma(t) - \gamma(s))\xi} dt ds \right\|_{L^{\frac{q}{2}}(\mathbb{R}^2)}$$

Step 2: Change variables $I \times I \rightarrow U \subset \mathbb{R}^2$, $(t, s) \mapsto \gamma(t) - \gamma(s) = x$. Choose I small enough such that the change of variables is invertible with Jacobian $J = \det(\frac{\partial(t,s)}{\partial x})$ satisfying $|J| \simeq |t - s|^{-1}$. Let $\gamma(t) = (x_1(t), x_2(t))$. Then

$$\left| \frac{\partial x}{\partial(t, s)} \right| = \left| \begin{pmatrix} -x'_1(s) & -x'_2(s) \\ x'_1(t) & x'_2(t) \end{pmatrix} \right| = |x_1(s)x'_2(t) - x'_1(s)x_2(t)| \simeq |t - s|$$

since in case of γ being the unit circle parametrized by arclength, this is equal to

$$|\cos s \sin t - \sin s \cos t| = |\sin(s - t)| \sim |s - t|.$$

Lets assume γ is already parametrized by arclength. To compare that situation with the unit circle, define

$$\theta(s) = \int_0^s k(t) dt,$$

where k is the (name?) curvature of γ . Then

$$\gamma'(s) = (\cos \theta(s), \sin \theta(s))$$

and due to

$$(\min k)|s - t| \lesssim |\theta(s) - \theta(t)| \leq \|k\|_{L^\infty} |s - t|$$

we can apply the above estimate.

Step 3: Hausdorff-Young

The integral now becomes

$$\left\| \int_U e^{ix\xi} F(x) dx \right\|_{L^{\frac{q}{2}}(\mathbb{R}^2)} \leq \|F\|_{L^r(\mathbb{R}^2)}$$

where

$$F(x) := \varphi(t) \overline{\varphi(s)} |J|$$

provided $\frac{q}{2} = r' \geq 2$.

Step 4: Fractional integration (Hölder-Littlewood-Sobolev)

We treat the following as an integral of $|\varphi|^r$ times a second function.

$$\|F\|_{L^r(\mathbb{R}^2)} \simeq \left(\int_I \int_I \frac{|\varphi(t)|^r |\varphi(s)|^r}{|t-s|^{r-1}} dt ds \right)^{\frac{1}{r}} \leq c \|\varphi\|_{L^p(I)}^2$$

provided $1 < r < 2$ and $1 + \frac{1}{(p/r)'} \geq r - 1 + \frac{1}{p/r}$. The first one is fine by assumption. The second is equivalent to $3p' \leq q$. \square

Recall, that a Kakeya set is a compact set $E \subset \mathbb{R}^d$ which contains a unit line segment in every possible direction. We have seen, that if $d \geq 2$, then there are Kakeya sets in \mathbb{R}^d of zero (Lebesgue) measure. The Kakeya (set) conjecture is: If $E \subset \mathbb{R}^d$ is a Kakeya set, then E has Hausdorff dimension d . We have seen it for $d = 2$, but remains an open problem in general d . However if

$$\dim E \geq \max\left\{\frac{d+2}{2}, (2-\sqrt{2})(d-4)+3\right\}$$

it is also done, when the first constant is due to Wolff '95 and the second due to Katz-Tao '02.

Kakeya maximal function Let $\delta > 0$, $e \in \mathbb{S}^{d-1}$, $a \in \mathbb{R}^d$. Let $T_e^\delta(a)$ be the cylinder centered at a , base diameter δ , oriented along e with length $2|e|$. Let $f \in L_{\text{loc}}^1(\mathbb{R}^d)$. Then we define

$$f_\delta^*(e) = \sup_{a \in \mathbb{R}^d} \frac{1}{|T_e^\delta|} \int_{T_e^\delta(a)} |f|$$

Remark. (i) trivially

$$\|f_\delta^*\|_{L^\infty(\mathbb{S}^{d-1})} \leq \|f\|_{L^\infty(\mathbb{R}^d)},$$

$$\|f_\delta^*\|_{L^\infty(\mathbb{S}^{d-1})} \leq \delta^{-(d-1)} \|f\|_{L^1(\mathbb{R}^d)}$$

(ii) If $d \geq 2$ and $p < \infty$ then there are no bounds of the form

$$\|f_\delta^*\|_{L^q(\mathbb{S}^{d-1})} \leq c \|f\|_{L^p(\mathbb{R}^d)}$$

with c independent of δ . Why: Let E be a Kakeya set with zero measure and E_δ a δ neighborhood of E , $f = 1_{E_\delta}$. Then

- $\forall e \in \mathbb{S}^{d-1} \quad f_\delta^*(e) = 1 \cdot \|f_\delta^*\|_{L^\infty(\mathbb{S}^{d-1})} \sim 1$
- $\|f\|_{L^p} = |E_\delta|^{\frac{1}{p}}$ which goes to 0 as $\delta \rightarrow 0^+$ if $p < \infty$.

