Geometric Aspects of Harmonic Analysis

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Q1

$$\left.\begin{array}{c} f : [a,b] \to \mathbb{R} \text{ integrable} \\ F(x) = \int_a^x f(t)\,\mathrm{d}t \end{array}\right] \underset{?}{\Rightarrow} F \text{ diff. (a.e. } x\text{)}, \ F' = f$$

Q2 Conditions of F (on $[a,b]$) s.t.

- $F'(x)$ exists a.e.
- F' integrable
- $\int_a^b F'(x)\,\mathrm{d}x = F(b) - F(a)$

?

Q1 Differentiation of the integral

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,\mathrm{d}t = \frac{1}{|I|}\int f = \operatorname{avg}_I f = {}_I\, f$$

$I = (x, x+b)$, $|I|$ Lebesgue measure of I.

Q1 equivalent to averaging problem: Given $f \in L^1(\mathbb{R}^d)$, is it true, that

$$\lim_{|B|\to 0,\ x\in B} = \frac{1}{|B|}\int_B f = f(x) \quad (x\text{-a.e.})?$$

$B \subset \mathbb{R}^d$ open ball

Yes, if f continuous $\forall \varepsilon \exists \delta |x-y| < \delta \Rightarrow |f(x) - f(y) < \varepsilon$. $x \in B$

$$|f(x) = ⨍ | = |⨍_B (f(y) - f(x))\,\mathrm{d}y| < \varepsilon \tag{1}$$

provided B is an opeb ball of radius $< \frac{\delta}{2}$ containing x

Yes, if f is integrable (not so easy). Hardy, Littlewood (1D, rearrangements; later Wiener for $d > 1$). $f \in L^1(\mathbb{R}^d)$

$$(Mf)(x) := \sup_{x \in B} \frac{1}{|B|}\int_B |f|$$

uncentered HL maximal function

Theorem. *Let f be integrable on $\mathbb{R}^d$. Then* *(i)* Mf *is measurable.*

(ii) $(Mf)(x) < \infty$ *a.e.* x

(iii)

$$\{x \in \mathbb{R}^d : (Mf)(x) > \alpha\}| < \frac{c}{\alpha}\|f\|_{L^1(\mathbb{R}^d)} \ (\forall x > 0). \tag{2}$$

$c = c_d = 3^d$, *independent of* f, α.

$f \neq f \in L^1 \Rightarrow Mf(x) \sim |x|^{-d}$ for large radius of x. So then $Mf \notin L^1$.

$$M : \begin{matrix} L^1 \not\to L^1 \\ L^1 \to L^{1,\infty} \end{matrix}$$

Proof. (i) easy $E_\alpha = \{x \in \mathbb{R}^d : (Mf)(x) > \alpha\}$ is open $(\forall x > 0)$ (because Mf is lowes semicontinuous)

(ii) $|\{x \in \mathbb{R}^d : (Mf)(x) = \infty\}| \subset |\{x \in \mathbb{R}^d : Mf(x) > \alpha\}|$, take $\alpha \to \infty$.

(iii) follows from an elemantary version of *Vitali covering*

□

Lemma. Let $B = \{B_1, B_2, \dots, B_N\}$ be a finite collection of open balls on $\mathbb{R}^d$. Then there exists a disjoint subcolletcino $B_{i_1}, B_{i_2}, \dots, B_{i_k}$ of B such that

$$|\bigcup_{j=1}^{n} B_j| \leq 3^d \sum_{j=1}^{k} |B_{ij}$$

Proof. (i) B_{i_1} = largest ball

(ii) Delete B_{i_1} and its neighbors

(iii) B_{i_2} = largest ball

(iv) repeat…

- Algorithm stops in at most N steps
- output has desired properties:
  - disjointness is clear
  - size $B \cap B' \neq \emptyset$, $r_{B'} \leq r_B$. B^* = ball with the same center as B but 3 times the radius. $\Rightarrow B' \subset B^*$. $|B^*| = 3^d|B|$

□

Back to (iii): Choose $\alpha > 0$, $E_\alpha = \{x \in \mathbb{R}^d : (Mf)(x) > \alpha\}$. Fr each

$$x \in E_\alpha \exists B = B_x := \frac{1}{|B_x|} \int_{B_x} |f(y)| \,\mathrm{d}y > \alpha$$

equivalent

$$|B_x| < \alpha^{-1} \int_{B_x} |f(y)| \,\mathrm{d}y$$

Fix $K \ll E_\alpha$ compact subset covered by $\bigcup_{x \in K} B_x$, $K \subset \bigcup_{l=1}]N B_l$

$$|K| \leq |\bigcup_{l=1}^{N} B_l| \underset{\text{Vitali}}{\leq} 3^d \sum_{j=1}^{k} |B_{ij}| \leq \frac{3^d}{\alpha} \in_{j=1}^{k} \int_{B_{i_j}} |f()| \,\mathrm{d}y = \frac{3^d}{\alpha} \int_{\bigcup_{j=1}^{k} B_{i_j}} |f(y)| \,\mathrm{d}y \leq \frac{3^d}{\alpha} \|f\|_{L^1(\mathbb{R}^d)}$$

Since K was choseen arbitrary (cpt.), it follows that

$$E_\alpha| \leq \frac{3^d}{\alpha} \|f\|_{L^1}$$

Can interpolate between weak type L^1-inequality and $L^\infty \to L^\infty$ (very easy).

Corollary (Lebesque differentiation theorem)**.** Let $f \in L^1(\mathbb{R}^d)$ Then

$$\lim_{|B|, \to 0, x \in B} \fint f = f(x) \quad x\text{-a.e.} \tag{3}$$

Proof.

$$E_\alpha = \{x \in \mathbb{R}^d : \limsup_{|B| \to 0, x \in B} |\fint_B f - f(x) > 2\alpha\}$$

ETS $|E_\alpha| = 0 \ \forall \alpha > 0$. Then $E = \bigcup_{n \in \mathbb{N}} E_{\frac{1}{n}} = 0$ and (3) holds on $E^\complement$.

Fix $\alpha > 0$, given $\varepsilon > 0$ choose $g \in C_0^0(\mathbb{R}^d)$ s.t. $\|f - g\|_{L^1} < \varepsilon$. Already seen

$$\lim_{|B| \Rightarrow 0, x \in B} \fint g = g(x) \ \forall x$$

$$\fint_B f - f(x) = \fint_B (f - g) + \fint_B g - g(x) + g(x) - f(x)$$

$$F_\alpha = \{x : M(f - r)(x) > \alpha\}$$
$$G_\alpha = \{x : |f(x) - g(x)| > \alpha\}$$

$E_\alpha \subset F_\alpha \cup G_\alpha$ since $u_1, u_2 > 0,\ u_1 + u_2 > 2\alpha \Rightarrow u_1 > \alpha \vee u_2 > \alpha$.

$$|G_\alpha| \leq \frac{1}{\alpha} \|f - g\|_{L^1} \quad \text{(Chebyshew)}$$

$$|F_\alpha| \leq \frac{c_d}{\alpha} \|f - g\|_{L^1} \quad \text{(weak type)}$$

$$|E_\alpha| \leq |F_\alpha| + |G_\alpha| \leq (\frac{c_d}{\alpha} + \frac{1}{\alpha}) \|f - g\|_{L^1} \leq \frac{c'_d \varepsilon}{\alpha}$$

Since $\varepsilon > 0$ was arbitrary $|E_\alpha| = 0$. □

$h \in L^1 \subset L^{1,\infty}$ by Chebyshew: $\infty > \|h\|_{l^1} = \int_{\mathbb{R}^d} |h(y)| \,\mathrm{d}y \geq \int_{h(y) \geq \alpha} |h(y)| \,\mathrm{d}y \geq \alpha |\{|h| > \alpha\}|$.

Would have been enough to replace $L^1(\mathbb{R}^d)$ by L^1_{loc}.

Sets $E \subset \mathbb{R}^d$ measurable, $x \in \mathbb{R}^d$ (not necc. in E) x is a point of Lebesque density of E if

$$\lim_{|B|\to 0, x\in B} \frac{|B \cap E|}{B} = 1$$

Corollary. Let $E \subset \mathbb{R}^d$ be measurable. Then

(i) Almost every $x \in E$ is a point of Lebesque density of E.

(ii) Almost every $x \notin E$ is not a point of Lebesque density.

Functions $f \in L^1_{\text{loc}}(\mathbb{R}^d)$.

$$Leb(f) := \{x \in \mathbb{R}^d : f(x) < \infty \text{ and } \lim_{|B|\to 0, x\in B} ⨍_B |f(y) - f(x)| \, \mathrm{d}y = 0\}$$

f continuous at $\bar{x} \Rightarrow \bar{x} \in \text{Leb}(f) \Rightarrow ⨍_B f \underset{|B|\to 0, x\in B}{\to} f(\bar{x})$ (all the inverse implications are wrong)

Corollary. $f \in L^1_{\text{loc}}(\mathbb{R}^d) \Rightarrow$ Almost every point belongs to $\text{Leb}(f)$.

(By checking the proof again?)

These things also works with other sets that ”shrink regularly to x than balls”. It gets worse however when one takes all parallel rectangles and even worse when arbitrarily oriented rectangles are allowed.

Q.2 Key: bounded varioation (BV) $F : [a,b] \to \mathbb{R}$, $P = \{a = t_0 < t_1 < ... < t_N = b\}$

$$V_F^P = \sum_{j=0}^{N} |F(t_j) - F(t_{j+1})|$$

is the variation of f over P. F is of bounded variation if

$$T_F(a,b) = T_F \sup_P V_F^P < \infty$$

$P \subset \tilde{P}$ partitions $\Rightarrow V_F^P \leq V_P^{\tilde{P}}$

Example. (i) f monotonic (increasing) and bounded, $|F| \leq M \Rightarrow F \in \text{BV}$

$$V_F^P = \sum_{j=1}^{N} |F(t_j)T = Nt_{j-1})| = F(b) - F(a) \leq 2M$$

(ii) F differentiable with F' bounded, $|F'| \leq M$, then by mean value theorem $f \in \text{BV}$. Or F LIpschitz

(iii) F α-Hölder ($\alpha < 1$) $6 \implies$ $F \in \text{BV}$. Take $F : [0,1] \to \mathbb{R}$, $x \mapsto \mathrm{d}(x,C)^\alpha$, where C is the cantor set. 2^{n-1} intervals of length 3^{-n}

$$\alpha > \frac{\log 2}{\log 3} \Rightarrow \sum_{n=1}^{\infty} 2^{n-1}(3^{-n})^\alpha < \infty$$

- *Total variation* of F on $[a, x]$ (where $a \le x \le b$) is

$$T_F(a, x) = \sup \sum_{j=0}^{N} |F(t_j) - F(t_{j-1})|$$

- *Positive variation* of F on $[a, 1]$ is

$$P_F(a, x) = \sup \sum_{(+)} (F(t_j) - F(t_{j-1})) \quad \text{all } j : F(t_j) \ge F(t_{j-1})$$

- *Negative variation* of F on $[a, 1]$ is

$$N_F(a, x) = \sup \sum_{(-)} -(F(t_j) - F(t_{j-1}))$$

Lemma. $f : [a, b] \to \mathbb{R}$. Then

(i) $F(x) = F(a) + P_F(a, x) - N_F(a, x)$

(ii) $T_F(a, x) = PF_(a, x) + N_F(a, x)$

$(\forall x \in [a, b])$

recall from measure theory: $f = f^+ - f^-$, $|f| = f^+ + f^-$

Proof. (i) given $\varepsilon > 0$, $\exists P = \{a = t_0 < t_1 < ... < t_N = x\}$

$$|PVF - \sum_{(+)} (F(t_j) - F(t_{j-1}))| < \varepsilon$$

$$|N_F - \sum_{(-)} -(F(t_j) - F(t_{j-1})))| < \varepsilon$$

Also

$$F(x) - F(a) = \sum_{(+)} F(t_j) - F(t_{j-1}) - \sum_{(-)} -(F(t_j) - F(t_{j-1}))$$

□

Corollary. $F : [a, b] \to \mathbb{R} \in \mathrm{BV}$ iff F is the differenco of two inclasing bounded functions

Theorem. $F : [a, b] \to \mathbb{R} \in \mathrm{BV} \Rightarrow F$ *differentiable a.e.*

Wlog f mononotic increasing, "Wlog" f continuous

Lemma (of the rising sun)**.** $G : \mathbb{R} \to \mathbb{R}$ continuous.

$$E = \{x \in \mathbb{R} : \exists h = h_x > 0 \; G(x + h) > G(x)\}$$

Then

(i) E is open $(E = \bigcup_{n=1}^{\infty} (a_n, b_n))$

(ii) $g(a_n) = G(b_n)$, provided $b_n - a_n < \infty$.

Proof. Let (a_n, b_n) be a finite interval in the decomposition. $a_k \notin E$ then $g(a_k) \geq G(b_k)$. Assume $G(a_k) > G(b_k)$. $\exists c \in (a_k, b_k)$ $g(c) = \frac{g(a_k)+g(b_k)}{2}$. Choose rightmost such c. $\exists d \in (c, b_k)$ $G(d) > G(c)$. But then by continuity c could not have been chosen rightmost, contradiction. □

Can replace $\mathbb{R}$ by $[a, b]$, but then only get for $a_0 = a$ that $G(a_0) \leq G(b_0)$

Proof. of theorem

$$\begin{aligned}
\Delta_h(F)(x) &= \frac{F(x+h) - F(x)}{h} \\
D^{\pm}(F)(x) &= \limsup_{h \to 0, h >< 0} \Delta_h(F)(x) \\
D_{\pm}(F)(x) &= \liminf_{h \to 0, h >< 0} \Delta_h(F)(x)
\end{aligned}$$

Dini numbers. Upshot: They are all the same and finite. $D_- \leq D^-$, $D_+ \leq D^+$ clear. ETS

(i) $D^+(F)(x) < \infty$ (a.e. x)

(ii) $D^+(F)(x) \leq D_-(F)(x)$ (a.e. x)

(ii) is equivalent to $D^-(F)(x) \leq D_+(F)(x)$ by replacing $F(x)$ by $-F(-x)$ somewhere. Then $D^+ \leq D_- \leq D^- \leq D_+ \leq D^+ < \infty$.

(i) relacc: F increasing ,bounded, continuous on $[a, b]$. Fix $\gamma > 0$,

$$E_\gamma := \{x : D^+(F)(x) > \gamma\}$$

- E_γ is measurable
- Apply rising sun to $G(x) = F(x) - \gamma x$

$$E_\gamma \subset E = \{x \in [a, b] : \exists h > 0 \; G(x+h) > G(x)\} = \bigcup_{k=1}^{\infty} (a_k, b_k)$$

The condition in the set is equivalent to

$$\begin{aligned}
&\iff \exists h > 0 \; F(x+h) - \gamma x - \gamma h > F(x) - \gamma x \\
&\iff \exists h > 0 \; \frac{F(x+h) - F(x)}{h} > \gamma \\
&\Leftarrow D^+(F)(x) > \gamma
\end{aligned}$$

$G(a_k) \leq G(b_k) \iff F(a_k) - \gamma a_k \leq F(b_k) - \gamma b_k \iff \gamma(b_k - a_k) \leq F(b_k) = F(a_k)$. Therefore

$$|E_\gamma| \leq |E| \leq \sum_{k=1}^{\infty} (b_k - a_k) \leq \frac{1}{\gamma} \sum_{k=1}^{\infty} (F(b_k) - F(a_k)) = \frac{1}{\gamma}(F(b) - F(a))$$

Take $\gamma \to \infty$, done.

(ii) see Stein-Shakarchi (vol 3)

□

Corollary. F increasing, continuous $\Rightarrow F'$ exists a.e., measurable, nonnegative and

$$\int_a^b F'(x)\,\mathrm{d}x \leq F(b) - F(a).$$

Proof. Let

$$G_h(x) = \frac{F(x + \frac{1}{n}) - F(x)}{\frac{1}{h}}$$

By the theorem, $G_h(x) \to F'(x)$ $(h \to 0)$ pointwise a.e. By Fatou

$$\int_{[a,b]} F' \leq \liminf_{n\to\infty} \int_a^b G_h(x)\,\mathrm{d}x = \liminf_{n\to\infty} ⨍_b^{b+\frac{1}{n}} F(x)\,\mathrm{d}x - ⨍_a^{a+\frac{1}{n}} F(x)\,\mathrm{d}x$$

□

Cannot do better than $\leq$: For the Devil's staircase the left hand side is 0 while the rightn hand side is 1.

Why is the sunrise Lemma a covering Lemma?

$$\left.\begin{array}{r} f_+^* = \sup \frac{1}{h}\int_x^{x+h} |f(y)|\,\mathrm{d}y \\ E_\alpha^+ = \{x \in \mathbb{R} : f_+^*(x) > \alpha\} \end{array}\right\} |E_\alpha^+| = \frac{1}{\alpha}\int_{E_\alpha^+} |f|$$

Why? Let

$$G(x) = \int_0^x |f(y)|\,\mathrm{d}y - \alpha x$$

$$x \in E_\alpha^+ \iff f_+^*(x) > \alpha \iff \exists h > 0\ \frac{1}{h}\int_x^{x+h} |f(y)|\,\mathrm{d}y > \alpha \iff \exists h > 0\ G(x+h) > G(x)$$

$$\{x \in \mathbb{R} : \exists h > 0\ G(x+h) > G(x)\} = \dot{\bigcup_{k\in\mathbb{N}}} (a_k, b_k), \quad G(a_k) = G(b_k)$$

$$|E_\alpha^+| = \sum_k (b_k - a_k) = \frac{1}{\alpha}\sum_k \int_{(a_k,b_k)} |f| = \frac{1}{\alpha}\int_{\bigcup_k (a_k,b_k)} |f| = \frac{1}{\alpha}\int_{|E_\alpha^+|} |f|.$$

Definition. $F : [a,b] \to \mathbb{R}$ is *absolutely continuous* if

$$\forall \varepsilon > 0 \exists \delta > 0 : \sum_{k'1}^{N} (B_k - a_k) < s$$

intervals $(a_k b_k)$ disjoint $(k = 1, ..., N) \Rightarrow$

$$\sum_{k=1}^{N} |F(b_k) - F(a_h)| < \varepsilon$$

Remark. (i) On a bounded Inetrval $I \subset \mathbb{R}$

$$C^1(I) \subset \mathrm{Lip}(I) \subset AC(I) \subset BV(I)$$

So they are diff. a.e.. All the inclusions are strict.

