

# Geometric Aspects of Harmonic Analysis

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**Q1**

$$\left. \begin{array}{l} f : [a, b] \rightarrow \mathbb{R} \text{ integrable} \\ F(x) = \int_a^x f(t) dt \end{array} \right] \stackrel{?}{\Rightarrow} F \text{ diff. (a.e. } x), F' = f$$

**Q2** Conditions of  $F$  (on  $[a, b]$ ) s.t.

- $F'(x)$  exists a.e.
- $F'$  integrable
- $\int_a^b F'(x) dx = F(b) - F(a)$

?

**Q1** Differentiation of the integral

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t) dt = \frac{1}{|I|} \int_I f = \text{avg}_I f = {}_I f$$

$I = (x, x+h)$ ,  $|I|$  Lebesgue measure of  $I$ .

Q1 equivalent to averaging problem: Given  $f \in L^1(\mathbb{R}^d)$ , is it true, that

$$\lim_{|B| \rightarrow 0, x \in B} \frac{1}{|B|} \int_B f = f(x) \quad (x\text{-a.e.})?$$

$B \subset \mathbb{R}^d$  open ball

Yes, if  $f$  continuous  $\forall \varepsilon \exists \delta |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon. x \in B$

$$|f(x) - \fint_B f| = \left| \fint_B (f(y) - f(x)) dy \right| < \varepsilon \quad (1)$$

provided  $B$  is an open ball of radius  $< \frac{\delta}{2}$  containing  $x$

Yes, if  $f$  is integrable (not so easy). Hardy, Littlewood (1D, rearrangements; later Wiener for  $d > 1$ ).  $f \in L^1(\mathbb{R}^d)$

$$(Mf)(x) := \sup_{x \in B} \frac{1}{|B|} \int_B |f|$$

uncentered HL maximal function

**Theorem.** Let  $f$  be integrable on  $\mathbb{R}^d$ . Then

(i)  $Mf$  is measurable.

(ii)  $(Mf)(x) < \infty$  a.e.  $x$

(iii)

$$|\{x \in \mathbb{R}^d : (Mf)(x) > \alpha\}| < \frac{c}{\alpha} \|f\|_{L^1(\mathbb{R}^d)} \quad (\forall \alpha > 0). \quad (2)$$

$c = c_d = 3^d$ , independent of  $f, \alpha$ .

$f \neq 0 \in L^1 \Rightarrow Mf(x) \sim |x|^{-d}$  for large radius of  $x$ . So then  $Mf \notin L^1$ .

$$M : \begin{matrix} L^1 \rightarrow L^1 \\ L^1 \rightarrow L^{1,\infty} \end{matrix}$$

*Proof.* (i) easy  $E_\alpha = \{x \in \mathbb{R}^d : (Mf)(x) > \alpha\}$  is open ( $\forall \alpha > 0$ ) (because  $Mf$  is lower semicontinuous)

(ii)  $|\{x \in \mathbb{R}^d : (Mf)(x) = \infty\}| \subset |\{x \in \mathbb{R}^d : Mf(x) > \alpha\}|$ , take  $\alpha \rightarrow \infty$ .

(iii) follows from an elementary version of *Vitali covering*

□

**Lemma.** Let  $B = \{B_1, B_2, \dots, B_N\}$  be a finite collection of open balls on  $\mathbb{R}^d$ . Then there exists a disjoint subcollection  $B_{i_1}, B_{i_2}, \dots, B_{i_k}$  of  $B$  such that

$$|\bigcup_{j=1}^n B_j| \leq 3^d \sum_{j=1}^k |B_{i_j}|$$

*Proof.* (i)  $B_{i_1}$  = largest ball

(ii) Delete  $B_{i_1}$  and its neighbors

(iii)  $B_{i_2}$  = largest ball

(iv) repeat...

- Algorithm stops in at most  $N$  steps
- output has desired properties:
  - disjointness is clear
  - size  $B \cap B' \neq \emptyset$ ,  $r_{B'} \leq r_B$ .  $B^*$  = ball with the same center as  $B$  but 3 times the radius.  $\Rightarrow B' \subset B^*$ .  $|B^*| = 3^d |B|$

□

**Back to (iii):** Choose  $\alpha > 0$ ,  $E_\alpha = \{x \in \mathbb{R}^d : (Mf)(x) > \alpha\}$ . For each

$$x \in E_\alpha \exists B = B_x := \frac{1}{|B_x|} \int_{B_x} |f(y)| dy > \alpha$$

equivalent

$$|B_x| < \alpha^{-1} \int_{B_x} |f(y)| dy$$

Fix  $K \ll E_\alpha$  compact subset covered by  $\bigcup_{x \in K} B_x$ ,  $K \subset \bigcup_{l=1}^N NB_l$

$$|K| \leq \left| \bigcup_{l=1}^N B_l \right| \stackrel{\text{Vitali}}{\leq} 3^d \sum_{j=1}^k |B_{i_j}| \leq \frac{3^d}{\alpha} \sum_{j=1}^k \int_{B_{i_j}} |f(y)| dy = \frac{3^d}{\alpha} \int_{\bigcup_{j=1}^k B_{i_j}} |f(y)| dy \leq \frac{3^d}{\alpha} \|f\|_{L^1(\mathbb{R}^d)}$$

Since  $K$  was chosen arbitrary (cpt.), it follows that

$$|E_\alpha| \leq \frac{3^d}{\alpha} \|f\|_{L^1}$$

Can interpolate between weak type  $L^1$ -inequality and  $L^\infty \rightarrow L^\infty$  (very easy).

**Corollary** (Lebesgue differentiation theorem). Let  $f \in L^1(\mathbb{R}^d)$ . Then

$$\lim_{|B| \rightarrow 0, x \in B} \oint_B f = f(x) \quad x\text{-a.e.} \quad (3)$$

*Proof.*

$$E_\alpha = \{x \in \mathbb{R}^d : \limsup_{|B| \rightarrow 0, x \in B} \oint_B f - f(x) > 2\alpha\}$$

ETS  $|E_\alpha| = 0 \forall \alpha > 0$ . Then  $E = \bigcup_{n \in \mathbb{N}} E_{\frac{1}{n}} = 0$  and (3) holds on  $E^c$ .

Fix  $\alpha > 0$ , given  $\varepsilon > 0$  choose  $g \in C_0^\infty(\mathbb{R}^d)$  s.t.  $\|f - g\|_{L^1} < \varepsilon$ . Already seen

$$\begin{aligned} \lim_{|B| \rightarrow 0, x \in B} \oint_B g &= g(x) \quad \forall x \\ \oint_B f - f(x) &= \oint_B (f - g) + \oint_B g - g(x) + g(x) - f(x) \end{aligned}$$

$$F_\alpha = \{x : M(f - g)(x) > \alpha\}$$

$$G_\alpha = \{x : |f(x) - g(x)| > \alpha\}$$

$E_\alpha \subset F_\alpha \cup G_\alpha$  since  $u_1, u_2 > 0$ ,  $u_1 + u_2 > 2\alpha \Rightarrow u_1 > \alpha \vee u_2 > \alpha$ .

$$|G_\alpha| \leq \frac{1}{\alpha} \|f - g\|_{L^1} \quad (\text{Chebyshev})$$

$$|F_\alpha| \leq \frac{c_d}{\alpha} \|f - g\|_{L^1} \quad (\text{weak type})$$

$$|E_\alpha| \leq |F_\alpha| + |G_\alpha| \leq \left(\frac{c_d}{\alpha} + \frac{1}{\alpha}\right) \|f - g\|_{L^1} \leq \frac{c'_d \varepsilon}{\alpha}$$

Since  $\varepsilon > 0$  was arbitrary  $|E_\alpha| = 0$ . □

$h \in L^1 \subset L^{1,\infty}$  by Chebyshev:  $\infty > \|h\|_{L^1} = \int_{\mathbb{R}^d} |h(y)| dy \geq \int_{h(y) \geq \alpha} |h(y)| dy \geq \alpha |\{h > \alpha\}|$ .

Would have been enough to replace  $L^1(\mathbb{R}^d)$  by  $L^1_{\text{loc}}$ .

**Sets**  $E \subset \mathbb{R}^d$  measurable,  $x \in \mathbb{R}^d$  (not necc. in  $E$ )  $x$  is a point of Lebesgue density of  $E$  if

$$\lim_{|B| \rightarrow 0, x \in B} \frac{|B \cap E|}{|B|} = 1$$

**Corollary.** Let  $E \subset \mathbb{R}^d$  be measurable. Then

- (i) Almost every  $x \in E$  is a point of Lebesgue density of  $E$ .
- (ii) Almost every  $x \notin E$  is not a point of Lebesgue density.

**Functions**  $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ .

$$\text{Leb}(f) := \{x \in \mathbb{R}^d : f(x) < \infty \text{ and } \lim_{|B| \rightarrow 0, x \in B} \int_B |f(y) - f(x)| dy = 0\}$$

$f$  continuous at  $\bar{x} \Rightarrow \bar{x} \in \text{Leb}(f) \Rightarrow \int_B f \rightarrow \int_B f(\bar{x})$  (all the inverse implications are wrong)

**Corollary.**  $f \in L^1_{\text{loc}}(\mathbb{R}^d) \Rightarrow$  Almost every point belongs to  $\text{Leb}(f)$ .

(By checking the proof again?)

These things also works with other sets that "shrink regularly to  $x$  than balls". It gets worse however when one takes all parallel rectangles and even worse when arbitrarily oriented rectangles are allowed.

**Q.2** Key: bounded variation (BV)  $F : [a, b] \rightarrow \mathbb{R}$ ,  $P = \{a = t_0 < t_1 < \dots < t_N = b\}$

$$V_F^P = \sum_{j=0}^N |F(t_j) - F(t_{j+1})|$$

is the variation of  $f$  over  $P$ .  $F$  is of bounded variation if

$$T_F(a, b) = \sup_P V_F^P < \infty$$

$$P \subset \tilde{P} \text{ partitions} \Rightarrow V_F^P \leq V_F^{\tilde{P}}$$

**Example.** (i)  $f$  monotonic (increasing) and bounded,  $|f| \leq M \Rightarrow F \in \text{BV}$

$$V_F^P = \sum_{j=1}^N |F(t_j) - F(t_{j-1})| = F(b) - F(a) \leq 2M$$

(ii)  $F$  differentiable with  $F'$  bounded,  $|F'| \leq M$ , then by mean value theorem  $f \in \text{BV}$ . Or  $F$  Lipschitz

(iii)  $F$   $\alpha$ -Hölder ( $\alpha < 1$ )  $\Rightarrow F \in \text{BV}$ . Take  $F : [0, 1] \rightarrow \mathbb{R}$ ,  $x \mapsto d(x, C)^\alpha$ , where  $C$  is the cantor set.  $2^{n-1}$  intervals of length  $3^{-n}$

$$\alpha > \frac{\log 2}{\log 3} \Rightarrow \sum_{n=1}^{\infty} 2^{n-1} (3^{-n})^\alpha < \infty$$

- *Total variation* of  $F$  on  $[a, x]$  (where  $a \leq x \leq b$ ) is

$$T_F(a, x) = \sup \sum_{j=0}^N |F(t_j) - F(t_{j-1})|$$

- *Positive variation* of  $F$  on  $[a, 1]$  is

$$P_F(a, x) = \sup_{(+)} \sum (F(t_j) - F(t_{j-1})) \quad \text{all } j : F(t_j) \geq F(t_{j-1})$$

- *Negative variation* of  $F$  on  $[a, 1]$  is

$$N_F(a, x) = \sup_{(-)} \sum -(F(t_j) - F(t_{j-1}))$$

**Lemma.**  $f : [a, b] \rightarrow \mathbb{R}$ . Then

$$(i) \quad F(x) = F(a) + P_F(a, x) - N_F(a, x)$$

$$(ii) \quad T_F(a, x) = P_F(a, x) + N_F(a, x)$$

( $\forall x \in [a, b]$ )

recall from measure theory:  $f = f^+ - f^-$ ,  $|f| = f^+ + f^-$

*Proof.* (i) given  $\varepsilon > 0$ ,  $\exists P = \{a = t_0 < t_1 < \dots < t_N = x\}$

$$|PVF - \sum_{(+)} (F(t_j) - F(t_{j-1}))| < \varepsilon$$

$$|N_F - \sum_{(-)} -(F(t_j) - F(t_{j-1}))| < \varepsilon$$

Also

$$F(x) - F(a) = \sum_{(+)} (F(t_j) - F(t_{j-1})) - \sum_{(-)} -(F(t_j) - F(t_{j-1}))$$

□

**Corollary.**  $F : [a, b] \rightarrow \mathbb{R} \in \text{BV}$  iff  $F$  is the difference of two increasing bounded functions

**Theorem.**  $F : [a, b] \rightarrow \mathbb{R} \in \text{BV} \Rightarrow F$  differentiable a.e.

Wlog  $f$  monotonically increasing, "Wlog"  $f$  continuous

**Lemma** (of the rising sun).  $G : \mathbb{R} \rightarrow \mathbb{R}$  continuous.

$$E = \{x \in \mathbb{R} : \exists h = h_x > 0 \ G(x+h) > G(x)\}$$

Then

$$(i) \quad E \text{ is open } (E = \bigcup_{n=1}^{\infty} (a_n, b_n))$$

$$(ii) \quad g(a_n) = G(b_n), \text{ provided } b_n - a_n < \infty.$$

*Proof.* Let  $(a_n, b_n)$  be a finite interval in the decomposition.  $a_k \notin E$  then  $g(a_k) \geq G(b_k)$ . Assume  $G(a_k) > G(b_k)$ .  $\exists c \in (a_k, b_k)$   $g(c) = \frac{g(a_k) + g(b_k)}{2}$ . Choose rightmost such  $c$ .  $\exists d \in (c, b_k)$   $G(d) > G(c)$ . But then by continuity  $c$  could not have been chosen rightmost, contradiction.  $\square$

Can replace  $\mathbb{R}$  by  $[a, b]$ , but then only get for  $a_0 = a$  that  $G(a_0) \leq G(b_0)$

*Proof.* of theorem

$$\begin{aligned}\Delta_h(F)(x) &= \frac{F(x+h) - F(x)}{h} \\ D^\pm(F)(x) &= \limsup_{h \rightarrow 0, h > < 0} \Delta_h(F)(x) \\ D_\pm(F)(x) &= \liminf_{h \rightarrow 0, h > < 0} \Delta_h(F)(x)\end{aligned}$$

Dini numbers. Upshot: They are all the same and finite.  $D_- \leq D^-$ ,  $D_+ \leq D^+$  clear. ETS

(i)  $D^+(F)(x) < \infty$  (a.e.  $x$ )

(ii)  $D^+(F)(x) \leq D_-(F)(x)$  (a.e.  $x$ )

(ii) is equivalent to  $D^-(F)(x) \leq D_+(F)(x)$  by replacing  $F(x)$  by  $-F(-x)$  somewhere. Then  $D^+ \leq D_- \leq D^- \leq D_+ \leq D^+ < \infty$ .

(i) relacc:  $F$  increasing, bounded, continuous on  $[a, b]$ . Fix  $\gamma > 0$ ,

$$E_\gamma := \{x : D^+(F)(x) > \gamma\}$$

- $E_\gamma$  is measurable
- Apply rising sun to  $G(x) = F(x) - \gamma x$

$$E_\gamma \subset E = \{x \in [a, b] : \exists h > 0 \ G(x+h) > G(x)\} = \bigcup_{k=1}^{\infty} (a_k, b_k)$$

The condition in the set is equivalent to

$$\begin{aligned}\Leftrightarrow \exists h > 0 \ F(x+h) - \gamma x - \gamma h > F(x) - \gamma x \\ \Leftrightarrow \exists h > 0 \ \frac{F(x+h) - F(x)}{h} > \gamma \\ \Leftrightarrow D^+(F)(x) > \gamma\end{aligned}$$

$$G(a_k) \leq G(b_k) \Leftrightarrow F(a_k) - \gamma a_k \leq F(b_k) - \gamma b_k \Leftrightarrow \gamma(b_k - a_k) \leq F(b_k) - F(a_k).$$

Therefore

$$|E_\gamma| \leq |E| \leq \sum_{k=1}^{\infty} (b_k - a_k) \leq \frac{1}{\gamma} \sum_{k=1}^{\infty} (F(b_k) - F(a_k)) = \frac{1}{\gamma} (F(b) - F(a))$$

Take  $\gamma \rightarrow \infty$ , done.

(ii) see Stein-Shakarchi (vol 3)

$\square$

**Corollary.**  $F$  increasing, continuous  $\Rightarrow F'$  exists a.e., measurable, nonnegative and

$$\int_a^b F'(x) dx \leq F(b) - F(a).$$

*Proof.* Let

$$G_h(x) = \frac{F(x + \frac{1}{n}) - F(x)}{\frac{1}{h}}$$

By the theorem,  $G_h(x) \rightarrow F'(x)$  ( $h \rightarrow 0$ ) pointwise a.e. By Fatou

$$\int_{[a,b]} F' \leq \liminf_{n \rightarrow \infty} \int_a^b G_h(x) dx = \liminf_{n \rightarrow \infty} \int_b^{b+\frac{1}{n}} F(x) dx - \int_a^{a+\frac{1}{n}} F(x) dx$$

□

Cannot do better than  $\leq$ : For the Devil's staircase the left hand side is 0 while the right hand side is 1.

Why is the sunrise Lemma a covering Lemma?

$$\left. \begin{aligned} f_+^* &= \sup \frac{1}{h} \int_x^{x+h} |f(y)| dy \\ E_\alpha^+ &= \{x \in \mathbb{R} : f_+^*(x) > \alpha\} \end{aligned} \right\} |E_\alpha^+| = \frac{1}{\alpha} \int_{E_\alpha^+} |f|$$

Why? Let

$$G(x) = \int_0^x |f(y)| dy - \alpha x$$

$$x \in E_\alpha^+ \iff f_+^*(x) > \alpha \iff \exists h > 0 \frac{1}{h} \int_x^{x+h} |f(y)| dy > \alpha \iff \exists h > 0 G(x+h) > G(x)$$

$$\{x \in \mathbb{R} : \exists h > 0 G(x+h) > G(x)\} = \bigcup_{k \in \mathbb{N}} (a_k, b_k), \quad G(a_k) = G(b_k)$$

$$|E_\alpha^+| = \sum_k (b_k - a_k) = \frac{1}{\alpha} \sum_k \int_{(a_k, b_k)} |f| = \frac{1}{\alpha} \int_{\bigcup_k (a_k, b_k)} |f| = \frac{1}{\alpha} \int_{|E_\alpha^+|} |f|.$$

**Definition.**  $F : [a, b] \rightarrow \mathbb{R}$  is *absolutely continuous* if

$$\forall \varepsilon > 0 \exists \delta > 0 : \sum_{k=1}^N (B_k - a_k) < \delta \implies \sum_{k=1}^N |F(b_k) - F(a_k)| < \varepsilon$$

intervals  $(a_k, b_k)$  disjoint ( $k = 1, \dots, N$ )  $\Rightarrow$

$$\sum_{k=1}^N |F(b_k) - F(a_k)| < \varepsilon$$

*Remark.* (i) On a bounded interval  $I \subset \mathbb{R}$

$$C^1(I) \subset \text{Lip}(I) \subset AC(I) \subset BV(I)$$

So they are diff. a.e.. All the inclusions are strict.

(ii) abs cont  $\Rightarrow$  unif. con.  $\Rightarrow$  cont.

(iii)  $f \in L^1_{\text{loc}}(\mathbb{R})$   $F(x) = \int_0^x f(t) dt$  Then  $F$  is absolutely continuous.  $(\forall \varepsilon \exists \delta |E| < \delta \Rightarrow \int_E |f| < \varepsilon)$

Upshot: AC functions are the ones which re diff a.e. and vrfy FTC.

**Theorem.**  $F \in AC(a, b) \Rightarrow F'$  exists a.e.,  $F' = 0$  a.e.  $\Rightarrow F$  constant

- Existence of  $F'$  clear  $\checkmark$
- $F' = 0$  a.e.  $\Rightarrow F$  constant: refinement of Vitali

**Definition.** A collection  $\mathcal{B} = \{B\}$  of (open) balls on  $\mathbb{R}^d$ . is a *Vitali covering* of a set  $E$  if

$$\forall x \in E \forall \eta > 0 \exists B \in \mathcal{B} : x \in B, |B| < \eta$$

**Lemma.**  $E \subset \mathbb{R}^d$  meas.  $|E| < \infty$ ,  $\mathcal{B}$  Vitali covering of  $E$ ,  $\delta > 0$ . Then there exist finitely many disjoint balls  $B_1, \dots, B_N \in \mathcal{B}$

$$\sum_{j=1}^N |B_j| \geq |E| - \delta$$

Recall elementary Vitali:  $\mathcal{B} = \{B_1, \dots, B_N\}$  finite collection of pen balls in  $\mathbb{R}^d \Rightarrow \exists$  disjoint subcollection  $B_{i_1}, \dots, B_{i_k}$  with

$$|\bigcup_{j=1}^B B_j| \leq 3^d \sum_{j=1}^k |B_{i_j}|$$

*Proof of Lemma.* wlog  $\delta > |E|$ . Vitali  $\Rightarrow \exists$  disjoint subcollection  $B_1, \dots, B_N \in \mathcal{B}$

$$\sum_{i=1}^{N_1} |B_i| \geq 3^{-d} \delta$$

Sequence of balls  $B_1, \dots, B_N$ . question: Is  $\sum_{j=1}^{N_1} |B_j| \geq |E| - \delta$ ? Yes: done with  $N = N_1$ . No: work harder.

$$\sum_{j=1}^{N_1} |B_j| < |E| - \delta$$

$$E_2 = E \setminus \bigcup_{j=1}^{N_1} B_j$$

$$|E_2| \geq |E| - \sum_{j=1}^{N_1} |B_j| > |E| - (|E| - \delta) = \delta$$

$\mathcal{B}$  Vitali covering  $\Rightarrow$  balls in  $\mathcal{B}$  disjoint from  $\bigcup_{i=1}^{N_1} B_i$  still covers  $E_2$ . Vitali  $\Rightarrow \exists$  finite disjoint subcollection of these balls  $B_{N_1+1}, \dots, B_{N_2}$

$$\sum_{N_1 < j < N_2} |B_j| \geq 3^{-d} \delta.$$

After  $k$  steps,  $B_1, \dots, B_{N_1}, \dots, B_{N_k}$  with

$$\sum_{j=1}^{N_k} |B_j| \geq k 3^{-d} \delta \geq |E| - \delta$$

iff  $k \geq 3^d \frac{|E| - \delta}{\delta}$ , stop. □



need to approximate with compact from inside somewhere and with open from outside somewhere else

**Corollary.** The balls can be arranged in such a way that

$$|E \cup \bigcup_{i=1}^N B_i| < 2\delta$$

*Proof.* Choose open  $O \supset E$  :  $|O \setminus E| < \delta$ .  $\mathcal{B}$  Vitali covering  $\Rightarrow$  wlog all balls in  $\mathcal{B}$  are contained in  $O$ .

$$(E \cup \bigcup_{i=1}^N B_i) \cup \bigcup_{i=1}^N NB_i \subset O. : |E \cup \bigcup_{i=1}^N B_i| \leq |O| - \sum_{i=1}^N |B_i| \leq |E| + \delta - (|E| - \delta) = 2\delta$$

□

$F \in AC$  Back to the real line: Goal:  $F' = 0$  a.e.  $\Rightarrow F$  constant. ETS  $F(a) = F(b)$

$$E = \{x \in (a, b) : F'(x) \text{ exists and } = 0\} \quad |E| = b - a$$

Fix  $\varepsilon > 0$ . For  $x \in E$ ,  $\lim_{h \rightarrow 0} \left| \frac{F(x+h) - F(x)}{h} \right| = 0$ .  $\forall \eta > 0 \exists$  open interval  $I = (a_x, b_x) \subset [a, b]$  containing  $x$ .  $|F(b_x) - F(a_x)| \leq \varepsilon(b_x - a_x)$  and  $b_x - a_x < \eta$ . The collection of these intervals (over all  $\eta > 0$ ) forms a Vitali covering of  $E$ . Lemma  $\Rightarrow$  Given  $\delta > 0$  can select finitely many, disjoint  $I_j = (a_j, b_j)_{j=1}^N$  such that

$$\sum_{j=1}^N |I_j| \geq |E| - \delta = b - a - \delta$$

But

$$\sum_{j=1}^N |F(b_j) - F(a_j)| \leq \varepsilon \sum_{j=1}^N (b_j - a_j) \leq \varepsilon(b - a)$$

$$[a, b] \supset \bigcup_{j=1}^N I_j = \bigcup_{k=1}^M [\alpha_k, \beta_k]$$

with total length  $\leq \delta$ .  $\therefore F \in AC$

$$\sum_{k=1}^M |F(\beta_k) - F(\alpha_k)| < \varepsilon$$

$$|F(b) - F(a)| \leq \sum |F(b_j) - F(a_j)| + \sum |F(\beta_k) - F(\alpha_k)| \leq \varepsilon(b - a) + \varepsilon,$$

done. □

**Theorem.**  $F \in AC(a, b)$ . Then

(i)  $F'$  exists a.e. and is integrable

(ii)

$$F(x) - F(a) = \int_a^x F'(t) dt \quad (\forall a \leq x \leq b)$$

Conversely, if  $f \in L^1(a, b)$ , then there exists  $F \in AC(a, b) : F' = f$  a.e..