Goal: $\delta^{-\varepsilon}$ estimate for f_δ^* , i.e. $\forall \varepsilon > 0 \exists C_\varepsilon < \infty$

$$\|f_\delta^*\|_{L^p(\mathbb{S}^{d-1})} \leq C_\varepsilon \delta^{-\varepsilon} \|f\|_{L^p(\mathbb{R}^d)}$$

for some $p < \infty$.

(iii) This inequality fails if $p < d$: $\forall e \in \mathbb{S}^{d-1} : B(0, \delta) \subset T_e^\delta(0)$. Let $f = 1_{B(0, \delta)}$. Then

$$f_\delta^*(e) \geq \frac{|B(\delta)|}{|T^\delta|} \sim \frac{\delta^d}{\delta^{d-1}} = \delta \implies \|f_\delta^*\|_{L^p(\mathbb{S}^{d-1})} \gtrsim \delta$$

$$\|f\|_{L^p} = |B(\delta)|^{\frac{1}{p}} = \delta^{\frac{d}{p}}$$

So we must have $p \geq d$ for the inequality to hold for all δ . **Keakeya maximal function conjecture:** It holds for $p = d$.

- $d = 2$: Cordoba '77, Bourgain '91 (seen: R_δ^* , $\|f_{*\delta}\|_{L^2(\mathbb{S}^1)} \lesssim \log \frac{1}{\delta} \|f\|_{L^2(\mathbb{R}^2)}$)
- $d \geq 3$ open,...
- If the inequality holds for some finite $p < \infty$, then Keakeya sets in \mathbb{R}^d have Hausdorff dimension d .

Restriction conjecture (on \mathbb{S}^{d-1}, σ)

$$\|\widehat{f\sigma}\|_{L^p(\mathbb{R}^d)} \lesssim_{p,d} \|f\|_{L^\infty(\mathbb{S}^{d-1})}$$

if $p > \frac{2d}{d-1}$. This is equivalent to

$$\|\widehat{f\sigma}\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{S}^{d-1})}$$

if $p > \frac{2d}{d-1}$, which is due to Bourgain.

Upshot: The restriction conjecture implies the Keakeya maximal function conjecture and hence the Keakeya set conjecture. (Unknown if these two are equivalent)

Theorem. *If the inequality is true, then the conjectured bound*

$$\|f_\delta^*\|_{L^d(\mathbb{S}^{d-1})} \lesssim_\varepsilon \delta^{-\varepsilon} \|f\|_{L^d(\mathbb{R}^d)}$$

is also true.

Lemma (Duality). Let $1 < p < \infty$, $p' = \frac{p}{p-1}$, $\{e_k\} \subset \mathbb{S}^{d-1}$. Assume maximal D -separated set and $\delta^{d-1} \sum_k d_k^{p-} \leq 1$ implies

$$\left\| \sum_k y_k 1_{T_{e_k}^\delta(a_k)} \right\|_{L^{p'}(\mathbb{R}^d)} \leq A.$$

Then there is a bound

$$\|f_\delta^*\|_{L^p(\mathbb{S}^{d-1})} \lesssim A \|f\|_{L^p(\mathbb{R}^d)}$$

Proof of lemma. $|e - e'| < \delta \implies f_\delta^*(e) \lesssim_d f_\delta^*(e')$ because any $T_e^\delta(a)$ can be covered by finitely many tubes $T_{e'}^\delta(a')$. Therefore

$$\|f_\delta^*\|_{L^p(\mathbb{S}^{d-1})} \leq \left(\sum_k \int_{D(e_k, \delta)} |F_\delta^*(e)|^p d\sigma_e \right)^{\frac{1}{p}} (\delta^{d-1} \sum_k \sum_k |f_\delta^o(e_k)|^p)^{\frac{1}{p}} = \delta^{d-1} \sum_k y_k |f_\delta^*(e_k)|$$

for some sequence (y_k) such that $\sum_k y_k^{p'} \delta^{d-1} = 1$, by duality of $l^p - l^{p'}$:

$$\|a\|_{l^p} = \sup_{\|b\|_{l^p}=1} \left| \sum a_n \bar{b}_n \right|$$

Therefore

$$\|f_\delta^*\|_{L^p(\mathbb{S}^{d-1})} \lesssim \delta^{d-1} \sum y_k \frac{1}{|T^\delta|} \int_{T_{e_k}^\delta(a_k)} |f|$$

for some choice of (a_k) that realizes the sup. This equals

$$\int_{\mathbb{R}^d} \left(\sum_k y_k 1_{T_{e_k}^\delta(a_k)} \right) |f| \leq \left\| \sum_k y_k 1_{T_{e_k}^\delta(a_k)} \right\|_{L^{p'}(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d)} \lesssim A \|f\|_{L^p(\mathbb{R}^d)}$$