(ii) abs cont $\Rightarrow$ unif. con. $\Rightarrow$ cont.

(iii) $f \in L^1_{\text{loc}}(\mathbb{R})$ $F(x) = \int_0^x f(t)\,\mathrm{d}t$ Then F is absolutely continuous. $(\forall \varepsilon \exists \delta |E| < \delta \Rightarrow \int_E |f| < \varepsilon)$
Upshot: AC functions are the ones which re diff a..e. and vrify FTC.

Theorem. $F \in AC(a, b) \Rightarrow F'$ *exists a.e.,* $F' = 0$ *a.e.* $\Rightarrow F$ *constant*

- Existence of F' clear $\surd$
- $F' = 0$ a.e. $\Rightarrow F$ constant: refinement of Vitali

Definition. A collection $\mathcal{B} = \{B\}$ of (open) balls ön $\mathbb{R}^d$. is a *Vitali covering* of a set E if

$$\forall x \in E \forall \eta > 0 \exists B \in \mathcal{B} : x \in B, |B| < \eta$$

Lemma. $E \subset \mathbb{R}^d$ meas. $|E| < \infty$, $\mathcal{B}$ Vitali covering of E, $\delta > 0$. Then there exist finitely many disjoint balls $B_1, ..., bvB_N \in \mathcal{B}$

$$\sum_{j=1}^{N} |B_j| \geq |E| - \delta$$

Recall elementary Vitali: $\mathcal{B} = \{B_1, ..., B_N\}$ finite collection of pen balls in $\mathbb{R}^d \Rightarrow \exists$ disjoint subcollection $B_{i_1}, ..., B_{i_k}$ with

$$|\bigcup_{j=1}^{B} B_j| \leq 3^d \sum_{j=1}^{k} |B_{i_j}|$$

Proof of Lemma. wlog $\delta > |E|$. Vitali $\Rightarrow \exists$ disjoint subcollection $B_1, ..., B_N \in \mathcal{B}$

$$\sum_{i=1}^{N_1} |B_i| \geq 3^{-d} \delta$$

Sequence of balls $B_1, ..., B_N$. question: Is $\sum_{j=1} |B_j| \geq |E| - \delta$? Yes: done with $N = N_1$. No: work harder.

$$\sum_{j=1}^{N_1} |B_j| < |E| - \delta$$

$$E_2 = E \bigcup_{j=1}^{N_1} \bar{B}_j$$

$$|E_2| \geq |E| - \sum_{j=1}^{N_1} |B_j| > |E| - (|E| - \delta) = \delta$$

$\mathcal{B}$ Vitali covering $\Rightarrow$ balls in $\mathcal{B}$ disjoint frm $\bigcup_{i=1}^{N_1} \bar{B}_i$ still covers E_2. Vitali $\Rightarrow \exists$ finite disjoint subcollection of these balls $B_{N_1+1}, ..., B_{N_2}$

$$\sum_{N_1 < j < N_2} |B_j| \geq 3^{-d} \delta.$$

After k steps, $B_1, ..., B_{N_1}, ...B_{N_1}, ..., B_{N_k}$ with

$$\sum_{j=1}^{N_k} \geq h 3^{-d} \delta \geq |E| - \delta$$

iff $k \geq 3^d \frac{|E| - \delta}{\delta}$, stop. $\square$

need to approximate with compact from inside somewhere and with open from outside somewhere else

Corollary. The balls can be arranged in such a way that

$$E \bigcup_{i=1}^{N} B_i| < 2\delta$$

Proof. Choose open $O \supset E : |O \ E| < \delta$. $\mathcal{B}$ Vitali covering $\Rightarrow$ wlog all balls in $\mathcal{B}$ are contained in O.

$$(E \bigcup_{i=1}^{N} B_i) \cup \bigcup_{i=1}]NB_i \subset O. : |E \bigcup_{i=1}^{N} B_i| \leq |O| - \sum_{i=1}^{N} |B_i| \leq |E| + \delta - (|E| - \delta) = 2\delta$$

□

$F \in AC$ *Back to the real lin:* Goal: $F' = 0$ a.e. $\Rightarrow F$ constant. ETS $F(a) = F(b)$

$$E = \{x \in (a,b) : F'(x) \text{ exists and } = 0\} \quad |E| = b - a\}$$

Fix $\varepsilon > 0$. For $x \in E$, $\lim_{h\to 0} |\frac{F(x+h)-F(x)}{h}| = 0$. $\forall \eta > 0 \exists$ open interval $I = (a_x, b_x) \subset [a,b]$ containing x. $F(b_x) - F(a_x)| \leq \varepsilon(b_x - a_x)$ and $b_x - a_x < \eta$. The collection of these intervals (over all $\eta > 0$) forms a Vitali covering of E. Lemma $\Rightarrow$ Given $\delta > 0$ can select finitely many, disjoint $I_j = (a_j, b_j)_{j=1}^N$ such that

$$\sum_{j=1}^{N} |I_j| \geq |E| - \delta = b - a - \delta$$

But

$$\sum_{j=1}^{N} |F(b_j) T F(a_j)| \leq \varepsilon \sum_{j=1}^{N} (b_j - a_j) \leq \varepsilon(b-a)$$

$$[a,b] \bigcup_{j=1}^{N} I_j = \bigcup_{k=1}^{M} [\alpha_k, \beta_k]$$

with total length $\leq \delta$. $.:F \in AC$

$$\sum_{k=1}^{M} |F(\beta_k) - F(\alpha_k)| < \varepsilon$$

$$|F(b) - F(a)| \leq \sum |F(b_j) - F(a_j)| + \sum |F(\beta_k) - F(\alpha_k)| \leq \varepsilon(b-a) + \varepsilon,$$

done. □

Theorem. $F \in AC(a,b)$. *Then*

(i) F' *exists a.e. and is integrable*

(ii)

$$F(x) - F(a) = \int_a^x F'(t)\,\mathrm{d}t \quad (\forall a \leq x \leq b)$$

Conversely, if $f \in L^1(,b)$*, then there exists* $F \in AC(a,b) : F' = f$ *a.e..*

Proof. $\Rightarrow$

(i) seen last lecture.

(ii)
$$G(x) := \int_a^x F'(t)\,\mathrm{d}t$$
$\therefore G \in AC \;\therefore F - G \in AC$. Lebesque diff. $\Rightarrow G'(x) = F'(x)$ (a.e. x) $\therefore\ (F-G)' = 0$ a.e.. Therefore $(F-G)(x) = (F-G)(a),\ F(x) - G(x) = F(a)$, equivalent to (*)

$\Leftarrow$
$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$
AC $\checkmark$. Leb. diff $\Rightarrow F' = f$ a.e. $\square$

Next: Monotone functions which are not nec. continuous. Wlog. F increasing, bounded on $[a,b]$.
$$F(x^-)F \lim_{y\to x, y<x} F(y) \quad F(x^+) := ...$$
$F(x^-) \le F(x) \le F(x^+)$, F cont. at x if $F(x^-) = F(x^+)$. Otherwise F has a jump discontinuity at x.

Obs: A (bounded) increasing finction F on $[a,b]$ has at most countable many jumps. There exists an injective map $\mathrm{Disc}(F) \to \mathbb{Q}$

$\therefore$ $\mathrm{Disc}(F) = \{x_n\}_{n=1}^\infty$ $\alpha_n = F(x_n^+) - F(x_n^-) =$ jump of F at x_n. $F(x_n^+) = F(x_n^-) + \alpha_n$ $F(x_n) = F(x_n^-) + \theta_n\alpha_n,\ \theta_n \in [0,1]$. $F(x) = \mu((-\infty, x])$. Corresponds to singular + abs. cont measures.
$$j_n(x) = \begin{cases} 0 & x < x_n \\ \phi_n & x = x_n \\ 1 & x > x_n \end{cases}$$
Jump function associated to F is
$$J_F(x) = \sum_{n=1}^\infty j_n(x)$$
$$\sum_{n=1}^\infty \alpha_n = \sum_{n=1}^\infty F(x_n^+) - F(x_n^-) \le F(b) - F(a) < \infty$$
because F incr, and F bounded.

Lemma. F increasing, bounded on $[a,b]$, $\mathrm{Disc}(F) = \{x_n\}_{n=1}^\infty$

(i) $J_F(x)$ is discontinuous precisely at $\{x_n\}_{n=1}^\infty$, has a jump at x_n equal to that if F.

(ii) Th function $F - J_F$ is increasing and continuous.

Proof. (i) $x \ne x_n (\forall n) \Rightarrow$ each j_n is continuous at $x \Rightarrow J_F$ is continuous at x because of uniform convergence. $x = x_N(\exists N) \Rightarrow J_F = \sum_{n=1}^N \alpha_n j_n(x) + \sum_{n>N}\alpha_n j_n(x)$. First sum has jump discontinuity x_N of size α_N

(ii) $F - J_F$ is continuous

$$F(x) - J_F(x) \leq F(y) T J_F(y) \iff J_F(y) - J_F(y) \leq F(y) - F(x)$$

, where

$$J_F(y) = \sum_{x<x_n\leq y} \alpha_n = \sum_{x<x_n\leq y} F(x_n^+) - F(x_n^-) \leq F(y) - F(x)$$

□

Since $F = (F - J_F) + J_F$ ETS J_F is diff a.e.. This was essential step of

$$\mu = \mu_{AC} + \mu_S + \mu_{PP}$$

$z(t) = (x(t), y(t))$. curve γ. $x, y : [a, b] \to \mathbb{R}$ continuous.
γ rectifiable if length

$$L(\gamma) = \sup \sum_{j=1}^{N} |z(t_j) - z(t_{j-1})| < \infty$$

sup over all partitions $P = \{a = t_0 < t_1 <_< t_N = b\}$ of $[a, b]$
When is

$$L(\gamma) = \int_a^b |z'(t)| \,\mathrm{d}t?$$

Lemma. γ is rectifiable iff x, y are of bounded variation (and cont.).

see $F = x + iy$
Assume γ rectifiable, let $L(A, B)$ length of $\gamma(A, B)$, $(a \leq A \leq B \leq b)$

(i) $L(A, B) = T_F(A, B)$ (where $F(t) = z(t)$)

(ii) $L(A, C) + L(C, B) = L(A, B)$ $(A \leq C \leq B)$

(iii) $A \mapsto L(A, B)$ (fix B) is continuous
$B \mapsto L(A, B)$ (fix A)
seen: $F \in \mathrm{BV}(a, b)$, cont. $\Rightarrow T_F$ cont.

Warning: $[0, 1] \ni t \mapsto (F(t), F(t))$, F Cantor. F cont. incr. $F(0) = 0,\ F(1) = 1,\ F' = 0$ a.e.

Theorem. $z : [a, b] \to \mathbb{R}^2, t \mapsto (x(t), y(t)) \sim$ *curve* γ. $x, y \in \mathrm{AC}(a, b)$. $\Rightarrow \gamma$ *rectifiable and*

$$L(\gamma) = \int_a^b |z'(t)| \,\mathrm{d}t$$

why? $F : [a, b] \to \mathbb{C}$ is $\mathrm{AC}(a, b)$

$$\Rightarrow T_F(a, b) = \int_a^b |F'(t)| \,\mathrm{d}t.$$

$$\sum_{j=1}^{N} |F(t_j) - F(t_{j-1})| = \sum_{j=1}^{N} |\int_{t_{j-1}}^{t_j} F'(t) \,\mathrm{d}t| \leq \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} |F'(t)| \,\mathrm{d}t = \int_a^b |F'(t)| \,\mathrm{d}t$$

First inequality by FTC. For $\geq$, write $F' = g + h$, g step function, h small in L^1 $G, H =$ def. integrals of g, h. Check $T_F \geq T_G, T_H$, T_H small, $T_G \geq \int_a^b |g(t)| \,\mathrm{d}t$.

Minkowski content of a curve simple, simple closed, quasi-simple curves.

trace of γ: $\Gamma = \{z(t) \in \mathbb{R}^2 : t \in [a,b]\}$.

Given $K \Subset \mathbb{R}^2$ and $\delta > 0$ define

$$K^\delta = \{x \in \mathbb{R}^2 : \mathrm{d}(x, K) < \delta\}$$

where $\mathrm{d}(x, K) = \inf_{k \in K} \mathrm{d}(x, k)$

Definition. The set K has (1D) Minkowski content if

$$\lim_{\delta \to 0} \frac{|K^\delta|}{2\delta}$$

exists (in $\mathbb{R}$), denoted $M(K)$.

Theorem. *Let $\Gamma = \{z(t) : a \leq t \leq b\}$ be (the trace of) a quasi-simple curve γ. Then Γ has Minkowski content iff γ is rectifiable (in which case $M(\Gamma) = L(\gamma)$).*

Upper Mink: content.

$$\limsup_{\delta \to 0^+} \frac{|K^\delta|}{2\delta} =: M^*(K)$$

lower

$$\liminf_{\delta \to 0^+} \frac{|K^\delta|}{2\delta} =: M_*(K).$$

Proposition. $T = \{z(t) : a \leq t \leq b\}$ quasi simple. If $M_*(\Gamma) < \infty$, then γ is rectifiable and $L(\gamma) \leq M_*(\Gamma)$.

Proposition. $\Gamma = \{z(t) : a \leq t \leq b\}$ rectifiable γ. Then $M^*(\Gamma) \leq L(\gamma)$.

Proof of Prop. 1, for simple curves. Obs: $\Gamma = \{z(t) : a \leq t \leq b\}$ any curve. $\Delta = |z(b) - z(a)|$. $|\Gamma^\delta| \geq 2\delta\Delta$.

Take any partition P of $[a, b)$. $L_P = \sum_{j=1}^N |z(t_j) - z(t_{j-1})|$. Given $\varepsilon > 0, \exists N$ proper closed subintervals $I_j = [a_j, b_j] \subset (t_{j-1}, t_j)$:

$$\sum_{j=1}^N |z(b_j) - z(a_j)| \leq L_P - \varepsilon$$

$I_1, ..., I_N$ disjoint $\Rightarrow \Gamma_1, ..., \Gamma_N$ disjoint because Γ is simple. $\Leftrightarrow \Gamma_1^\delta, ..., \Gamma_N^\delta$ disjoint, provided $\delta > 0$ small enough.

$$\bigcup_{j=1}^N \Gamma_j^\delta \subset \Gamma^\delta$$

$$|\Gamma^\delta| \geq \sum_{j=1}^N |\Gamma_j^\delta| \geq 2\delta \sum_{j=1}^N |z(t_j) - z(t_{j-1})| \geq 2\delta(L_p - \varepsilon)$$

$\square$

Isoperimetric inequality (soft) $\gamma : [a,b] \to \mathbb{R}^2,\ \gamma \in C^1(a,b) : \gamma'(s) \neq 0 \forall s,\ \gamma(a) = \gamma(b)$. Arclength parametrization: $\gamma : [0,L] \to \mathbb{R}^2,\ |\gamma'(s)| = 1 \forall s$.

Theorem. *$\Gamma \subset \mathbb{R}^2$ simple colsed C^2 curve of length L. A area of the region enclosed by Γ.*

$$A = \frac{1}{2}| \int_\Gamma > (x\,\mathrm{d}y - y\,\mathrm{d}x)| = \frac{1}{2}| \int_0^L (x(s)y'(s) - x'(s)y(s))\,\mathrm{d}s.$$

Then $4\pi A \leq L^2$. Equality iff Γ is a circle.

Proof. wlog (rescale) $L = 2\pi$.: WTS $A \leq \pi$, equality iff Γ is a circle of radius 1.
$\gamma : [0,2\pi] \to \mathbb{R}^2,\ s \mapsto \gamma(s) = (x(s), y(s))$ arclength par. $x'(s)^2 + y'(s)^2 = 1 \forall s$.:

$$\frac{1}{2\pi} \int_0^{2\pi} (x'(s)^2 + y'(s)^2)\,\mathrm{d}s = 1$$

Γ closed $\Rightarrow x(s), y(s)$ 2η-periodic.

$$x'(s) = \sum_n a_n ine^{ins}$$

$$y'(s) = \sum_n b_n ine^{ins}$$

Parseval $\Rightarrow$

$$\sum_{n\in\mathbb{Z}} n^2(|a_n|^2 + |b_n|^2) = 1$$

$$A = \frac{1}{2} \int_0^{2\pi} (x(s)y'(s) - x'(s)y(s))\,\mathrm{d}s| = \pi | \sum_{n\in\mathbb{Z}} n(a_n\bar{b}_n - b_n\bar{a}_n)|$$

by bilinear Parseval

$$|a_n\bar{b}_n - b_n\bar{a}_n| \leq 2|a_n||b_n| \leq |a_n|^2 + |b_n|^2$$

$$A \leq \pi \sum_{n\in\mathbb{Z}} n^2(|a_n|^2 + |b_n|^2) \leq \pi$$

□

Cases of equality: $A = \pi \Rightarrow$

$$x(s) = a_{-1}e^{-is} + a_0 + a_1e^{is}$$

$$y(s) = b_{-1}e^{-is} + b_0 + b_1e^{is}$$

x, y real-valued $\Rightarrow a_1 = \bar{a}_{-1},\ b_1 = \bar{b}_{-1}$. (**) $\Rightarrow 2(|a_1|^2 + |b_1|^2) = 1$.
(***) $\Rightarrow$

$$|a_1| = |b_1| = \frac{1}{2}.: \quad a_1 = \frac{1}{2}e^{i\alpha} \quad b_1 = \frac{1}{2}e^{i\beta} \quad (\alpha, \beta \in \mathbb{R})$$

$$1 = 2|a_1\bar{b}_1 - \bar{a}_1b_1| = \sin(\alpha - \beta)|.: \quad \alpha - \beta = \frac{k\pi}{2} \quad (\text{odd } k)$$

$$x(s) = a_0 + \cos(s + \alpha)$$

$$y(s) = b_0 \pm \sin(s + \alpha)$$

$\pm$ dep. on parity of $\frac{k-1}{2}$.