*Proof.*  $\Rightarrow$

(i) seen last lecture.

(ii)

$$G(x) := \int_a^x F'(t) dt$$

$\therefore G \in AC \therefore F - G \in AC$ . Lebesgue diff.  $\Rightarrow G'(x) = F'(x)$  (a.e.  $x$ )  $\therefore (F - G)' = 0$  a.e..  
Therefore  $(F - G)(x) = (F - G)(a)$ ,  $F(x) - G(x) = F(a)$ , equivalent to (\*)

$\Leftarrow$

$$F(x) = \int_a^x f(t) dt$$

AC  $\checkmark$  Leb. diff  $\Rightarrow F' = f$  a.e.  $\square$

*Next:* Monotone functions which are not nec. continuous. Wlog.  $F$  increasing, bounded on  $[a, b]$ .

$$F(x^-)F \lim_{y \rightarrow x, y < x} F(y) \quad F(x^+) := \dots$$

$F(x^-) \leq F(x) \leq F(x^+)$ ,  $F$  cont. at  $x$  if  $F(x^-) = F(x^+)$ . Otherwise  $F$  has a jump discontinuity at  $x$ .

*Obs:* A (bounded) increasing function  $F$  on  $[a, b]$  has at most countable many jumps. There exists an injective map  $\text{Disc}(F) \rightarrow \mathbb{Q}$

$\therefore \text{Disc}(F) = \{x_n\}_{n=1}^\infty$   $\alpha_n = F(x_n^+) - F(x_n^-) = \text{jump of } F \text{ at } x_n$ .  $F(x_n^+) = F(x_n^-) + \alpha_n$   
 $F(x_n) = F(x_n^-) + \theta_n \alpha_n$ ,  $\theta_n \in [0, 1]$ .  $F(x) = \mu((-\infty, x])$ . Corresponds to singular + abs. cont measures.

$$j_n(x) = \begin{cases} 0 & x < x_n \\ \phi_n & x = x_n \\ 1 & x > x_n \end{cases}$$

Jump function associated to  $F$  is

$$J_F(x) = \sum_{n=1}^\infty j_n(x)$$

$$\sum_{n=1}^\infty \alpha_n = \sum_{n=1}^\infty F(x_n^+) - F(x_n^-) \leq F(b) - F(a) < \infty$$

because  $F$  incr, and  $F$  bounded.

**Lemma.**  $F$  increasing, bounded on  $[a, b]$ ,  $\text{Disc}(F) = \{x_n\}_{n=1}^\infty$

(i)  $J_F(x)$  is discontinuous precisely at  $\{x_n\}_{n=1}^\infty$ , has a jump at  $x_n$  equal to that of  $F$ .

(ii) The function  $F - J_F$  is increasing and continuous.

*Proof.* (i)  $x \neq x_n (\forall n) \Rightarrow$  each  $j_n$  is continuous at  $x \Rightarrow J_F$  is continuous at  $x$  because of uniform convergence.  $x = x_N (\exists N) \Rightarrow J_F = \sum_{n=1}^N \alpha_n j_n(x) + \sum_{n>N} \alpha_n j_n(x)$ . First sum has jump discontinuity  $x_N$  of size  $\alpha_N$

(ii)  $F - J_F$  is continuous

$$F(x) - J_F(x) \leq F(y) - J_F(y) \iff J_F(y) - J_F(x) \leq F(y) - F(x)$$

, where

$$J_F(y) = \sum_{x < x_n \leq y} \alpha_n = \sum_{x < x_n \leq y} F(x_n^+) - F(x_n^-) \leq F(y) - F(x)$$

□

Since  $F = (F - J_F) + J_F$  ETS  $J_F$  is diff a.e.. This was essential step of

$$\mu = \mu_{AC} + \mu_S + \mu_{PP}$$

$z(t) = (x(t), y(t))$ . curve  $\gamma$ .  $x, y : [a, b] \rightarrow \mathbb{R}$  continuous.

$\gamma$  rectifiable if length

$$L(\gamma) = \sup \sum_{j=1}^N |z(t_j) - z(t_{j-1})| < \infty$$

sup over all partitions  $P = \{a = t_0 < t_1 < \dots < t_N = b\}$  of  $[a, b]$

When is

$$L(\gamma) = \int_a^b |z'(t)| dt?$$

**Lemma.**  $\gamma$  is rectifiable iff  $x, y$  are of bounded variation (and cont.).

see  $F = x + iy$

Assume  $\gamma$  rectifiable, let  $L(A, B)$  length of  $\gamma(A, B)$ , ( $a \leq A \leq B \leq b$ )

(i)  $L(A, B) = T_F(A, B)$  (where  $F(t) = z(t)$ )

(ii)  $L(A, C) + L(C, B) = L(A, B)$  ( $A \leq C \leq B$ )

(iii)  $A \mapsto L(A, B)$  (fix  $B$ ) is continuous

$B \mapsto L(A, B)$  (fix  $A$ )

seen:  $F \in \text{BV}(a, b)$ , cont.  $\Rightarrow T_F$  cont.

Warning:  $[0, 1] \ni t \mapsto (F(t), F(t))$ ,  $F$  Cantor.  $F$  cont. incr.  $F(0) = 0$ ,  $F(1) = 1$ ,  $F' = 0$  a.e.

**Theorem.**  $z : [a, b] \rightarrow \mathbb{R}^2, t \mapsto (x(t), y(t)) \sim \text{curve } \gamma$ .  $x, y \in \text{AC}(a, b)$ .  $\Rightarrow \gamma$  rectifiable and

$$L(\gamma) = \int_a^b |z'(t)| dt$$

why?  $F : [a, b] \rightarrow \mathbb{C}$  is  $\text{AC}(a, b)$

$$\Rightarrow T_F(a, b) = \int_a^b |F'(t)| dt.$$

$$\sum_{j=1}^N |F(t_j) - F(t_{j-1})| = \sum_{j=1}^N \left| \int_{t_{j-1}}^{t_j} F'(t) dt \right| \leq \sum_{j=1}^N \int_{t_{j-1}}^{t_j} |F'(t)| dt = \int_a^b |F'(t)| dt$$

First inequality by FTC. For  $\geq$ , write  $F' = g + h$ ,  $g$  step function,  $h$  small in  $L^1$   $G, H = \text{def.}$  integrals of  $g, h$ . Check  $T_F \geq T_G, T_H$ ,  $T_H$  small,  $T_G \geq \int_a^b |g(t)| dt$ .

**Minkowski content of a curve** simple, simple closed, quasi-simple curves.

trace of  $\gamma$ :  $\Gamma = \{z(t) \in \mathbb{R}^2 : t \in [a, b]\}$ .

Given  $K \in \mathbb{R}^2$  and  $\delta > 0$  define

$$K^\delta = \{x \in \mathbb{R}^2 : d(x, K) < \delta\}$$

where  $d(x, K) = \inf_{k \in K} d(x, k)$

**Definition.** The set  $K$  has (1D) Minkowski content if

$$\lim_{\delta \rightarrow 0} \frac{|K^\delta|}{2\delta}$$

exists (in  $\mathbb{R}$ ), denoted  $M(K)$ .

**Theorem.** Let  $\Gamma = \{z(t) : a \leq t \leq b\}$  be (the trace of) a quasi-simple curve  $\gamma$ . Then  $\Gamma$  has Minkowski content iff  $\gamma$  is rectifiable (in which case  $M(\Gamma) = L(\gamma)$ ).

Upper Mink: content.

$$\limsup_{\delta \rightarrow 0^+} \frac{|K^\delta|}{2\delta} =: M^*(K)$$

lower

$$\liminf_{\delta \rightarrow 0^+} \frac{|K^\delta|}{2\delta} =: M_*(K).$$

**Proposition.**  $\Gamma = \{z(t) : a \leq t \leq b\}$  quasi simple. If  $M_*(\Gamma) < \infty$ , then  $\gamma$  is rectifiable and  $L(\gamma) \leq M_*(\Gamma)$ .

**Proposition.**  $\Gamma = \{z(t) : a \leq t \leq b\}$  rectifiable  $\gamma$ . Then  $M^*(\Gamma) \leq L(\gamma)$ .

*Proof of Prop. 1, for simple curves.* Obs:  $\Gamma = \{z(t) : a \leq t \leq b\}$  any curve.  $\Delta = |z(b) - z(a)|$ .  $|\Gamma^\delta| \geq 2\delta\Delta$ .

Take any partition  $P$  of  $[a, b]$ .  $L_P = \sum_{j=1}^N |z(t_j) - z(t_{j-1})|$ . Given  $\varepsilon > 0$ ,  $\exists N$  proper closed subintervals  $I_j = [a_j, b_j] \subset (t_{j-1}, t_j)$ :

$$\sum_{j=1}^N |z(b_j) - z(a_j)| \leq L_P - \varepsilon$$

$I_1, \dots, I_N$  disjoint  $\Rightarrow \Gamma_1, \dots, \Gamma_N$  disjoint because  $\Gamma$  is simple.  $\Leftrightarrow \Gamma_1^\delta, \dots, \Gamma_N^\delta$  disjoint, provided  $\delta > 0$  small enough.

$$\bigcup_{j=1}^N \Gamma_j^\delta \subset \Gamma^\delta$$

$$|\Gamma^\delta| \geq \sum_{j=1}^N |\Gamma_j^\delta| \geq 2\delta \sum_{j=1}^N |z(t_j) - z(t_{j-1})| \geq 2\delta(L_P - \varepsilon)$$

□

**Isoperimetric inequality (soft)**  $\gamma : [a, b] \rightarrow \mathbb{R}^2$ ,  $\gamma \in C^1(a, b) : \gamma'(s) \neq 0 \forall s$ ,  $\gamma(a) = \gamma(b)$ .  
 Arclength parametrization:  $\gamma : [0, L] \rightarrow \mathbb{R}^2$ ,  $|\gamma'(s)| = 1 \forall s$ .

**Theorem.**  $\Gamma \subset \mathbb{R}^2$  simple closed  $C^2$  curve of length  $L$ .  $A$  area of the region enclosed by  $\Gamma$ .

$$A = \frac{1}{2} \left| \int_{\Gamma} (x dy - y dx) \right| = \frac{1}{2} \left| \int_0^L (x(s)y'(s) - x'(s)y(s)) ds \right|$$

Then  $4\pi A \leq L^2$ . Equality iff  $\Gamma$  is a circle.

*Proof.* wlog (rescale)  $L = 2\pi$ .: WTS  $A \leq \pi$ , equality iff  $\Gamma$  is a circle of radius 1.

$\gamma : [0, 2\pi] \rightarrow \mathbb{R}^2$ ,  $s \mapsto \gamma(s) = (x(s), y(s))$  arclength par.  $x'(s)^2 + y'(s)^2 = 1 \forall s$ .

$$\frac{1}{2\pi} \int_0^{2\pi} (x'(s)^2 + y'(s)^2) ds = 1$$

$\Gamma$  closed  $\Rightarrow x(s), y(s)$   $2\pi$ -periodic.

$$x'(s) = \sum_n a_n i n e^{i n s}$$

$$y'(s) = \sum_n b_n i n e^{i n s}$$

Parseval  $\Rightarrow$

$$\sum_{n \in \mathbb{Z}} n^2 (|a_n|^2 + |b_n|^2) = 1$$

$$A = \frac{1}{2} \int_0^{2\pi} (x(s)y'(s) - x'(s)y(s)) ds = \pi \left| \sum_{n \in \mathbb{Z}} n (a_n \bar{b}_n - b_n \bar{a}_n) \right|$$

by bilinear Parseval

$$|a_n \bar{b}_n - b_n \bar{a}_n| \leq 2|a_n||b_n| \leq |a_n|^2 + |b_n|^2$$

$$A \leq \pi \sum_{n \in \mathbb{Z}} n^2 (|a_n|^2 + |b_n|^2) \leq \pi$$

□

**Cases of equality:**  $A = \pi \Rightarrow$

$$x(s) = a_{-1}e^{-is} + a_0 + a_1e^{is}$$

$$y(s) = b_{-1}e^{-is} + b_0 + b_1e^{is}$$

$x, y$  real-valued  $\Rightarrow a_1 = \bar{a}_{-1}$ ,  $b_1 = \bar{b}_{-1}$ . (\*\*) $\Rightarrow 2(|a_1|^2 + |b_1|^2) = 1$ .

(\*\*\*)  $\Rightarrow$

$$|a_1| = |b_1| = \frac{1}{2} : a_1 = \frac{1}{2}e^{i\alpha} \quad b_1 = \frac{1}{2}e^{i\beta} \quad (\alpha, \beta \in \mathbb{R})$$

$$1 = 2|a_1 \bar{b}_1 - \bar{a}_1 b_1| = \sin(\alpha - \beta) : \alpha - \beta = \frac{k\pi}{2} \quad (\text{odd } k)$$

$$x(s) = a_0 + \cos(s + \alpha)$$

$$y(s) = b_0 \pm \sin(s + \alpha)$$

$\pm$  dep. on parity of  $\frac{k-1}{2}$ .

**Isoperimetric inequality (hard)**  $\Omega \subset \mathbb{R}^2$  bounded, open,  $\partial\Omega = \bar{\Omega} - \Omega =: \Gamma$  rectifiable curve (not nec. simple) with length  $l(\Gamma)$ .

**Theorem.**

$$4\pi|\Omega| \leq l(\Omega)^2$$

*Proof.* inner:

$$\Omega_-^\delta = \{x \in \mathbb{R}^2 : d(x, \mathbb{R}^2 - \Omega) \geq \delta\}$$

outer:

$$\Omega_+^\delta = \{x \in \mathbb{R}^2 : d(x, \bar{\Omega}) < \delta\}$$

$$\Gamma^\delta = \{x : d(x, \Gamma) < \delta\}$$

$$\Omega_+^\delta = \Omega_-^\delta \dot{\cup} \Gamma^\delta > \delta$$

$A, B \subset \mathbb{R}^d$ ,  $A + B = \{a + b : a \in A, b \in B\}$  Note:  $\Omega + B_\delta \subset \Omega_+^\delta$ ,  $\Omega_-^\delta + B_\delta \subset \Omega$ .

Brunn-Minkowski:  $A, B \subset \mathbb{R}^2$  meas.,  $A + B$  meas.

$$|A + B|^{\frac{1}{2}} \geq |A|^{\frac{1}{2}} + |B|^{\frac{1}{2}}$$

$$|\Omega_-^\delta| \geq (|\Omega|^{\frac{1}{2}} + |B_\delta|^{\frac{1}{2}})^2 \geq |\Omega| + 2|\Omega|^{\frac{1}{2}} \underbrace{|B_\delta|^{\frac{1}{2}}}_{=(\pi\delta^2)^{\frac{1}{2}}}$$

$$|\Omega| \geq (|\Omega_-^\delta|^{\frac{1}{2}} + |B_\delta|^{\frac{1}{2}})^2 \geq |\Omega_-^\delta| + 2|\Omega_-^\delta|^{\frac{1}{2}}|B_\delta|^{\frac{1}{2}}$$

$$|\Gamma^\delta| \geq |\Omega| + 2|\Omega|^{\frac{1}{2}}\sqrt{\pi} - |\Omega_-^\delta| + |\Omega_-^\delta|^{\frac{1}{2}}\sqrt{\pi}$$

$$\limsup_{\delta \rightarrow 0^+} \frac{|\Gamma^\delta|}{2\delta} \geq 2|\Omega|^{\frac{1}{2}}\sqrt{\pi}$$

$$4\pi|\Omega| \leq M^*(\Gamma)^2 \leq l(\Gamma)^2$$

Note, that only in the very last inequality did we use the rectifiability of  $\Gamma$ . □

**Brunn-Minkowski ineq.** ( $\mathbb{R}^d$ )  $A, B \subset \mathbb{R}^d$  measurable.  $A + B = \{a + b : a \in A, b \in B\}$ .  $\lambda A = \{\lambda a : a \in A\}$  ( $\lambda > 0$ ).

Q.: Can  $|A + B|$  be controlled in terms of  $|A|, |B|$ ? No! There exist sets  $A, B$   $|A| = |B| = 0$  with  $|A + B| > 0$ . Example  $[0, 1] \times [0, 1]$ . Another example  $A = B = C \subset [0, 1]$  Cantor set. Then  $A + B = [0, 2]$ .

Q.: Can  $|A + B|^\alpha \geq c_\alpha(|A|^\alpha + |B|^\alpha)$  hold? (for some  $\alpha > 0$  with  $c_\alpha < \infty$ , indep of  $A, B$ ) Best possible  $c_\alpha = 1$ .

What about  $\alpha$ ? Convex sets play a role.  $A$  = convex,  $B = \lambda A$ .  $|B| = |\lambda A| = \lambda^d |A|$ .  $|A + B| = |A + \lambda A| = |(1 + \lambda)A| = (1 + \lambda)^d |A|$  because  $A$  is convex.

( $\lambda_1 A + \lambda_2 A = (\lambda_1 + \lambda_2)A$  iff  $A$  is convex.)

$|A + B|^\alpha \geq |A|^\alpha + |B|^\alpha$  iff  $(1 + \lambda)^{d\alpha} \geq 1 + \lambda^{d\alpha} \Rightarrow \alpha \geq \frac{1}{d}$ .

$(a + b)^\gamma \geq a^\gamma + b^\gamma \forall a, b \geq 0, \gamma \geq 1$ .

Candidate inequality:

$$|A + B|^{\frac{1}{d}} \geq |A|^{\frac{1}{d}} + |B|^{\frac{1}{d}}$$

(BM)

$A, B$  measurable  $\Rightarrow A + B$  measurable. Take  $[0, 1] \times \text{nonmeasurable}$ .

(i)  $A, B$  closed  $\Rightarrow A + B$  measurable

(ii)  $A, B$  compact  $\Rightarrow A + B$  compact

(iii)  $A, B$  open  $\Rightarrow A + B$  open

**Theorem.** (BM) holds if  $A, B, A + B$  measurable.

(i)  $A, B$  rectangles with sidelengths  $\{a_j\}_{j=1}^\infty, \{b_j\}_{j=1}^\infty$

(ii)  $A, B$  unions of finitely many rectangles with disjoint interiors.

(iii)  $A, B$  open sets of finite measure

(iv)  $A, B$  compact

(v)  $A, B, A + B$  measurable.

*Proof.* (i) (BM) becomes

$$\prod_{j=1}^d (a_j + b_j)^{\frac{1}{d}} \geq \prod_{j=1}^d a_j^{\frac{1}{d}} + \prod_{j=1}^d b_j^{\frac{1}{d}}$$

$a_j \rightarrow \lambda_1 a_j, b_j \rightarrow \lambda_1 b_j$ . Both sides are multiplied by  $(\lambda_1 \lambda_2 \dots \lambda_d)^{\frac{1}{d}}$ : wlog can assume  $a_j + b_j = 1 \forall j$  (Choose  $\lambda_j = a_j + b_j$ )

AMGM:

$$\prod_{j=1}^d a_j^{\frac{1}{d}} \leq \frac{1}{d} \sum_{j=1}^d a_j$$

$$\prod_{j=1}^d b_j^{\frac{1}{d}} \leq \frac{1}{d} \sum_{j=1}^d b_j$$

$$\prod_{j=1}^d a_j^{\frac{1}{d}} + \prod_{j=1}^d b_j^{\frac{1}{d}} \leq \frac{1}{d} \sum_{j=1}^d (a_j + b_j) = 1$$

(ii) Induction on  $n$  = number of rectangles in  $A$  and  $B$ . Choose pair of disjoint rectangles  $R_1, R_2$  in  $A$ . Can rotate s.t.  $R_1$  and  $R_2$  are separated by hyperplane  $\{x_j = 0\}$ .  $R_1$  lies in  $A_+ = A \cap \{x_j \geq 0\}$ ,  $A_- = A \cap \{x_j \leq 0\}$ .

Rem.: Both  $A_+, A_-$  contain at least one less rectangle than  $A$ ,  $A = A_+ \cup A_-$  and  $A_B \cap A_-$  has measure zero.

Now: translate  $B$  s.t.  $B_-$  and  $B_+$  satisfy

$$\frac{|B_\pm|}{|B|} = \frac{|A_\pm|}{|A|}$$

$(A_+ + B_+) \cup (A_- + B_-) \subset A + B$  Number of rectangles in  $A_+$  and  $B_+$ , number of rectangles in  $A_-$  and  $B_-$  is  $< n$ .

$$\begin{aligned} |A + B| &\geq |A_+ + B_+| + |A_- + B_-| \geq (|A_+|^{\frac{1}{d}} + |B_+|^{\frac{1}{d}})^d + (|A_-|^{\frac{1}{d}} + |B_-|^{\frac{1}{d}})^d \\ &= (|A_+|(1 + (\frac{|B_+|}{|A_+|})^{\frac{1}{d}}))^d + |A_-|(1 + (\frac{|B_-|}{|A_-|})^{\frac{1}{d}})^d = (|A_+| + |A_-|)(1 + (\frac{|B|}{|A|})^{\frac{1}{d}})^d \\ &= (|A|^{\frac{1}{d}} + |B|^{\frac{1}{d}})^d. \end{aligned}$$

(iii) Open sets of finite measure  $A, B$ .  $\forall \varepsilon > 0 \exists A_\varepsilon, B_\varepsilon$  finet unions of parallel rectangles with disjoint interiors.  $A_\varepsilon \subset A, B_\varepsilon \subset B, |A| \leq |A_\varepsilon| + \varepsilon, |B| \leq |B_\varepsilon| + \varepsilon$ .

$$|A + B| \geq |A_\varepsilon + B_\varepsilon| \geq (|A_\varepsilon|^{\frac{1}{d}} + |B_\varepsilon|^{\frac{1}{d}})^d \geq ((|A| - \varepsilon)^{\frac{1}{d}} + (|B| - \varepsilon)^{\frac{1}{d}})^d. \text{ Let } \varepsilon \rightarrow 0^+, \text{ done.}$$

(iv)  $A, B$  compact. Let  $A^\varepsilon = \{x : d(x, A) < \varepsilon\}$ .  $A + B \subset A^\varepsilon + B^\varepsilon \subset (A + B)^{2\varepsilon}$

(v)  $A, B, A + B$  measurable: use inner regularity of Lebesgue measure.

□

*Remark.*  $A, B$  open sets of finite positive measure. Equality in (BM) iff  $A, B$  convex and similar.  $\exists \delta > 0 \exists h \in \mathbb{R}^d : A = \delta B + h$  ( $A$  convex iff  $\lambda_1 A + \lambda_2 A = (\lambda_1 + \lambda_2)A$ )

**Consequences for isoperimetric inequality**  $A \subset \mathbb{R}^d$  bounded open with smooth boundary.  $(\partial A, B \subset \mathbb{R}^d \text{ ball } |B| = |A|)$

$$|\partial A| = \lim_{\varepsilon \rightarrow 0^+} \frac{|A + \varepsilon B| - |A|}{\varepsilon}$$

Isoper ineq.:  $|\partial A| \geq |\partial B|$ .