□

Proof of theorem. Choose maximl δ -seperated set $\{e_j\} \subset \mathbb{S}^{d-1}$ (of cardinality $\sim \delta^{-(d-1)}$). Take a cap of size δ . Encapsulate it by a cylinder $C_{j,\delta}$ of size δ, δ^2 . Recall Knapp: functions $f_j \in L^\infty(\mathbb{S}^{d-1}) : \text{supp}(f_j) \subset c_j$ such that $\|f_j\|_{L^\infty(\mathbb{S}^{d-1})} = 1$ and

$$|\widehat{f_j \sigma}(x)| \gtrsim |c_j| \sim \delta^{d-1}$$

if $x \in \tau_j$. Idea: take functions of the form

$$f_\omega = \sum_j \omega_j y_j f_j$$

where $j_j \geq 0$, $\omega_j \pm 1$ iid.

Khinchine's inequality: $(\omega_n)_{n=1}^N$ iid variables taking valuas ± 1 with equal probability. Then

$$\mathbb{E} \left(\left| \sum_{n=1}^N a_n \omega_n \right|^p \right) \simeq_p \left(\sum_{n=1}^N |a_n|^2 \right)^{\frac{1}{2}}$$

where $(a_n) \subset \mathbb{C}$. (We now write q instead of p)

$$\|f_\omega\|_{L^q(\mathbb{S}^{d-1})}^q = \sum_j \|y_j f_j\|_{L^q(\mathbb{S}^{d-1})}^q = a \sum_j y_j^q |c_j| \sim \sum_j y_j^q \delta^{d-1}$$

$$\mathbb{E}_\omega (\|\widehat{f_\omega \sigma}\|_{L^q(\mathbb{R}^d)}^q) = \int_{\mathbb{R}^d} E_\omega (|\widehat{f_\omega \sigma}(x)|^q) dx \simeq \int_{\mathbb{R}^d} \left(\sum y_j^2 |\widehat{f_j \sigma}(x)|^2 \right)^{\frac{q}{2}} dx \gtrsim \delta^{q(d-1)} \int_{\mathbb{R}^d} \left| \sum_j y_j^2 1_{\tau_j}(x) \right|^{\frac{q}{2}} dx$$

Now by the restriction inequality the last term is up to a constant bounded by

$$\sum y_j^q \delta^{d-1}$$

for every $q > \frac{2d}{d-1}$. Now we rewrite. Set $\phi_j = y_j^2, p' = \frac{q}{2}$. Then it is equivalent to

$$\|\xi_j 1_{\tau_j}\|_{L^{\frac{q}{2}}(\mathbb{R}^d)} \lesssim \delta^{-2(d-1)}$$

provided

$$\sum_j \xi_j^{p-} \delta^{d-1} \leq 1$$

for all $p' > \frac{d}{d-1}$. Now dilate by δ^2

$$\left\| \sum \xi_j 1_{T_j} \right\|_{L^{p'}(\mathbb{R}^d)} \lesssim \delta^{2(\frac{d}{p'} - (d-1))}$$

if $\sum_j \xi_j^{p'} \delta^{d-1} \leq 1$ for all $p' > \frac{d}{d-1}$. As $p' \downarrow \frac{d}{d-1}$, $\frac{d}{p'} - (d-1) \uparrow 0$. Therefore

$$\forall \varepsilon > 0 \left\| \sum \xi_j 1_{T_j} \right\|_{L^{p'}(\mathbb{R}^d)} \lesssim_\varepsilon \delta^{-\varepsilon}$$

if $\delta^{d-1} \sum \xi_j^{p'} \leq 1$ provided p' is close enough to $\frac{d}{d-1}$. Now by the duality lemm

$$\forall \varepsilon > 0 \|f_\delta^*\|_{L^p(\mathbb{S}^{d-1})_\varepsilon} \delta^{-\varepsilon} \|f\|_{L^p(\mathbb{R}^d)},$$

provided $p(< d)$ is close enough to d . Now with trivial $\|f_\delta^*\|_{L^\infty} \lesssim \|f\|_{L^\infty}$ to finish the argument. \square