Isoperimetric inequality (hard) $\Omega \subset \mathbb{R}^2$ bounded, open, $\partial\Omega = \bar{\Omega} - \Omega =: \Gamma$ rectifiable curve (not nec. simple) with length $l(\Gamma)$.

Theorem.

$$4\pi|\Omega| \leq l(\Omega)^2$$

Proof. inner:

$$\Omega_-^\delta = \{x \in \mathbb{R}^2 : \mathrm{d}(x, \mathbb{R}^2\ \Omega) \geq \delta\}$$

outer:

$$\Omega_+^\delta = \{x \in \mathbb{R}^2 : \mathrm{d}(x, \bar{\Omega}) < \delta\}$$

$$\Gamma^\delta = \{x : \mathrm{d}(x, \Gamma) < \delta\}$$

$$\Omega_+^\delta = \Omega_-^\delta \dot{\cup} \Gamma >^\delta$$

$A, B \subset \mathbb{R}^d,\ A + B = \{a + b : a \in A, b \in B\}$ Note: $\Omega + B_\delta \subset \Omega_+^\delta,\ \Omega_-^\delta + B_\delta \subset \Omega$.

Brunn-Minkowski: $A, B \subset \mathbb{R}^2$ meas., $A + B$ meas.

$$|A + B|^{\frac{1}{2}} \geq |A|^{\frac{1}{2}} + |B|^{\frac{1}{2}}$$

$$|\Omega_-^\delta| \geq (|\Omega|^{\frac{1}{2}} + |B_\delta|^{\frac{1}{2}})^2 \geq |\Omega| + 2|\Omega|^{\frac{1}{2}} \underbrace{|B_\delta|^{\frac{1}{2}}}_{=(\pi\delta^2)^{\frac{1}{2}}}$$

$$|\Omega| \geq (|\Omega_-^\delta|^{\frac{1}{2}} + |B_\delta|^{\frac{1}{2}})^2 \geq |\Omega_-^\delta| + 2|\Omega_-^\delta|^{\frac{1}{2}}|B_\delta|^{\frac{1}{2}}$$

$$|\Gamma^\delta| \geq |\Omega| + 2|\Omega|^{\frac{1}{2}}\sqrt{\pi} - |\Omega| + |\Omega_-^\delta|^{\frac{1}{2}}\sqrt{\pi}$$

$$\limsup_{\delta \to 0^+} \frac{|\Gamma^\delta|}{2\delta} \geq 2|\Omega|^{\frac{1}{2}}\sqrt{\pi}$$

$$4\pi|\Omega| \leq M^*(\Gamma)^2 \leq l(\Gamma)^2$$

Note, that only in the very last inequality did we use the rectifiability of Γ. □

Brunn-Minkowski ineq. ($\mathbb{R}^d$) $A, b \subset \mathbb{R}^d$ measurable. $A + B = \{a + b : a \in A, , b \in B\}$. $\lambda A = \{\lambda a : a \in A\}$ $(\lambda > 0)$.

Q.: Can $|A + B|$ be controlled in terms of $|A|, |B|$? No! There exist sets A, B $|A| = |B| = 0$ with $|A+B| > 0$. Example $[0, 1] \times [0, 1]$. Another example $A = B = C \subset [0, 1]$ Cantor set. Then $A + B = [0, 2]$.

Q.: Can $|A+B|^\alpha \geq c_\alpha(|A|^\alpha + |B|^\alpha)$ hold? (for some $\alpha > 0$ with $c_\alpha < \infty$, indep of A, B) Best possible $c_\alpha = 1$.

What about α? Convex sets play a role. $A =$ convex, $B = \lambda A$. $|B| = |\lambda A| = \lambda^d|A|$. $|A + B| = |A + \lambda A| = |(1 + \lambda)A| = (1 + \lambda)^d|A|$ because A is convex.

($\lambda_1 A + \lambda_2 A = (\lambda_1 + \lambda_2)A$ iff A is convex.)

$|A + B|^\alpha \geq |A|^\alpha + |B|^\alpha$ iff $(1 + \lambda)^{d\alpha} \geq 1 + \lambda^{d\alpha} \Rightarrow \alpha \geq \frac{1}{d}$.

$(a + b)^\gamma \geq a^\gamma + b^\gamma \forall a, b \geq 0,\ \gamma \geq 1$.

Candidate inequality:

$$|A + B|^{\frac{1}{d}} \geq |A|^{\frac{1}{d}} + |B|^{\frac{1}{d}}$$

(BM)

A, B measurable $6 \Longrightarrow$ $A + B$ measurable. Take $[0, 1] \times$ nonmeasurable.

(i) A, B closed $\Rightarrow A + B$ measurable

(ii) A, B compact $\Rightarrow A + B$ compact

(iii) A, B open $\Rightarrow A + B$ open

Theorem. *(BM) holds if $A, B, A + B$ measurable.*

(i) A, B rectangles with sidelengths $\{a_j\}_{j=1}^{\infty}, \ \{b_j\}_{j=1}^{\infty}$

(ii) A, B unions of fifinely many rectangles with disjoint interiors.

(iii) A, B open sets of finite measure

(iv) A, B compact

(v) $A, B, A + B$ measurable.

Proof. (i) (BM) becomes

$$\prod_{j=1}^{d}(a_j + b_j)^{\frac{1}{d}} \geq \prod_{j=1}^{d} a_j^{\frac{1}{d}} + \prod_{j=1}^{d} b_j^{\frac{1}{d}}$$

$a_j \to \lambda_l a_j, \ b_j \to \lambda_j b_j$. Both sides are multiplied by $(\lambda_1\lambda_2 ... \lambda_d)^{\frac{1}{d}}$: wglog can assume $a_j + b_j = 1 \forall j$ (Choose $\lambda_j = a_j + b_j$)

AMGM:

$$\prod_{j=1}^{d} a_j^{\frac{1}{d}} \leq \frac{1}{d}\sum_{j_1}^{d} a_j$$

$$\prod_{j=1}^{d} b_j^{\frac{1}{d}} \leq \frac{1}{d}\sum_{j_1}^{d} b_j$$

$$\prod a_j^{\frac{1}{d}} + \prod b_j^{\frac{1}{d}} \leq \frac{1}{d}\sum_{j=1}^{d}(a_j + b_j) = 1$$

(ii) Induction on $n =$ numeber of rectangles in A and B. Choose pair of disjoint rectangles R_1, R_2 in A. Can rotate s.t. R_1 and R_2 are separated by hyperplane $\{x_j = 0\}$. R_1 lies in $A_+ = A \cap \{x_j \geq 0\}, \ A_I = A \cap \{x_j \leq 0\}$.

Rem.: Both A_+, A_- contain at leas one less rectangle than A, $A = A_+ \subset A_-$ and $A_B \cap A_-$ has measure zero.

Now: translate B s.t. B_- and B_+ satisfy

$$\frac{|B_\pm|}{|B|} = \frac{|A_\pm|}{|A|}$$

$(A_+ + B_+) \cup (A_- + B_-) \subset A + B$ Number of rectangles in A_+ and B_+, number of rectangles in A_- and B_- is $< n$.

$$\begin{aligned}
|A + B| &\geq |A_+ + B_-| + |A_- B +_-| \geq (|A_+|^{\frac{1}{d}} + |B_+|^{\frac{1}{d}})^d + (|A_-|^{\frac{1}{d}} + |B_-|^{\frac{1}{d}})^d \\
&= (|A_+|(1 + (\frac{|B_+|}{|A_+|})^{\frac{1}{d}})^d + |A_-|(1 + (\frac{|B_-|}{|A_-|})^{\frac{1}{d}})^d = (|A_+| + |A_-|)(1 + (\frac{|B|}{|A|})^{\frac{1}{d}})^d \\
&= (|A|^{\frac{1}{d}} + |B|^{\frac{1}{d}})^d.
\end{aligned}$$

(iii) Open sets of finite measure A, B. $\forall \varepsilon > 0 \exists A_\varepsilon, B_\varepsilon$ finet unions of parallel rectangles with disjoint interiors. $A_\varepsilon \subset A, B_\alpha \subset B$, $|A| \leq |A_\varepsilon| + \varepsilon,\ |B| \leq |B_\varepsilon| + \varepsilon$.

$|A+B| \geq |A_\varepsilon + B_\varepsilon| \geq (|A_\varepsilon|^{\frac{1}{d}} + |B_\varepsilon|^{\frac{1}{d}})^d \geq ((|A|-\varepsilon)^{\frac{1}{d}} + (|B|-\varepsilon)^{\frac{1}{d}})^d$. Let $\varepsilon \to 0^+$, done.

(iv) A, B compact. Let $A^\varepsilon = \{x : d(x, A) < \varepsilon\}$. $A + B \subset A^\varepsilon + B^\varepsilon \subset (A+B)^{2\varepsilon}$

(v) $A, B, A+B$ measurable: usi inner regularity of Lebesque measure.

□

Remark. A, B open sets of finite positive measure. Equality in (BM) iff A, B convex and similar. $\exists \delta > 0 \exists h \in \mathbb{R}^d : A = \delta B + h$ (A convex iff $\lambda_i A + \lambda_2 A = (\lambda_1 + \lambda_2) A$)

Consequences for isoperimetric inequality $A \subset \mathbb{R}^d$ bounded open with smooth boundary. (∂A, $B \subset \mathbb{R}^d$ ball $|B| = |A|$)

$$|\partial A| = \lim_{\varepsilon \to 0^+} \frac{|A + \varepsilon B| - |A|}{\varepsilon}$$

Isoper ineq.: $|\partial A| \geq |\partial B|$.

Proof.

$$\frac{|A+\varepsilon B| - |A|}{\varepsilon} \geq \frac{(|A|^{\frac{1}{d}} + |\varepsilon B|^{\frac{1}{d}})^d - |A|}{\varepsilon} = \frac{(1+\varepsilon)^d - 1}{\varepsilon} |B| \to d|B| = |\partial B|$$

for $\varepsilon \to 0$. □

Better: $A \subset \mathbb{R}^d$ has finite perimeter ($\iff 1_A \in \mathrm{BV}(U),\ U \subset \mathbb{R}^d$ bdd open)

$$\frac{\mathcal{H}^{d-1}(\partial A)}{|A|^{\frac{d-1}{d}}} \geq \frac{\mathcal{H}^{d-1}(S^{d-1})}{|B^d(0,1)|^{\frac{d-1}{d}}}$$

Hausdorff measure Q: How does a set replicate under scaling? $E \to nE = E_1 \cup ... \cup E_m$ disjoint congruent copies of E. Examples: line $m = n^1$, square $m = n^2$, cube $m = n^3$, Cantor set $3C = C_1 \cup C_2$ $2 = 3^\alpha \iff \alpha = \frac{\log 3}{\log 2}$

$\#(\varepsilon)$ =least # of segments that arise from such poygonal lines. Γ rectifiable iff $\#(\varepsilon) \sim \varepsilon^{-1}$ as $\varepsilon \to 0^+$. If $\#(\varepsilon) \sim \varepsilon^{-\alpha}$ $(\alpha > 1)$ In this case, say "Γ has dim α". Snowflake has $\alpha = \frac{\log 4}{\log 3} > 1$.

Upshot: E $\alpha > 1$. $m_\alpha(E) = \alpha$-dimensional mass of E among sets of "dimension" α.

- $\alpha > \dim(E) \Rightarrow m_\alpha(E) = 0$
- $\alpha < \dim(E) \Rightarrow m_\alpha(E) = \infty$
- $\alpha = \dim(E)$ interesting

R. Gardener Bulletin AMS more about Brunn-Minkowski, geometrically, including more proofs, e.g. with induction of the dimension.

Hausdorff measure $E \subset \mathbb{R}^d$ any subset.

$$m_\alpha^*(E) := \lim_{\delta\to 0^+} \underbrace{\inf\{\sum_k (\operatorname{diam} F_k)^\alpha : E \subset \bigcup_{k=1}^\infty F_k \;\; \operatorname{diam}(F_k) \le \Delta\}}_{H_\alpha^\delta(E)}$$

exterior/outer α-dim Hausdorff measure.

Remark. $H_\alpha^\delta(E) \le H_\alpha^\delta(E) \le m_\alpha^*(E) (\forall \delta > 0)$. $H_\alpha^\delta(E)$ increases when δ ecreases. $\therefore m_\alpha^*(E) = \lim_{\delta\to 0^+} H_\alpha^\delta(E)$ exists

Remark. Coverings must be by sets of arb. small measure. (If we allowed the δ to be arbitrary then two parallel lines would get the same 1d-measure as one of them.)

Remark (Skaling). "The measure of a set should scale like its dimension". E.g.: $\Gamma \subset \mathbb{R}^d$ smoot cureve of length L *sim* $\lambda\Gamma$ has length λL. $Q \subset \mathbb{R}^d$ cube *sum* λQ has measure $\lambda^d |Q|$. $|F|$ scaled by $\lambda \Rightarrow (\operatorname{diam} F)^\alpha$ scaled by λ^α

Properties

(i) $E_1 \subset E_2 \Rightarrow m_\alpha^*(E_1) \le m_\alpha^*(E_2)$

(ii) $\{E_j\} \subset \mathbb{R}^d$ countable family of sets $\Rightarrow m_\alpha^*(\bigcup_{j=1}^\infty E_j) \le \sum_{j=1}^\infty m_\alpha^*(E_j)$

(iii) (Finite additility) $\inf_{x\in E_1, y \in E_1} |x-y| = \operatorname{d}(E_1, E_2) > 0 \Rightarrow m_\alpha^*(E_1 \cup E_2) = m_\alpha^*(E_1) + m_\alpha^*(E_2)$

Proof. ETS $\ge$. Fix $0 < \varepsilon < \operatorname{d}(E_1, E_2)$. Given any cover of $E_1 \cup E_2$ with sets $F_1, F_, \ldots$ of diam $\le \delta < \varepsilon$, let $F_j' = F_j \cap E_1$, $F_j'' = F_j \cap E_2$.

$$\sum (\operatorname{diam}_j F_j')^\alpha + \sum_j (\operatorname{diam} F_j'')^\alpha \le \sum_k \operatorname{diam}(F_k)^\alpha$$

Take inf over all covers, let $\delta \to 0^+$, done. □

m_α^* satisfies all properties of a Caratheodory outer measure $\therefore m_\alpha^*$ is a countabley additive maisure when restricted to Borel sets, call it m_α= α-dim Hausdorff measure.

(iv) $\{E_j\}$ countable family of disjoint Borel sets $\Rightarrow$

$$m_\alpha(\dot{\bigcup}_{j=1}^\infty E_j) = \sum_{j=1}^\infty m_\alpha(E_j)$$

(v) Hausdorff masure is invariant under translation and rotations. It scales like:

$$m_\alpha(\lambda E) = \lambda^\alpha m_\alpha(E)$$

(vi) $m_0(E) = \#E$, $m_1(E) = |E|$ (=1D LEbesgue measure of E), $E \subset \mathbb{R}$ Borel.

(vii) $E \subset \mathbb{R}^d$ Borel, $m_\alpha(E) \simeq |E|$

Proof. (i) Isodiametric inequality: $|E| \le v_d(\frac{\operatorname{diam} E}{2})^d$, v_d volume of the unit ball in $\mathbb{R}^d$. Prove first for sets $E = -E$ and then something hard.

(ii) Covering argument: Given $\varepsilon, \delta > 0$, there exists a covering of E by balls $\{B_j\}$: $\text{diam}B_j < \delta$, $\sum_j |B_j| \leq |E| + \varepsilon$

$$H_d^\delta(E) \leq \sum_j (\text{ diam}B_j)^d = c_d \sum_j |B_j| \leq c_d(|E| + \varepsilon),$$

let $\delta, \varepsilon \to 0^+$, get one of the inequalities.

$\square$

(viii) if $m^*_\alpha(E) < \infty$ and $\beta > \alpha$, then $m^*_\beta(E) = 0$. If $m^*_\alpha(E) > 0$ and $\beta < \alpha$, then $m^(_\beta E) = \infty$.

Proof. $\text{diam}F < \delta, \beta > \alpha \Rightarrow (\text{ diam}F)^\beta = (\text{ diam}F)^{\beta-\alpha}(\text{ diam}F)^\alpha < \delta^{\beta-\alpha}(\text{ diam}F)^\alpha$ $\square$

Consequence: Given $E \subset \mathbb{R}^d$ Borel, $\exists!\, \alpha$ such that

$$m_\beta(E) = \begin{cases} \infty & \beta < \alpha \\ 0\beta > \alpha \end{cases}$$

$$\alpha = \sup\{\beta : m_\beta(E) = \infty\} = \inf\{\beta : m_\beta(E) = 0\} := \text{Hausdorff dimension of } E = \dim E$$

At the critical value $\alpha = \dim E$ $0 \leq m_\alpha(E) \leq \infty$. If E is bounded and the enequalities are strict, we say that E has strict Hausdorff dimension α.

Theorem. *The Cantor set $C \subset [0, 1)$ has strict Hausdorff dimensios $\frac{\log 2}{\log 3}$.*

ETS: $0 < m_\alpha(C) \leq 1$

Proof. $m_\alpha(C) \leq 1$*:* $C = \bigcap C_k$ where each C_k is a finite unionof 2^k inetrvals of length 3^{-k}.. Given $\delta > 0$ coose k large enough tuch tht $3^{-k} < \delta$. C_k covers C and Consists of 2^k intervals of diameter $3^{-k} < \delta$. $H^\delta_\alpha(C) \leq 2^k(3^{-k})^\alpha = 1$, let $\delta \to 0^+$, done.

$m_\alpha(C) > 0$:

Lemma. $E \Subset d$ compact, $f : E \to \mathbb{R}$ γ-Hölder,

$$|f(x) - f(y)| \leq m|x - y|^\gamma \quad (\forall x, y \in E) \quad 0 < \gamma \leq 1$$

Then

(i) $m_\beta(f(E)) \leq M^\beta m_\alpha(E)$ if $\beta = \frac{\alpha}{\gamma}$.