*Proof.*

$$\frac{|A + \varepsilon B| - |A|}{\varepsilon} \geq \frac{(|A|^{\frac{1}{d}} + |\varepsilon B|^{\frac{1}{d}})^d - |A|}{\varepsilon} = \frac{(1 + \varepsilon)^d - 1}{\varepsilon} |B| \rightarrow d|B| = |\partial B|$$

for  $\varepsilon \rightarrow 0$ .

□

Better:  $A \subset \mathbb{R}^d$  has finite perimeter ( $\iff 1_A \in \text{BV}(U), U \subset \mathbb{R}^d$  bdd open)

$$\frac{\mathcal{H}^{d-1}(\partial A)}{|A|^{\frac{d-1}{d}}} \geq \frac{\mathcal{H}^{d-1}(S^{d-1})}{|B^d(0, 1)|^{\frac{d-1}{d}}}$$

**Hausdorff measure** Q: How does a set replicate under scaling?  $E \rightarrow nE = E_1 \cup \dots \cup E_m$  disjoint congruent copies of  $E$ . Examples: line  $m = n^1$ , square  $m = n^2$ , cube  $m = n^3$ , Cantor set  $3C = C_1 \cup C_2$   $2 = 3^\alpha \iff \alpha = \frac{\log 3}{\log 2}$

$\#(\varepsilon)$  = least  $\#$  of segments that arise from such polygonal lines.  $\Gamma$  rectifiable iff  $\#(\varepsilon) \sim \varepsilon^{-1}$  as  $\varepsilon \rightarrow 0^+$ . If  $\#(\varepsilon) \sim \varepsilon^{-\alpha}$  ( $\alpha > 1$ ) In this case, say " $\Gamma$  has dim  $\alpha$ ". Snowflake has  $\alpha = \frac{\log 4}{\log 3} > 1$ .

Upshot:  $E$   $\alpha > 1$ .  $m_\alpha(E) = \alpha$ -dimensional mass of  $E$  among sets of "dimension"  $\alpha$ .

- $\alpha > \dim(E) \Rightarrow m_\alpha(E) = 0$
- $\alpha < \dim(E) \Rightarrow m_\alpha(E) = \infty$
- $\alpha = \dim(E)$  interesting

R. Gardner Bulletin AMS more about Brunn-Minkowski, geometrically, including more proofs, e.g. with induction of the dimension.



**Hausdorff measure**  $E \subset \mathbb{R}^d$  any subset.

$$m_\alpha^*(E) := \lim_{\delta \rightarrow 0^+} \inf \left\{ \underbrace{\sum_k (\text{diam} F_k)^\alpha : E \subset \bigcup_{k=1}^{\infty} F_k, \text{diam}(F_k) \leq \delta}_{H_\alpha^\delta(E)} \right\}$$

exterior/outer  $\alpha$ -dim Hausdorff measure.

*Remark.*  $H_\alpha^\delta(E) \leq H_\alpha^\delta(E) \leq m_\alpha^*(E) (\forall \delta > 0)$ .  $H_\alpha^\delta(E)$  increases when  $\delta$  decreases.  $\therefore m_\alpha^*(E) = \lim_{\delta \rightarrow 0^+} H_\alpha^\delta(E)$  exists

*Remark.* Coverings must be by sets of arb. small measure. (If we allowed the  $\delta$  to be arbitrary then two parallel lines would get the same 1d-measure as one of them.)

*Remark (Scaling).* "The measure of a set should scale like its dimension". E.g.:  $\Gamma \subset \mathbb{R}^d$  smooth curve of length  $L$  *sim*  $\lambda\Gamma$  has length  $\lambda L$ .  $Q \subset \mathbb{R}^d$  cube *sum*  $\lambda Q$  has measure  $\lambda^d |Q|$ .  $|F|$  scaled by  $\lambda \Rightarrow (\text{diam} F)^\alpha$  scaled by  $\lambda^\alpha$

### Properties

- (i)  $E_1 \subset E_2 \Rightarrow m_\alpha^*(E_1) \leq m_\alpha^*(E_2)$
- (ii)  $\{E_j\} \subset \mathbb{R}^d$  countable family of sets  $\Rightarrow m_\alpha^*(\bigcup_{j=1}^{\infty} E_j) \leq \sum_{j=1}^{\infty} m_\alpha^*(E_j)$
- (iii) (Finite additivity)  $\inf_{x \in E_1, y \in E_2} |x - y| = d(E_1, E_2) > 0 \Rightarrow m_\alpha^*(E_1 \cup E_2) = m_\alpha^*(E_1) + m_\alpha^*(E_2)$

*Proof.* ETS  $\geq$ . Fix  $0 < \varepsilon < d(E_1, E_2)$ . Given any cover of  $E_1 \cup E_2$  with sets  $F_1, F_2, \dots$  of  $\text{diam} \leq \delta < \varepsilon$ , let  $F'_j = F_j \cap E_1$ ,  $F''_j = F_j \cap E_2$ .

$$\sum (\text{diam}_j F'_j)^\alpha + \sum (\text{diam}_j F''_j)^\alpha \leq \sum_k (\text{diam}(F_k))^\alpha$$

Take inf over all covers, let  $\delta \rightarrow 0^+$ , done. □

$m_\alpha^*$  satisfies all properties of a Caratheodory outer measure  $\therefore m_\alpha^*$  is a countably additive measure when restricted to Borel sets, call it  $m_\alpha = \alpha$ -dim Hausdorff measure.

- (iv)  $\{E_j\}$  countable family of disjoint Borel sets  $\Rightarrow$

$$m_\alpha(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} m_\alpha(E_j)$$

- (v) Hausdorff measure is invariant under translation and rotations. It scales like:

$$m_\alpha(\lambda E) = \lambda^\alpha m_\alpha(E)$$

- (vi)  $m_0(E) = \#E$ ,  $m_1(E) = |E|$  (=1D Lebesgue measure of  $E$ ),  $E \subset \mathbb{R}$  Borel.

- (vii)  $E \subset \mathbb{R}^d$  Borel,  $m_\alpha(E) \simeq |E|$

*Proof.* (i) Isodiametric inequality:  $|E| \leq v_d \left(\frac{\text{diam} E}{2}\right)^d$ ,  $v_d$  volume of the unit ball in  $\mathbb{R}^d$ .  
Prove first for sets  $E = -E$  and then something hard.

- (ii) Covering argument: Given  $\varepsilon, \delta > 0$ , there exists a covering of  $E$  by balls  $\{B_j\}$ :  
 $\text{diam} B_j < \delta$ ,  $\sum_j |B_j| \leq |E| + \varepsilon$

$$H_d^\delta(E) \leq \sum_j (\text{diam} B_j)^d = c_d \sum_j |B_j| \leq c_d(|E| + \varepsilon),$$

let  $\delta, \varepsilon \rightarrow 0^+$ , get one of the inequalities.

□

- (viii) if  $m_\alpha^*(E) < \infty$  and  $\beta > \alpha$ , then  $m_\beta^*(E) = 0$ . If  $m_\alpha^*(E) > 0$  and  $\beta < \alpha$ , then  $m_\beta^*(E) = \infty$ .

*Proof.*  $\text{diam} F < \delta, \beta > \alpha \Rightarrow (\text{diam} F)^\beta = (\text{diam} F)^{\beta-\alpha} (\text{diam} F)^\alpha < \delta^{\beta-\alpha} (\text{diam} F)^\alpha$  □

Consequence: Given  $E \subset \mathbb{R}^d$  Borel,  $\exists! \alpha$  such that

$$m_\beta(E) = \begin{cases} \infty & \beta < \alpha \\ 0 & \beta > \alpha \end{cases}$$

$$\alpha = \sup\{\beta : m_\beta(E) = \infty\} = \inf\{\beta : m_\beta(E) = 0\} := \text{Hausdorff dimension of } E = \dim E$$

At the critical value  $\alpha = \dim E$   $0 \leq m_\alpha(E) \leq \infty$ . If  $E$  is bounded and the inequalities are strict, we say that  $E$  has strict Hausdorff dimension  $\alpha$ .

**Theorem.** The Cantor set  $C \subset [0, 1]$  has strict Hausdorff dimension  $\frac{\log 2}{\log 3}$ .

ETS:  $0 < m_\alpha(C) \leq 1$

*Proof.*  $m_\alpha(C) \leq 1$ :  $C = \bigcap C_k$  where each  $C_k$  is a finite union of  $2^k$  intervals of length  $3^{-k}$ . Given  $\delta > 0$  choose  $k$  large enough such that  $3^{-k} < \delta$ .  $C_k$  covers  $C$  and consists of  $2^k$  intervals of diameter  $3^{-k} < \delta$ .  $H_\alpha^\delta(C) \leq 2^k (3^{-k})^\alpha = 1$ , let  $\delta \rightarrow 0^+$ , done.

$$m_\alpha(C) > 0:$$

**Lemma.**  $E \subseteq \mathbb{R}^d$  compact,  $f : E \rightarrow \mathbb{R}$   $\gamma$ -Hölder,

$$|f(x) - f(y)| \leq m|x - y|^\gamma \quad (\forall x, y \in E) \quad 0 < \gamma \leq 1$$

Then

$$(i) \quad m_\beta(f(E)) \leq M^\beta m_\alpha(E) \text{ if } \beta = \frac{\alpha}{\gamma}.$$

$$(ii) \quad \dim f(E) \leq \frac{1}{\gamma} \dim(E)$$

*Proof.*  $\{F_k\}$  countable family of sets that covers  $E$  :  $\{f(F_k \cap E)\}$  covers  $f(E)$ .  $\text{diam} f(F_k \cap E) \leq M(\text{diam} F_k)^\gamma$ .

$$\sum_k (\text{diam} f(F_k \cap E))^{\frac{\alpha}{\gamma}} \leq M^{\frac{\alpha}{\gamma}} \sum_k (\text{diam} F_k)^\alpha,$$

done. and 1 implies 2. □

**Lemma.** The Cantor-Lebesgue function  $F : C \rightarrow [0, 1]$  is  $\gamma = \frac{\log 2}{\log 3}$ -Hölder.

*Proof.* Goal:  $|F(x) - F(y)| \leq c|x - y|^\gamma \forall x, y \in C$ .

$F_n$  increases at most  $2^{-n}$  on an interval of length  $3^{-n}$ .  $\therefore$  slope  $\leq (\frac{3}{2})^n \therefore |F_n(x) - F_n(y)| \leq (\frac{3}{2})^n |x - y|$ .  $|F_n(x) - F(x)| \leq 2^{-n}$ . Given  $x, y$  chose  $n$ :  $3^n |x - y| \sim 1$ ,  $3^\gamma = 2$ .

$$|F(x) - F(y)| \leq |F_n(x) - F_n(y)| + |F_n(x) - F(x)| + |F_n(y) - F(y)| \leq (\frac{3}{2})^n |x - y| + 2 \cdot 2^{-n} \leq c 2^{-n} = c(3^{-n})^\gamma \leq c' |x - y|^\gamma$$

□

Apply Lemma 1 with  $E = C$ ,  $f = F$ ,  $\gamma = \frac{\log 2}{\log 3} \Rightarrow 1 = m_1([0, 1]) \leq M m_\alpha(C)$ ,  $\dim C = \frac{\log 2}{\log 3}$

□

## Rectifiable curves

**Theorem.**  $\gamma : [a, b] \rightarrow \mathbb{R}^d$  continuous and simple. Then  $\gamma$  is rectifiable iff  $\Gamma = \{\gamma(t) : a \leq t \leq b\}$  has strict Hausdorff dimension equal to 1.  $m_1(\Gamma) = l(\gamma)$ .

*Proof.*  $\Rightarrow$ : Let  $\gamma$  be rectifiable of length  $L$ . Consider arclength parametrization  $\tilde{\gamma}$ .  $\Gamma = \{\tilde{\gamma}(s) : 0 \leq s \leq L\}$ .

$$|\tilde{\gamma}(s_1) - \tilde{\gamma}(s_2)| \leq |s_1 - s_2|$$

By Lemma 1 (i)  $m_1(\Gamma) \leq L$ . Wh  $m_1(\Gamma) \geq L$ ?

$$\Gamma_j = \{\gamma(t) : t_j \leq t \leq t_{j+1}\}$$

$$\Gamma = \bigcup_{j=1}^{N-1} \Gamma_j \quad m_1(\Gamma) = \sum_{j=0}^{N-1} m_1(\Gamma_j)$$

Claim:  $m_1(\Gamma_j) \geq l_j := |\gamma(t_j) - \gamma(t_{j+1})|$

*Proof.*  $\pi : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2$   $(x, y) \mapsto x$  Lipschitz,  $\pi(\Gamma_j) \subset [0, l_j]$  Lemma 1 (i) implies the claim.

□

$$\therefore m_1(\Gamma) \geq \sum m_1(\Gamma_j) \geq \sum l_j, \quad L := \sup_p \sum l_j : m_1(\Gamma) \geq L, \text{ done.}$$

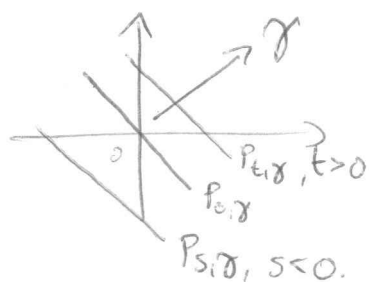
□

## Radon transforms

$$R(f)(t, \gamma) = \int_{P_{t, \gamma}} f. \quad \text{where ...}$$

$$f: \mathbb{R}^d \rightarrow \mathbb{R}, \quad t \in \mathbb{R}, \quad \gamma \in S^{d-1} \subseteq \mathbb{R}^d$$

$$P_{t, \gamma} = \{x \in \mathbb{R}^d : \underbrace{x \cdot \gamma}_{\text{inner pr.}} = t\} \quad \text{hyperplane}$$



$P_{t, \gamma}$  equipped with natural  $(d-1)$ -dim. Leb. measure, denoted by  $m_{d-1}$  (coincides with  $(d-1)$ -Hausdorff measure)

Remarks (i)  $f \in C_0^\infty(\mathbb{R}^d) \Rightarrow f$  integrable on every  $P_{t, \gamma} \Rightarrow R(f)(t, \gamma)$  defined for every  $(t, \gamma)$ .  
( $R(f)$  cont. fct. of  $(t, \gamma)$ ,  
cpctly supp. in  $t$ )

(ii)  $f \in L^1(\mathbb{R}^d) \Rightarrow f$  may fail to be measurable/integrable on some  $P_{t, \gamma}$  ( $\rightarrow R(f)(t, \gamma)$  not defined.)

(iii)  $f = \chi_E$  ( $E \subset \mathbb{R}^d$  mb.)  $\Rightarrow R(f)(t, \gamma) = m_{d-1}(E_{t, \gamma})$   
if  $E_{t, \gamma}$  measurable.  
 $E_{t, \gamma} = E \cap P_{t, \gamma}$

Look instead at maximal Radon transform:

$$R^*(f)(\gamma) = \sup_{t \in \mathbb{R}} |R(f)(t, \gamma)|.$$

$\rightarrow$  Want to study  $L^p$ -mapping properties of  $R$  in order to study regularity of subsets of  $\mathbb{R}^d$ .

Thm. 1  $f \in C^0(\mathbb{R}^d)$ ,  $n \geq 3$ :

(\*)  $\int_{S^{d-1}} R^*(f)(\gamma) d\sigma_\gamma \leq C(\|f\|_{L^1(\mathbb{R}^d)} + \|f\|_{L^2(\mathbb{R}^d)}).$

~~the f needs~~

Rem. i) necessary conditions:

•)  $f \in L^1$ :  $f(x) = (1+|x|^{d-1})^{-1} \in (L^2 \setminus L^1)(\mathbb{R}^d)$  if  $d \geq 3$   
 $f$  is not integrable in any plane  $P_{t,\gamma}$ .  
( $\rightarrow f \in L^1$  gives global control.)

•)  $f \in L^2$ :  $f_\varepsilon(x) = (|x|+\varepsilon)^{-d+\delta}$  if  $|x| \leq 1$ ,  $\delta \in (0,1)$  fixed.  
Let  $\varepsilon \rightarrow 0^+$  to see that (\*) fails if  $\|\cdot\|_{L^2}$   
on the RHS is not there  
( $\rightarrow f \in L^2$  gives local control.)

Key: Interplay between Radon and Fourier transform.  
 $t \mapsto \lambda \in \mathbb{R}$  dual variable.

Fourier transform:

$$\hat{R}(f)(\lambda, \gamma) = \int_{-\infty}^{\infty} R(f)(t, \gamma) e^{-2\pi i \lambda t} dt$$

Lemma 1  $f \in C^0(\mathbb{R}^d)$ ,  $\gamma \in S^{d-1}$ :

$$\hat{R}(f)(\lambda, \gamma) = \hat{f}(\lambda \gamma).$$

Proof  $\hat{f}(\lambda \gamma) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot (\lambda \gamma)} dx$

$$= \int_{-\infty}^{\infty} \left( \int_{\mathbb{R}^{d-1}} f(u, t) du \right) e^{-2\pi i \lambda t} dt = \int_{-\infty}^{\infty} \left( \int_{P_{t,\gamma}} f \right) e^{-2\pi i \lambda t} dt.$$

Choose coordinates

$$x = (u, t), \quad t = x \cdot \gamma = x_d \in \mathbb{R},$$

$$u = (x_1, \dots, x_{d-1}) \in \mathbb{R}^{d-1}.$$

□

Lemma 2  $f \in C_0^\infty(\mathbb{R}^d)$ : (2)

$$\int_{S^{d-1}} \left( \int_{-\infty}^{\infty} |\hat{F}(f)(\lambda, y)|^2 |\lambda|^{d-1} d\lambda \right) d\sigma_y = 2 \int_{\mathbb{R}^d} |f(x)|^2 dx$$

Proof  $2 \int_{\mathbb{R}^d} |f(x)|^2 dx = 2 \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 d\xi$   
 $= 2 \int_{S^{d-1}} \left( \int_0^\infty |\hat{f}(\lambda y)|^2 \lambda^{d-1} d\lambda \right) d\sigma_y$   
 $= 2 \int_{S^{d-1}} \left( \int_{-\infty}^\infty |\hat{f}(\lambda y)|^2 |\lambda|^{d-1} d\lambda \right) d\sigma_y$   
 done by Lemma 2.  $\square$

Lemma 3 Suppose  $F(t) = \int_{-\infty}^{\infty} \hat{F}(\lambda) e^{2\pi i \lambda t} d\lambda$   
 Where  $\sup_{\lambda \in \mathbb{R}} |\hat{F}(\lambda)| \leq A$  and  $\int_{-\infty}^{\infty} |\hat{F}(\lambda)|^2 |\lambda|^{d-1} d\lambda \leq B^2$ .

Then  $\sup_{t \in \mathbb{R}} |F(t)| \leq C(A+B)$  (This is essential in the proof of Thm. 1)

Moreover, if  $0 < \alpha < \frac{1}{2}$ , then  $|F(t_1) - F(t_2)| \leq C_\alpha |t_1 - t_2|^\alpha \cdot (A+B)$

( $\rightarrow \alpha$ -Hölder)

Proof.

$$F(t) = \underbrace{\int_{|\lambda| \leq 1} \hat{F}(\lambda) e^{2\pi i \lambda t} d\lambda}_{\leq CA} + \underbrace{\int_{|\lambda| > 1} \hat{F}(\lambda) e^{2\pi i \lambda t} d\lambda}_{\leq \int_{|\lambda| > 1} |\hat{F}(\lambda)| d\lambda}$$

$$\stackrel{CS}{\leq} \left( \int_{|\lambda| > 1} |\hat{F}(\lambda)|^2 |\lambda|^{d-1} d\lambda \right)^{1/2} \cdot \underbrace{\left( \int_{|\lambda| > 1} |\lambda|^{-d+1} d\lambda \right)^{1/2}}_{< \infty \text{ if } -d+1 < -1 \Rightarrow d \geq 3. \checkmark} \leq B$$

$\Rightarrow$  first estimate.

$$\begin{aligned}
|F(t_1) - F(t_2)| &= \int_{-\infty}^{\infty} \hat{F}(\lambda) \underbrace{(e^{2\pi i \lambda t_1} - e^{2\pi i \lambda t_2})}_{1 - |e^{2\pi i \lambda(t_1 - t_2)}|} d\lambda \\
&\leq \int_{|\lambda| \leq 1} \hat{F}(\lambda) d\lambda + \int_{|\lambda| > 1} |\hat{F}(\lambda)| |\lambda|^{-\alpha} d\lambda \\
&\leq C_\alpha A |t_1 - t_2|^\alpha \quad (\text{Lipschitz}) \\
&\quad + \int_{|\lambda| > 1} |\hat{F}(\lambda)| |\lambda|^{-\alpha} d\lambda \\
&\leq \left( \int_{|\lambda| > 1} |\hat{F}(\lambda)|^2 |\lambda|^{-d+1} d\lambda \right)^{1/2} \\
&\quad \cdot \left( \int_{|\lambda| > 1} |\lambda|^{-d+1+2\alpha} d\lambda \right)^{1/2} \\
&\leq \infty \text{ if } \alpha < \frac{1}{2} \text{ for } d \leq 3.
\end{aligned}$$

□

Proof of Thm. For each  $\gamma \in S^{d-1}$ , let  $F(t) = R(f)(t, \gamma)$

$$\Rightarrow \sup_{t \in \mathbb{R}} |F(t)| = R^*(f)(\gamma).$$

$$\text{Let } A(\gamma) = \sup_{\lambda \in \mathbb{R}} |\hat{F}(\lambda)|, \quad B^2(\gamma) = \int_{-\infty}^{\infty} |\hat{F}(\lambda)|^2 |\lambda|^{d-1} d\lambda$$

$$\text{Lemma 3 } \stackrel{\text{check assumptions!}}{\Rightarrow} \sup_{t \in \mathbb{R}} |F(t)| \leq C(A(\gamma) + B(\gamma))$$

$$\text{Lemma 1 } \Rightarrow \hat{F}(\lambda) = \hat{R}(f)(\lambda, \gamma) = \hat{f}(\lambda \gamma) \Rightarrow A(\gamma) \leq \|f\|_{L^1(\mathbb{R}^d)}$$

$$\text{Lemma 2 } \Rightarrow \int_{S^{d-1}} B^2(\gamma) d\sigma_\gamma = 2 \|f\|_{L^2(\mathbb{R}^d)}^2.$$

$$\text{We have } \sup_{t \in \mathbb{R}} |F(t)|^2 \leq C^2 (A^2(\gamma) + B^2(\gamma))$$

Integrate both sides:

$$\int_{S^{d-1}} R^*(f)(\gamma)^2 d\sigma_\gamma \lesssim \underbrace{\int_{S^{d-1}} A^2(\gamma) d\sigma_\gamma}_{\approx \|f\|_1^2} + \underbrace{\int_{S^{d-1}} B^2(\gamma) d\sigma_\gamma}_{\approx \|f\|_{L^2}^2}$$

Use Hölder, because on cpt. space:

$$\lesssim (\|f\|_{L^1}^2 + \|f\|_{L^2}^2)^2$$

$$\|R^*(f)\|_{L^1}^2 \lesssim \int_{S^{d-1}} R^*(f)(\gamma)^2 d\sigma_\gamma.$$

□

Regularity of sets when  $d \geq 3$ .

(3)

$E \subset \mathbb{R}^d$  meas.  $E_{t,\gamma} = E \cap P_{t,\gamma}$  ( $t$  varies,  $\gamma$  fixed)

Fubini  $\Rightarrow E_{t,\gamma}$  is  $m_{d-1}$ -measurable for a.e.  $t$

$t \mapsto m_{d-1}(E_{t,\gamma})$  is a measurable fctn of  $t$ .

Thm. 2  $E \subset \mathbb{R}^d$  ( $d \geq 3$ ) of finite measure.

Then for a.e.  $\gamma \in S^{d-1}$ :

(i)  $E_{t,\gamma}$  is  $m_{d-1}$ -measurable for every  $t$ .

(ii)  $t \mapsto m_{d-1}(E_{t,\gamma})$  is a cont. fct. of  $t$ .