(ii) $\dim f(E) \leq \frac{1}{\gamma} \dim(E)$

Proof. $\{F_k\}$ countable family of sets that overs $E \therefore \{f(F_k \cap E)\}$ covers $f(E)$. $\text{diam}f(F_k \cap E) \leq M(\text{ diam}F_k)^\gamma$.

$$\sum_k (\text{ diam}f(E \aleph F_k))^{\frac{\alpha}{\gamma}} \leq M^{\frac{\alpha}{\gamma}} \sum_k (\text{ diam}F_k)^\alpha,$$

done. and 1 implies 2. $\square$

Lemma. The Cantor-Lebesgue function $F : C \to [0, 1]$ is $\gamma = \frac{\log 2}{\log 3}$-Hölder.

Proof. Goal: $|F(x)-F(y)| \leq c|x-y|^\gamma \ \forall x, y \in C$.

F_n increases at most 2^{-n} on an interval of length 3^{-n}. $\therefore$ slope $\leq (\frac{3}{2})^n$ $\therefore |F_n(x)-F_n(y)| \leq (\frac{3}{2})^n|x-y|$. $|F_n(x)-F(x)| \leq 2^{-n}$. Given x, y chose n: $3^n|x-y| \sim 1$, $3^\gamma = 2$.

$$|F(x)-F(y)| \leq |F_n(x)-F_n(y)|+|F_n(x)-F(x)|+|F_n(y)-F(y)| \leq (\frac{3}{2})^n|x-y|+2\cdot 2^{-n} \leq c2^{-n} = c(3^{-n})^\gamma \leq c'|x-y|^\gamma$$

□

Apply LEmma 1 with $E = C$, $f = F$, $\gamma = \frac{\log 2}{\log 3} \Rightarrow 1 = m_1([0,1]) \leq Mm_\alpha(C)$, $\dim C = \frac{\log 2}{\log 3}$ □

Rectifiable curves

Theorem. *$\gamma : [a,b] \to \mathbb{R}^d$ continuous and simple. Then γ is rectifiable iff $\Gamma = \{\gamma(t) : a \leq t \leq b\}$ has strict Hausdorff dimension equal to 1. $m_1(\Gamma) = l(\gamma)$.*

Proof. $\Rightarrow$: Let γ be rectifiable of length L. Consider acrlength parametrization $\tilde{\gamma}$. $\Gamma = \{\tilde{\gamma}(s) : 0 \leq s \leq L\}$.

$$|\tilde{\gamma}(s_1) - \tilde{\gamma}(s_2)| \leq |s_1 - s_2|$$

By Lemma 1 (i) $m_1(\Gamma) \leq L$. Wh $m_1(\Gamma) \geq L$?

$$\Gamma_j = \{\gamma(t) : t_j \leq t \leq t_{j+1}\}$$

$$\Gamma = \bigcup_{j=1}^{N-1} \Gamma_j \quad m_1(\Gamma) = \sum_{j=0}^{N-1} m_1(\Gamma_j)$$

Claim: $m_1(\Gamma_j) \geq l_j := |\gamma(t_j) - \gamma(t_{j+1})|$

Proof. $\pi : \mathbb{R}^2 \infty \mathbb{R}$ $(x,y) \mapsto x$ Lipschitz, $\pi(\Gamma_j) \subset [0, l_j]$ Lemma 1 (i) implies the claim. □

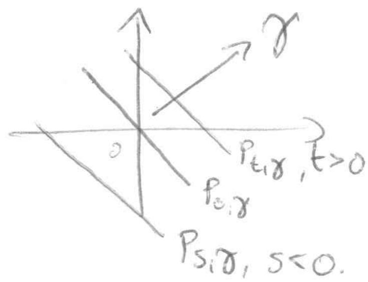
$\therefore m_1(\Gamma) \geq \sum m_1(\Gamma_j) \geq \sum l_j$, $L := \sup_p \sum l_j \therefore m_1(\Gamma) \geq L$, done. □

Radon transforms

$R(f)(t,\gamma) = \int_{P_{t,\gamma}} f.$ where...

$f: \mathbb{R}^d \to \mathbb{R}$, $t \in \mathbb{R}$, $\gamma \in S^{d-1} \subseteq \mathbb{R}^d$

$P_{t,\gamma} = \{x \in \mathbb{R}^d : \underbrace{x \cdot \gamma}_{\text{inner pr.}} = t\}$ hyperplane



$P_{t,\gamma}$ equipped with natural $(d-1)$-dim. Leb. measure, (coincides with $(d-1)$-Hausdorff measure) denoted by m_{d-1}

Remarks (i) $f \in C_0^0(\mathbb{R}^d) \Rightarrow f$ integrable on every $P_{t,\gamma}$
$\Rightarrow R(f)(t,\gamma)$ defined for every (t,γ).
($R(f)$ cont. fct. of (t,γ), cpctly spp. in t

(ii) $f \in L^1(\mathbb{R}^d) \Rightarrow f$ may fail to be measurable/integrable on some $P_{t,\gamma}$ ($\to R(f)(t,\gamma)$ not defined.)

(iii) $f = \chi_E$ ($E \subset \mathbb{R}^d$ mb.) $\Rightarrow R(f)(t,\gamma) = m_{d-1}(E_{t,\gamma})$
$E_{t,\gamma} = E \cap P_{t,\gamma}$ if $E_{t,\gamma}$ measurable.

Look instead at maximal Radon transform:

$$R^*(f)(\gamma) = \sup_{t \in \mathbb{R}} |R(f)(t,\gamma)|.$$

$\to$ Want to study L^p-mapping properties of R in order to study regularity of subsets of $\mathbb{R}^d$.

Thm. 1 $f \in C_0^\infty(\mathbb{R}^d)$, $n \geq 3$:

$$(*) \quad \int_{S^{d-1}} \mathcal{R}^*(f)(\gamma)\, d\sigma_\gamma \leq C\left(\|f\|_{L^1(\mathbb{R}^d)} + \|f\|_{L^2(\mathbb{R}^d)}\right).$$

~~i.e. f needs~~

Rem. i) necessary conditions:

•) $f \in L^1$: $f(x) = (1+|x|^{d-1})^{-1} \in (L^2 \setminus L^1)(\mathbb{R}^d)$ if $d \geq 3$

f is not integrable in any plane $P_{t,\gamma}$.

($\to f \in L^1$ gives global control.)

•) $f \in L^2$: $f_\varepsilon(x) = (|x|+\varepsilon)^{-d+\delta}$ if $|x| \leq 1$, $\delta \in (0,1)$ fixed.

Let $\varepsilon \to 0^+$ to see that $(*)$ fails if $\|\cdot\|_{L^2}$ on the RHS is not there

($\to f \in L^2$ gives local control.)

Key: Interplay between Radon and Fourier transform.

$t \mapsto \lambda \in \mathbb{R}$ dual variable.

Fourier transform:

$$\hat{R}(f)(\lambda,\gamma) = \int_{-\infty}^{\infty} R(f)(t,\gamma)\, e^{-2\pi i \lambda t}\, dt$$

Lemma 1 $f \in C_0^\infty(\mathbb{R}^d)$, $\gamma \in S^{d-1}$:

$$\hat{R}(f)(\lambda,\gamma) = \hat{f}(\lambda\gamma).$$

Proof $\hat{f}(\lambda\gamma) = \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i x\cdot(\lambda\gamma)}\, dx$

$$= \int_{-\infty}^{\infty}\left(\int_{\mathbb{R}^{d-1}} f(u,t)\, du\right) e^{-2\pi i \lambda t}\, dt. = \int_{-\infty}^{\infty}\left(\int_{P_{t,\gamma}} f\right) e^{-2\pi i \lambda t}\, dt.$$

Choose coordinates $x = (u,t)$, $t = x\cdot\gamma = x_d \in \mathbb{R}$, $u = (x_1,\dots,x_{d-1}) \in \mathbb{R}^{d-1}$.

$\square$

(2

Lemma 2 $f \in C_0^0(\mathbb{R}^d)$:

$$\int_{S^{d-1}} \left(\int_{-\infty}^{\infty} |\hat{\mathcal{R}}(f)(\lambda,\gamma)|^2 |\lambda|^{d-1} d\lambda \right) d\sigma_\gamma = 2 \int_{\mathbb{R}^d} |f(x)|^2 dx$$

Proof $2 \int_{\mathbb{R}^d} |f(x)|^2 dx = 2 \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 d\xi$

$$= 2 \int_{S^{d-1}} \left(\int_0^\infty |\hat{f}(\lambda\gamma)|^2 \lambda^{d-1} d\lambda \right) d\sigma_\gamma .$$

$$= 2 \int_{S^{d-1}} \left(\int_{-\infty}^0 |\hat{f}(\lambda\gamma)|^2 |\lambda|^{d-1} d\lambda \right) d\sigma_\gamma$$

done by Lemma 2. $\square$

Lemma 3

Suppose $F(t) = \int_{-\infty}^{\infty} \hat{F}(\lambda) e^{2\pi i \lambda t} d\lambda$

Where $\sup_{\lambda \in \mathbb{R}} |\hat{F}(\lambda)| \leq A$ and $\int_{-\infty}^{\infty} |\hat{F}(\lambda)|^2 |\lambda|^{d-1} d\lambda \leq B^2$.

Then $\sup_{t \in \mathbb{R}} |F(t)| \leq c(A+B)$ (This is essential in the proof of Thm. 1)

Moreover, if $0 < \alpha < \frac{1}{2}$, then $|F(t_1) - F(t_2)| \leq C_\alpha |t_1 - t_2|^\alpha \cdot (A+B)$

($\to$ α-Hölder)

Proof.

$$F(t) = \underbrace{\int_{|\lambda|<1} \hat{F}(\lambda) e^{2\pi i \lambda t} d\lambda}_{\leq CA} + \underbrace{\int_{|\lambda|>1} \hat{F}(\lambda) e^{2\pi i \lambda t} d\lambda}_{\leq \int_{|\lambda|>1} \hat{F}(\lambda) d\lambda}$$

$$\overset{CS}{\leq} \underbrace{\left(\int_{|\lambda|>1} |\hat{F}(\lambda)|^2 |\lambda|^{d-1} d\lambda \right)^{1/2}}_{\leq B} \cdot \underbrace{\left(\int_{|\lambda|>1} |\lambda|^{-d+1} d\lambda \right)^{1/2}}_{<\infty \text{ if } -d+1 < -1 \iff d \geq 3. \checkmark}$$

$\Rightarrow$ first estimate.

$$|F(t_1)-F(t_2)| = \int_{-\infty}^{\infty} \hat{F}(\lambda)\underbrace{(e^{2\pi i\lambda t_1}-e^{2\pi i \lambda t_2})}_{|\cdot|\leq C_\alpha |t_1-t_2|^\alpha|\lambda|^\alpha \text{ if } \alpha\in(0,1]}\,d\lambda$$

$(x \to e^{ix}$ Lipschitz$)$

$$= \int_{|\lambda|\leq 1} \Big\} \leq C_\alpha A\, |t_1-t_2|^\alpha$$

$$+ \int_{|\lambda|>1} \Big\} \leq |t_1-t_2|^\alpha \underbrace{\int_{|\lambda|>1} |\hat{F}(\lambda)|\,|\lambda|^\alpha d\lambda}_{\leq \left(\int |\hat{F}(\lambda)|^2|\lambda|^{d-1}d\lambda\right)^{1/2} \cdot \left(\underbrace{\int_{|\lambda|>1} |\lambda|^{-d+1+2\alpha} d\lambda}_{<\infty \text{ if } \alpha<\frac{1}{2} \text{ for } d\leq 3.}\right)}$$

$\square$

Proof of Thm. For each $\gamma\in S^{d-1}$, let $F(t)=R(f)(t,\gamma)$

$\Rightarrow \sup_{t\in\mathbb{R}} |F(t)| = R^*(f)(\gamma)$.

Let $A(\gamma)=\sup_{\lambda\in\mathbb{R}} |\hat{F}(\lambda)|$, $B^2(\gamma)=\int_{-\infty}^{\infty}|\hat{F}(\lambda)|^2|\lambda|^{d-1}d\lambda$

Lemma 3 $\underset{\text{assumptions!}}{\overset{\text{check}}{\Rightarrow}} \sup_{t\in\mathbb{R}} |F(t)| \leq C(A(\gamma)+B(\gamma))$

Lemma 1 $\Rightarrow \hat{F}(\lambda) = \hat{R}(f)(\lambda,\gamma)=\hat{f}(\lambda\gamma) \Rightarrow A(\gamma)\leq \|f\|_{L^1(\mathbb{R}^d)}$

Lemma 2 $\Rightarrow \int_{S^{d-1}} B^2(\gamma)\,d\sigma_\gamma = 2\|f\|^2_{L^2(\mathbb{R}^d)}$.

We have $\sup_{t\in\mathbb{R}} |F(t)|^2 \leq C'(A^2(\gamma)+B^2(\gamma))$

Integrate both sides:

$$\int_{S^{d-1}} R^*(f)(\gamma)^2 d\sigma_\gamma \lesssim \underbrace{\underbrace{\int A^2(\gamma)\,d\sigma_\gamma}_{\lesssim \|f\|_1^2} + \underbrace{\int B^2(\gamma)\,d\sigma_\gamma}_{\simeq \|f\|^2_{L^2}}}_{\lesssim (\|f\|_{L^1}+\|f\|_{L^2})^2}$$

Use Hölder, because on cpt. space:

$\|R^*(f)\|^2_{L^1} \lesssim \int_{S^{d-1}} R^*(f)(\gamma)^2 d\sigma_\gamma$.

$\square$

Regularity of sets when $d \geq 3$.

$E \subset \mathbb{R}^d$ meas. $E_{t,\gamma} = E \cap P_{t,\gamma}$ (t varies, γ fixed)

Fubini $\Rightarrow$ $E_{t,\gamma}$ is m_{d-1}-measurable for a.e. t

$t \mapsto m_{d-1}(E_{t,\gamma})$ is a measurable fctn of t.

<u>Thm. 2</u> $E \subset \mathbb{R}^d$ ($d \geq 3$) of finite measure.

Then for a.e. $\gamma \in S^{d-1}$:

(i) $E_{t,\gamma}$ is m_{d-1}-measurable for <u>every</u> t.

(ii) $t \mapsto m_{d-1}(E_{t,\gamma})$ is a <u>cont.</u> fct. of t.

Moreover, this fctn is α-Hölder $\forall \alpha \in (0, \frac{1}{2})$.

<u>Cor.</u> $d \geq 3$, $E \subset \mathbb{R}^d$ of Lebesgue measure zero.

Then, for a.e. $\gamma \in S^{d-1}$, the slice $E_{t,\gamma}$ has zero measure for <u>every</u> $t \in \mathbb{R}$.

<u>Prop.</u> $d \geq 3$, $f \in (L^1 \cap L^2)(\mathbb{R}^d)$. Then for a.e. $\gamma \in S^{d-1}$:

<u>i)</u> f is meas. and int. on the plane for every $t \in \mathbb{R}$.

<u>ii)</u> $t \mapsto \mathcal{R}(f)(t,\gamma)$ is cont. and α-Hölder if $\alpha < \frac{1}{2}$.

Moreover, estimate $(*)$ from Thm. 1 holds for f.

<u>Rem.</u> Prop. implies Thm. 2 by taking char. fctn of E.

$\mathcal{R}(\chi_E)(t,\gamma) = m_{d-1}(E_{t,\gamma})$.

We skip the proof of Prop. (follows from Thm. 1 using some delicate measure theory.)

What about $d = 2$?

Given $f \in L^1(\mathbb{R}^2)$, define

$$\mathcal{R}_\delta(f)(t,\gamma) = \frac{1}{2\delta} \int_{t-\delta}^{t+\delta} \mathcal{R}(f)(s,\gamma)\, ds. \quad \text{(averaged version of } \mathcal{R}\text{.)}$$

(integration over thickened Line (= hyperplane).)

$$= \frac{1}{2\delta} \int_{\{t-\delta \leq x\cdot\gamma \leq t+\delta\}} f(x)\, dx.$$

Thm. 3 $f \in C_0^0(\mathbb{R}^2)$, $0 < \delta \leq 1/2$.

$$\int_{S^1} R_\delta^*(f)(\gamma)\, d\sigma_\gamma \lesssim \left(\log \tfrac{1}{\delta}\right)^{1/2} \cdot \left(\|f\|_{L^1(\mathbb{R}^2)} + \|f\|_{L^2(\mathbb{R}^2)} \right)$$

Theorem. *$f \in C_0^0(\mathbb{R}^2)$, $0 < \delta \le \frac{1}{2}$. Then*

$$\int_{S^1} R_\delta^*(f)(\gamma)\,\mathrm{d}\sigma_\gamma \lesssim (\log \frac{1}{\delta})^{\frac{1}{2}}(\|f\|_{L^1(\mathbb{R}^2)} + \|f\|_{L^2(\mathbb{R}^2)})$$

Proof of Theorem. Modified version of lemma 3: Setting

$$F_\delta(t) = \int_{-\infty}^{\infty} \hat{F}(\lambda)(\frac{e^{2\pi i(t+\delta)\lambda} - e^{2\pi i(t-\delta)\lambda}}{2\pi i\lambda(2\delta)})\,\mathrm{d}\lambda$$

Suppose $\sup_{\lambda\in\mathbb{R}} |\hat{F}(\lambda)| \le A$ and $\int_{-\infty}^{\infty} |\hat{F}(\lambda)|^2 |\lambda|^{d-1}\,\mathrm{d}\lambda \le B^2$.

Claim:

$$\sup_t |F_\delta(t)| \lesssim (\log \frac{1}{\delta})^{\frac{1}{2}}(A+B)$$

$$F_\delta(t) = \int_{-\infty} \infty = \int_{|\lambda|\le 1} + \int_{|\lambda|>1} \le cA + \int_{1<|\lambda|\le\frac{1}{\delta}} |\hat{F}(\lambda)|\,\mathrm{d}\lambda + \frac{c}{\delta}\int_{|\lambda|>\frac{1}{\delta}} |\hat{F}(\lambda)||\lambda|^{-1}\,\mathrm{d}\lambda = I + II$$

CS:

$$I \lesssim (\int_{\mathbb{R}} |\hat{F}(\lambda)|^2|\lambda|\,\mathrm{d}\lambda)^{\frac{1}{2}}(\int_{1<|\lambda|\le\frac{1}{\delta}} |\lambda|^{-1}\,\mathrm{d}\lambda)^{\frac{1}{2}} \le B(\log\frac{1}{\delta})^{\frac{1}{2}}$$

$$II \lesssim \frac{c}{\delta}(\int_{\mathbb{R}^2} |\hat{F}(\lambda)|^2|\lambda|\,\mathrm{d}\lambda)^{\frac{1}{2}}(\int_{|\lambda|>\frac{1}{\delta}} |\lambda|^{-3}\,\mathrm{d}\lambda)^{\frac{1}{2}} \lesssim B$$

□

Theorem. *There exists a subset $K \subset \mathbb{R}^2$ such that*

(i) K is compact

(ii) K has Lebesgue measure zero

(iii) K contains a translate of every unit line segment

Theorem. *Suppose F is any set that satisfies conditions (i) and (iii) from Theorem 1. Then F has Hausdorff dimension 2.*

Proof of Theorem 2. Let F be a Kakeya set. Fix $0 < \alpha < 2$. Let $F \subset \bigcup_{i=1}^\infty B_i$ be a covering with balls B_i of diameter $\le \delta$. It is enough to show

$$\sum(\operatorname{diam} B_i)^\alpha \ge c_\alpha > 0$$

for $\alpha < 2$.

Case 1: Assume $\operatorname{diam} B_1 = \delta \le \frac{1}{2}$ and let $N < \infty$ be the number of balls in the covering. WTS $N\delta^\alpha \ge c_\alpha$. $B_i^* =$ double of B_i. $F^* = \bigcup_i B_i^*$. $|F^*| \le \sum |B_i^*| = cN\delta^2$. F Kakeya $\Rightarrow \forall\gamma \in S^1 \exists s_\gamma \perp \gamma$ unit lime segment: $s_\gamma \subset F$. $s_\gamma^\delta \subset F^*$. $\therefore R_\delta^*(\chi_{F^*})(\gamma) \ge 1$ $(\forall\gamma \in S^1)$. Take $f = \chi_{F^*}$ in (*). Since $L^2 \subset L^1$,

$$\|\chi_{F^*}\|_{L^1} \lesssim \|\chi_{F^*}\|_{L^2} = |F^*|^{\frac{1}{2}} \lesssim N^{\frac{1}{2}}\delta.$$

(*) $\Rightarrow 0 < c \le (\log\frac{1}{\delta})^{\frac{1}{2}} N^{\frac{1}{2}}\delta$. This implies $N\delta^\alpha \ge c_\alpha > 0$.

Case 2: General case. $F \subset \bigcup_{i=1}^{\infty} B_i$ with each ball B_i of diameter ≤ 1. For each $k \in \mathbb{N}$, let N_k be the number of balls in $\{B_i\}$ with diameter $B_k \sim 2^{-k}$, i.e. $\in [2^{-k-1}, 2^{-k}]$. WTS

$$\sum_{k=0}^{\infty} N_k 2^{-k\alpha} \geq c_\alpha > 0.$$

ETS $\exists k' : N_{k'} 2^{-k'\alpha} \geq c_\alpha$.

$$F_k = F \cap (\bigcup_{\text{diam} B_i \sim 2^{-k}} B_i)$$

$$F_k^* = \bigcup_{\text{diam} B_v \sim 2^{-k}} B_i^*$$

$$|F_k^*| \leq c N_k 2^{-2k} \quad \forall k$$

F Kakeya $\Rightarrow \forall \gamma \in S^2 \exists s_\gamma \perp \gamma : s_\gamma \subset F$ (in particular $m_1(s_\gamma \cap F) = 1$).

Key: For some k, a large proportion of s_γ belongs to F_k. Pick $\{a_k\}_{k=0}^{\infty}$ such that $0 \leq a_k < 1$, $\sum a_k = 1$, (a_k) dos not nend to 0 too quickly, e.g. $a_k = c_\varepsilon 2^{-k\varepsilon}$ (for sufficiently small ε.

Claim:

$$\exists k : m_1(s_\gamma \cap F_k) \geq a_k.$$

Otherwise $m_1(s_\gamma \cap F) \leq \sum_k m_1(s_\gamma \cap F_k) < \sum a_k = 1$, contradicts (**)

For this value of k,

$$R^*_{2^{-k}}(\chi_{F_k^*})(\gamma) \geq a_k.$$

Since this choice of k depends on γ, let

$$E_k = \{\gamma \in S^1 : R^*_{2^{-k}}(\chi_{F_k^*})(\gamma) \geq a_k\}.$$

$S^1 = \bigcup_{k_1}^{\infty} E_k$. Therefore $\exists k' : |E_{k'}| \geq 2\pi a_{k'}$.

$$2\pi a_{k'}^2 = 2\pi a_{k'} a_{k'} \leq \int_{E_{k'}} a_{k'} \, \mathrm{d}\sigma \leq_{S^1} R^*_{2^{-k'}}(\chi_{F_{k'}^*})(\gamma) \, \mathrm{d}\sigma_\gamma$$

$$2^{-2k\varepsilon} \sim a_{k'}^2 \leq c(\log 2^{k'})^{\frac{1}{2}} |F_{k'}^*|^{\frac{1}{2}} \leq c(\log 2^{k'})^{\frac{1}{2}} N_{k'}^{\frac{1}{2}} 2^{-k'}$$

$\Rightarrow N_{k'} 2^{-\alpha k'} \geq c_\alpha$, provided $4\varepsilon < 2 - \alpha$. □

Construction of a Kakeya set I (Stein-Shakarchi, III)

Thinner Cantor set, always taking away the half.

Take two of them, E_0, E_1, where E_1 has twice the length. Put E_0 on $y = 1$ and E_1 on $y = 0$. Let F be the union of all line segments that join a point in E_0 with one in E_1.

Construction of an ε-Kakeya set (Stein)

Theorem. *Given $\varepsilon > 0$, $\exists N = N_\varepsilon$ and 2^N rectangles $R_1, ..., R_{2^N}$ with side lengths 1×2^{-N} such that*

(i)

$$|\bigcup_{j=1}^{2^N} R_j| < \varepsilon$$

(ii) the reaches $\tilde{R}_j$ are mutually disjoint, i.e.

$$|\bigcup_{j=1}^{2^N} \tilde{R}_j| = 1$$

Proof. Fix $\alpha \in (\frac{1}{2}, 1)$. Symmetric triangle ABC with M opposite C. Push the right part into the left part and call the resulting body $\Phi(T)$. It consists of heart $\Phi_h(T)$ and arms $\Phi_a(T)$. Then

$$|\Phi_h(T)| = \alpha^2 |T|$$

$$|\Phi_a(T)| = 2(1-\alpha)^2 |T|$$

Conclusion

$$|\Phi(T)| = (\alpha^2 + 2(1-\alpha)^2)|T|$$

n-fold iteration (Peron trees): Split not into two but 2^n parts and do everything pairwise. Key: right side of $\Phi_h(A_0A_2C)$ // left side of $\Phi_n(A_2A_4C)$ // CA_2

Then look at heart/arms again.

$$|\text{arms of } \Psi_1(ABC)| \leq 2(1-\alpha)^2|T|.$$

$$|\text{heart of } \Psi_1(ABC)| = \alpha^2|T|$$

$$\therefore |\Psi_1(ABC)| = (\alpha^2 + 2(1-\alpha)^2)|T|.$$

Iterate: Carry out this process on the heart of $\Psi_1(ABC)$ with n replaced by $n-1$, given are the union of 2^{n-1} triangles.

Then retranslate all 2^n original triangles to obtain figure $\Psi_2(ABC)$.

$$|\text{heart of } \Psi_2(ABC) = \alpha^2\alpha^2|T|$$

$$|\text{additional arms of } \Psi_2(ABC)| \leq 2(1-\alpha)^2\alpha^2|T|$$

$$\begin{aligned} |\Psi_n(ABC)| &\leq (\alpha^{2n} + 2(1-\alpha)^2 + 2(1-\alpha)^2\alpha^2 + \dots + 2(1-\alpha)^2\alpha^{2n-2}) \\ &\leq \alpha^{2n} + 2(1-\alpha)^2 \underbrace{\sum_{n=0}^{\infty} \alpha^{2n}}_{=\frac{1}{1-\alpha^2}} \\ &\leq \alpha^{2n} + 2(1-\alpha) \end{aligned}$$

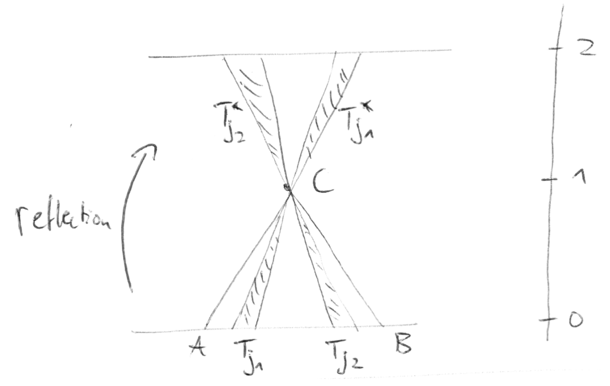


Figure 1: Obtaining mutually disjoint reaches by reflecting in C.

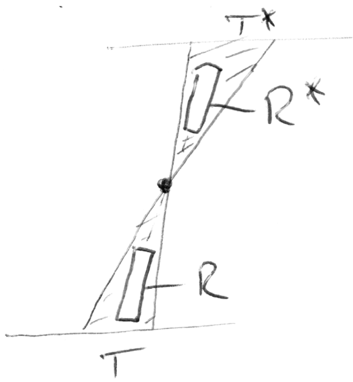


Figure 2: Go from triangles to rectangles by placing rectangles into the triangles with half the length.

$\square$

Application Maximal functions and counterexamples. Q: Given a collection $\mathcal{C} = \{C\}$ of sets, for which class of functions do we have

$$\lim_{\operatorname{diam}(C)\to 0,\ c\in\mathcal{C}} \frac{1}{|C|}\int_C f(x-y)\,\mathrm{d}y = f(x) \quad x-\text{a.e.?}$$

Seen:

$$(M_{\mathcal{C}}f)(x) = \sup_{c\in\mathcal{C}} \frac{1}{|C|}\int_C |f(x)-y)|\,\mathrm{d}y$$

$\mathcal{C} = \{\text{balls}\}$, weak-type (1,1) inequality for $M_{\mathcal{C}} \Rightarrow$ a.e. convergence of averages. A converse also holds!

$\{\mathrm{d}\mu_j\}_{j=1}^{\infty}$ collection of finite, nonnegative measures on $\mathbb{R}^d$: $\operatorname{supp}(\mu_j) \subset K \Subset \mathbb{R}^d$. Define the maximal operator

$$(Mf)(x) = \sup_j |f * \mu_j|(x).$$

Proposition. $1 \le p < \infty$. Assume for each $f \in L^p(\mathbb{R}^d)$ that $(Mf)(x) < \infty$ for some set of x having positive measure. Then $f \mapsto Mf$ is of weak-type (p,p), i.e.

$$\exists A < \infty : |\{x \in \mathbb{R}^d : (Mf)(x) > \alpha\}| \le \frac{A}{\alpha^p}\|f\|_{L^p} \quad (\forall \alpha > 0)$$

Lemma. $\{E_j\}$ collection of subsets of a fixed compact set:

$$\sum_{j=1}^{\infty} |E_j| = \infty.$$

Then there exists a sequence of translates $F_j = E_j + x_j$:

$$\limsup F_j = \bigcap_{k=1}^{\infty}(\bigcup_{j=k}^{\infty} F_j) = \mathbb{R}^n \quad \text{(a.e.)}$$

The above set equals $\{x \in \mathbb{R}^d : x \in F_j \text{ infinitely often}\}$.

$$\liminf F_j = \bigcup_{k=1}^{\infty} (\bigcup_{j=k}^{\infty} F_j)$$

is a subset.

Proof of Lemma. $Q \subset \mathbb{R}^d$ unit cube. $A_1, A_2 \subset Q$. Then $\exists h \in \mathbb{R}^d : |A_1 \cap (A_2 - h)| \geq 2^{-d}|A_1||A_2|$. Why?

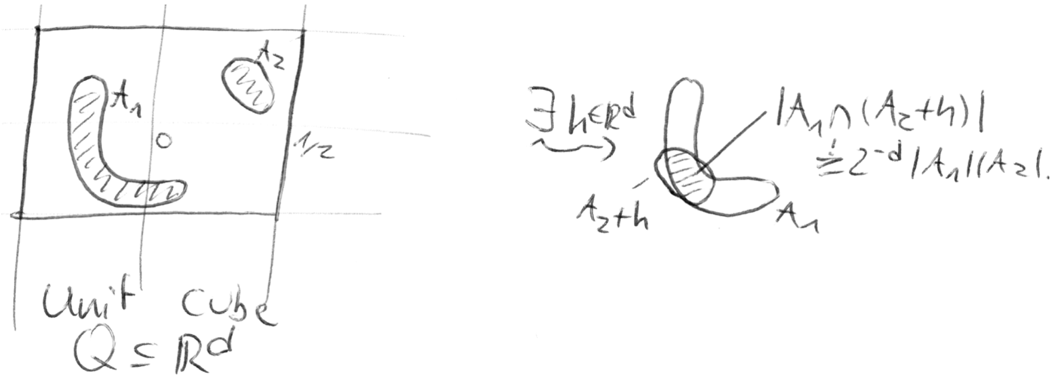


Figure 3: Translation of subsets A_1 and A_2 of a unit cube Q.

$$\eta(x) = \int_{\mathbb{R}^d} \chi_{A_1}(y)\chi_{A_2}(x+y)\,\mathrm{d}y \sim \chi_{A_1} * \chi_{A_1}(x)$$

$$\int_{\mathbb{R}^d} |A_1||A_1|$$

$$\operatorname{supp}(\eta) \subset Q^*$$

$|Q^*| = 2^d$.

$$\exists h \in Q^* : \eta(h) \geq \operatorname{avg}_{Q^*}(\eta) = \frac{1}{|Q^*|} \int_{\mathbb{R}^d} \eta = \frac{|A_1||A_2|}{2^d}$$

Wlog $\operatorname{supp}(E_j) \subset Q$.

Step 2: There exist translates $F_j = E_j + x_j$ that cover Q at least once.

$$Q \subset \bigcup_j F_j$$

Why? $F_1 = E_1$. Suppose (inductively) that $F_1, ..., F_{j-1}$ have been constructed. Let $A_1 = Q \cap (F_1 \cup ... \cup F_{j-1})^\complement$ and $A_2 = E_j$. Step 1 $\Rightarrow$ $\exists h : |A_1 \cap (A_2 - h)| \geq 2^{-d}|A_1||A_2|$. Set $F_j = A_2 - h = E_j - h$. Let $p_j = |Q \cap (F_1 \cup ... \cup F_j)|$. Then

$$\begin{aligned} p_j &= p_{j-1} + |\underbrace{Q \cap (F_1 \cup ... \cup F_{j-1})^\complement}_{A_1} \cap \underbrace{F_j}_{A_2 - h}| = p_{j-1} + |A_1 \cap (A_2 - h)| \\ &\geq p_{j-1} + 2^{-d}|A_1||E_j| = p_{j-1} + 2^{-d}(1 - p_{j-1})|E_j| \end{aligned}$$

$$\therefore p_j - p_{j-1} \geq 2^{-d}(1 - p_{j-1})|E_j|$$

$$\sum_{j=2}^{\infty}(p_j - p_{j-1}) = \lim_{j\to\infty} p_j - p_1 \therefore \lim_j p_j = 1$$

Step 3: Decompose (twice) $\{E_j\}$ into a countable infinite number of subcollections so that on each subcollection the sum of the measures diverges. $\square$

Proof of Proposition. Take a ball B such that $B \supset Q + K$. $\text{supp}(F) \subset Q \Rightarrow \text{supp}(F * \mu_j) \subset\Rightarrow \text{supp}(Mf) \subset B$. Key: Estimate (*) (the violation of the weak type estimate) holds if $\text{supp}(f) \subset Q$. For each $k, \exists \alpha_k > 0 \exists g_k \subset L^p : \text{supp}(g_k) \subset Q$ such that

$$|\{x \in B : Mg_k(x) > \alpha_k\}| \geq \frac{2^k}{\alpha_k^p} \|g_k\|_{L^p}^p$$

Replace g_k by $\tilde{g}_k = \frac{k}{\alpha_k} g_k$.

$$\frac{2^k}{k^p} \leq \frac{|\{x \in B : M\tilde{g}_k(x) > k\}|}{\|\tilde{g}_k\|_{L^p}^p} \to \infty \quad \text{as} k \to \infty$$

$\therefore$ There exists a sequence $\{f_k\} \subset L^p$ and a sequence of constants $R_k \to \infty$ such that with $E_k = \{x \in B : Mf_k(x) > R_k\}$ we get

$$\sum_k |E_k| = \infty \qquad \sum_k \|f_k\|_{L^p}^p < \infty.$$

Remark. $\mathrm{d}\mu_j \geq 0$ wlog $f_k \geq 0$.