Moreover, this fctn is  $\alpha$ -Hölder  $\forall \alpha \in (0, \frac{1}{2})$ .

Cor.  $d \geq 3$ ,  $E \subset \mathbb{R}^d$  of Lebesgue measure zero.

Then, for a.e.  $\gamma \in S^{d-1}$ , the slice  $E_{t,\gamma}$  has zero measure for every  $t \in \mathbb{R}$ .

Prop.  $d \geq 3$ ,  $f \in (L^1 \cap L^2)(\mathbb{R}^d)$ . Then for a.e.

$\gamma \in S^{d-1}$ :

i)  $f$  is meas. and int. on the plane for every  $t \in \mathbb{R}$ .

ii)  $t \mapsto \mathcal{R}(f)(t,\gamma)$  is cont. and  $\alpha$ -Hölder if  $\alpha < \frac{1}{2}$ .

Moreover, estimate (\*) from Thm. 1 holds for  $f$ .

Rem. Prop. implies Thm. 2 by taking char. fctn of  $E$ .

$$\mathcal{R}(\chi_E)(t,\gamma) = m_{d-1}(E_{t,\gamma}).$$

We skip the proof of Prop. (follows from Thm. 1 using some delicate measure theory)

What about  $d=2$ ?

Given  $f \in L^1(\mathbb{R}^2)$ , define

$$\mathcal{R}_\delta(f)(t,\gamma) = \frac{1}{2\delta} \int_{t-\delta}^{t+\delta} \mathcal{R}(f)(s,\gamma) ds. \quad (\text{averaged version of } \mathcal{R})$$

(integration over thickened line = hypplane.)

$$= \frac{1}{2\delta} \int_{\{t-\delta \leq x \cdot \gamma \leq t+\delta\}} f(x) dx.$$



Thm. 3  $f \in C_0^\infty(\mathbb{R}^2)$ ,  $0 < \delta \leq 1/2$ .

$$\int_{S^1} R_\delta^*(f)(\gamma) d\sigma_\gamma \lesssim (\log \frac{1}{\delta})^{1/2} \cdot (\|f\|_{L^1(\mathbb{R}^2)} + \|f\|_{L^2(\mathbb{R}^2)})$$

**Theorem.**  $f \in C_0^0(\mathbb{R}^2)$ ,  $0 < \delta \leq \frac{1}{2}$ . Then

$$\int_{S^1} R_\delta^*(f)(\gamma) d\sigma_\gamma \lesssim (\log \frac{1}{\delta})^{\frac{1}{2}} (\|f\|_{L^1(\mathbb{R}^2)} + \|f\|_{L^2(\mathbb{R}^2)})$$

*Proof of Theorem.* Modified version of lemma 3: Setting

$$F_\delta(t) = \int_{-\infty}^{\infty} \hat{F}(\lambda) \left( \frac{e^{2\pi i(t+\delta)\lambda} - e^{2\pi i(t-\delta)\lambda}}{2\pi i\lambda(2\delta)} \right) d\lambda$$

Suppose  $\sup_{\lambda \in \mathbb{R}} |\hat{F}(\lambda)| \leq A$  and  $\int_{-\infty}^{\infty} |\hat{F}(\lambda)|^2 |\lambda|^{d-1} d\lambda \leq B^2$ .

Claim:

$$\sup_t |F_\delta(t)| \lesssim (\log \frac{1}{\delta})^{\frac{1}{2}} (A + B)$$

$$F_\delta(t) = \int_{-\infty}^{\infty} \hat{F}(\lambda) \left( \frac{e^{2\pi i(t+\delta)\lambda} - e^{2\pi i(t-\delta)\lambda}}{2\pi i\lambda(2\delta)} \right) d\lambda = \int_{|\lambda| \leq 1} + \int_{|\lambda| > 1} \leq cA + \int_{1 < |\lambda| \leq \frac{1}{\delta}} |\hat{F}(\lambda)| d\lambda + \frac{c}{\delta} \int_{|\lambda| > \frac{1}{\delta}} |\hat{F}(\lambda)| |\lambda|^{-1} d\lambda = I + II$$

CS:

$$I \lesssim \left( \int_{\mathbb{R}} |\hat{F}(\lambda)|^2 |\lambda| d\lambda \right)^{\frac{1}{2}} \left( \int_{1 < |\lambda| \leq \frac{1}{\delta}} |\lambda|^{-1} d\lambda \right)^{\frac{1}{2}} \leq B (\log \frac{1}{\delta})^{\frac{1}{2}}$$

$$II \lesssim \frac{c}{\delta} \left( \int_{\mathbb{R}^2} |\hat{F}(\lambda)|^2 |\lambda| d\lambda \right)^{\frac{1}{2}} \left( \int_{|\lambda| > \frac{1}{\delta}} |\lambda|^{-3} d\lambda \right)^{\frac{1}{2}} \lesssim B$$

□

**Theorem.** There exists a subset  $K \subset \mathbb{R}^2$  such that

- (i)  $K$  is compact
- (ii)  $K$  has Lebesgue measure zero
- (iii)  $K$  contains a translate of every unit line segment

**Theorem.** Suppose  $F$  is any set that satisfies conditions (i) and (iii) from Theorem 1. Then  $F$  has Hausdorff dimension 2.

*Proof of Theorem 2.* Let  $F$  be a Kakeya set. Fix  $0 < \alpha < 2$ . Let  $F \subset \bigcup_{i=1}^{\infty} B_i$  be a covering with balls  $B_i$  of diameter  $\leq \delta$ . It is enough to show

$$\sum (\text{diam} B_i)^\alpha \geq c_\alpha > 0$$

for  $\alpha < 2$ .

Case 1: Assume  $\text{diam} B_1 = \delta \leq \frac{1}{2}$  and let  $N < \infty$  be the number of balls in the covering. WTS  $N\delta^\alpha \geq c_\alpha$ .  $B_i^*$  = double of  $B_i$ .  $F^* = \bigcup_i B_i^*$ .  $|F^*| \leq \sum |B_i^*| = cN\delta^2$ .  $F$  Kakeya  $\Rightarrow \forall \gamma \in S^1 \exists s_\gamma \perp \gamma$  unit line segment:  $s_\gamma \subset F$ .  $s_\gamma^\delta \subset F^*$ .  $\therefore R_\delta^*(\chi_{F^*})(\gamma) \geq 1$  ( $\forall \gamma \in S^1$ ). Take  $f = \chi_{F^*}$  in (\*). Since  $L^2 \subset L^1$ ,

$$\|\chi_{F^*}\|_{L^1} \lesssim \|\chi_{F^*}\|_{L^2} = |F^*|^{\frac{1}{2}} \lesssim N^{\frac{1}{2}} \delta.$$

(\*)  $\Rightarrow 0 < c \leq (\log \frac{1}{\delta})^{\frac{1}{2}} N^{\frac{1}{2}} \delta$ . This implies  $N\delta^\alpha \geq c_\alpha > 0$ .

Case 2: General case.  $F \subset \bigcup_{i=1}^{\infty} B_i$  with each ball  $B_i$  of diameter  $\leq 1$ . For each  $k \in \mathbb{N}$ , let  $N_k$  be the number of balls in  $\{B_i\}$  with diameter  $B_k \sim 2^{-k}$ , i.e.  $\in [2^{-k-1}, 2^{-k}]$ . WTS

$$\sum_{k=0}^{\infty} N_k 2^{-k\alpha} \geq c_{\alpha} > 0.$$

ETS  $\exists k' : N_{k'} 2^{-k'\alpha} \geq c_{\alpha}$ .

$$F_k = F \cap \left( \bigcup_{\text{diam } B_i \sim 2^{-k}} B_i \right)$$

$$F_k^* = \bigcup_{\text{diam } B_v \sim 2^{-k}} B_v^*$$

$$|F_k^*| \leq c N_k 2^{-2k} \quad \forall k$$

$F$  Keakeya  $\Rightarrow \forall \gamma \in S^2 \exists s_{\gamma} \perp \gamma : s_{\gamma} \subset F$  (in particular  $m_1(s_{\gamma} \cap F) = 1$ ).

Key: For some  $k$ , a large proportion of  $s_{\gamma}$  belongs to  $F_k$ . Pick  $\{a_k\}_{k=0}^{\infty}$  such that  $0 \leq a_k < 1$ ,  $\sum a_k = 1$ ,  $(a_k)$  does not tend to 0 too quickly, e.g.  $a_k = c_{\varepsilon} 2^{-k\varepsilon}$  (for sufficiently small  $\varepsilon$ ).

Claim:

$$\exists k : m_1(s_{\gamma} \cap F_k) \geq a_k.$$

Otherwise  $m_1(s_{\gamma} \cap F) \leq \sum_k m_1(s_{\gamma} \cap F_k) < \sum a_k = 1$ , contradicts (\*\*)

For this value of  $k$ ,

$$R_{2^{-k}}^*(\chi_{F_k^*})(\gamma) \geq a_k.$$

Since this choice of  $k$  depends on  $\gamma$ , let

$$E_k = \{\gamma \in S^1 : R_{2^{-k}}^*(\chi_{F_k^*})(\gamma) \geq a_k\}.$$

$S^1 = \bigcup_{k=1}^{\infty} E_k$ . Therefore  $\exists k' : |E_{k'}| \geq 2\pi a_{k'}$ .

$$2\pi a_{k'}^2 = 2\pi a_{k'} a_{k'} \leq \int_{E_{k'}} a_{k'} d\sigma \leq_{S^1} \int_{E_{k'}} R_{2^{-k'}}^*(\chi_{F_{k'}^*})(\gamma) d\sigma_{\gamma}$$

$$2^{-2k\varepsilon} \sim a_{k'}^2 \leq c(\log 2^{k'})^{\frac{1}{2}} |F_{k'}^*|^{\frac{1}{2}} \leq c(\log 2^{k'})^{\frac{1}{2}} N_{k'}^{\frac{1}{2}} 2^{-k'}$$

$\Rightarrow N_{k'} 2^{-\alpha k'} \geq c_{\alpha}$ , provided  $4\varepsilon < 2 - \alpha$ . □

### Construction of a Keakeya set I (Stein-Shakarchi, III)

Thinner Cantor set, always taking away the half.

Take two of them,  $E_0, E_1$ , where  $E_1$  has twice the length. Put  $E_0$  on  $y = 1$  and  $E_1$  on  $y = 0$ . Let  $F$  be the union of all line segments that join a point in  $E_0$  with one in  $E_1$ .

### Construction of an $\varepsilon$ -Keakeya set (Stein)

**Theorem.** Given  $\varepsilon > 0$ ,  $\exists N = N_{\varepsilon}$  and  $2^N$  rectangles  $R_1, \dots, R_{2^N}$  with side lengths  $1 \times 2^{-N}$  such that

(i)

$$\left| \bigcup_{j=1}^{2^N} R_j \right| < \varepsilon$$

(ii) the reaches  $\tilde{R}_j$  are mutually disjoint, i.e.

$$|\bigcup_{j=1}^{2^N} \tilde{R}_j| = 1$$

*Proof.* Fix  $\alpha \in (\frac{1}{2}, 1)$ . Symmetric triangle  $ABC$  with  $M$  opposite  $C$ . Push the right part into the left part and call the resulting body  $\Phi(T)$ . It consists of heart  $\Phi_h(T)$  and arms  $\Phi_a(T)$ . Then

$$|\Phi_h(T)| = \alpha^2 |T|$$

$$|\Phi_a(T)| = 2(1 - \alpha)^2 |T|$$

Conclusion

$$|\Phi(T)| = (\alpha^2 + 2(1 - \alpha)^2) |T|$$

$n$ -fold iteration (Peron trees): Split not into two but  $2^n$  parts and do everything pairwise. Key: right side of  $\Phi_h(A_0 A_2 C)$  // left side of  $\Phi_n(A_2 A_4 C)$  //  $CA_2$

Then look at heart/arms again.

$$|\text{arms of } \Psi_1(ABC)| \leq 2(1 - \alpha)^2 |T|.$$

$$|\text{heart of } \Psi_1(ABC)| = \alpha^2 |T|$$

$$\therefore |\Psi_1(ABC)| = (\alpha^2 + 2(1 - \alpha)^2) |T|.$$

Iterate: Carry out this process on the heart of  $\Psi_1(ABC)$  with  $n$  replaced by  $n - 1$ , given are the union of  $2^{n-1}$  triangles.

Then retranslate all  $2^n$  original triangles to obtain figure  $\Psi_2(ABC)$ .

$$|\text{heart of } \Psi_2(ABC)| = \alpha^2 \alpha^2 |T|$$

$$|\text{additional arms of } \Psi_2(ABC)| \leq 2(1 - \alpha)^2 \alpha^2 |T|$$

$$\begin{aligned} |\Psi_n(ABC)| &\leq (\alpha^{2n} + 2(1 - \alpha)^2 + 2(1 - \alpha)^2 \alpha^2 + \dots + 2(1 - \alpha)^2 \alpha^{2n-2}) \\ &\leq \alpha^{2n} + 2(1 - \alpha)^2 \underbrace{\sum_{n=0}^{\infty} \alpha^{2n}}_{= \frac{1}{1 - \alpha^2}} \\ &\leq \alpha^{2n} + 2(1 - \alpha) \end{aligned}$$

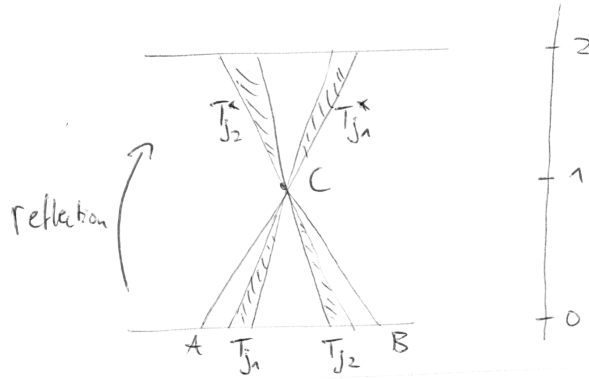


Figure 1: Obtaining mutually disjoint reaches by reflecting in  $C$ .

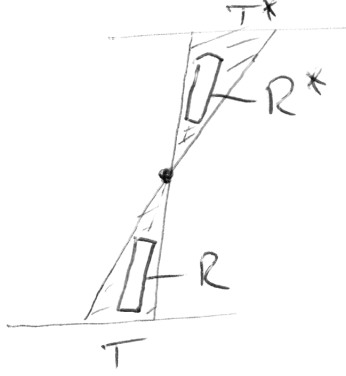


Figure 2: Go from triangles to rectangles by placing rectangles into the triangles with half the length.

□

**Application** Maximal functions and counterexamples. Q: Given a collection  $\mathcal{C} = \{C\}$  of sets, for which class of functions do we have

$$\lim_{\text{diam}(C) \rightarrow 0, c \in \mathcal{C}} \frac{1}{|C|} \int_C f(x-y) dy = f(x) \quad x - \text{a.e.}?$$

Seen:

$$(M_{\mathcal{C}} f)(x) = \sup_{c \in \mathcal{C}} \frac{1}{|C|} \int_C |f(x-y)| dy$$

$\mathcal{C} = \{\text{balls}\}$ , weak-type (1,1) inequality for  $M_{\mathcal{C}} \Rightarrow$  a.e. convergence of averages. A converse also holds!

$\{\mu_j\}_{j=1}^{\infty}$  collection of finite, nonnegative measures on  $\mathbb{R}^d : \text{supp}(\mu_j) \subset K \Subset \mathbb{R}^d$ . Define the maximal operator

$$(Mf)(x) = \sup_j |f * \mu_j|(x).$$

**Proposition.**  $1 \leq p < \infty$ . Assume for each  $f \in L^p(\mathbb{R}^d)$  that  $(Mf)(x) < \infty$  for some set of  $x$  having positive measure. Then  $f \mapsto Mf$  is of weak-type  $(p, p)$ , i.e.

$$\exists A < \infty : |\{x \in \mathbb{R}^d : (Mf)(x) > \alpha\}| \leq \frac{A}{\alpha^p} \|f\|_{L^p}^p \quad (\forall \alpha > 0)$$

**Lemma.**  $\{E_j\}$  collection of subsets of a fixed compact set:

$$\sum_{j=1}^{\infty} |E_j| = \infty.$$

Then there exists a sequence of translates  $F_j = E_j + x_j$ :

$$\limsup F_j = \bigcap_{k=1}^{\infty} \left( \bigcup_{j=k}^{\infty} F_j \right) = \mathbb{R}^n \quad (\text{a.e.})$$

The above set equals  $\{x \in \mathbb{R}^d : x \in F_j \text{ infinitely often}\}$ .

$$\liminf F_j = \bigcup_{k=1}^{\infty} \left( \bigcap_{j=k}^{\infty} F_j \right)$$

is a subset.

*Proof of Lemma.*  $Q \subset \mathbb{R}^d$  unit cube.  $A_1, A_2 \subset Q$ . Then  $\exists h \in \mathbb{R}^d : |A_1 \cap (A_2 - h)| \geq 2^{-d} |A_1| |A_2|$ . Why?

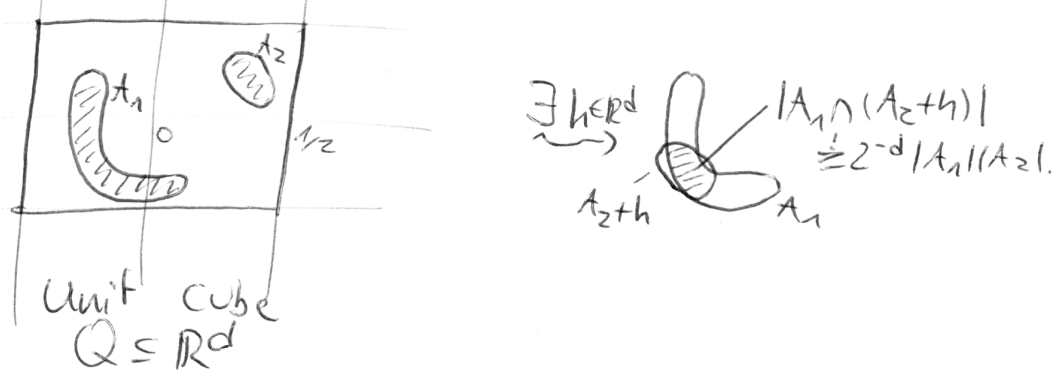


Figure 3: Translation of subsets  $A_1$  and  $A_2$  of a unit cube  $Q$ .

$$\eta(x) = \int_{\mathbb{R}^d} \chi_{A_1}(y) \chi_{A_2}(x+y) dy \sim \chi_{A_1} * \chi_{A_2}(x)$$

$$\int_{\mathbb{R}^d} |A_1| |A_2|$$

$$\text{supp}(\eta) \subset Q^*$$

$$|Q^*| = 2^d.$$

$$\exists h \in Q^* : \eta(h) \geq \text{avg}_{Q^*}(\eta) = \frac{1}{|Q^*|} \int_{\mathbb{R}^d} \eta = \frac{|A_1| |A_2|}{2^d}$$

Wlog  $\text{supp}(\eta) \subset Q$ .

Step 2: There exist translates  $F_j = E_j + x_j$  that cover  $Q$  at least once.

$$Q \subset \bigcup_j F_j$$

Why?  $F_1 = E_1$ . Suppose (inductively) that  $F_1, \dots, F_{j-1}$  have been constructed. Let  $A_1 = Q \cap (F_1 \cup \dots \cup F_{j-1})^c$  and  $A_2 = E_j$ . Step 1  $\Rightarrow \exists h : |A_1 \cap (A_2 - h)| \geq 2^{-d} |A_1| |A_2|$ . Set  $F_j = A_2 - h = E_j - h$ . Let  $p_j = |Q \cap (F_1 \cup \dots \cup F_j)|$ . Then

$$p_j = p_{j-1} + \underbrace{|Q \cap (F_1 \cup \dots \cup F_{j-1})^c|}_{A_1} \underbrace{|E_j - h|}_{A_2 - h} = p_{j-1} + |A_1 \cap (A_2 - h)|$$

$$\geq p_{j-1} + 2^{-d} |A_1| |E_j| = p_{j-1} + 2^{-d} (1 - p_{j-1}) |E_j|$$

$$\therefore p_j - p_{j-1} \geq 2^{-d} (1 - p_{j-1}) |E_j|$$

$$\sum_{j=2}^{\infty} (p_j - p_{j-1}) = \lim_{j \rightarrow \infty} p_j - p_1 = \lim_j p_j = 1$$

Step 3: Decompose (twice)  $\{E_j\}$  into a countable infinite number of subcollections so that on each subcollection the sum of the measures diverges.  $\square$

*Proof of Proposition.* Take a ball  $B$  such that  $B \supset Q + K$ .  $\text{supp}(F) \subset Q \Rightarrow \text{supp}(F * \mu_j) \subset \text{supp}(Mf) \subset B$ . Key: Estimate (\*) (the violation of the weak type estimate) holds if  $\text{supp}(f) \subset Q$ . For each  $k, \exists \alpha_k > 0 \exists g_k \in L^p : \text{supp}(g_k) \subset Q$  such that

$$|\{x \in B : M g_k(x) > \alpha_k\}| \geq \frac{2^k}{\alpha_k^p} \|g_k\|_{L^p}^p$$

Replace  $g_k$  by  $\tilde{g}_k = \frac{k}{\alpha_k} g_k$ .

$$\frac{2^k}{k^p} \leq \frac{|\{x \in B : M \tilde{g}_k(x) > k\}|}{\|\tilde{g}_k\|_{L^p}^p} \rightarrow \infty \quad \text{as } k \rightarrow \infty$$

$\therefore$  There exists a sequence  $\{f_k\} \subset L^p$  and a sequence of constants  $R_k \rightarrow \infty$  such that with  $E_k = \{x \in B : M f_k(x) > R_k\}$  we get

$$\sum_k |E_k| = \infty \quad \sum_k \|f_k\|_{L^p}^p < \infty.$$

*Remark.*  $d\mu_j \geq 0$  wlog  $f_k \geq 0$ .

By the lemma  $\exists \{x_k\}$  such that  $F_k = E_k + x_k$  satisfy  $\limsup F_k = \mathbb{R}^d$  (a.e). Let

$$\tilde{f}_k(x) = f_k(x + x_k), \quad F(x) = \sup_k \tilde{f}_k(x)$$

Then

$$M(F) = \sup_j |F * \mu_j| = \sup_j |(\sup_k \tilde{f}_k) * \mu_j| \geq \sup_k \sup_j |\tilde{f}_k * \mu_j| = \sup_k M(\tilde{f}_k)$$

Also  $M(\tilde{f}_k) > R_k$  on  $F_k$ .  $M(F) = \infty$  a.e. Check  $f \in L^p$ :

$$|F|^p = |\sup_k \tilde{f}_k|^p \leq \sum_p |\tilde{f}_k|^p$$

$$\|F\|_{L^p}^p \leq \sum_k \|f_k\|_{L^p}^p < \infty$$

Full conclusion

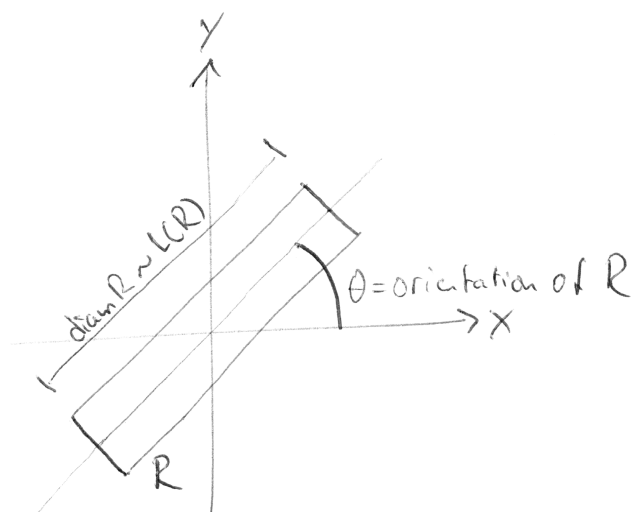
$$f = \sum f \chi_{Q_j} =: \sum f_j$$

$$M(f) \leq \sum_j M(f_j)$$

$\square$

**Example.** Rectangles with arbitrary orientation

$$\mathcal{C} = \mathcal{R} = \{\text{all rectangles in } \mathbb{R}^2 \text{ centered at } 0\}$$



**Corollary.** Given  $1 \leq p < \infty$ ,  $\exists f \in L^p(\mathbb{R}^n)$  such that

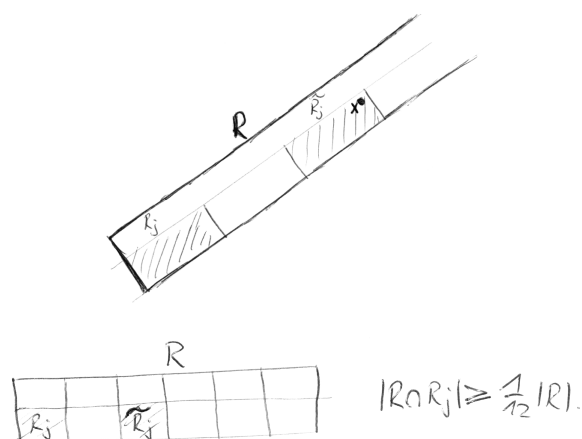
$$\limsup_{\text{diam}(R) \rightarrow 0, R \in \mathcal{R}} \frac{1}{|R|} \int_R f(x-y) dy = \infty \quad (x - \text{a.e.})$$

Idea: Use the  $\varepsilon$ -Kakeya set to show that  $M$  is not weak  $(p, p)$

$$(Mf)(x) = \sup_{\text{diam}(R) < 8} \frac{1}{|R|} \left| \int_R f(x-y) dy \right|$$

Let  $E = \bigcup_{j=1}^{2^N} R_j$  as before.  $\|\chi_E\|_{L^p}^p = |E| < \varepsilon$ . If  $x \in \tilde{R}_j$ , then  $\exists$  rectangle  $R$  such that

- $R$  is centered at  $x$
- $\text{diam}(R) \leq 8$
- $|R \cap R_j| \geq \frac{1}{12}|R|$





$$y \in x - E = -(E - x), \quad y - x \in -E, \quad x - y \in E$$

$$M(\chi_E)(x) \geq \int_{R-x} \chi_E(x-y) dy = \frac{|(R-x) \cap (E-x)|}{|R|} \geq \frac{1}{12}$$

Conclusion:  $M\chi_E \geq \frac{1}{12}$  on the set  $\bigcup_{j=1}^{2^N} \tilde{R}_j$  (of measure 1)

$$\forall A > 0 \exists \text{ set } E : |\{x \in \mathbb{R}^d : M\chi_E > A\}| \leq A\alpha^{-p} \|\chi_E\|_{L^p}^p$$

does not hold!  $\therefore M$  is not of weak typ  $(p, p)$ .

Note, that this is not the complete proof. Therefore still have to replace 8 by  $\delta$ .

**Bochner-Ries summability** Q: In which way does Fourier inversion hold? (In  $L^p(\mathbb{R}^d)$ )

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i x \xi} d\xi = \lim_{R \rightarrow \infty} \underbrace{\int_{|\xi| < R} \hat{f}(\xi) \left(1 - \frac{|\xi|^2}{R^2}\right)^\delta e^{2\pi i x \xi} d\xi}_{f * k_\delta} ?$$

for suitable  $\delta \geq 0$ .

$$\delta > \frac{d-1}{2} \Rightarrow k_\delta \in L^1(\mathbb{R}^d)$$

$$\delta \leq \frac{d-1}{2}$$

$$\delta = 0 \text{ today}$$

$d = 1$ : The second equality holds in  $L^p$ -norm ( $1 < p < \infty$ )  $\iff$   $L^p$  boundedness of Hilbert transform

It also holds a.e. ( $p = 2$  Carleson '66  $1 < p < \infty$  Hunt '68)

$d \geq 2$  Let us only consider norm convergence

$$f \mapsto (S^\delta f)(x) = \int_{|\xi| \leq 1} \hat{f}(\xi) (1 - |\xi|^2)^\delta e^{2\pi i x \xi} d\xi$$

$$Sf(x) = \int_{|\xi| \leq 1} \hat{f}(\xi) e^{2\pi i x \xi} d\xi$$

$$\widehat{Sf} = 1_{|\xi| \leq 1} \hat{f}$$

$$(\widehat{Hf} = i\pi \operatorname{sgn} \xi \hat{f})$$

$$\|Sf\|_{L^2(\mathbb{R}^d)} = \|\widehat{Sf}\|_{L^2} = \|1_{|\xi| \leq 1} \hat{f}\|_{L^2} \leq \|\hat{f}\|_{L^2} = \|f\|_{L^2}$$

He said something about every bounded operator can be written like this or so? Fourier multiplier  $S$  is bounded iff multiplier function is bounded.

**Theorem** (C.Fefferman '71 – Annals of mathematics "The multiplier problem for the ball").  
Suppose  $q \geq 2$  and  $p \neq 2$ . Then the operator  $S$  (initially defined on  $L^p \cap L^2$ ) is not extendable to a bounded operator from  $L^p(\mathbb{R}^d)$  to itself.

*Proof.* Let  $B \subset \mathbb{R}^d$  ball, let  $S_B$  be the multiplier operator associated to  $B$ :

$$\widehat{S_B f} = 1_B \hat{f}.$$

Given  $u \in S^{d-1} \subset \mathbb{R}^d$ , let  $S^u$  be the multiplier operator associated to the half-space with normal  $u$ :

$$\widehat{S^u f} = 1_{\{\xi u > 0\}} \hat{f} \quad (S^u f)(x) = \int_{\xi u > 0} \hat{f}(\xi) e^{2\pi i x \xi} d\xi$$

Upshot:  $L^p$ -bound for  $S$  implies an  $L^p$  vector-valued inequality for  $S_B$ s and  $S_u$ s.

**Lemma** (Y. Méyer). Suppose

$$\|Sf\|_{L^p} \leq A_p \|f\|_{L^p}$$

$f \in (L^2 \cap L^p)(\mathbb{R}^d)$  holds for some  $p \in [1, \infty]$ . Suppose  $f_1, \dots, f_M \in L^2 \cap L^p$ ,  $u_1, \dots, u_M \in S^{d-1} \subset \mathbb{R}^d$ . Then

$$\left\| \left( \sum_{j=1}^M |S^{u_j}(f_j)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^d)} \leq A_p \left\| \left( \sum_{j=1}^M |f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^d)} \quad (4)$$

where  $A_p$  is the same constant as above.

*Proof of Lemma.* Step 1:  $B = B_R$  = ball of radius  $R$  centered at 0. Then

$$\|S_B(f)\|_{L^p} \leq A_p \|f\|_{L^p} \quad (g \in L^2 \cap L^p) \quad (5)$$

Why? Scaling:

$$\delta_R(g)(x) = g\left(\frac{x}{R}\right)$$

Check  $\delta_{R^{-1}} \circ S \circ \delta_R = S_{B_R}$  since  $S = S_{B_1}$ .  $\hat{\delta}_\rho(g)(\xi) = R^d \hat{g}(R\xi)$ .

Step 2:  $M$  balls. ( $p < \infty$ )  $f = (f_1, \dots, f_M)$  given  $M$ -tuple of functions.  $T(f) := (Tf_1, \dots, Tf_M)$ . Given a unit vector  $\omega = (\omega_1, \dots, \omega_M) \in \mathbb{C}^M$ , let

$$S_\omega(f) = \sum_{j=1}^M \bar{\omega}_j S_B(f_j) = S_B\left(\sum_j \bar{\omega}_j f_j\right) = S_B(f_\omega) \quad f_\omega = \sum_{j=1}^M \bar{\omega}_j f_j$$

$$(5) \Rightarrow \int_{\mathbb{R}^d} |S_\omega f(x)|^p dx \leq A_p^k \int_{\mathbb{R}^d} |f_\omega(x)|^p dx \quad (6)$$

$$(S_\omega f)(x) = ?. \quad x, y \in \mathbb{C}^M, \quad \langle x, y \rangle = \sum_{i=1}^M x_i \bar{y}_i$$

$$\begin{aligned} S_\omega(f)(x) &= S_B(f_\omega)(x) = S_B\left(\sum_{j=1}^M \bar{\omega}_j f_j\right)(x) = \sum_{j=1}^M \bar{\omega}_j S_B(f_j)(x) = |\langle S_B(f)(x), \omega \rangle| \\ &= |S_B(f)(x)| \left| \left\langle \frac{S_B(f)(x)}{|S_B(f)(x)|}, \omega \right\rangle \right| = \left( \sum_{j=1}^M |S_B(f_j)(x)|^2 \right)^{\frac{1}{2}} |\varphi(\omega, S_B(f)(x))| \end{aligned}$$

Integrate both sides of (6) with respect to  $\omega$  (before integrating in  $x$ ).

$$\begin{aligned} \text{LHS} &= \int_{\mathbb{R}^d} \left( \int_{|\omega|=1} |S_\omega(f)(x)|^p d\omega \right) dx \\ &= \int_{\mathbb{R}^d} \left( \sum_{j=1}^M |S_B(f_j)(x)|^2 \right)^{\frac{p}{2}} \underbrace{\left( \int_{|\omega|=1} |\Phi(\omega, S_B(f)(x))|^p d\omega \right)}_{\gamma_p} dx \end{aligned}$$

$$0 \neq \gamma_p = \int_{|\omega|=1} |\Phi(\omega, 1)|^p d\omega$$

For fixed  $\nu \in S^{d-1}$   $\int_{S^{d-1}} |\langle \omega, \nu \rangle|^p d\sigma_\omega = \omega_{d-2} \int_{-1}^1 (1-t^2)^{\frac{d-3}{2}} dt$ .

Similarly,

$$\text{RHS} = \int_{\mathbb{R}^d} \left( \sum_{j=1}^M |f_j(x)|^2 \right)^{\frac{p}{2}} dx \gamma_p$$

$$(6) \Rightarrow \left\| \left( \sum_{j=1}^M |S_B(f_j)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^d)} \leq A_p \left\| \left( \sum_{j=1}^M |f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^d)}$$

Step 3: From balls to half-spaces

$B_R^u$  = ball of radius  $R$  centered at  $Ru$ . Upshot:  $B_R^u \rightarrow \{\xi u > 0\}$  as  $R \rightarrow \infty$ .

Translation

$$(T_y f)(x) = f(x - y)$$

$$\widehat{T_y f}(\xi) = e^{i\xi y} \hat{f}(\xi)$$

$$S_{B_R^u}(f)(x) = e^{2\pi i u R x} S_{B_R}(f e^{-2\pi i u R x})$$

(4) implies

$$\left\| \left( \sum |S_{B_R^{u_j}}(f_j)|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \leq A_p \left\| \left( \sum |f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^p}$$

Let  $R \rightarrow \infty$  to finish:

$$\therefore S_{B_R^{u_j}}(f_j) \rightarrow S^{u_j}(f_j) \quad R \rightarrow \infty \text{ (in } L^2)$$

$\therefore$  there exists an almost everywhere converging subsequence, done.  $\square$

$\widehat{Sf} = 1_{B(0,1)} \hat{f}$ .  $S$  is not bounded in  $L^p(\mathbb{R}^d)$  unless  $d = 1$  or  $p = 2$ . Focus on multiplier operator for the half-space  $(S^u)$ ,  $d = 1$ .

$$(S^+ f)(x) = \int_0^\infty \hat{f}(\xi) e^{2\pi i x \xi} d\xi \quad (f \in L^2)$$

$$|(S^+ f)(x)| \geq \frac{c}{|x|} \quad \text{if } |x| \geq \frac{1}{2} \tag{7}$$

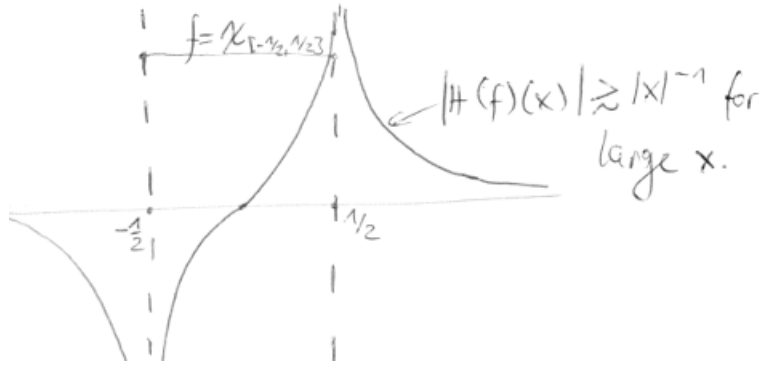
*Proof.*

$$(S^+ f)(x) = \lim_{\varepsilon \rightarrow 0} \int_0^\infty \hat{f}(\xi) e^{2\pi i (x+i\varepsilon)\xi} d\xi \quad \text{in } L^2$$

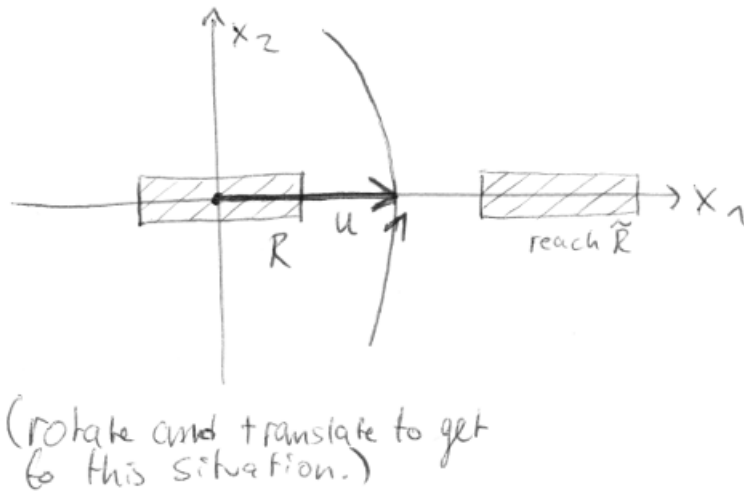
$$\int_{-\infty}^\infty \left( \int_0^\infty e^{-2\pi i y \xi} e^{2\pi i (x+i\varepsilon)\xi} d\xi \right) f(y) dy = \frac{1}{2\pi i} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{y - x - i\varepsilon} dy$$

This has absolute value  $\lesssim c|x|$  if  $|x| \geq \frac{1}{2}$ .

Alternative proof:  $|H(f)(x)| \geq |x|^{-1}$  for large  $x$ ,  $S^+ = \frac{1}{2}(I + iH)$ .



□



For  $R = (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2^{N+1}}, \frac{1}{2^{N+1}})$

$$1_R = 1_{(-\frac{1}{2}, \frac{1}{2})} \otimes 1_{(-2^{-(N+1)}, 2^{-(N+1)})}$$

If  $u$  points in the direction of  $x_1$  then

$$(S^u 1_R)(x_1, x_2) = (S^+ 1_{(-\frac{1}{2}, \frac{1}{2})})(x_1) 1_{(-2^{-(N+1)}, 2^{-(N+1)})}(x_2)$$

$$(7) \Rightarrow |S^u(1_R)| \geq c' 1_{\tilde{R}}$$

Similarly for any  $1 \times 2^{-N}$  rectangle  $R_j$ .  $u_j \in S^1$  in the positive direction of the longest side of  $R_j$ . Rotate and translate, then we get

$$|S^{u_j}(1_{R_j})| \geq c' 1_{\tilde{R}_j} \quad (8)$$

Take  $R_1, \dots, R_{2^N}$  to be the collection given by  $\varepsilon$ -Kakeya construction, plug that into result from lemma, get a contradiction.

Key:  $p < 2$  and  $d = 2$ .

Lemma with  $f_j = 1_{R_j}$  and  $M = 2^N$

$$\begin{aligned} c' &\leq \left\| \left( \sum_{j=1}^{2^N} |S^{u_j}(1_{R_j})|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)} \quad (8), \quad \left| \bigcup \tilde{R}_j \right| = 1 \\ &\leq A_p \left\| \left( \sum_{j=1}^{2^N} |1_{R_j}|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)} = A_p \underbrace{\left( \int_E \left( \sum_{j=1}^{2^N} |1_{R_j}|^2 \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}}}_I \end{aligned}$$

$$\begin{aligned} I &\leq |E|^{\frac{1}{q}} \left( \int \left( \sum |1_{R_j}|^2 \right)^{\frac{p}{2} \frac{2}{p}} dx \right)^{\frac{1}{p} \frac{p}{2}} \quad \text{Hölder} \\ &= |E|^{\frac{1}{q}} \sum_{j=1}^{2^N} |R_j| = |E|^{\frac{1}{q}} \end{aligned}$$

$$E = \bigcup_{j=1}^{2^N} R_j, \quad |E| < \varepsilon, \quad \frac{1}{q} + \frac{1}{p/2} = 1, \quad \frac{1}{q} = 1 - \frac{p}{2}, \quad \frac{1}{pq} = \frac{1}{p} \left(1 - \frac{p}{2}\right) > 0$$

In the end we get

$$c' \leq A_p \varepsilon^{\frac{1}{pq}}.$$

Let  $\varepsilon \rightarrow 0^+$  to finish.

$d > 2$ :

$$f_j \left( \underset{\in \mathbb{R}^d}{x} \right) = f_j(x_1, x_2, \underset{\in \mathbb{R}^{d-2}}{x'}) = 1_{R_j}(x_1, x_2) f(x') \quad f \in S(\mathbb{R}^{d-2})$$

$p > 2$ :

$$\langle Sf, g \rangle = \langle \widehat{Sf}, \hat{g} \rangle = \langle 1_B \hat{f}, \hat{g} \rangle = \langle \hat{f}, 1_B \hat{g} \rangle = \langle f, Sg \rangle \quad S = S^*$$

□

**Oscillatory integrals in harmonic analysis** Stein (VIII,IX), Stein-Shakarchi (Chapter 8), Sogge

- averaging operators
- restriction theory
- Bochner-Riesz summability

**Motivation (in  $\mathbb{R}^3$ )**

$$(Af)(x) = \int_{\mathbb{S}^2} f(x-y) d\sigma_y = \frac{1}{4\pi} f * \sigma(x)$$

$\sigma$  surface measure in  $\mathbb{S}^2$ .

Smoothing properties:

$$\left\| \frac{\partial}{\partial x_j} A(f) \right\|_{L^2(\mathbb{R}^3)} \lesssim \|f\|_{L^2(\mathbb{R}^3)} \quad (j = 1, 2, 3) \quad (9)$$

$f \in L^2 \Rightarrow$

$$\|f * \sigma\|_{L^2} \leq \|f\|_{L^2} \underbrace{|\sigma|(\mathbb{R}^3)}_{< \infty}$$

(use Minkowski integral inequality)  $\therefore f * \sigma \in L^2$  if  $f \in L^2$ .

Idea  $\widehat{(f * \sigma)} = \widehat{f} \widehat{\sigma}$

$$\widehat{\left(\frac{\partial}{\partial x_j} A(f)\right)}(\xi) = \xi_j \widehat{f}(\xi) \widehat{\sigma}(\xi),$$

so we should study  $\widehat{\sigma}$ .

without loss of generality  $\xi = (0, 0, |\xi|)$  because the integral is spherically symmetric.

$$\widehat{\sigma}(\xi) = \int_{\mathbb{S}^2} e^{-2\pi i \omega \xi} d\sigma_\omega = 2\pi \int_0^\pi e^{-2\pi i |\xi| \cos \theta} \sin \theta d\theta = 2\pi \int_{-1}^1 e^{2\pi i |\xi| t} dt = \frac{2 \sin(2\pi |\xi|)}{|\xi|}$$

$$t = \cos \theta \quad dt = -\sin \theta d\theta$$

$$\therefore |\widehat{\sigma}(\xi)| \lesssim (1 + |\xi|)^{-1}$$

(9) follows from this and Plancherel.

This also generalizes to higher dimensions. There we get a factor  $(1-t^2)^{\frac{d-3}{2}}$  in the last integral.

### Oscillatory integrals

$$I(\lambda) = \int_{\mathbb{R}^d} e^{i\lambda \phi(x)} \psi(x) dx$$

where  $\lambda \in \mathbb{R}$  is the oscillatory parameter,  $\phi \in C^\infty(\mathbb{R}^d, \mathbb{R})$  the phase and  $\psi(x) \in C$  the amplitude.

Q.: How does  $I(\lambda)$  behave for large  $|\lambda|$ ? General principle: Main contribution comes from the critical points of the phase,  $x_0 : \nabla \phi(x_0) = 0$ .

**Principle of non-stationary phase**  $\phi \in C^\infty, \psi \in C_0^\infty : |\nabla \phi(x)| > 0 (\forall x \in \text{supp } \psi)$ . Then for any  $N \in \mathbb{N}$

$$|I(\lambda)| \leq c_N |\lambda|^{-N}$$

*Proof.* involves integration by parts □

$d = 1$ :

$$I_1(\lambda) = \int_a^b e^{i\lambda \phi(x)} dx$$

$$0 < a < b < \infty \quad \psi(x) = \chi_{[a,b]}(x) \quad \text{which is rough!}$$

This means we will not get such a fast decay

**Lemma** (van der Corput (I)).  $\phi \in C^2, \phi'$  monotonic,  $|\phi'(x)| \geq 1 (\forall x \in [a, b])$ . Then

$$|I_1(\lambda)| \leq \frac{3}{|\lambda|} \quad (\forall \lambda > 0)$$

*Remark.* (i) 3 is neither important nor sharp; independence of  $a, b, \phi$  is the key!

(ii) Order of decrease in  $\lambda$  is sharp ( $\phi(x) = x : I_1(\lambda) = \frac{e^{i\lambda b} - e^{i\lambda a}}{i\lambda}$ )

(iii) monotonicity of  $\phi'$  is essential

*Proof.* Integrate by parts(...) □

What if critical points are present? ( $d = 1$ )  
 $x_0 : \phi'(x_0) = 0$  (critical point) and  $\phi''(x_0) \neq 0$  (non degenerate), e.g.  $\phi(x) = x^2$ ,  $x_0 = 0$ . In this case

$$\int_{\mathbb{R}} e^{i\lambda x^2} \psi(x) dx = c_0 \lambda^{-\frac{1}{2}} + \mathcal{O}(|\lambda|^{-\frac{3}{2}}) = \sum_{k=0}^N a_k \lambda^{-\frac{1}{2}-k} + \mathcal{O}(|\lambda|^{-\frac{3}{2}-N}) \quad (\forall N, \lambda \rightarrow \infty)$$

**Lemma** (van der Corput (II)).  $\phi \in C^2[a, b]$ ,  $|\phi''(x)| \geq 1$  ( $\forall x \in [a, b]$ ). Then

$$|I_1(\lambda)| \leq \frac{8}{\lambda^{\frac{1}{2}}} \quad (\forall \lambda > 0)$$

*Remark.* More generally:  $\mathcal{O}(|\lambda|^{\frac{1}{k}})$  if  $|\phi^{(k)}| \geq 1$ .

*Proof.* Integration by parts not needed. Instead split up region in small area around critical point with properly chosen size, and rest, and then use results from above.  $\square$

**Corollary.** Same assumptions as van der Corput (II).  $\psi \in C^1[a, b]$ .

$$|\int e^{i\lambda \phi(x)} \psi(x) dx| \leq c_\psi \lambda^{-\frac{1}{2}}$$

Application: Asymptotics of Bessel functions

$$J_m(r) = \frac{1}{2\pi} \int_0^{2\pi} e^{ir \sin x} e^{-imx} dx \quad \phi(x) = \sin x, \quad \psi(x) = e^{-imx} \quad (m \in \mathbb{Z})$$

**Corollary.**

$$|J_m(r)| \leq cr^{-\frac{1}{2}} \quad r \rightarrow \infty$$

Recall: Averaging operator in  $\mathbb{R}^d$  ( $d > 1$ ) is

$$(Af)(x) = (f * \sigma)(x) \quad \sigma \text{ surface measure on } \mathbb{S}^{d-1}$$

**Theorem.**  $f \mapsto A(f)$  is bounded from  $L^2(\mathbb{R}^d)$  to  $L_k^2(\mathbb{R}^d)$  with  $= \frac{d-1}{2}$ .