By the lemma $\exists \{x_k\}$ such that $F_k = E_k + x_k$ satisfy $\limsup F_k = \mathbb{R}^d$ (a.e). Let

$$\tilde{f}_k(x) = f_k(x + x_k), \qquad F(x) = \sup_k \tilde{f}_k(x)$$

Then

$$M(F) = \sup_j |F * \mu_j| = \sup_j |(\sup_k \tilde{f}_k) * \mu_j| \geq \sup_k \sup_j |\tilde{f}_k * \mu_j| = \sup_k M(\tilde{f}_k)$$

Also $M(\tilde{f}_k) > R_k$ on $F_k \therefore M(F) = \infty$ a.e. Check $f \in L^p$:

$$|F|^p = |\sup_k \tilde{f}_k|^p \leq \sum_p |\tilde{f}_k|^p$$

$$\|F\|_{L^p}^p \leq \sum_k \|f_k u|_{L^p}^p < \infty$$

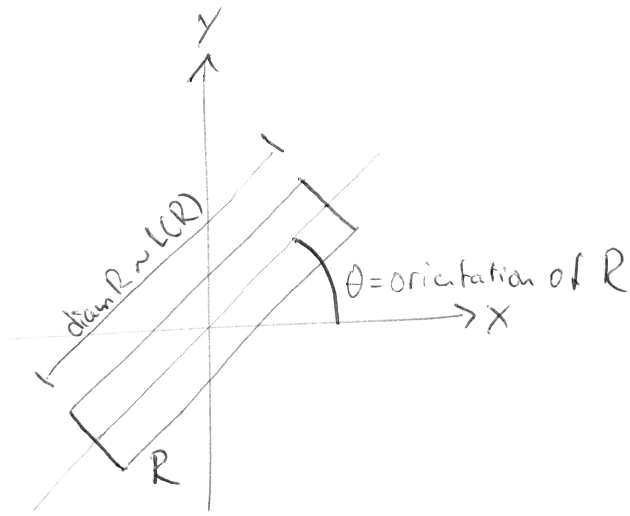
Full conclusion

$$f = \sum f\chi_{Q_j} =: \sum f_j$$
$$M(f) \leq \sum_j M(f_j)$$

$\square$

Example. Rectangles with arbitrary orientation

$$\mathcal{C} = \mathcal{R} = \{\textit{all} \text{ rectangles in } \mathbb{R}^2 \text{ centered at } 0\}$$



Corollary. Given $1 \leq p < \infty, \exists f \in L^p(\mathbb{R}^n)$ such that

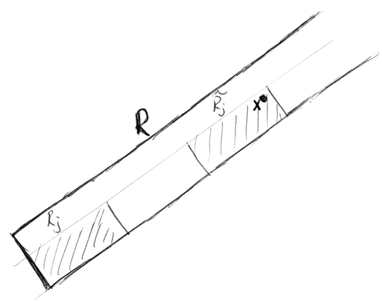
$$\limsup_{\operatorname{diam}(R)\to 0 R\in\mathcal{R}} \frac{1}{|R|}\int_R f(x-y)\,\mathrm{d}y = \infty \quad (x - \text{a.e.})$$

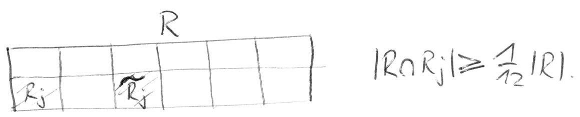
Idea: Use the ε-Kakeya set to show that M is not weak (p,p)

$$(Mf)(x) = \sup_{\operatorname{diam}(R)<8} \frac{1}{|R|}\Big|\int_R f(x-y)\,\mathrm{d}y\Big|$$

Let $E = \bigcup_{j=1}^{2^N} R_j$ as before. $\|\chi_E\|_{L^p}^p = |E| < \varepsilon$. If $x \in \tilde{R}_j$, then $\exists$ rectangle R such that

- R is centered at x
- $\operatorname{diam}(R) \leq 8$
- $|R \cap R_j| \geq \frac{1}{12}|R|$





$y \in x - E = -(E-x),\ y - x \in -E,\ x - y \in E$

$$M(\chi_E)(x) \geq \fint_{R-x} \chi_E(x-y)\,\mathrm{d}y = \frac{|(R-x)\cap(E-x)|}{|R|} \geq \frac{1}{12}$$

Conclusion: $M\chi_E \geq \frac{1}{12}$ on the set $\bigcup_{j-1}^{2^N} \tilde{R}_j$ (of measure 1)

$$\forall A > 0\ \exists \text{ set } E : |\{x \in \mathbb{R}^d : M\chi_E > \alpha\}| \leq A\alpha^{-p}\|\chi_E\|_{L]p}^p$$

does not hold! $\therefore M$ is not of weak typ (p,p).

Note, that this is not the complete proof. Therefore still have to replace 8 by δ.

Bochner-Ries summability Q: In which way does Fourier inversion hold? (In $L^p(\mathbb{R}^d)$)

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi)e^{2\pi i x\xi}\,\mathrm{d}\xi = \lim_{R\to\infty} \underbrace{\int_{|\xi|<R} \hat{f}(\xi)(1-\frac{|\xi|^2}{R^2})^\delta e^{2\pi i x\xi}\,\mathrm{d}\xi}_{f * k_\delta}?$$

for suitable $\delta \geq 0$.

$$\delta > \frac{d-1}{2} \Rightarrow k_\delta \in L^1(\mathbb{R}^d)$$

$$\delta \leq \frac{d-1}{2}$$

$$\delta = 0 \text{ today}$$

$d = 1$: The second equality holds in L^p-norm $(1 < p < \infty)$ $\Longleftarrow$ L^p boundedness of Hilbert transform

It also holds a.e. ($p = 2$ Carleson '66 $1 < p < \infty$ Hunt '68)

$d \geq 2$ Let us only consider norm convergence

$$f \mapsto (S^\delta f)(x) = \int_{|\xi|\leq 1} \hat{f}(\xi)(1-|\xi|^2)^\delta e^{2\pi i x\xi}\,\mathrm{d}\xi$$

$$Sf(x) = \int_{|\xi|\leq 1} \hat{f}(\xi)e^{2\pi i x\xi}\,\mathrm{d}\xi$$

$$\widehat{Sf} = 1_{|\xi|\leq 1}\hat{f}$$

$(\widehat{Hf} = i\pi\ \mathrm{sgn}\xi\hat{f})$

$$\|Sf\|_{L^2(\mathbb{R}^d)} = \|\widehat{Sf}\|_{L^2} = \|1_{|\xi|\leq 1}\hat{f}\|_{L^2} \leq \|\hat{f}\|_{L^2} = \|f\|_{L^2}$$

He said something about every bounded operator can be written like this or so? Fourier multiplier S is bounded iff multiplier function is bounded.

Theorem (C.Fefferman '71 – Annals of mathematics "The multiplier problem for the ball")**.** *Suppose $q \geq 2$ and $p \neq 2$. Then the operator S (initially defined on $L^p \cap L^2$) is not extendable to a bounded operator from $L^p(\mathbb{R}^d)$ to itself.*

Proof. Let $B \subset \mathbb{R}^d$ ball, let S_B be the multiplier orator associated to B:

$$\widehat{S_B f} = 1_B \hat{f}.$$

Given $u \in S^{d-1} \subset \mathbb{R}^d$, let S^n be the multiplier operator associated to the half-space with normal u:

$$\widehat{S^u f} = 1_{\{\xi u > 0\}} \hat{f} \qquad (S^u f)(x) = \int_{\xi u > 0} \hat{f}(\xi) e^{2\pi i x \xi} \, \mathrm{d}\xi$$

Upshot: L^p-bound for S implies an L^p vector-valued inequality for S_Bs and S_us.

Lemma (Y. Méyer)**.** Suppose

$$\|Sf\|_{L^p} \leq A_p \|f\|_{L^p}$$

$f \in (L^2 \cap L^p)(\mathbb{R}^d)$ holds for some $p \in [1, \infty]$. Suppose $f_1, ..., f_M \in L^2 \cap L^p$, $u_1, ..., u_M \in S^{d-1} \subset \mathbb{R}^d$. Then

$$\|(\sum_{j=1}^{M} ||S^{u_j}(f_j)|^2)^{\frac{1}{2}}\|_{L^p(\mathbb{R}^d)} \leq A_p \|(\sum_{j=1}^{M} |f_j|^2)^{\frac{1}{2}}\|_{L^p(\mathbb{R}^d)} \tag{4}$$

where A_p is the same constant as above.

Proof of Lemma. Step 1: $B = B_R =$ ball of radius R centered at 0. Then

$$\|S_B(f)\|_{L^p} \leq A_p \|f\|_{L^p} \quad (g \in L^2 \cap L^p) \tag{5}$$

Why? Scaling:

$$\delta_R(g)(x) = g(\frac{x}{R})$$

Check $\delta_{R^{-1}} \circ S \circ \delta_R = S_{B_R}$ since $S = S_{B_1}$. $\hat{\delta}_\rho(g)(\xi) = R^d \hat{g}(R\xi)$.

Step 2: M balls. $(p < \infty)$ $f = (f_1, ..., f_M)$ given M-tuple of functions. $T(f) := (Tf_1, ..., Tf_M)$. Given a unit vector $\omega = (\omega_1, ..., \omega_M) \in \mathbb{C}^M$, let

$$S_\omega(f) = \sum_{j=1}^{M} \bar{\omega}_j S_B(f_j) = S_B(\sum_j \bar{\omega}_j f_j) = S_B(f_\omega) \qquad f_\omega = \sum_{j=1}^{M} \bar{\omega}_j f_j$$

$$(5) \Rightarrow \int_{\mathbb{R}^d} |S_\omega f(x)|^p \, \mathrm{d}x \leq A_p^k \int_{\mathbb{R}^d} |f_\omega(x)|^p \, \mathrm{d}x \tag{6}$$

$(S_\omega f)(x) = ?$. $x, y \in \mathbb{C}^M$, $\langle x, y \rangle = \sum_{i=1}^{M} x_j \bar{y}_j$

$$\begin{aligned} S_\omega(f)(x) = S_B(f_\omega)(x) = S_B(\sum_{j=1}^{M} \bar{\omega}_j f_j)(x) = \sum_{j=1}^{M} \bar{\omega}_j S_B(f_j)(x)| = |\langle S_B(f)(x), \omega \rangle| \\ = |S_B(f)(x)| |\langle \frac{S_B(f)(x)}{|S_B(f)(x)|}, \omega, \rangle| = (\sum_{j=1}^{M} |S_B(f_j)(x)|^2)^{\frac{1}{2}} |\varphi(\omega, S_B(f)(x))| \end{aligned}$$

Integrate both sides of (6) with respect to ω (before integrating in x).

$$\begin{aligned} \text{LHS} &= \int_{\mathbb{R}^d} (\int_{|\omega|=1} |S_\omega(f)(x)|^p \, \mathrm{d}\omega) \, \mathrm{d}x \\ &= \int_{\mathbb{R}^d} (\sum_{j=1}^{M} |S_B(f_j)(x)|^2)^{\frac{p}{2}} \underbrace{(\int_{|\omega|=1} |\Phi(\omega, S_B(f)(x)|^p \, \mathrm{d}\omega) \, \mathrm{d}x))}_{\gamma_p} \, \mathrm{d}x \end{aligned}$$

$$0 \neq \gamma_p = \int_{|\omega|=1} |\Phi(\omega, 1)|^p \,\mathrm{d}\omega$$

For fixed $\nu \in S^{d-1}$ $\int_{S^{d-1}} |\langle \omega, \nu \rangle|^p \,\mathrm{d}\sigma_\omega = \omega_{d-2} \int_{-1}^{1} (1-t^2)^{\frac{d-3}{2}} \,\mathrm{d}t$.

Similarly,

$$\mathrm{RHS} = \int_{\mathbb{R}^d} (\sum_{j=1}^{M} |f_j(x)^2)^{\frac{p}{2}} \,\mathrm{d}x \gamma_p$$

$$(6) \Rightarrow \|(\sum_{j=1}^{M} |S_B(f_j)|^2)^{\frac{1}{2}}\|_{L^p(\mathbb{R}^d)} \leq A_p \|(\sum_{j=1}^{M} |f_j|^2)^{\frac{1}{2}}\|_{L^p(\mathbb{R}^d)}$$

Step 3: From balls to half-spaces

$B_R^u =$ ball of radius R centered at Ru. Upshot: $B_R^u \to \{\xi u > 0\}$ as $R \to \infty$.

Translation

$$(T_y f)(x) = f(x-y)$$

$$\widehat{T_y f}(\xi) = e^{i\xi y} \hat{f}(\xi)$$

$$S_{B_R^u}(f)(x) = e^{2\pi i u R x} S_{B_R}(f e^{-2\pi i u R x})$$

(4) implies

$$\|(\sum |S_{B_R^{u_j}}(f_j)|^2)^{\frac{1}{2}}\|_{L^p} \leq A_p \|(\sum |f_j|^2)^{\frac{1}{2}}\|_{L^p}$$

Let $R \to \infty$ to finish:

$$\therefore S_{B_R^{u_j}}(f_j) \to S^{u_j}(f_j) \quad R \to \infty (\text{ in } L^2)$$

$\therefore$ there exists an almost everywhere converging subsequence, done. $\square$

$\widehat{Sf} = 1_{B(0,1)} \hat{f}$. S is not bounded in $L^p(\mathbb{R}^d)$ unless $d = 1$ or $p = 2$. Focus on multiplier operator for the half-space (S^u), $d = 1$.

$$(S^+ f)(x) = \int_0^\infty \hat{f}(\xi) e^{2\pi i x \xi} \,\mathrm{d}\xi \quad (f \in L^2)$$

$$|(S^+ f)(x)| \geq \frac{c}{|x|} \quad \text{if } |x| \geq \frac{1}{2} \tag{7}$$

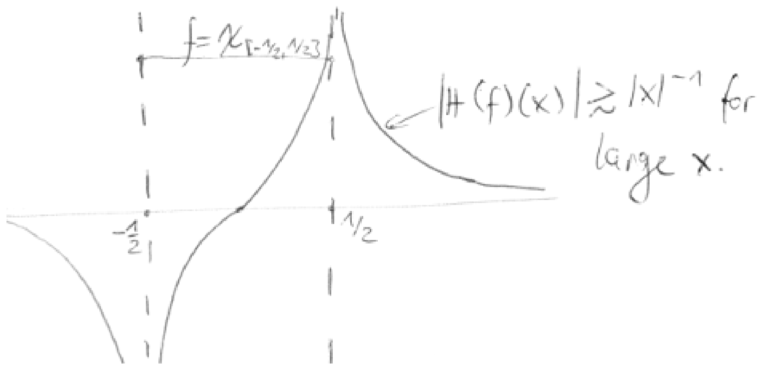
Proof.

$$(S^+ f)(x) = \lim_{\varepsilon \to 0} \int_0^\infty \hat{f}(\xi) e^{2\pi i (x + i\varepsilon)\xi} \,\mathrm{d}\xi \quad \text{in } L^2$$

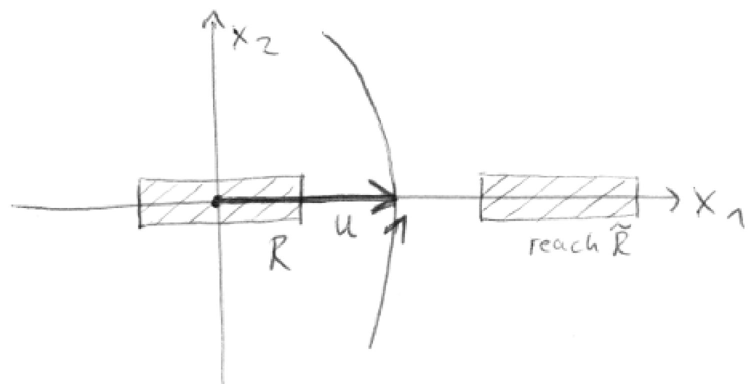
$$\int_{-\infty}^{\infty} (\int_0^\infty e^{-2\pi i y \xi} e^{2\pi i (x+i\varepsilon)\xi} \,\mathrm{d}\xi) f(y) \,\mathrm{d}y = \frac{1}{2\pi i} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{y - x - i\varepsilon} \,\mathrm{d}y$$

This has absolute value $\lesssim c|x|$ if $|x| \geq \frac{1}{2}$.

Alternative proof: $|H(f)(x)| \geq |x|^{-1}$ for large x, $S^+ = \frac{1}{2}(I + iH)$.



$\square$



(rotate and translate to get to this situation.)

For $R = (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2^{N+1}}, \frac{1}{2^{N+1}})$

$$1_R = 1_{(-\frac{1}{2}, \frac{1}{2})} \otimes 1_{(-2^{-(N+1)}, 2^{-(N+1)})}$$

If u points in the direction of x_1 then

$$(S^u 1_R)(x_1, x_2) = (S^+ 1_{(-\frac{1}{2}, \frac{1}{2})})(x_1) 1_{(-2^{-(N+1)}, 2^{-(N+1)})}(x_2)$$

$$(7) \Rightarrow |S^u(1_R)| \geq c' 1_{\tilde{R}}$$

Similarly for any 1×2^{-N} rectangle R_j. $u_j \in S^1$ in the positive direction of the longest side of R_j. Rotate and translate, then we get

$$|S^{u_j}(1_{R_j})| \geq c' 1_{\tilde{R}_j} \tag{8}$$

Take $R_1, ..., R_{2^N}$ to be the collection given by ε-Kakeya construction, plug that into result from lemma, get a contradiction.

Key: $p < 2$ and $d = 2$.

Lemma with $f_j = 1_{R_j}$ and $M = 2^N$

$$c' \leq \|(\sum_{j=1}^{2^N} |S^{u_j}(1_{R_j})|^2)^{\frac{1}{2}}\|_{L^p(\mathbb{R}^2)} \qquad (8),\ |\bigcup \tilde{R}_j| = 1$$
$$\leq A_p \|(\sum_{j=1}^{2^N} |1_{R_j}|^2)^{\frac{1}{2}}\|_{L^p(\mathbb{R}^2)} = A_p \underbrace{(\int_E (\sum_{j=1}^{2^N} |1_{R_j}|^2)^{\frac{p}{2}} \,\mathrm{d}x)^{\frac{1}{p}}}_{I}$$

$$I \leq |E|^{\frac{1}{q}} (\int (\sum |1_{R_j}|^2)^{\frac{p}{2}\frac{2}{p}} \,\mathrm{d}x)^{\frac{1}{p}\frac{p}{2}} \qquad \text{Hölder}$$
$$= |E|^{\frac{1}{q}} \sum_{j=1}^{2^N} |R_j| = |E|^{\frac{1}{q}}$$

$$E = \bigcup_{j=1}^{2^N} R_j, \quad |E| < \varepsilon, \quad \frac{1}{q} + \frac{1}{p/2} = 1, \quad \frac{1}{q} = 1 - \frac{p}{2}, \quad \frac{1}{pq} = \frac{1}{p}(1 - \frac{p}{2}) > 0$$

In the end we get

$$c' \leq A_p \varepsilon^{\frac{1}{pq}}.$$

Let $\varepsilon \to 0^+$ to finish.