*Proof.*

$$\begin{aligned} \hat{\sigma}(\xi) &= 2\pi |\xi|^{-\frac{d}{2}+1} \underbrace{J_{\frac{d}{2}-1}(2\pi|\xi|)}_{=\mathcal{O}(|\xi|^{-\frac{1}{2}}), |\xi| \rightarrow \infty} \\ \therefore |\hat{\sigma}(\xi)| &= \mathcal{O}(|\xi|^{-\frac{d-1}{2}}) \quad |\xi| \rightarrow \infty \end{aligned}$$

$\square$

"What is van der Corput's lemma in higher dimension?" (Carbery-Wright, 2000)

$$I(\lambda) = \int_{\mathbb{R}^d} e^{i\lambda \phi(x)} \psi(x) dx$$

( $\phi$  smooth,  $\psi$  smooth, compactly supported)  
 nondegeneracy hypothesis

$$\det(\nabla^2 \phi)(x) \neq 0 \quad \forall x \in \text{supp}(\psi)$$

**Theorem.** Under above assumptions

$$|I(\lambda)| = \mathcal{O}(|\lambda|^{-\frac{1}{2}}) \quad \lambda \rightarrow \infty$$

*Remark.* (i) Decay rate is sharp

(ii) Proof uses  $TT^*$  method:  $|I(\lambda)|^2 = I(\lambda)\overline{I(\lambda)}$

(iii) variant:  $\text{rk}(\nabla^2\phi) \geq m$  for some  $0 < m \leq d$  on  $\text{supp}(\psi)$ . Then

$$|I(\lambda)| = \mathcal{O}(|\lambda|^{-\frac{m}{2}})$$

**Application:** Fourier transform of surface-carried measures

Recall:  $(\mathbb{S}^{d-1}, \sigma)$

$$|\hat{\sigma}(\xi)| \lesssim (1 + |\xi|)^{-\frac{d-1}{2}}$$

(not a Bessel coincidence)

(local)  $C^\infty$ -hypersurface  $M$ . After translation and rotation  $x_0 = 0$ ,  $T_{x_0}M = \{x_d = 0\}$ .  $M$  can be represented as

$$M = \{(x', x_d) \in B \subset \mathbb{R}^d : x_d = \varphi(x')\}$$

Can arrange  $\varphi(0) = 0 = (\nabla_{x'}\varphi)(x')|_{x'=0}$ .

$$\varphi(x') = \frac{1}{2} \sum_{k,j=1}^{d-1} \underbrace{\frac{\partial^2 \varphi}{\partial x_k \partial x_j}}_{(a_{jk})} x_k x_j + \mathcal{O}(|x'|^3) = \frac{1}{2} \sum_{j=0}^{d-1} k_j x_j^2 + \mathcal{O}(|x'|^3)$$

$(a_{jk})$   $(d-1) \times (d-1)$   $\mathbb{R}$ -valued symmetric matrix  $\therefore$  diagonalizable.  $k_j$  principal curvatures of  $M$  at  $x_0$ .  $k := \prod_{j=1}^{d-1} k_j$  is the Gaussian curvature of  $M$  at  $x_0$  ( $k = \det(\nabla^2\varphi)$ )

E.g.

(i)  $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ .  $k_j = 1$  ( $\forall j$ )  $\therefore k = 1$

(ii)  $\{x_3 = \underbrace{x_1^2 - x_2^2}_{\varphi(x_1, x_2)}\} \subset \mathbb{R}^3$ ,  $\frac{1}{2}\nabla^2\varphi(x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(iii)  $\{x_1^2 = |x'|^2 : x \neq 0\}$ ,  $x' \in \mathbb{R}^{d-1}$ .  $d-2$  identical nonvanishing principal curvatures  $x_d^{-2} + 1$  vanishing principal curvature.

**surface measure  $\sigma$**

$$\int_M f d\sigma = \int_{\mathbb{R}^{d-1}} f(x', \varphi(x')) \underbrace{\sqrt{1 + |\nabla_{x'}\varphi(x')|^2} dx'}_{d\sigma \text{ in our coordinate sys.}}$$

$$d\mu = \psi d\sigma, \quad \psi \in C_0^\infty(M, \sigma)$$

is a surface carried measure.

$$\hat{\mu}(\xi) = \int_M e^{-2\pi i x \xi} d\mu(x) = \int_M e^{-2\pi i x \xi} \psi(x) d\sigma_x$$

is bounded on  $\mathbb{R}^d$  because  $|\mu|(\mathbb{R}^d) < \infty$ .



**Theorem.** Hypersurface  $M \subset \mathbb{R}^d$  with nonvanishing Gaussian curvature at each point of  $\text{supp}(\psi)$ . Then

$$|\hat{\mu}(\xi)| = \mathcal{O}(|\xi|^{-\frac{d-1}{2}}), \quad |\xi| \rightarrow \infty$$

**Corollary.** If  $M$  has at last  $m$  non vanishing principal curvatures (at each point of  $\text{supp}(\psi)$ ), then

$$|\hat{\mu}(\xi)| = \mathcal{O}(|\xi|^{-\frac{m}{2}}), \quad |\xi| \rightarrow \infty$$

Last time: oscillatory integrals and averaging operators

$$(Af)(x) = (F * \sigma)(x) = \int_{\mathbb{S}^{d-1}} f(x-y) d\sigma_y \quad (d > 1)$$

Smoothin property:  $f \mapsto A(f)$  is bounded from  $L^2(\mathbb{R}^d)$  to  $L_k^2(\mathbb{R}^d)$  with  $k = \frac{d-1}{2}$ . Here we used

$$|\hat{\sigma}(\xi)| \lesssim |\xi|^{-\frac{d-1}{2}}.$$

A few weeks ago:

$$R(f)(t, \gamma) = \int_{P_{t, \gamma}} f$$

where  $P_{t, \gamma} = \{x \in \mathbb{R}^d : x\gamma = t\}$ .

$$R^*(f)(\gamma) = \sup_{t \in \mathbb{R}} |R(f)(t, \gamma)|$$

if  $d \geq 3$  then

$$\int_{\mathbb{S}^{d-1}} R^*(f)(\gamma) d\sigma_\gamma \lesssim \|f\|_{L^2} + \|f\|_{L^2}$$

This estimate was based on

$$\int_{\mathbb{S}^{d-1}} \int_{-\infty}^{\infty} |\hat{R}(f)(\lambda, \gamma)|^2 |\lambda|^{d-1} d\lambda d\sigma_\gamma = 2 \int_{\mathbb{R}^d} |f(x)|^2 dx$$

due to  $\hat{R}(f)(\lambda, \gamma) = \hat{f}(\lambda\gamma)$ . Consider  $d = 3$  then this becomes

$$\int_{\mathbb{S}^2} \int_{\mathbb{R}} \left| \frac{d}{dt} R(f)(t, \gamma) \right|^2 dt d\sigma_\gamma = 8\pi^2 \int_{\mathbb{R}^3} |f(x)|^2 dx \quad (10)$$

by Plancherel  $t \leftrightarrow \lambda$ . Note, that something like this also holds for higher dimensions.

Now consider the following "linearized" version of the Radon transform:

$$R_B(f) = \int_{\mathbb{R}^{d-1}} f(y', x_d - B(x', y')) dy' = \int_{M_x} f$$

where  $x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$ ,  $y = (y', y_d)$  and  $B : \mathbb{R}^{d-1} \times \mathbb{R}^{d-1} \rightarrow \mathbb{R}$  is a nondegenerate bilinear form, and

$$M_x = \{(y', y_d) \mid y_d = x_d - B(x', y')\}.$$

E.g.  $B(x', y') = \langle x', y' \rangle$  (usual inner product on  $\mathbb{R}^d$ ).  $y_d = x_d - \langle x', y' \rangle \iff \langle x', y' \rangle + y_d = x_d \iff \langle (x', 1), (y', y_d) \rangle = x_d$ . The map

$$\begin{aligned} \mathbb{R}^d &\rightarrow \{\text{affine hyperplanes on } \mathbb{R}^d\} \\ x &\mapsto M_x \end{aligned}$$

is injective and surjective onto {hyperplanes not orthogonal to  $M_0 = \{x_d = 0\}$ }. The excerpted collection of hyperplanes is lower dimensional, so we can think of  $R_B$  as a substitute for  $R$ .

An analogue of (10) is

$$\int_{\mathbb{R}^d} \left| \frac{\partial}{\partial x_3} R_B(f)(x) \right|^2 dx = c_B \int_{\mathbb{R}^3} |f(x)|^2 dx$$

for  $f \in C_0^0(\mathbb{R})$

*Proof.*

$$\begin{aligned} \int_{\mathbb{R}^3} \left| \frac{\partial}{\partial x_3} R_B(f)(x) \right|^2 dx &= \int_{\mathbb{R}} \int_{\mathbb{R}^2} \left| \underbrace{\left( \frac{\partial}{\partial x_3} R_B(f) \right)^\wedge(x', \xi_3)}_{=2\pi\xi_3 \int_{\mathbb{R}^2} e^{-2\pi i \xi_3 B(x', y')} \hat{f}(y', \xi_3) dy'} \right|^2 dx' d\xi_3 \quad x = (x', x_3) \end{aligned}$$

The last equality follows from

$$\begin{aligned} \hat{R}_B(f)(x', \xi_3) &= \int_{\mathbb{R}} e^{-2\pi i \xi_3 x_3} R_B(f)(x', x_3) dx_3 \\ &= \int_{\mathbb{R}^2} e^{-2\pi i \xi_3 x_3} \int_{\mathbb{R}} f(y', \underbrace{x_3 - B(x', y')}_{y_3}) dy' dx_3 \\ &= \int_{\mathbb{R}^{2+1}} e^{-2\pi i \xi_3 (y_3 + B(x', y'))} f(y', y_3) dy' dy_3 \\ &= \int_{\mathbb{R}^2} e^{-2\pi i \xi_3 B(x', y')} \underbrace{\left( \int_{\mathbb{R}} e^{-2\pi i \xi_3 y_3} f(y', y_3) dy_3 \right)}_{\hat{f}(y', \xi_3)} dy' \end{aligned}$$

Since  $B$  is nondegenetare  $\exists C : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  linear, invertible such that  $B(x', y') = \langle C(x'), y' \rangle$ . Changle variables  $\xi_3 C(x') = u \in \mathbb{R}^2$  is well defined since  $C$  is invertible.  $\therefore \xi_3 B(x', y') = \langle \xi_3 C(x'), y' \rangle = \langle u, y' \rangle \therefore \xi_3^2 |\det C| dx' = du$ . Then the first integral becomes

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} e^{-2\pi i u y'} \hat{f}(y', \xi_3) dy' \right|^2 \frac{du}{|\det C|} d\xi_3 \simeq \int \int |\hat{f}(y', \xi_3)|^2 dy' d\xi_3 \simeq \int_{\mathbb{R}^3} |f(y)|^2 dy$$

by 2D-Plancherel  $y' \leftrightarrow u$  and 1D-Plancherel  $\xi_3 \leftrightarrow y_3$ .  $\square$

**Rotational curvature** Both the averaging operator  $A$  and the Radon transform  $R_B$  are of the form  $f \mapsto \int_{M_x} f(y) d_x(y)$ , where for each  $x \in \mathbb{R}^d$  we have a manifold  $M_x$  (depending smoothly on  $x$ ) over which we integrate.

$A :$   $M_x = x + M_0$   $M_0$  curved

$R_B :$   $M_x = \{y = (y', y_d) \mid y' \in \mathbb{R}^{d-1}, y_d = x_d - B(x', y')\}$  flat but  $M_x$  rotates as  $x$  varies.

Start with a smooth "double defining" function  $\rho = \rho(x, y)$  given in a ball in  $\mathbb{R}^d \times \mathbb{R}^d$ . Its rotational matrix is

$$M = M(\rho) = \begin{pmatrix} \rho & \frac{\partial \rho}{\partial y_1} & \cdots & \frac{\partial \rho}{\partial y_d} \\ \frac{\partial \rho}{\partial x_1} & & & \\ \vdots & \left( \frac{\partial^2 \rho}{\partial x_j \partial y_k} \right)_{j,k=1}^d & & \\ \frac{\partial \rho}{\partial x_d} & & & \end{pmatrix}$$

containing the mixed Hessian. The rotational curvature of  $\rho$  is

$$\text{rotcurv}(\rho) := \det(M(\rho))$$

We want  $\rho = 0 \Rightarrow \text{rotcurv}(\rho) \neq 0$ .  $M_x = \{y : \rho(x, y) = 0\}$ . The fact that  $\nabla_y \rho \neq 0$  if  $\rho = 0$  implies that  $M_x$  is a smooth surface (or something like that).

Examples/Properties:

- (i) Translational invariant case:  $\rho(x, y) = \rho(x - y)$ .  $M_x = M_0 + x$  and  $\text{rotcurv}(\rho) \neq 0$  iff  $M_0$  has nonvanishing Gaussian curvature.
- (ii) Case of  $R_B$ :  $\rho(x, y) = y_d - x_d + B(x', y')$ .  $\text{rotcurv}(\rho) \neq 0$  iff  $B$  is nondegenerate.
- (iii)  $\tilde{\rho}(x, y) = a(x, y)\rho(x, y)$  with  $a(x, y) \neq 0$ . Then  $\tilde{\rho}$  is another defining function for  $\{M_x\}$ , and  $\text{rotcurv}(\tilde{\rho}) = a^{d+1} \text{rotcurv}(\rho)$
- (iv)  $x \mapsto \psi_1(x)$ ,  $y \mapsto \psi_2(y)$  local diffeomorphisms of  $\mathbb{R}^d$ . For  $\tilde{\rho}(x, y) = \rho(\psi_1(x), \psi_2(y))$  then  $\text{rotcurv}(\tilde{\rho}) = J_1(x)J_2(y) \text{rotcurv}(\rho)$  with  $J_k = \det \text{jac}(\psi_k)$ ,  $k = 1, 2$

Define the general averaging operator  $A$  by

$$A(f)(x) = \int_{M_x} f(y) \psi_0(x, y) d\sigma_x(y)$$

initially for  $f \in C_0^0(\mathbb{R}^d)$ .  $M_x = \{y \mid \rho(x, y) = 0\}$  with induced Lebesgue measure  $d\sigma_x$ .  $\rho$  is a double defining function with  $\text{rotcurv}(\rho) \neq 0$ .  $\psi_0 \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ .

**Theorem.** The operator  $A$  extends to a bounded linear map from  $L^2(\mathbb{R}^d)$  to  $L_k^2(\mathbb{R}^d)$  where  $k = \frac{d-1}{2}$ .

*Proof.* Step 1: Oscillatory integral operators (FIOs)

Step 2:  $L^2$  estimate via dyadic decomposition of "almost-orthogonal" parts.

Step 1: Define

$$T_\lambda(f)(x) = \int_{\mathbb{R}^d} e^{i\lambda\varphi(x, y)} \psi(x, y) f(y) dy$$

where  $\varphi \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$  and  $\psi \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$  with

$$\det(\nabla_{x, y}^2 \varphi) = \det\left(\frac{\partial^2 \varphi}{\partial x_k \partial y_j}\right)_{k, j=1}^d \neq 0 \quad \text{on } \text{supp}(\psi)$$

last week:

$$I(\lambda) = \int_{\mathbb{R}^d} e^{i\lambda\varphi(y)} \psi(y) dy \Rightarrow |I(\lambda)| \lesssim |\lambda|^{-\frac{d}{2}}$$

if  $\det \nabla^2 \varphi \neq 0$  on  $\text{supp}(\psi)$ . We used  $|I(\lambda)|^2 = I(\lambda) \overline{I(\lambda)}$  where then appeared the term  $\varphi(u + y) - \varphi(y)$ .

**Proposition.** Under the above assumptions,

$$\|T_\lambda\|_{L^2 \rightarrow L^2} \leq c\lambda^{-\frac{d}{2}} \quad \forall \lambda > 0$$

*Proof.* Similar to its scalar version, omitted. □

Consequence: For the corresponding oscillatory integral operator involving  $\rho$

$$S_\lambda(f)(x) = \int_{\mathbb{R} \times \mathbb{R}^d} e^{i\lambda y_0 \rho(x,y)} \psi(x, y_0, y) f(y) dy_0 dy$$

with  $(y_0, y) \in \mathbb{R} \times \mathbb{R}^d$  and  $\psi \in C_0^\infty$  is supported away from  $y_0 = 0$ .

**Corollary.** If  $\rho = 0 \Rightarrow \text{rotcurv}(\rho) \neq 0$  then

$$\|S_\lambda\|_{L^2 \rightarrow L^2} \leq c\lambda^{-\frac{d+1}{2}}$$

*Proof of Corollary.*  $\bar{x} = (x_0, x), \bar{y} = (y_0, y) \in \mathbb{R} \times \mathbb{R}^d$ . Set  $\varphi(\bar{x}, \bar{y}) = x_0 y_0 \rho(x, y)$  then  $\det(\nabla_{x,y}^2 \varphi) = (x_0 y_0)^{d+1} \text{rotcurv}(\rho)$ . Define

$$F_\lambda(x_0, x) = F_\lambda(\bar{x}) = \int_{\mathbb{R}^{d+1}} e^{i\lambda \varphi(\bar{x}, \bar{y})} \psi_1(x_0, x, y_0, y) f(y) dy_0 dy$$

with  $\psi_1(1, \lambda, y_0, y) = \psi(x, y_0, y)$  (from  $S_\lambda$ ). Then  $S_\lambda(f)(x) = F_\lambda(1, x)$ .

Observation: If  $I \subset \mathbb{R}$  interval of length 1,  $g \in C^1(I)$ ,  $x_0 \in I$  then

$$|g(u_0)|^2 \leq 2 \left( \int_I |g(u)|^2 du + \int_I |g'(u)|^2 du \right)$$

Apply this observation with  $I = [1, 2]$ ,  $u_0 = 1$ ,  $g(u) = F_\lambda(u, x)$  to get:

$$\int_{\mathbb{R}^d} |S_\lambda(f)(x)|^2 dx \leq 2 \left( \int |F_\lambda(x_0, x)|^2 dx + \int \left| \frac{\partial}{\partial x_0} F_\lambda(x_0, x) \right|^2 dx_0 dx \right)$$

The first integral is  $\lesssim \lambda^{-(d+1)} \|f\|_{L^2}^2$  by Proposition with  $\mathbb{R}^{d+1}$  instead of  $\mathbb{R}^d$ .

$$\frac{\partial}{\partial x_0} (e^{i\lambda x_0 y_0 \rho(x,y)}) = \frac{\partial}{\partial y_0} (e^{i\lambda x_0 y_0 \rho(x_0, y_0)}) \frac{y_0}{x_0}$$

we integrate by parts somewhere and use our form of  $\varphi$ . Therefore the second summand also satisfies the desired estimate.  $\square$

Step 2: Dyadic decomposition of  $A$ .

Co-area formula (see Evans-Cariery (?), Stein-Shakarchi IV, Exercice 8, Ch. 8): Fix  $h \in S(\mathbb{R})$  such that  $\int h = 1$ ,  $M = \{x \in \mathbb{R}^d \mid \rho(x) = 0\}$ . Then

$$\int_M f \frac{d\sigma}{|\nabla \rho|} = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{\mathbb{R}^d} h\left(\frac{\rho(x)}{\varepsilon}\right) f(x) dx.$$

E.g.  $\rho(x) = |x| - 1$ ,  $M = \mathbb{S}^{d-1}$ ,  $\frac{\nabla}{\rho}(x) = x|x|, \therefore |\nabla \rho| = 1$

$$\int_{\mathbb{S}^{d-1}} f d\varepsilon = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{\mathbb{R}^d} h\left(\frac{|x| - 1}{\varepsilon}\right) F(x) dx$$

where  $f = F|_{\mathbb{S}^{d-1}}$ .

$$A(f)(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\mathbb{R}^d} h(\rho(x, y)\varepsilon) \psi(x, y) f(y) dy$$

where  $\psi(x, y) = \psi_0(x, y)|\nabla_y \rho| \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ . Let  $\gamma \in C_0^\infty(\mathbb{R})$  such that  $\gamma = 1$  on  $[-\frac{1}{2}, \frac{1}{2}]$  and 0 on  $[-1, 1]^c$ . Let  $h = \hat{\gamma}$ . Then

$$h(\rho) = \int_{\mathbb{R}} e^{2\pi i \xi u} \gamma(u) du.$$

Then

$$\int h = \int \hat{\gamma} = \gamma(0) = 1$$

and

$$\int_{\mathbb{R}} e^{2\pi i u \rho} \gamma(\varepsilon u) du = (\delta_\varepsilon \gamma)^\vee(\rho) = \varepsilon^{-1} \gamma^\vee(\varepsilon^{-1} \rho) = \varepsilon^{-1} h(\varepsilon^{-1} \rho)$$

where  $\delta_\varepsilon \gamma(u) = \gamma(\varepsilon u)$ . Choose  $\varepsilon = 2^{-r}$ ,  $r \in \mathbb{N}$ . Note

$$\gamma(2^{-r} u) = \gamma(u) + \sum_{k=1}^r (\gamma(\frac{u}{2^k}) - \gamma(\frac{u}{2^{k-1}}))$$

Let  $r \rightarrow \infty$  to get

$$1 = \gamma(u) + \sum_{k=1}^{\infty} \eta(\frac{u}{2^k})$$

where  $\eta(\cdot) = \gamma(\cdot) - \gamma(2\cdot)$  because  $\gamma$  is continuous. Then  $\eta \in C_0^\infty(\mathbb{R})$ ,  $\text{supp}(\eta) \subset \{\frac{1}{4} \leq |u| \leq 1\}$ . Whenever  $f$  is continuous we get by Fourier inversion that

$$\begin{aligned} A(f)(x) &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\mathbb{R}^d} \int_{\mathbb{R}} e^{2\pi i u \frac{\rho(x, y)}{\varepsilon}} \gamma(u) \psi(x, y) f(y) du dy \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^d} \int_{\mathbb{R}} e^{2\pi i u \rho(x, y) \gamma(\varepsilon u)} \psi(x, y) f(y) du dy \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}} e^{2\pi i u \rho(x, y) \gamma(u)} \psi(x, y) f(y) du dy \\ &= \sum_{k=1}^{\infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}} e^{2\pi i u \rho(x, y) \eta(\frac{u}{2^k})} \psi(x, y) f(y) du dy \end{aligned}$$

since  $\gamma(\varepsilon u) \rightarrow \gamma(0) = 1$   $\varepsilon \rightarrow 0$ . Call the summands  $A_k(f)(x)$ . Properties of  $A_k$ :

(i)  $f \in L^2(\mathbb{R}^d) \Rightarrow$

$$A_k(f) \in C_0^\infty(\mathbb{R}^d)$$

(ii)

$$\|A_k(f)\|_{L^2} \leq c 2^{-k(\frac{d-1}{2})} \|f\|_{L^2}.$$

Recall  $\|S_\lambda\|_{L^2 \rightarrow L^2} \lesssim \lambda^{-\frac{d+1}{2}i}$  and change variables in the definition of  $A_k(f)$ .

(iii)  $\exists m : |j - k| \geq m \forall N$

$$\|(A_k^* A_j)(f)\|_{L^1} \lesssim_N 2^{-N \max(k, j)} \|f\|_{L^2}.$$

Similarly for  $A_k A_j^*$ . For the proof, invoke nonstationary phase. Also, recall  $|I(\lambda)|^2 = I(\lambda) \overline{I(\lambda)}$ .

(iv)  $A_k^{(\alpha)} = (\frac{\partial}{\partial x})^\alpha A_k$ . Then

$$\|A_k^{(\alpha)}\|_{L^2 \rightarrow L^2} \lesssim 2^{k|\alpha|} 2^{-k(\frac{d-1}{2})}$$

and

$$\|A_k^{(\alpha)} (A_j^{(\alpha)})^*\|_{L^2 \rightarrow L^2} \lesssim_{\alpha, N} 2^{-N \max(k, j)}.$$

Step 3: Almost-orthogonality

Assume that  $\{T_k\}_{k=1}^\infty$  is a sequence of bounded operators on  $L^2(\mathbb{R}^d)$  and that  $\{a(k)\}_{k \in \mathbb{Z}}$  are positive constants with

$$A = \sum_{k \in \mathbb{Z}} a(k) < \infty.$$

**Lemma** (Cotlar-Knapp-Stein). Assume for  $\|T_k T_j^*\|_{L^2 \rightarrow L^2}$  that  $\|T_k^* T_j\| \leq a(k-j)^2$ . Then, for every  $r$ ,

$$\left\| \sum_{k=0}^r T_k \right\| \leq A.$$

Note, that the bound  $A$  is independent of  $r$ .

Write  $T = \sum_{k=0}^r T_k$ . Recall  $\|T\|^2 = \|T^* T\|$  since  $\|AB\| \leq \|A\| \|B\|$  and  $\|Tx\|^2 = \langle Tx, Tx \rangle = \langle x, T^* T x \rangle \leq \|T^* T\| \|x\|^2$  and plug in an extremizing sequence of  $\|Tx\|$  for  $x$ . Then  $\|T\|^4 = (\|T\|^2)^2 = \|T^* T\|^2 = \|(T^* T)^2\|$  since  $T^* T$  is self adjoint. By induction we get

$$\|T\|^{2n} = \|(T^* T)^n\|.$$

$$(T^* T)^n = \sum_{i_1, i_2, \dots, i_{2n}} (T_{i_1} T_{i_2}^* \dots T_{i_{2n-1}} T_{i_{2n}}^*)$$

(i)

$$\|(T_{i_1} T_{i_2}^*) \dots (T_{i_{2n-1}} T_{i_{2n}}^*)\| \leq a(i_1 - i_2)^2 a(i_3 - i_4)^2 \dots a(i_{2n-1} - i_{2n})^2$$

(ii)

$$\|T_{i_1} (T_{i_2} T_{i_3}^*) \dots (T_{i_{2n-2}} T_{i_{2n-1}}^*) T_{i_{2n}}\| \leq A a(i_2 - i_3)^2 a(i_4 - i_5)^2 \dots a(i_{2n-2} - i_{2n-1})^2 A$$

Take geometric mean of (i) and (ii) and get

$$\|T_{i_1} T_{i_2} \dots T_{i_{2n-1}} T_{i_{2n}}^*\| \leq A a(i_1 - i_2) a(i_2 - i_3) \dots a(i_{2n-1} - i_{2n})$$

Now sum the whole thing in  $i_1, i_2, \dots, i_{2n-1}$ . Then sum by sum, each of the factors turns into an  $A$ . In the end the sum in  $i_{2n}$  gives a factor  $r+1$ . So,

$$\sum_{i_1, i_2, \dots, i_{2n}} \|T_{i_1} T_{i_2}^* \dots T_{i_{2n-1}} T_{i_{2n}}^*\| \leq A^{2n} (r+1)$$

$$\therefore \|T\| \leq A(1+r)^{\frac{1}{n}} \rightarrow A \quad n \rightarrow \infty$$

(This is called 'Tensor power trick')

Putting everything together: Case 1:  $d$  odd ( $\therefore \frac{d-1}{2} \in \mathbb{Z}$ ). ETS  $\forall |\alpha| \leq \frac{d-1}{2} \quad \forall f \in L^2(\mathbb{R}^d)$

- $\partial_x^\alpha A(f)$  exists (in the sense of distributions) and is an  $L^2$  function
- $f \mapsto \partial_x^\alpha A(f)$  is bounded on  $L^2$ .

For each  $r$ , set

$$\partial_x^\alpha \sum_{k=0}^r =: \sum_{k=0}^r T_k \quad (T_k = A_k^{(\alpha)})$$

The estimates 1 and 2 imply that the hypotheses of CKS are satisfied with  $a(k) = c_n 2^{-|k|N}$  ( $\forall N, \therefore$  can choose  $N = 1$ ).

$$\therefore \|\partial_x^\alpha \sum_{k=0}^r A_k(f)\|_{L^2} \leq A \|f\|_{L^2}$$

where  $A = \sum_{k \in \mathbb{Z}} a(k)$ , provided  $|\alpha| \leq \frac{d-1}{2}$ .

(i) with  $\alpha = 0$  we get

$$\lim_{r \rightarrow \infty} \sum_{k=0}^r A_k(f) = A(f)$$

in  $L^2$  and therefore in the weak sense.

$$\lim_{r \rightarrow \infty} \partial_x^\alpha \sum_{k=1}^r A_k(f) = \partial_x^\alpha A(f)$$

in the weak sense and therefore in  $L^2$ .

Conclusion

$$\|\partial_x^\alpha A(f)\|_{L^2} \leq A\|f\|_{L^2}$$

whenever  $f \in C_0^0(\mathbb{R}^d)$ ,  $|\alpha| \leq \frac{d-1}{2}$  (and  $d$  is odd)

□

today:

$$(A_t f)(x) = \int_{|y|=1} f(x - ty) d\sigma(y) = (f * \sigma_t)(x)$$

which is the average of  $f$  over the sphere of radius  $t$  centered at  $x$ ,

$$\int_{|x|=t} g(x) d\sigma_t(x) := \int_{|x|=1} g(tx) d\sigma(x).$$

This definition works fine provided  $g$  is continuous.

Question: We would like to know for general  $f$  whether  $(A_t f)(x) \rightarrow f(x)$  ( $x$ -a.e.) as  $t \rightarrow 0$ . Recall that we already did this for balls instead of spheres.

Is  $t \mapsto (A_t f)(x)$  continuous (for any  $x$ )? Is  $\sup_{t_1 < t < t_2} |A_t f(x)|$  measurable in  $x$ ? This actually has to be discussed before we can think about the first question.

A priori estimate for the spherical maximal averages:

**Theorem.** Let  $f \in C_0^0(\mathbb{R}^d)$ ,  $d \geq 4$ . Then

$$\left\| \sup_{t>0} |(A_t f)(x)| \right\|_{L_x^2(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)}.$$

Note that this can be improved up to  $d \geq 2$ , but fails for  $d = 1$ .

Key: Let  $f \in C_0^1(\mathbb{R}^d)$ . Then

$$(Sf)(x) = \left( \int_0^\infty \left| \frac{\partial A_t f}{\partial t}(x) \right|^2 t dt \right)^{\frac{1}{2}}.$$

This also holds for  $C_0^0$  by density.

**Lemma.**

$$\sup_{t>0} |(A_t f)(x)| \leq (Mf)(x) + c(Sf)(x)$$

where  $Mf$  is the standard Hardy-Littlewood maximal function over centered balls.

*Proof fo Lemma 1.*

$$(A_t f)(x) = t^{-d} \int_0^t \frac{\partial}{\partial s} [s^d (A_s f)(x)] ds = I + II$$

using that the integrand equals

$$ds^{d-1}A_sf)(x) + s^d \frac{\partial(A_sf)}{\partial s}(x)$$

$$I = dt^{-d} \int_0^t s^{d-1} \left( \int_{|y|=1} f(x-sy) d\sigma(y) \right) ds = dt^{-d} \int_{B(x,t)=\{y:|x-y|\leq t\}} f(y) dy = \frac{1}{\frac{t^d}{d}} \int_{B(x,t)} f \leq \sup_{t>0} \int_{B(x,t)} f = (Mf)(x)$$

due to our choice of normalization  $|B(x,t)| = \int_0^t \underbrace{\omega_{d-1}}_1 s^{d-1} ds = \frac{t^d}{d}$ .

$$II = t^{-d} \int_0^t s^{d-\frac{1}{2}} \left( \frac{\partial A_sf}{\partial s}(x) s^{\frac{1}{2}} \right) ds \leq \left( \int_0^\infty \left| \frac{\partial(A_sf)(x)}{\partial s} \right|^2 s ds \right)^{\frac{1}{2}} \cdot t^{-d} \left( \int_0^t s^{2d-1} ds \right)^{\frac{1}{2}} \lesssim_d (Sf)(x)$$

□

**Lemma.** If  $d \geq 4$  then

$$\|Sf\|_{L^2(\mathbb{R}^d)} \leq A\|f\|_{L^2(\mathbb{R}^d)}$$

*Proof.*  $A_sf = f * \sigma_s$ .

$$(\hat{A_sf})(\xi) = \hat{f}(\xi) \hat{\sigma}_s(\xi) = \hat{f}(\xi) \hat{\sigma}(s\xi)$$

since

$$\hat{\sigma}_s(\xi) = \int_{|x|=1} e^{2\pi i x \xi} d\sigma_s(x) = \int_{|x|=1} e^{2\pi i s x \xi} d\sigma(x) = \hat{\sigma}(s\xi).$$

It follows that

$$\frac{\partial(\hat{A_sf})}{\partial s}(\xi) = \frac{\mu(s\xi)}{s} \hat{f}(\xi)$$

where

$$\mu(\xi) = \sum_{j=1}^d \xi_j \frac{\partial \hat{\sigma}}{\partial \xi_j}(\xi) = \langle \xi, \nabla \hat{\sigma}(\xi) \rangle$$

because we differentiate with respect to a variable independent from the Fourier transform.

Key:

$$|\mu(\xi)| \leq A \min\{|\xi|, |\xi|^{-\frac{d-3}{2}}\}$$

*Proof.*

$$\frac{\partial \hat{\sigma}}{\partial \xi_j}(\xi) = 2\pi \int_{|x|=1} x_j e^{2\pi i x \xi} d\sigma(x)$$

Since  $x_j \in [-1, 1]$  we get

$$\left| \frac{\partial \hat{\sigma}}{\partial \xi_j}(\xi) \right| \leq A(1 + |\xi|)^{-\frac{d-1}{2}}$$

The statement follows from this and Cauchy-Schwarz applied to the definition of  $\mu(\xi)$ . □

By Plancherel we get

$$\int_{\mathbb{R}^d} \left| \frac{\partial(A_sf)}{\partial s}(x) \right|^2 dx = \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \frac{|(s\xi)|^2}{s^2} d\xi$$



Now we multiply by  $s$  and integrate with respect to  $s$  and get

$$\int_{\mathbb{R}^d} |Sf(x)|^2 dx = \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \underbrace{\left( \int_0^\infty \frac{|\mu(s\xi)|^2}{s} ds \right)}_* d\xi$$

It is enough to show, that  $*$  is bounded with respect to  $\xi$ .

$$\int_0^\infty \frac{|\mu(s\xi)|^2}{s} ds = \int_0^{\frac{1}{|\xi|}} + \int_{\frac{1}{|\xi|}}^\infty \leq A(|\xi|^2 \int_0^{\frac{1}{|\xi|}} s^2 \frac{ds}{s} + |\xi|^{-(d-3)} \int_{\frac{1}{|\xi|}}^\infty s^{-(d-3)} \frac{ds}{s}) \leq C <$$

where  $C$  is independent of  $\xi$ . The whole thing is true iff  $-(d-3) - 1 < -1$ ,  $d \leq 4$ , since this takes care of the second summand and the first summand is no problem.  $\square$

Now the theorem follows from the Lemmas and the  $L^2$  boundedness of the maximal function. Now look at  $d_1$ . Then the theorem fails because

$$A_t f(x) = \frac{1}{2}(f(x+t) + f(x-t))$$

and we can take as a counterexample a function nonnegative which blows up close to 0 but is in  $L^p$  for every  $p < \infty$  such that  $\sup_{t>0} (A_t f)(x) = \infty$  everywhere.

He said something with the wave equation and its smoothing properties for  $1 \leq d \leq 3$ .

$d \geq 2$ : The maximal operator  $f \mapsto \sup_{t>0} (A_t(f))$  is bounded on  $L^p(\mathbb{R}^d)$  if  $p > \frac{d}{d-1}$ .

- For  $d \geq 3$ , this is in Stein's Chapter XI paragraph 3. Ideas: rotational curvature, FIOs, dyadic decomposition, almost orthogonality
- For  $d = 2$ , this is a theorem of Bourgain (1986) with alternative proofs by Sogge (1991): Cinematic curvature. Mockenhaupt-Seeger-Sogge (1993): Local smoothing for wave equation.  $d = 2$  is also in Stein's Chapter XI paragraph 4D
- No such result holds in  $L^p(\mathbb{R}^d)$  if  $p \leq \frac{d}{d-1}$ : Let

$$f(y) = \frac{|y|^{1-d}}{\log \frac{1}{|y|}} 1_{\{|y| \leq \frac{1}{2}\}}(y)$$

Then  $f \in L^p$  if  $p \leq \frac{d}{d-1}$ : First, forget about the log for a moment. It is only there to take care of the endpoint.

$$\|f\|_{L^p(\mathbb{R}^d)}^p \simeq \int_0^{\frac{1}{2}} \frac{r^{(1-d)p}}{(\log \frac{1}{r})^p} r^{d-1} dr = \int_0^{\frac{1}{2}} r^{(1-d)(p-1)} dr$$

$$(1-d)(p-1) > -1 \quad (d-1)(p-1) < 1 \quad p < \frac{1}{d-1} + 1 = \frac{d}{d-1}$$

For any  $x$ , the quantity  $(A_t f)(x)$  is unbounded when  $f \sim |x|$ :

$$\therefore \sup_t (A_t f)(x) = \infty \quad \text{everywhere}$$

Next: Averages with respect to a fixed curve:

$$(Mf)(x) = \sup_{h>0} \frac{1}{2h} \left| \int_{-h}^h f(x - (t, t^2)) dt \right|$$

**Maximal operator along a parabola**  $f : \mathbb{R}^2 \rightarrow \mathbb{C}$

$$\tilde{A}f(x) = \frac{1}{2t} \int_{-t}^t f(x - \gamma(s)) \, ds$$

Maximal operator  $\sup_{t>0} \tilde{A}_t f(x)$ .  
 $2^k < t \leq 2^{k+1}$ .

$$\frac{1}{2t} \int_{-t}^t f(x - \gamma(s)) \, ds \leq \frac{1}{22^k} \int_{-2^{k+1}}^{2^{k+1}} |f(x - \gamma(s))| \, ds \leq 4 \underbrace{\frac{1}{2^{k-1}} \int |f(x - \gamma(s))| \eta(2^{-k-1}s) \, ds}_{A_{-k-1}f(x)}$$

where  $\eta : \mathbb{R} \rightarrow \mathbb{R}_+$  smooth such that

$$\eta(s) = \begin{cases} 1 & |s| \leq 1 \\ 0 & |s| \geq 2 \end{cases}$$

Then the maximal operator we consider equal to

$$\sup_{k \in \mathbb{Z}} |A_k f(x)|$$

up to a factor (right?)

Now we consider  $\gamma(s) = (s, s^2)$ . Then

$$A_j f(x) = 2^j \int f(x_1 - s, x_2 - s^2) \eta(2^j s) \, ds = 2^{j+k} \int f(x_1 - 2^k \tilde{s}, x_2 - 2^{2k} \tilde{s}^2) \eta(2^j \tilde{s}) \, d\tilde{s} = 2^{j+k} \int f(2^h(2^{-k}x_1 - \tilde{s}), 2^{2k}(2^{-2k}x_2 - \tilde{s}^2)) \eta(2^j \tilde{s}) \, d\tilde{s}$$

where  $s = 2^{-k} \tilde{s}$  and

$$\begin{aligned} P_k f(x) &= f(2^k x_1, 2^{2k} x_2) \\ \Rightarrow M &= P_{-k} M P_k \end{aligned}$$

Compare  $A_j$  with

$$B_j f(x) = 2^{2j} \int f(x - y) \psi(2^j y_1, 2^{2j} y_2) \, dy$$

where  $\psi$  is compactly supported and smooth and

$$\int_{\mathbb{R}^2} \psi = \int_{\mathbb{R}} \eta$$

$$\psi_j(x) = 2^{3j} \psi(2^j x, 2^{2j} x_2)$$

is supported in a ball of radius  $2^{-j}$  with respect to the metric

$$\rho(x, y) = \max(|x_1 - y_1|, |x_2 - y_2|^{\frac{1}{2}})$$

$$B_j f(x) \lesssim M_\rho f(x)$$

where  $M$  is the maximal function with respect to  $(\mathbb{R}^2, \rho, \text{Leb. measure})$ , doubling metric measure space.  $M_\rho$  is  $L^p$ -bounded for some reason. Therefore

$$f \mapsto \sup_j |B_j f|$$

is bounded on  $L^p(\mathbb{R}^2)$ ,  $1 < p \leq \infty$ . Need to estimate

$$\sup_j |(A_j - B_j)f(x)| \leq \left( \sum_j |(A_j - B_j)f(x)|^2 \right)^{\frac{1}{2}}$$

$$\int f d\mu_j = 2^j \int f(s, s^2) \eta(2^j s) ds$$

$$A_j f = \mu_j * f, \quad B_j f = \psi_j * f$$

Define

$$\sigma_j := \mu_j - \psi_j$$

Goal: Show that

$$\left\| \left( \sum_j |\sigma_j * f|^2 \right)^{\frac{1}{2}} \right\|_p \lesssim \|f\|_p$$

By interpolation it suffices to show this for  $1 < p \leq 2$ .  $p = 2$ : Let  $f \in S$ . Then by Plancherel

$$\left\| \left( \sum_j |\sigma_j * f|^2 \right)^{\frac{1}{2}} \right\|_2^2 = \int \sum_j |\sigma_j * f|^2 = \sum_j \int |\sigma_j * f|^2 = \sum_j \int |\hat{\sigma}_j|^2 |\hat{f}|^2 \leq \int |\hat{f}(\xi)|^2 \sum_j |\hat{\sigma}_j(\xi)|^2 \lesssim \int |\hat{f}|^2 = \|f\|_2^2$$

if we show that  $\sum_j |\hat{\sigma}_j(\xi)|^2$  is bounded by a constant. Since the parabola has nonvanishing curvature we have

$$|\hat{\mu}_0(\xi)| \lesssim |\xi|^{-\frac{1}{2}}$$

Since  $\psi_0$  is smooth we have for all  $N$

$$|\hat{\psi}_0(\xi)| \lesssim |\xi|^{-N}.$$

That implies

$$|\hat{\sigma}_0(\xi)| = |\hat{\mu}_0(\xi) - \hat{\psi}_0(\xi)| \lesssim |\xi|^{-\frac{1}{2}}.$$

Also,

$$\int_{\mathbb{R}^2} d(\mu_0 - \psi_0) = 0 \Rightarrow \hat{\sigma}_0(0) = 0$$

and since  $\sigma_0$  has compact support and

$$\int_{\mathbb{R}^2} d|\sigma_0| < \infty$$

$\hat{\sigma}_0(\xi)$  is differentiable in zero and therefore

$$|\hat{\sigma}_0(\xi)| \lesssim |\xi|.$$

Now

$$\hat{\mu}_j(\xi) = 2^j \int e^{-2\pi i(\xi_1 s + \xi_2 s^2)} \eta(2^j s) ds = \int e^{-2\pi i(2^{-j}\xi \tilde{s} + 2^{-2j}\xi_2 \tilde{s}^2)} \eta(\tilde{s}) d\tilde{s} = \hat{\mu}_0(2^{-j}\xi_1, 2^{-2j}\xi_2)$$

Also,

$$|\hat{\sigma}_0(\xi)| \leq \min(\rho(\xi), \rho(\xi)^{-\frac{1}{2}})$$

$$|\hat{\sigma}_j(\xi)| = |\hat{\sigma}_0(2^{-j}\xi_2, 2^{-2j}\xi_2)| \lesssim \min(2^{-j}\rho(\xi), (2^{-j}\rho(\xi))^{-\frac{1}{2}})$$

Now we sum

$$\sum_j |\sigma_j(\xi)|^2 \lesssim \sum_{j \in \mathbb{Z}} \min((2^{-j}\rho(\xi))^2, (2^{-j}\rho(\xi))^{-1}) \lesssim 1$$

$$1 < p < 2.$$

**Lemma.** Let  $1 < q, p_0 < \infty$  with  $\frac{1}{1q} = |\frac{1}{2} - \frac{1}{p_0}|$ . Assume

$$\sigma^* f := \sup_k |\sigma_k| * |f|$$

is bounded on  $L^q$ . Let  $(g_k) \in L^{p_0}(l^2)$ . Then

$$\|(\sum_k |\sigma_k * g_k|^2)^{\frac{1}{2}}\|_{p_0} \lesssim \|(\sum_k |g_k|^2)^{\frac{1}{2}}\|_{p_0}$$

Something with an operator

*Remark.* For fixed  $q$  there are two  $p_0$ s. If  $p_0 < 2$  then

$$\frac{1}{2q} = \frac{1}{p_0} - \frac{1}{2} \Rightarrow p_0 = \frac{1}{\frac{1}{2q} + \frac{1}{2}} = \frac{2q}{1+q} < q$$

If  $p_0 > 2$ , then

$$\frac{1}{2q} = \frac{1}{2} - \frac{1}{p_0} \Rightarrow \frac{1}{p_0} = \frac{1}{2} - \frac{1}{2q} \Rightarrow \frac{2}{p_0} = 1 - \frac{1}{q} \Rightarrow q = (\frac{p_0}{2})'$$

*Proof.* Result for  $p_0 < 2$  follows from  $p_0 > 2$  by duality because for  $L^{p_0}(l^2) \rightarrow L^{p_0}(l^2)$  the adjoint maps on  $(L^{p_0}(l^2))' = L^{p_0'}(l^2)$ . For  $g \in L^{p_0}(l^2)$ ,  $f \in L^{p_0'}(l^2)$  we use that

$$\langle (Tg)_k, f \rangle = \int \sum_k (Tg)_k f_k = \int (\sigma_k * g) f_k = \int g_k (\tilde{\sigma}_k * f_k)$$

where  $\tilde{\sigma}_k$  is  $\sigma_k$  reflected at 0.

$p_0 > 2$ .

$$\|(\sum_k |\sigma_k * g_k|^2)^{\frac{1}{2}}\|_{p_0}^2 = \|\sum_k |\sigma_k * g_k|^2\|_{\frac{p_0}{2}} = \int f \sum_k |\sigma_k * g_k|^2$$

for some  $f$  with  $1 = \|f\|_{(\frac{p_0}{2})'} = \|f\|_q$ .

$$(|\sigma_k| * |g_k(x)|)^2 = (\int |g_k(x-y)| d|\sigma_k|)^2 \leq (\int |g_k(x-y)|^2 d|\sigma_k|) (\underbrace{\int d|\sigma_k|}_{\lesssim 1})$$

Now the whole sum becomes

$$\begin{aligned} &\lesssim \int |f| \sum_k |\sigma_k| * |g_k|^2 = \int \sum_k (|\tilde{\sigma}_k| * |f|) |g_k|^2 \leq \int (\tilde{\sigma}^* f) \sum_k |g_k|^2 \\ &\leq \underbrace{\|\sigma^* f\|_q}_{\lesssim 1} \|\sum_k |g_k|^2\|_{\frac{p_0}{2}} \lesssim \|(\sum_k |g_k|^2)^{\frac{1}{2}}\|_{p_0}^2 \end{aligned}$$

□

**Theorem.** Let  $1 < q < \infty$ ,  $\frac{2q}{1+q} < p \leq 2$ . Assume  $\sigma^*$  is bounded on  $L^q(\mathbb{R}^2)$ , and  $\sigma_j$  are measures with

$$\hat{\sigma}_j(\xi) \lesssim \min(2^j \rho(\xi), (2^j \rho(\xi))^{-1})^\varepsilon$$

for some  $\varepsilon > 0$  and

$$\int d|\hat{\sigma}_j| \lesssim 1.$$

Then

$$\|(\sum_j |\sigma_j * f|^2)^{\frac{1}{2}}\|_p \lesssim \|f\|_p.$$

By construction  $q_s$  and ranges for  $p$  iteratively we can cover the whole interval.

This theorem can actually be applied to different situations, e.g. dyadic spheres. One might not get optimal results though. Something with maximal functions for general spheres or so...

Let  $S_k$  be Littlewood-Paley projections, i.e.

$$\hat{S}_k f = \hat{\varphi}_k f$$

where  $\varphi_k$  is a smooth and bounded function with

$$\text{supp } \hat{\varphi}_k \subset \{\xi : 2^{-k-1} \leq \rho(\xi) \leq 2^{-k+1}\}$$

and

$$\sum_k \hat{\varphi}_k(\xi) = 1$$

for  $\xi \neq 0$ , so that

$$f = \sum_k S_k f$$

for  $f \in L^p$ ,  $1 < p < \infty$ . Then (or: and?) we have for the square function

$$\|(\sum_k |S_k f|^2)^{\frac{1}{2}}\|_p \lesssim \|f\|_p$$

for  $1 < p < \infty$ .