$d > 2$:

$$f_j(\underbrace{x}_{\in \mathbb{R}^d}) = f_j(x_1, x_2, \underbrace{x'}_{\in \mathbb{R}^{d-2}}) = 1_{R_j}(x_1, x_2) f(x') \quad f \in S(\mathbb{R}^{d-2})$$

$p > 2$:

$$\langle Sf, g\rangle = \langle \widehat{Sf}, \hat{g}\rangle = \langle 1_B \hat{f}, \hat{g}\rangle = \langle \hat{f}, 1_B \hat{g}\rangle = \langle f, Sg\rangle \quad S = S^*$$

□

Oscillatory integrals in harmonic analysis Stein (VIII,IX), Stein-Shakarchi (Chapter 8), Sogge

- averaging operators
- restriction theory
- Bochner-Riesz summability

Motivation (in $\mathbb{R}^3$)

$$(Af)(x) = ⨍_{\mathbb{S}^2} f(x-y) \,\mathrm{d}\sigma_y = \frac{1}{4\pi} f * \sigma(x)$$

σ surface measure in $\mathbb{S}^2$.

Smoothing properties:

$$\|\frac{\partial}{\partial x_j} A(f)\|_{L^2(\mathbb{R}^3)} \lesssim \|f\|_{L^2(\mathbb{R}^3)} \quad (j = 1, 2, 3) \tag{9}$$

$f \in L^2 \Rightarrow$

$$\|f * \sigma\|_{L^2} \leq \|f\|_{L^2} \underbrace{|\sigma|(\mathbb{R}^3)}_{<\infty}$$

(use Minkowski integral inequality) $\therefore f * \sigma \in L^2$ if $f \in L^2$.

Idea $\widehat{(f * \sigma)} = \hat{f}\hat{\sigma}$

$$(\widehat{\frac{\partial}{\partial x_j} A(f)})(\xi) = \xi_j \hat{f}(\xi)\hat{\sigma}(\xi),$$

so we should study $\hat{\sigma}$.

without loss of generality $\xi = (0, 0, |\xi|)$ because the integral is spherically symmetric.

$$\hat{\sigma}(\xi) = \int_{\mathbb{S}^2} e^{-2\pi i \omega \xi} \, \mathrm{d}\sigma_\omega = 2\pi \int_0^\pi e^{-2\pi i |\xi| \cos\theta} \sin\theta \, \mathrm{d} = 2\pi \int_{-1}^1 e^{2\pi i |\xi| t} \, \mathrm{d}t = \frac{2\sin(2\pi|\xi|)}{|\xi|}$$

$$t = \cos\theta \quad \mathrm{d}t = \sin\theta \, \mathrm{d}\theta$$

$$\therefore |\hat{\sigma}(\xi)| \lesssim (1 + |\xi|)^{-1}$$

(9) follows from this and Plancherel.

This also generalizes to higher dimensions. There we get a factor $(1-t^2)^{\frac{d-3}{2}}$ in the last integral.

Oscillatory integrals

$$I(\lambda) = \int_{\mathbb{R}^d} e^{i\lambda\phi(x)} \psi(x) \, \mathrm{d}x$$

where $\lambda \in \mathbb{R}$ is the oscillatory parameter, $\phi \in C^\infty(\mathbb{R}^d, \mathbb{R})$ the phase and $\psi(x) \in C$ the amplitude.

Q.: How does $I(\lambda)$ behave for large $|\lambda|$? General principle: Main contribution comes from the critical points of the phase, $x_0 : \nabla\phi(x_0) = 0$.

Principle of non-stationary phase $\phi \in C^\infty$, $\psi \in C_0^\infty : |\nabla\phi(x)| > 0$ ($\forall x \in \operatorname{supp}\psi$). Then for any $N \in \mathbb{N}$

$$|I(\lambda)| \leq c_N |\lambda|^{-N}$$

Proof. involves integration by parts □

$d = 1$:

$$I_1(\lambda) = \int_a^b e^{i\lambda\phi(x)} \, \mathrm{d}x$$

$$0 < a < b < \infty \quad \psi(x) = \chi_{[a,b]}(x) \quad \text{which is rough!}$$

This means we will not get such a fast decay

Lemma (van der Corput (I))**.** $\phi \in C^2$, ϕ' monotonic, $|\phi'(x)| \geq 1$ ($\forall x \in [a, b]$). Then

$$|I_1(\lambda)| \leq \frac{3}{|\lambda|} \quad (\forall \lambda > 0)$$

Remark. (i) 3 is neither important nor sharp; independence of a, b, ϕ is the key!

(ii) Order of decrease in λ is sharp ($\phi(x) = x \therefore I_1(\lambda) = \frac{e^{i\lambda b} - e^{i\lambda a}}{i\lambda}$)

(iii) monotonicity of ϕ' is essential

Proof. Integrate by parts(...) □

What if critical points are present? ($d = 1$)

$x_0 : \phi'(x_0) = 0$ (critical point) and $\phi''(x_0) \neq 0$ (non degenerate), e.g. $\phi(x) = x^2,\ x_0 = 0$. In this case

$$\int_{\mathbb{R}} e^{i\lambda x^2} \psi(x)\,\mathrm{d}x = c_0 \lambda^{\frac{-1}{2}} + \mathcal{O}(|\lambda|^{-\frac{3}{2}}) = \sum_{k=0}^{N} a_x \lambda^{-\frac{1}{2}-k} + \mathcal{O}(|\lambda|^{-\frac{3}{2}-N}) \quad (\forall N,\ \lambda \to \infty)$$

Lemma (van der Corput (II))**.** $\phi \in C^2[a,b],\ |\phi''(x)| \geq 1$ ($\forall x \in [a,b]$). Then

$$|I_1(\lambda)| \leq \frac{8}{\lambda^{\frac{1}{2}}} \quad (\forall \lambda > 0)$$

Remark. More generally: $\mathcal{O}(|\lambda|^{\frac{1}{k}})$ if $|\phi^{(k)}| \geq 1$.

Proof. Integration by parts not needed. Instead split up region in small area around critical point with properly chosen size, and rest, and then use results from above. $\square$

Corollary. Same assumptions as van der Corput (II). $\psi \in C^1[a,b]$.

$$|\int e^{i\lambda\phi(x)} \psi(x)\,\mathrm{d}x| \leq c_\psi \lambda^{-\frac{1}{2}}$$

Application: Asymptotics of Bessel functions

$$J_m(r) = \frac{1}{2\pi} \int_0^{2\pi} e^{ir\sin x} e^{-imx}\,\mathrm{d}x \quad \phi(x) = \sin x, \quad \psi(x) = e^{-imx} \quad (m \in \mathbb{Z})$$

Corollary.

$$|J_m(r)| \leq c r^{-\frac{1}{2}} \quad r \to \infty$$

Recall: Averaging operator in $\mathbb{R}^d$ ($d > 1$) is

$$(Af)(x) = (f * \sigma)(x) \quad \sigma \text{ surface measure on } \mathbb{S}^{d-1}$$

Theorem. *$f \mapsto A(f)$ is bounded from $L^2(\mathbb{R}^d)$ to $L^2_k(\mathbb{R}^d)$ with $= \frac{d-1}{2}$.*

Proof.

$$\hat{\sigma}(\xi) = 2\pi|\xi|^{-\frac{d}{2}+1} \underbrace{J_{\frac{d}{2}-1}(2\pi|\xi|)}_{=\mathcal{O}(|\xi|^{-\frac{1}{2}}),\ |\xi|\to\infty}$$

$$\therefore |\hat{\sigma}(\xi)| = \mathcal{O}(|\xi|^{-\frac{d-1}{2}}) \quad |\xi| \to \infty$$

$\square$

"What is van der Corput's lemma in higher dimension?" (Carbery-Wright, 2000)

$$I(\lambda) = \int_{\mathbb{R}^d} e^{i\lambda\phi(x)} \psi(x)\,\mathrm{d}x$$

(ϕ smooth, ψ smooth, compactly supported)

nondegeneracy hypothesis

$$\det(\nabla^2\phi)(x) \neq 0 \quad \forall x \in \mathrm{supp}(\psi)$$

Theorem. *Under above assumptions*

$$|I(\lambda)| = \mathcal{O}(|\lambda|^{-\frac{1}{2}}) \quad \lambda \to \infty$$

Remark. (i) Decay rate is sharp

(ii) Proof uses TT^* method: $|I(\lambda)|^2 = I(\lambda)\overline{I(\lambda)}$

(iii) variant: $\mathrm{rk}(\nabla^2\phi) \geq m$ for some $0 < m \leq d$ on $\mathrm{supp}(\psi)$. Then

$$|I(\lambda)| = \mathcal{O}(|\lambda|^{-\frac{m}{2}})$$

Application: Fourier transform of surface-carried measures

Recall: $(\mathbb{S}^{d-1}, \sigma)$

$$|\hat{\sigma}(\xi)| \lesssim (1+|\xi|)^{-\frac{d-1}{2}}$$

(not a Bessel coincidence)

(local) C^∞-hypersurface M. After translation and rotation $x_0 = 0$, $T_{x_0}M = \{x_d = 0\}$. M can be represented as

$$M = \{(x', x_d) \in B \subset \mathbb{R}^d : x_d = \varphi(x')\}$$

Can arrange $\varphi(0) = 0 = (\nabla_{x'}\varphi)(x')|_{x'=0}$.

$$\varphi(x') = \frac{1}{2}\sum_{k,j=1}^{d-1} \underbrace{\frac{\partial^2\varphi}{\partial x_k \partial x_j}}_{(a_{jk})} x_k x_j + \mathcal{O}(|x'|^3) = \frac{1}{2}\sum_{j=0}^{d-1} k_j x_j^2 + \mathcal{O}(|x'|^3)$$

(a_{jk}) $(d-1)\times(d-1)$ $\mathbb{R}$-valued symmetric matrix $\therefore$ diagonizable. k_j principal curvatures of M at x_0. $k := \prod_{j=1}^{d-1} k_j$ is the Gaussian curvature of M at x_0 ($k = \det(\nabla^2\varphi)$)

E.g.

(i) $\mathbb{S}^{d-1} \subset \mathbb{R}^d$. $k_j = 1$ $(\forall j) \therefore k = 1$

(ii) $\{x_3 = \underbrace{x_1^2 - x_2^2}_{\varphi(x_1,x_2)}\} \subset \mathbb{R}^3$, $\frac{1}{2}\nabla^2\varphi(x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(iii) $\{x_1^2 = |x'|^2 : x \neq 0\}$, $x' \in \mathbb{R}^{d-1}$. $d-2$ identical nonvanishing principal curvatures x_d^{-2} +1 vanishing principal curvature.

surface measure σ

$$\int_M f \,\mathrm{d}\sigma = \int_{\mathbb{R}^{d-1}} f(x', \varphi(x')) \underbrace{\sqrt{1 + |\nabla_{x'}\varphi(x')|^2}\,\mathrm{d}x'}_{\mathrm{d}\sigma \text{ in our coordinate sys.}}$$

$$\mathrm{d}\mu = \psi \,\mathrm{d}\sigma, \quad \psi \in C_0^\infty(M, \sigma)$$

is a surface carried measure.

$$\hat{\mu}(\xi) = \int_M e^{-2\pi i x\xi}\,\mathrm{d}\mu(x) = \int_M e^{-2\pi i x\xi}\psi(x)\,\mathrm{d}\sigma_x$$

is bounded on $\mathbb{R}^d$ because $|\mu|(\mathbb{R}^d) < \infty$.

Theorem. *Hypersurface $M \subset \mathbb{R}^d$ with nonvanishing Gaussion curvature at each point of* supp(ψ)*. Then*

$$|\hat{\mu}(\xi)| = \mathcal{O}(|\xi|^{-\frac{d-1}{2}}), \quad |\xi| \to \infty$$

Corollary. If M has at last m non vanishing principal curvatures (at each point of supp(ψ)), then

$$|\hat{\mu}(\xi))| = \mathcal{O}(|\xi|^{-\frac{m}{2}}), \quad |\xi| \to \infty$$

Last time: oscillatory integrals and averaging operators

$$(Af)(x) = (F * \sigma)(x) = \fint_{\mathbb{S}^{d-1}} f(x-y)\,\mathrm{d}\sigma_y \quad (d>1)$$

Smoothin property: $f \mapsto A(f)$ is bounded from $L^2(\mathbb{R}^d)$ to $L^2_k(\mathbb{R}^d)$ with $k = \frac{d-1}{2}$. Here we used

$$|\hat{\sigma}(\xi)| \lesssim |\xi|^{-\frac{d-1}{2}}.$$

A few weeks ago:

$$R(f)(t.\gamma) = \int_{P_{t,\gamma}} f$$

where $P_{t,\gamma} = \{x \in \mathbb{R}^d : x\gamma = t\}$.

$$R^*(f)(\gamma) = \sup_{t\in\mathbb{R}} |R(f)(t,\gamma)|$$

if $d \geq 3$ then

$$\int_{\mathbb{S}^{d-1}} R^*(f)(\gamma)\,\mathrm{d}\sigma_\gamma \lesssim \|f\|_{L^2} + \|f\|_{L^2}$$

This estimate was based on

$$\int_{\mathbb{S}^{d-1}} \int_{-\infty}^{\infty} |\hat{R}(f)(\lambda,\gamma)|^2 |\lambda|^{d-1}\,\mathrm{d}\lambda\,\mathrm{d}\sigma_\gamma = 2\int_{\mathbb{R}^d} |f(x)|^2\,\mathrm{d}x$$

due to $\hat{R}(f)(\lambda,\gamma) = \hat{f}(\lambda\gamma)$. Consider $d=3$ then this becomes

$$\int_{\mathbb{S}^2} \int_{\mathbb{R}} |\frac{\mathrm{d}}{\mathrm{d}t} R(f)(t,\gamma)|^2\,\mathrm{d}t\,\mathrm{d}\sigma_\gamma = 8\pi^2 \int_{\mathbb{R}^3} |f(x)|^2\,\mathrm{d}x \tag{10}$$

by Plancherel $t \leftrightarrow \lambda$. Note, that something like this also holds for higher dimensions.

Now consider the following "linearized" version of the Radon transform:

$$R_B(f) = \int_{\mathbb{R}^{d-1}} f(y', x_d - B(x',y'))\,\mathrm{d}y' = \int_{M_x} f$$

where $x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R},\ y = (y', y_d)$ and $B : \mathbb{R}^{d-1} \times \mathbb{R}^{d-1} \to \mathbb{R}$ is a nondegenerate bilinear form, and

$$M_x = \{(y', y_d) \mid y_d = x_d - B(x', y')\}.$$

E.g. $B(x',y') = \langle x', y'\rangle$ (usual inner product on $\mathbb{R}^d$). $y_d = x_d - \langle x', y'\rangle \iff \langle x', y'\rangle + y_d = x_d \iff \langle (x',1), (y', y_d)\rangle = x_d$. The map

$$\begin{aligned} \mathbb{R}^d &\to \{\text{affine hyperplanes on } \mathbb{R}^d\} \\ x &\mapsto M_x \end{aligned}$$

is injective and surjective onto {hyperplanes not orthogonal to $M_0 = \{x_d = 0\}\}$. The excerpted collection of hyperplanes is lower dimensional, so we can think of R_B as a substitute for R.

An analogue of (10) is

$$\int_{\mathbb{R}^d} |\frac{\partial}{\partial x_3} R_B(f)(x)|^2 \,\mathrm{d}x = c_B \int_{\mathbb{R}^3} |f(x)|^2 \,\mathrm{d}x$$

for $f \in C_0^0(\mathbb{R})$

Proof.

$$\int_{\mathbb{R}^3} |\frac{\partial}{\partial x_3} R_B(f)(x)|^2 \,\mathrm{d}x = \int_{\mathbb{R}} \int_{\mathbb{R}^2} | \underbrace{(\frac{\partial}{\partial x_3} R_B(f))^\wedge(x', \xi_3)}_{=2\pi\xi_3 \int_{\mathbb{R}^2} e^{-2\pi i \xi_3 B(x',y')} \hat{f}(y',\xi_3)\,\mathrm{d}y'} |^2 \,\mathrm{d}x' \,\mathrm{d}\xi_3 \qquad x = (x', x_3)$$

The last equality follows from

$$\begin{aligned}
\hat{R}_B(f)(x', \xi_3) &= \int_{\mathbb{R}} e^{-2\pi i \xi_3 x_3} R_B(f)(x', x_3) \,\mathrm{d}x_3 \\
&= \int e^{-2\pi i \xi_3 x_3} \int_{\mathbb{R}^2} f(y', \underbrace{x_3 - B(x', y')}_{y_3}) \,\mathrm{d}y' \,\mathrm{d}x_3 \\
&= \int \int_{\mathbb{R}^{2+1}} e^{-2\pi i \xi_3 (y_3 + B(x',y'))} f(y' y_3) \,\mathrm{d}y' \,\mathrm{d}y_3 \\
&= \int_{\mathbb{R}^2} e^{-2\pi i \xi_3 B(x',y')} \underbrace{(\int_{\mathbb{R}} e^{-2\pi i \xi_3 y_3} f(y', y_3) \,\mathrm{d}y_3)}_{\hat{f}(y',\xi_3)} \,\mathrm{d}y'
\end{aligned}$$

Since B is nondegenetare $\exists C : \mathbb{R}^2 \to \mathbb{R}^2$ linear, invertible such that $B(x', y') = \langle C(x'), y' \rangle$. Change variables $\xi_3 C(x') = u \in \mathbb{R}^2$ is well defined since C is invertible. $\therefore \xi_3 B(x', y') = \langle \xi_3 C(x'), y' \rangle = \langle u, y' \rangle \therefore \xi_3^2 |\det C| \,\mathrm{d}x' = \mathrm{d}u$. Then the first integral becomes

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} | \int_{\mathbb{R}^2} e^{-2\pi i u y'} \hat{f}(y', \xi_3) \,\mathrm{d}y' |^2 \frac{\mathrm{d}u}{|\det C|} \,\mathrm{d}\xi_3 \simeq \int \int |\hat{f}(y', \xi_3)|^2 \,\mathrm{d}y' \,\mathrm{d}\xi_3 \simeq \int_{\mathbb{R}^3} |f(y)|^2 \,\mathrm{d}y$$

by 2D-Plancherel $y' \leftrightarrow u$ and 1D-Plancherel $\xi_3 \leftrightarrow y_3$. ☐

Rotational curvature Both the averaging operator A and the Radon transform R_B are of the form $f \mapsto \int_{M_x} f(y) \,\mathrm{d}_x(y)$, where for each $x \in \mathbb{R}^d$ we have a manifold M_x (depending smoothly on x) over which we integrate.

$$\begin{aligned}
A:&\quad M_x = x + M_0 \quad M_0 \text{ curved} \\
R_B:&\quad M_x = \{y = (y', y_d) \mid y' \in \mathbb{R}^{d-1},\ y_d = x_d - B(x', y')\} \quad \text{flat but } M_x \text{ rotates as } x \text{ varies.}
\end{aligned}$$

Start with a smooth "double defining" function $\rho = \rho(x, y)$ given in a ball in $\mathbb{R}^d \times \mathbb{R}^d$. Its rotational matrix is

$$M = M(\rho) = \begin{pmatrix} \rho & \frac{\partial \rho}{\partial y_1} & \cdots & \frac{\partial \rho}{\partial y_d} \\ \frac{\partial \rho}{\partial x_1} & & & \\ \vdots & (\frac{\partial^2 \rho}{\partial x_j \partial y_k})_{j,k=1}^d & & \\ \frac{\partial \rho}{\partial x_d} & & & \end{pmatrix}$$

containing the mixed Hessian. The rotational curvature of ρ is

$$\mathrm{rotcurv}(\rho) := \det(M(\rho))$$

We want $\rho = 0 \Rightarrow \mathrm{rotcurv}(\rho) \neq 0$. $M_x = \{y : \rho(x,y) = 0\}$. The fact that $\nabla_y \rho \neq 0$ if $\rho = 0$ implies that M_x is a smooth surface (or something like that).

Examples/Properties:

(i) Translatior invariant case: $\rho(x,y) = \rho(x-y)$. $M_x = M_0 + x$ and $\mathrm{rotcurv}(\rho) \neq 0$ iff M_0 has nonvanishing Gaussian curvature.

(ii) Case of R_B: $\rho(x,y) = y_d - x_d + B(x', y')$. $\mathrm{rotcurv}(\rho) \neq 0$ iff B is nondegenerate.

(iii) $\tilde{\rho}(x,y) = a(x,y)\rho(x,y)$ with $a(x,y) \neq 0$. Then $\tilde{\rho}$ is another defining function for $\{M_x\}$, and $\mathrm{rotcurv}(\tilde{\rho}) = a^{d+1}\,\mathrm{rotcurv}(\rho)$

(iv) $x \mapsto \psi_1(x)$, $y \mapsto \psi_2(y)$ local diffeomorphisms of $\mathbb{R}^d$. For $\tilde{\rho}(x,y) = \rho(\psi_1(x), \psi_2(y))$ then $\mathrm{rotcurv}(\tilde{\rho}) = J_1(x) J_2(y)\,\mathrm{rotcurv}(\rho)$ with $J_k = \det \mathrm{jac}(\psi_k)$, $k = 1,2$

Define the general averaging operator A by

$$A(f)(x) = \int_{M_x} f(y)\psi_0(x,y)\,\mathrm{d}\sigma_x(y)$$

initially for $f \in C_0^0(\mathbb{R}^d)$. $M_x = \{y \mid \rho(x,y) = 0\}$ with induced Lebesgue measure $\mathrm{d}\sigma_x$. ρ is a double defining function with $\mathrm{rotcurv}(\rho) \neq 0$. $\psi_0 \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$.

Theorem. *The operator A extends to a bounded linear map from $L^2(\mathbb{R}^d)$ to $L^2_k(\mathbb{R}^d)$ where $k = \frac{d-1}{2}$.*

Proof. Step 1: Oscillatory integral operators (FIOs)

Step 2: L^2 estimate via dyadic decomposition of "almost-orthogonal" parts.

Step 1: Define

$$T_\lambda(f)(x) = \int_{\mathbb{R}^d} e^{i\lambda\varphi(x,y)}\psi(x,y)f(y)\,\mathrm{d}y$$

where $\varphi \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and $\psi \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ with

$$\det(\nabla^2_{x,y}\varphi) = \det\left(\frac{\partial^2\varphi}{\partial x_k \partial y_j}\right)_{k,j=1}^d \neq 0 \quad \text{on } \mathrm{supp}(\psi)$$

last week:

$$I(\lambda) = \int_{\mathbb{R}^d} e^{i\lambda\varphi(y)}\psi(y)\,\mathrm{d}y \Rightarrow |I(\lambda)| \lesssim |\lambda|^{-\frac{d}{2}}$$

if $\det \nabla^2\varphi \neq 0$ on $\mathrm{supp}(\psi)$. We used $|I(\lambda)|^2 = I(\lambda)\overline{I(\lambda)}$ where then appeared the term $\varphi(u+y) - \varphi(y)$.

Proposition. Under the above assumptions,

$$\|T_\lambda\|_{L^2 \to L^2} \leq c\lambda^{-\frac{d}{2}} \quad \forall \lambda > 0$$

Proof. Similar to its scalar version, omitted. $\square$

Consequence: For the corresponding oscillatory integral operator involving ρ

$$S_\lambda(f)(x) = \int_{\mathbb{R}\times\mathbb{R}^d} e^{i\lambda y_0 \rho(x,y)} \psi(x, y_0, y) f(y) \, \mathrm{d}y_0 \, \mathrm{d}y$$

with $(y_0, y) \in \mathbb{R} \times \mathbb{R}^d$ and $\psi \in C_0^\infty$ is supported away from $y_0 = 0$.

Corollary. If $\rho = 0 \Rightarrow \operatorname{rotcurv}(\rho) \neq 0$ then

$$\|S_\lambda\|_{L^2 \to L^2} \leq c\lambda^{-\frac{d+1}{2}}$$

Proof of Corollary. $\bar{x} = (x_0, x), \bar{y} = (y_0, y) \in \mathbb{R}\times\mathbb{R}^d$. Set $\varphi(\bar{x}, \bar{y}) = x_0 y_0 \rho(x, y)$ then $\det(\nabla^2_{x,y}\varphi) = (x_0 y_0)^{d+1} \operatorname{rotcurv}(\rho)$. Define

$$F_\lambda(x_0, x) = F_\lambda(\bar{x}) = \int_{\mathbb{R}^{d+1}} e^{i\lambda\varphi(\bar{x},\bar{y})} \psi_1(x_0, x, y_0, y) f(y) \, \mathrm{d}y_0 \, \mathrm{d}y$$

with $\psi_1(1, \lambda, y_0, y) = \psi(x, y_0, y)$ (from S_λ). Then $S_\lambda(f)(x) = F_\lambda(1, x)$.

Observation: If $I \subset \mathbb{R}$ interval of length 1, $g \in C^1(I),\ x_0 \in I$ then

$$|g(u_0)|^2 \leq 2(\int_I |g(u)|^2 \, \mathrm{d}u + \int_I |g'(u)|^2 \, \mathrm{d}u)$$

Apply this observation with $I = [1, 2],\ u_0 = 1,\ g(u) = F_\lambda(u, x)$ to get:

$$\int_{\mathbb{R}^d} |S_\lambda(f)(x)|^2 \, \mathrm{d}x \leq 2(\int |F_\lambda(x_0, x)|^2 \, \mathrm{d}x + \int |\frac{\partial}{\partial x_0} F_\lambda(x_0, x)|^2 \, \mathrm{d}x_0 \, \mathrm{d}x)$$

The first integral is $\lesssim \lambda^{-(d+1)} \|f\|^2_{L^2}$ by Proposition with $\mathbb{R}^{d+1}$ instead of $\mathbb{R}^d$).

$$\frac{\partial}{\partial x_0}(e^{i\lambda x_0 y_0 \rho(x,y)}) = \frac{\partial}{\partial y_0}(e^{i\lambda x_0 y_0 \rho(x_0,y_0)}) \frac{y_0}{x_0}$$

we integrate by parts somewhere and use our form of φ. Therefore the second summand also satisfies the desired estimate. $\square$

Step 2: Dyadic decomposition of A.

Co-area formula (see Evans-Cariery (?), Stein-Shakarchi IV, Exercise 8 ,Ch. 8): Fix $h \in S(\mathbb{R})$ sucht that $\int h = 1,\ M = \{x \in \mathbb{R}^d \mid \rho(x) = 0\}$. Then

$$\int_M f \frac{\mathrm{d}\sigma}{|\nabla\rho|} = \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon} \int_{\mathbb{R}^d} h(\frac{\rho(x)}{\varepsilon}) f(x) \, \mathrm{d}x.$$

E.g. $\rho(x) = |x| - 1 \therefore M = \mathbb{S}^{d-1},\ \frac{\nabla}{\rho}(x) = x|x|, \therefore |\nabla\rho| = 1$

$$\int_{\mathbb{S}^{d-1}} f \, \mathrm{d}\varepsilon = \lim_{\varepsilon\to *^+} \frac{1}{\varepsilon} \int_{\mathbb{R}^d} h(\frac{|x| - 1}{\varepsilon}) F(x) \, \mathrm{d}x$$

where $f = F|_{\mathbb{S}^{d-1}}$.

$$A(f)(x) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{\mathbb{R}^d} h(\rho(x, y)\varepsilon) \psi(x, y) f(y) \, \mathrm{d}y$$

where $\psi(x,y)=\psi_0(x,y)|\nabla_y\rho|\in C_0^\infty(\mathbb{R}^d\times\mathbb{R}^d)$. Let $\gamma\in C_0^\infty(\mathbb{R})$ such that $\gamma=1$ on $[-\frac{1}{2},\frac{1}{2}]$ and 0 on $[-1,1]^{\complement}$. Let $h=\widehat{\gamma}$. Then

$$h(\rho)=\int_{\mathbb{R}} e^{2\pi i\xi u}\gamma(u)\,\mathrm{d}u.$$

Then

$$\int h=\int\widehat{\gamma}=\gamma(0)=1$$

and

$$\int_{\mathbb{R}} e^{2\pi iu\rho}\gamma(\varepsilon u)\,\mathrm{d}u=(\delta_\varepsilon\gamma)^\vee(\rho)=\varepsilon^{-1}\gamma^\vee(\varepsilon^{-1}\rho)=\varepsilon^{-1}h(\varepsilon^{-1}\rho)$$

where $\delta_\varepsilon\gamma)(u)=\gamma(\varepsilon u)$. Choose $\varepsilon=2^{-r},\ r\in\mathbb{N}$. Note

$$\gamma(2^{-r}u)=\gamma(u)+\sum_{k=1}^{r}(\gamma(\frac{u}{2^k}-\gamma(\frac{u}{2^{k-1}}))$$

Let $r\to\infty$ to get

$$1=\gamma(u)+\sum_{k=1}^{\infty}\eta(\frac{u}{2^k})$$

where $\eta(\cdot)=\gamma(\cdot)-\gamma(2\cdot)$ because γ is continuous. Then $\eta\in C_0^\infty(\mathbb{R}),\ \operatorname{supp}(\eta)\subset\{\frac{1}{4}\le|u|\le 1\}$. Whenever f is continuous we get by Fourier inversion that

$$\begin{aligned}A(f)(x)&=\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\int_{\mathbb{R}^d}\int_{\mathbb{R}} e^{2\pi iu\frac{\rho(x,y)}{\varepsilon}}\gamma(u)\psi(x,y)f(y)\,\mathrm{d}u\,\mathrm{d}y\\&=\lim_{\varepsilon\to0}\int_{\mathbb{R}^d}\int_{\mathbb{R}} e^{2\pi iu\rho(x,y)\gamma(\varepsilon u)}\psi(x,y)f(y)\,\mathrm{d}u\,\mathrm{d}y\\&=\int_{\mathbb{R}^d}\int_{\mathbb{R}} e^{2\pi iu\rho(x,y)}\gamma(u)\psi(x,y)f(y)\,\mathrm{d}u\,\mathrm{d}y\\&=\sum_{k=1}^{\infty}\int_{\mathbb{R}^d}\int_{\mathbb{R}} e^{2\pi iu\rho(x,y)}\eta(\frac{u}{2^k})\psi(x,y)f(y)\,\mathrm{d}u\,\mathrm{d}y\end{aligned}$$

since $\gamma(\varepsilon u)\to\gamma(0)=1\ \varepsilon\to0$. Call the summands $A_k(f)(x)$. Properties of A_k:

(i) $f\in L^2(\mathbb{R}^d)\Rightarrow$

$$A_k(f)\in C_0^\infty(\mathbb{R}^d)$$

(ii)

$$\|A_k(f)\|_{L^2}\le c2^{-k(\frac{d-1}{2})}\|f\|_{L^2}.$$

Recall $\|S_\lambda\|_{L^2\to L^2}\lesssim\lambda^{-\frac{d+1}{2}i}$ and change variables in the definition of $A_k(f)$.

(iii) $\exists m:|j-k|\ge m\ \forall N$

$$\|(A_k^*A_j)(f)\|_{L^1}\lesssim_N 2^{-N\max(k,j)}\|f\|_{L^2}.$$

Similarly for $A_kA_j^*$. For the proof, invoke nonstationary phase. Also, recall $|I(\lambda)|^2=I(\lambda)\overline{I(\lambda)}$.

(iv) $A_k^{(\alpha)}=(\frac{\partial}{\partial x})^\alpha A_k$. Then

$$\|A_k^{(\alpha)}\|_{L^2\to L^2}\lesssim 2^{k|\alpha|}2^{-k(\frac{d-1}{2})}$$

and

$$\|A_k^{(\alpha)}(A_j^{(\alpha)})^*\|_{L^2\to L^2}\lesssim_{\alpha,N}2^{-N\max(k,j)}.$$

Step 3: Almost-orthogonality

Assume that $\{T_k\}_{k=1}^{\infty}$ is a sequence of bounded operators on $L^2(\mathbb{R}^d)$ and that $\{a(k)\}_{k\in\mathbb{Z}}$ are positive constants with

$$A = \sum_{k\in\mathbb{Z}} a(k) < \infty.$$

Lemma (Cotlar-Knapp-Stein)**.** Assume for $\|T_kT_j^*\|_{L^2\to L^2}$ that $\|T_k^*T_j\| \leq a(k-j)^2$. Then, for every r,

$$\|\sum_{k=0}^{r} T_k\| \leq A.$$

Note, that the bound A is independent of r.

Write $T = \sum_{k=0}^{r} T_k$. Recall $\|T\|^2 = \|T^*T\|$ since $\|AB\| \leq \|A\|\|B\|$ and $\|Tx\|^2 = \langle Tx, Tx\rangle = \langle x, T^*Tx\rangle \leq \|T^*T\|\|x\|^2$ and plug in an extremizing sequence of $\|Tx\|$ for x. Then $\|T\|^4 = (\|T\|^2)^2 = \|T^*T\|^2 = \|(T^*T)^2\|$ since T^*T is self adjoint. By induction we get

$$\|T\|^{2n} = \|(T^*T)^n\|.$$

$$(T^*T)^n = \sum_{i_1,i_2,\dots,i_{2n}} (T_{i_1}T_{i_2}^*\dots T_{i_{2n-1}}T_{i_{2n}}^*)$$

(i)

$$\|(T_{i_1}T_{i_2}^*)\dots(T_{i_{2n-1}}T_{2_{2n}}^*)\| \leq a(i_1-i_2)^2a(i_3-i_4)^2\dots a(2_{2n-1})^2$$

(ii)

$$\|T_{i_1}(T_{i_2}T_{i_3})\dots(T_{i_{2n-2}}T_{i_{2n-1}})T_{i_{2n}}\| \leq Aa(i_2-i_3)^2a(i_4-i_5)^2\dots a(i_{2n-2}-i_{2n-1})^2A$$

Take geometric mean of (i) and (ii) and get

$$\|T_{i_1}T_{i_2}\dots T_{i_{2n-1}}T_{i_{2n}}^*\| \leq Aa(i_1-a_2)a(i_2-i_3)\dots a(i_{2n-1}-i_{2n})$$

Now sum the whole thing in $i_1, i_2, \dots, i_{2n-1}$. Then sum by sum, each of the factors turns into an A. In the end the sum in i_{2n} gives a factor $r+1$. So,

$$\sum_{i_1,i_2,\dots i_{2n}} \|T_{i_1}T_{i_2}^*\dots T_{i_{2n-1}}T_{i_{2n}}^*\| \leq A^{2n}(r+1)$$

$$\therefore \|T\| \leq A(1+r)^{\frac{1}{n}} \to A \quad n\to\infty$$

(This is called 'Tensor power trick')

Putting everything together: Case 1: d odd ($\therefore \frac{d-1}{2} \in \mathbb{Z}$). ETS $\forall|\alpha| \leq \frac{d-1}{2} \quad \forall f \in L^2(\mathbb{R}^d)$

- $\partial_x^\alpha A(f)$ exists (in the sense of distributions) and is an L^2 function
- $f \mapsto \partial_x^\alpha A(f)$ is bounded on L^2.

For each r, set

$$\partial_x^\alpha \sum_{k=0}^{r} =: \sum_{k=0}^{r} T_k \quad (T_k = A_k^{(\alpha)})$$

The estimates 1 and 2 imply that the hypotheses of CKS are satisfied with $a(k) = c_n 2^{-|k|N}$ ($\forall N, \therefore$ can choose $N=1$).

$$\therefore \|\partial_x^\alpha \sum_{k=0}^{r} A_k(f)\|_{L^2} \leq A\|f\|_{L^2}$$

where $A = \sum_{k\in\mathbb{Z}}$, provided $|\alpha| \leq \frac{d-1}{2}$.

(i) with $\alpha = 0$ we get

$$\lim_{r\to\infty} \sum_{k=0}^{r} A_k(f) = A(f)$$

in L^2 and therefore in the weak sense.

$$\lim_{r\to\infty} \partial_x^\alpha \sum_{k=1}^{r} A_k(f) = \partial_x^\alpha A(f)$$

in the weak sense and therefore in L^2.

Conclusion

$$\|\partial_x^\alpha A(f)\|_{L^2} \leq A\|f\|_{L^2}$$

whenever $f \in C_0^0(\mathbb{R}^d),\ |\alpha| \leq \frac{d-1}{2}$ (and d is odd) $\square$