*Proof of Theorem.* By the lemma

$$\|(\sum_j |\sigma_j * \sum_k S_{j+k} f|^2)^{\frac{1}{2}}\|_p \leq \sum_k \|(\sum_j |\sigma_j * S_{j+k} f|^2)^{\frac{1}{2}}\|_p$$

Now the summands are

$$\lesssim \|(\sum_j |S_{j+k} f|^2)^{\frac{1}{2}}\|_p \lesssim \|f\|_p$$

where  $p = \frac{2q}{1+q}$ . But this is not enough. So we do

$$\|(\sum_j |\sigma_j * S_{j+k} f|^2)^{\frac{1}{2}}\|_2^2 = \sum_j \|\sigma_j * S_{j+k} f\|_2^2 = \sum_j \|\hat{\sigma}_j \hat{\varphi}_{j+k} \hat{f}\|_2^2 \lesssim 2^{-2\varepsilon|k|} \sum_j \|\hat{\varphi}_{j+k} \hat{f}\|_2^2 \lesssim 2^{-2\varepsilon|k|} \|\hat{f}\|_2^2$$

since  $\hat{\varphi}_{j+k}$  is supported where  $\rho(\xi) \sim 2^{-j-k}$  and  $\hat{\sigma}_j \lesssim 2^{-\varepsilon|k|}$  on  $\text{supp } \hat{\varphi}_{j+k}$ . Now we use Marcinkiewicz interpolation:

$$\begin{aligned} \frac{1-\theta}{\frac{2q}{1+q}} + \frac{\theta}{2} &= \frac{1}{p_\theta} \\ \Rightarrow I_k &\lesssim 1^{1-\theta} (2^{-2\varepsilon|k|})^\theta \|f\|_{p_\theta} \end{aligned}$$

which is summable in  $k$ . For the missing part see Duoandikoetxea and Rubio de Francia 1986  $\square$

**Fourier restriction theory** Given  $f : \mathbb{S}^{d-1} \rightarrow \mathbb{C}$ , consider the Fourier transform  $\widehat{f\sigma}(\xi) = \int_{\mathbb{S}^{d-1}} f(x) e^{-2\pi i x \xi} d\sigma(x)$  for  $\xi \in \mathbb{R}^d$  (tempered distribution which turns out to be a function)

- $f$  smooth  $\Rightarrow \widehat{f\sigma}$  decays at  $\infty$ , e.g.

$$|\widehat{f\sigma}(\xi)| \leq c \|f\|_{C^2} (1 + |\xi|)^{-\frac{d-1}{2}}$$

via stationary phase

- $f$  bounded then no pointwise decay holds in general. E.g.

$$f_k(x) = e^{2\pi i k x}$$

Consider  $\xi = k$  then  $|\widehat{f_k\sigma}(\xi)| = \sigma(\mathbb{S}^{d-1}) \simeq 1$ , no decay here. Now take

$$f = \sum_{j \geq 0} \frac{f_{k_j}}{j^2}$$

where  $|k_j| \rightarrow \infty$  sufficiently fast, e.g.  $k_j = j!$ . Easy to check:  $f$  is continuous but  $|\widehat{f\sigma}(\xi)| \leq c(1 + |\xi|)^{-\varepsilon}$  does not hold for any  $c < \infty$ ,  $\varepsilon > 0$ . (uniformly in  $\xi$ ).

- Problem of distinguished origin disappears if we take  $L^q$  norms (There was some discussion on the wording going on which I did not understand)

Restriction conjecture (Stein): Prove, that if  $f \in L^\infty(\mathbb{S}^{d-1})$ , then

$$\|\widehat{f\sigma}\|_{L^q(\mathbb{R}^d)} \leq c_{q,d} \|f\|_{L^\infty(\mathbb{S}^{d-1}, \sigma)}$$

for very  $q > \frac{2d}{d-1}$ .

- Range of exponents would be sharp: take  $f \equiv 1 \in L^\infty(\mathbb{S}^{d-1})$

$$\|\widehat{\sigma}\|_{L^q(\mathbb{R}^d)}^q = \int_{\mathbb{R}^d} |\widehat{\sigma}(\xi)|^q d\xi \lesssim \int_{\mathbb{R}^d} (1 + |\xi|)^{-\frac{d-1}{2}q} d\xi \simeq_d \int_0^\infty (1 + r)^{-\frac{d-1}{2}q} r^{d-1} dr$$

The last expression is  $< \infty$  iff  $-\frac{d-1}{2}q + (d-1) < -1$  iff  $q > \frac{2d}{d-1}$ .

Corresponding problem for  $L^2$  densities was solved by Tomas-Stein inequality ( $\sim 1975$ ). If  $f \in L^2(\mathbb{S}^{d-1})$ , then

$$\|\widehat{f\sigma}\|_{L^q(\mathbb{R}^d)} \lesssim_{q,d} \|f\|_{L^2(\mathbb{S}^{d-1})}$$

if  $q \geq \frac{2d+2}{d-1}$ , and this range is the best possible. Note, that this was already proven last term and can be found in Stein/Shakarchi

- Assumptions are of the form  $q > q_0$  or  $g \geq q_0$ . Why?

$$\|\widehat{f\sigma}\|_{L^\infty(\mathbb{R}^d)} = \sup_{\xi \in \mathbb{R}^d} \left| \int_{\mathbb{S}^{d-1}} f(x) e^{-2\pi i x \xi} d\sigma_x \right| \leq \int_{\mathbb{S}^{d-1}} |f(x)| d\sigma_x = \|f\|_{L^1(\mathbb{S}^{d-1})} \lesssim \|f\|_{L^2(\mathbb{S}^{d-1})}$$

+ Riesz-Thorin because the sphere is compact

- $q \geq \frac{2d+2}{d-1}$  is the best possible for  $L^2$  densities: Knapp counterexample.

$$C_\delta = \{x \in \mathbb{S}^{d-1} : 1 - xe_d \leq \delta^2\}$$

where  $e_d = (0, \dots, 0, 1)$ . Since  $|x - e_d|^2 = 2(1 - xe_d)$ ,

$$|x - e_d| \leq c\delta \Rightarrow x \in C_\delta \Rightarrow |x - e_d| \leq Cd$$

for appropriate constants  $0 < c < C < \infty$ . Let  $f = 1_{C_\delta}$ . Right hand side is easy:

$$\|f\|_{L^2(\mathbb{S}^{d-1})} = |C_\delta|^{\frac{1}{2}} \simeq_d \delta q^{\frac{d-1}{2}}$$

Left hand is trickier. Note: the support of  $f\sigma$  is contained in a cylindrical box  $B_\delta$  centered at  $e_d$  with length  $\sim \delta^2$  in the  $e_d$  direction and  $\sim \delta$  in the  $(d-1)$  orthogonal direction.

$$\|\widehat{f\sigma}\|_{L^q(\mathbb{R}^d)}$$

Uncertainty principle: If a function is supported in a box, then its fourier transform is more or less constant on the dual box, which is the same but with inverse lengths. Idea: look at  $\widehat{f\sigma}$  on the dual box  $B_\delta^*$  (centered at 0), u.e. suppose  $|\xi_d| \leq c_1^{-1}\delta^{-1}$  and  $|\xi_j| \leq c_1^{-1}\delta^{-1}$  if  $j < d$ ,  $c_1$  is a large constant to be chosen. If  $\xi \in B_\delta^*$ , then

$$|\widehat{f\sigma}(\xi)| = \left| \int_{C_\delta} e^{2\pi i x \xi} d\sigma_x \right| = \left| \int e^{2\pi i (x - e_d) \xi} d\sigma_x \right| \geq \left| \int_{C_\delta} \cos(2\pi (x - e_d) \xi) d\sigma_x \right|$$

Conditions on  $\xi$  imply that

$$2\pi(x - e_d)\xi \leq \frac{\pi}{3}$$

if  $C_1$  large enough. Therefore

$$|\widehat{f\sigma}(\xi)| \geq \frac{1}{2}|C_d| \simeq \delta^{d-1}$$

How large is  $B_\delta^*$ ?

$$|B_\delta^*| \simeq \delta^{-2}(\delta^{-1})^{d-1} = \delta^{-(d+1)}$$

Conclusion:

$$\|\widehat{f\sigma}\|_{L^1(\mathbb{R}^d)} \gtrsim \left( \int_{B_\delta^*} |\widehat{f\sigma}(\xi)|^q d\xi \right)^{\frac{1}{q}} \gtrsim \delta^{wd} - 1 \delta^{-\frac{d+1}{q}}$$

$$\delta^{d-1} \delta^{-\frac{d+1}{q}} \leq \|\widehat{f\sigma}\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{S}^{d-1})} \simeq \delta^{\frac{d-1}{2}}$$

$$\therefore d-1 - \frac{d+1}{q} \geq \frac{d-1}{2} \Leftrightarrow \frac{d-1}{2} \geq \frac{d+1}{q} \Leftrightarrow q \geq 2d+2d-1$$

So, if you violate both of the two obstructions, then the inequality should hold (which obstructions? Maybe  $\infty$ -bound on the domain and on the range?)

Technical tool: convolution of Schwartz function with a (compactly supported) measure.  $\phi \in S(\mathbb{R}^d)$ ,  $\mu \in M(\mathbb{R}^d)$  then

$$(\phi * \mu)(x) = \int \phi(x - y) d\mu(y)$$

Notation  $\check{\mu} = \hat{\mu}(\cdot)$

**Lemma.** (i)

$$\widehat{\check{\phi}\mu} = \phi * \hat{\mu}$$

(ii)

$$\widehat{\phi\mu} = \hat{\varphi} * \hat{\mu}$$

*Proof of (b), (a) is similar.* Enough to show  $\forall \psi \in S(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \widehat{\phi\mu}\psi \, dx = \int_{\mathbb{R}^d} (\hat{\varphi} * \hat{\mu})\psi \, dx$$

$$\int_{\mathbb{R}^d} \widehat{\phi\mu}\psi \, dx = \int \widehat{\psi\varphi} \, d\mu = \int \widehat{\check{\phi} * \psi} \, d\mu = \int_{\mathbb{R}^d} (\check{\varphi} * \psi)\hat{\mu} \, dx = \int_{\mathbb{R}^d} (\hat{\phi} * \hat{\mu})\psi \, dx$$

due to duality, Fourier inversion on  $S$ , duality and definition of  $*$  + Fubini.  $\square$

**Lemma.**  $f, g \in S, \mu \in M(\mathbb{R}^d)$ . Then

$$\int \widehat{f\hat{g}} \, d\mu = \int_{\mathbb{R}^d} (\hat{\mu} * \bar{g})f \, dx$$

*Proof.*

$$\int \widehat{f\hat{g}} \, d\mu = \int_{\mathbb{R}^d} \widehat{f\hat{g}\mu} \, dx = \int_{\mathbb{R}^d} f(\bar{g} * \hat{\mu}) \, dx$$

by duality and lemma 1  $\square$

**Lemma.**  $\mu$  finite positive measure. Then the following are equivalent

- (i)  $\|\widehat{f\mu}\|_{L^q} \leq c\|f\|_{L^2(\mu)} \quad \forall f \in L^2(\mu)$
- (ii)  $\|\hat{g}\|_{L^2(\mu)} \leq c\|g\|_{L^{q'}} \quad \forall g \in S$
- (iii)  $\|f * \hat{\mu}\|_{L^q} \leq c^2\|f\|_{L^{q'}} \quad \forall f \in S$

Note, that

- $T : L^2 \rightarrow L^q, f \mapsto \widehat{f\mu}$  (extension) iff
- $T^* : L^{q'} \rightarrow L^2, g \mapsto \hat{g}|_{\text{supp } \mu}$  (restriction) iff
- $TT^* : L^{q'} \rightarrow L^q, f \mapsto f * \hat{\mu}$

Again, there were some words about the expression 'extension', which I did not understand.

*Proof of Tomas-Stein, up to endpoint.* Will show

$$q > \frac{2d+2}{d-1} \Rightarrow \|f * \hat{\mu}\|_{L^q(\mathbb{R}^d)} \lesssim_{q,d} \|f\|_{L^{q'}(\mathbb{R}^d)}$$

Relevant properties of  $\sigma$ :

- (i)  $|\hat{\sigma}(\xi)| \lesssim (1 + |\xi|)^{-\frac{d-1}{2}}$
- (ii)  $\sigma(D(x, r)) \simeq r^{d-1}$



Note, that every measure satisfying this will have a Tomas-Stein, because these are the only properties we need.

Let  $\phi \in C_0^\infty(\mathbb{R}^d)$ ,  $\text{supp}(\Phi) \subset \{x : \frac{1}{4} \leq |x| \leq 1\}$ ,  $\sum_{j \geq 0} \varphi(\frac{x}{2^j}) = 1$  if  $|x| \geq 1$ .

Cut up  $\hat{\sigma}$  as follows:

$$\hat{\sigma} = k_{-\infty} + \sum_{j=0}^{\infty} k_j$$

with

$$k_j(x) = \phi(\frac{x}{2^j}) \hat{\sigma}(x)$$

and

$$k_{-\infty}(x) = (1 - \sum_{j \geq 0} \phi(\frac{x}{2^j})) \hat{\sigma}(x)$$

Easy:  $k_{-\infty} \in C_0^\infty$ :

$$\|f * k_{-\infty}\|_{L^q} \lesssim \|f\|_{L^p} \quad \forall p \leq q$$

Since  $q > 2$  we can take  $p = q'$  (Why does it hold for  $q'$ , then? And don't we want to show it only for  $q'$  anyways?).

Trickier:  $(k_j)_{j=0}^\infty$ . Upshot: Estimate the convolution with  $k_j$   $L^1 \rightarrow L^\infty$ ,  $L^2 \rightarrow L^2$ .

First  $L^1 \rightarrow L^\infty$ .

$$\|f * k_j\|_{L^\infty} \leq \|k_j\|_{L^\infty} \|f\|_{L^1}$$

where

$$\|k_j\|_{L^\infty} \sim 2^{-j \frac{d-1}{2}}$$

as a consequence of property (i).

$$L^2 \rightarrow L^2$$

$$\|f * k_j\|_{L^2} = \|\widehat{f} \widehat{k}_j\|_{L^2} \leq \|\widehat{k}_j\|_{L^\infty} \|f\|_{L^2}$$

Claim:  $\|\widehat{k}_j\|_{L^\infty} \sim 2^j$  (Next lecture)

Interpolate (Riesz-Thorin)

$$\|f * k_j\|_{L^q} \lesssim 2^{-j \frac{d-1}{2} (1-\theta)} 2^{j\theta} \|f\|_{L^{q'}}$$

$$\frac{1}{q} = \frac{1-\theta}{\infty} + \frac{\theta}{2} \therefore q = \frac{2}{\theta}$$

which is equivalent to

$$\|f * k_j\|_{L^q} \lesssim 2^{j(\frac{d+1}{q} - \frac{d-1}{2})} \|f\|_{L^{q'}}$$

for any  $q \in [2, \infty]$ . The exponent is less than 0 iff  $q > \frac{2d+2}{d-1}$

□

**Fourier restriction theory 2**  $f \in L^1(\mathbb{R}^d)$  implies  $\widehat{f}$  is uniformly continuous (so,  $\widehat{f}$  can be restricted to any set)

$f \in L^2(\mathbb{R}^d)$  iff  $\widehat{f} \in L^2$ , (therefore  $\widehat{f}$  cannot be restricted to a set of Lebesgue measure 0)

**Question 1** What happens for intermediate  $p \in (1, 2)$ ?

Let  $M \subset \mathbb{R}^d$  be smooth compact hypersurface equipped with  $d\mu = \psi d\sigma$ ,  $\psi \in C_0^\infty$  and  $d\sigma$  surface measure on  $M$ . Given  $1 < p < 2$ , for which exponents  $q$  does

$$\left( \int_M |\widehat{f}(\xi)|^q d\mu_\xi \right)^{\frac{1}{q}} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$$

hold? A complete answer for  $q = 2$  is given by Tomas-Stein:  $M$  compact hypersurface whose Gauss curvature does not vanish on  $\text{supp}(M)$ . Then the restriction inequality holds provided  $q = 2$  and  $1 \leq p \leq \frac{2d+2}{d+3}$ . Note, that this is the dual version.

**Question 2** What happens for  $q < 2$ ? Dimensional analysis implies

$$1 \leq p < \frac{2d}{d+1}$$

and Knapp-Type examples

$$q \leq \left(\frac{d-1}{d+1}\right)p'$$

The restriction conjecture states that these conditions are also sufficient.

*End of proof of Tomas-Stein.* Back to  $(\mathbb{S}^{d-1}, \sigma)$ . Strategy:

$$\|f * \hat{\sigma}\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^{q'}(\mathbb{R}^d)}$$

if  $q > \frac{2d+2}{d-1}$ .  
Why is

$$\begin{aligned} \|\hat{k}_j\|_{L^\infty} &\lesssim 2^j \\ k_j &= \phi_{2^{-j}} \hat{\sigma} \end{aligned}$$

implies  $\hat{k}_j = \psi^{2^{-j}} * \sigma$  where  $\psi^\varepsilon(x) = \varepsilon^{-d} \psi(\varepsilon^{-1}x)$  and  $\psi = \hat{\phi} \in S$ . Now

$$|\hat{k}_j(\xi)| \lesssim_N 2^{jd} \int_{\mathbb{S}^{d-1}} (1 + 2^j |\xi - \eta|)^{-N} d\sigma(\eta) \quad \forall N \in \mathbb{N}$$

Therefore

$$|\hat{h}_j(\xi)| \lesssim 2^{jd} \int_{D(\xi, 2^{-j})} (1 + 2^j |\xi - \eta|)^{-N} d\sigma(\eta) + \sum_{k \geq 0} \int_{D(\xi, 2^{k+1-j}) \setminus D(\xi, 2^{k-j})} (1 + 2^j |\xi - \eta|)^{-N} d\sigma(\eta)$$

The first on will dominate because of the rapid decay of  $\psi$ .

$$\lesssim 2^{jd} \underbrace{[\sigma(D(\xi, 2^j))]}_{\sim 2^{-j(d-1)}} + \sum_{k \geq 0} 2^{-Nk} \underbrace{[\sigma(D(\xi, 2^{k+1-j}) \setminus D(\xi, 2^{k-j}))]}_{\sim 2^{-(d-1)(k-j)}} \lesssim 2^j$$

just choose  $N = d$  such that the sum becomes a geometric series.  $\square$

*Remark.* The  $L^2 \rightarrow L^2$  bound in the previous argument was based only on dimensionality considerations. Therefore there should be an  $L^2$  bound for  $\widehat{f\nu}$  valid under very general conditions.

**Theorem.** Let  $\nu$  be a positive finite measure where

$$\nu(D(x, r)) \leq cr^\alpha$$

Then

$$\|\widehat{f\nu}\|_{L^2(D(0, R))} \leq cR^{\frac{d-\alpha}{2}} \|f\|_{L^2(d\nu)}$$

The proof relies on Schur's test:  $(x, \mu)$ ,  $(Y, \nu)$  measure spaces,  $K(x, y)$  measurable on  $X \times Y$ . If for all  $y$  respectively  $x$

$$\int_X |K(x, y)| d\mu(x) \leq A, \quad \int_Y |K(x, y)| d\nu(y) \leq B,$$

then for

$$(T_K f)(x) = \int_Y K(x, y) f(y) d\nu(y)$$

we have

$$\|T_K\|_{L^2 \rightarrow L^2} \leq \sqrt{AB}.$$

*Proof of the theorem.* Let  $\phi \in S(\mathbb{R}^d)$  be radial such that  $\phi \geq 1$  on the unit disc  $\widehat{\phi}$  has compact support.  $\phi_\varepsilon(x) := \phi(\varepsilon x)$ . Then

$$\|\widehat{f\nu}\|_{L^2(D(0,R))} \leq \|\phi_{R^{-1}}(x)\widehat{f\nu}(-x)\|_{L^2_X(\mathbb{R}^d)} = \|\widehat{\phi_{R^{-1}}} * (f\nu)\|_{L^2(\mathbb{R}^d)} = \left\| \int_Y R^d \widehat{\phi}(R(x-y)) f(y) d\nu(y) \right\|_{L^2}$$

We have the estimates

$$\int_{\mathbb{R}^d} R^d \widehat{\phi}(R(x-y)) |dx| = \|\phi\|_{L^1} < \infty$$

by change of variables.

$$\int_Y R^d |\widehat{\phi}(R(xt\otimes))| d\nu(y) \lesssim R^{d-\alpha}$$

by the hypothesis of  $\nu$  and compact support of  $\widehat{\phi}$ . Now apply Schur.  $\square$

### Proof of the restriction conjecture in $d = 2$ (Zygmund '70, Fefferman '72)

**Theorem.** Let  $\gamma : I \rightarrow \mathbb{R}^2$  be a smooth curve with  $\gamma' \neq 0$  and  $\gamma'' \neq 0$  on some finite interval  $I$ . Let  $4 < q \leq \infty$  and  $3p' \leq q$ . Then  $\forall \varphi \in L^p(I)$

$$\left\| \int_I \varphi(t) e^{i\gamma(t)\xi} dt \right\|_{L^q_\xi(\mathbb{R}^2)} \lesssim_{p,q} \|\varphi\|_{L^p(I)}$$

*Proof.* Step 1: 'Even integer' trick.

$$\left\| \int_I \varphi(t) e^{i\gamma(t)\xi} dt \right\|_{L^q_\xi(\mathbb{R}^2)}^2 = \left\| \int_I \int_I \varphi(t) \overline{\varphi(s)} e^{i(\gamma(t) - \gamma(s))\xi} dt ds \right\|_{L^{\frac{q}{2}}(\mathbb{R}^2)}$$

Step 2: Change variables  $I \times I \rightarrow U \subset \mathbb{R}^2$ ,  $(t, s) \mapsto \gamma(t) - \gamma(s) = x$ . Choose  $I$  small enough such that the change of variables is invertible with Jacobian  $J = \det\left(\frac{\partial(t, s)}{\partial x}\right)$  satisfying  $|J| \simeq |t - s|^{-1}$ . Let  $\gamma(t) = (x_1(t), x_2(t))$ . Then

$$\left| \frac{\partial}{\partial(t, s)} \right| = \left| \begin{pmatrix} -x'_1(s) & -x'_2(s) \\ x'_1(t) & x'_2(t) \end{pmatrix} \right| = |x_1(s)x'_2(t) - x'_2(s)x_1(t)| \simeq |t - s|$$

since

$$|\cos s \sin t - \sin s \cos t| = |\sin(s - t)| \sim |s - t|$$

Define

$$\theta(s) = \int_0^s k(t) dt$$

Then

$$(\min k)|s - t| \lesssim \|\theta(s) - \theta(t)\| \leq \|k\|_{L^\infty} |s - t|$$

$$\gamma'(s) = (\cos \theta(s), \sin \theta(s))$$

Step 3: Hausdorff-Young

The integral now becomes

$$\left\| \int_U e^{ix\xi} F(x) dx \right\|_{L^{\frac{q}{2}}(\mathbb{R}^2)} \leq \|F\|_{L^r(\mathbb{R}^2)}$$

where

$$F(x) := \varphi(t) \overline{\varphi(s)} |J|$$

provided  $\frac{q}{2} = r' \geq 2$ .

Step 4: Fractional integration (Hölder-Littlewood-Sobolev)

We treat the following as an integral of  $|\varphi|^r$  against a second function.

$$\|F\|_{L^r(\mathbb{R}^2)} \simeq \left( \int_I \int_I \frac{|\varphi(t)|^r |\varphi(s)|^r}{|t-s|^{r-1}} dt ds \right)^{\frac{1}{r}} \leq c \|\varphi\|_{L^p(I)}^2$$

provided  $1 < r < 2$  and  $1 + \frac{1}{(p/r)'} = r - 1 + \frac{1}{p/r}$ . The first one is fine by assumption. The second is equivalent to  $3p' = q$ .  $\square$