

Geometric Aspects of Harmonic Analysis

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Q1

$$\left. \begin{array}{l} f : [a, b] \rightarrow \mathbb{R} \text{ integrable} \\ F(x) = \int_a^x f(t) dt \end{array} \right] \stackrel{?}{\Rightarrow} F \text{ diff. (a.e. } x), F' = f$$

Q2 Conditions of F (on $[a, b]$) s.t.

- $F'(x)$ exists a.e.
- F' integrable
- $\int_a^b F'(x) dx = F(b) - F(a)$

?

Q1 Differentiation of the integral

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t) dt = \frac{1}{|I|} \int_I f = \text{avg}_I f = {}_I f$$

$I = (x, x+h)$, $|I|$ Lebesgue measure of I .

Q1 equivalent to averaging problem: Given $f \in L^1(\mathbb{R}^d)$, is it true, that

$$\lim_{|B| \rightarrow 0, x \in B} \frac{1}{|B|} \int_B f = f(x) \quad (x\text{-a.e.})?$$

$B \subset \mathbb{R}^d$ open ball

Yes, if f continuous $\forall \varepsilon \exists \delta |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon. x \in B$

$$|f(x) - \fint_B f| = \left| \fint_B (f(y) - f(x)) dy \right| < \varepsilon \quad (1)$$

provided B is an open ball of radius $< \frac{\delta}{2}$ containing x

Yes, if f is integrable (not so easy). Hardy, Littlewood (1D, rearrangements; later Wiener for $d > 1$). $f \in L^1(\mathbb{R}^d)$

$$(Mf)(x) := \sup_{x \in B} \frac{1}{|B|} \int_B |f|$$

uncentered HL maximal function

Theorem. Let f be integrable on \mathbb{R}^d . Then

(i) Mf is measurable.

(ii) $(Mf)(x) < \infty$ a.e. x

(iii)

$$|\{x \in \mathbb{R}^d : (Mf)(x) > \alpha\}| < \frac{c}{\alpha} \|f\|_{L^1(\mathbb{R}^d)} \quad (\forall \alpha > 0). \quad (2)$$

$c = c_d = 3^d$, independent of f, α .

$f \neq 0 \in L^1 \Rightarrow Mf(x) \sim |x|^{-d}$ for large radius of x . So then $Mf \notin L^1$.

$$M : \begin{matrix} L^1 \rightarrow L^1 \\ L^1 \rightarrow L^{1,\infty} \end{matrix}$$

Proof. (i) easy $E_\alpha = \{x \in \mathbb{R}^d : (Mf)(x) > \alpha\}$ is open ($\forall \alpha > 0$) (because Mf is lower semicontinuous)

(ii) $|\{x \in \mathbb{R}^d : (Mf)(x) = \infty\}| \subset |\{x \in \mathbb{R}^d : Mf(x) > \alpha\}|$, take $\alpha \rightarrow \infty$.

(iii) follows from an elementary version of *Vitali covering*

□

Lemma. Let $B = \{B_1, B_2, \dots, B_N\}$ be a finite collection of open balls on \mathbb{R}^d . Then there exists a disjoint subcollection $B_{i_1}, B_{i_2}, \dots, B_{i_k}$ of B such that

$$|\bigcup_{j=1}^n B_j| \leq 3^d \sum_{j=1}^k |B_{i_j}|$$

Proof. (i) B_{i_1} = largest ball

(ii) Delete B_{i_1} and its neighbors

(iii) B_{i_2} = largest ball

(iv) repeat...

- Algorithm stops in at most N steps
- output has desired properties:
 - disjointness is clear
 - size $B \cap B' \neq \emptyset$, $r_{B'} \leq r_B$. B^* = ball with the same center as B but 3 times the radius. $\Rightarrow B' \subset B^*$. $|B^*| = 3^d |B|$

□

Back to (iii): Choose $\alpha > 0$, $E_\alpha = \{x \in \mathbb{R}^d : (Mf)(x) > \alpha\}$. For each

$$x \in E_\alpha \exists B = B_x := \frac{1}{|B_x|} \int_{B_x} |f(y)| dy > \alpha$$

equivalent

$$|B_x| < \alpha^{-1} \int_{B_x} |f(y)| dy$$

Fix $K \ll E_\alpha$ compact subset covered by $\bigcup_{x \in K} B_x$, $K \subset \bigcup_{l=1}^N NB_l$

$$|K| \leq \left| \bigcup_{l=1}^N B_l \right| \stackrel{\text{Vitali}}{\leq} 3^d \sum_{j=1}^k |B_{i_j}| \leq \frac{3^d}{\alpha} \sum_{j=1}^k \int_{B_{i_j}} |f(y)| dy = \frac{3^d}{\alpha} \int_{\bigcup_{j=1}^k B_{i_j}} |f(y)| dy \leq \frac{3^d}{\alpha} \|f\|_{L^1(\mathbb{R}^d)}$$

Since K was chosen arbitrary (cpt.), it follows that

$$|E_\alpha| \leq \frac{3^d}{\alpha} \|f\|_{L^1}$$

Can interpolate between weak type L^1 -inequality and $L^\infty \rightarrow L^\infty$ (very easy).

Corollary (Lebesgue differentiation theorem). Let $f \in L^1(\mathbb{R}^d)$. Then

$$\lim_{|B| \rightarrow 0, x \in B} \oint_B f = f(x) \quad x\text{-a.e.} \quad (3)$$

Proof.

$$E_\alpha = \{x \in \mathbb{R}^d : \limsup_{|B| \rightarrow 0, x \in B} \oint_B f - f(x) > 2\alpha\}$$

ETS $|E_\alpha| = 0 \forall \alpha > 0$. Then $E = \bigcup_{n \in \mathbb{N}} E_{\frac{1}{n}} = 0$ and (3) holds on E^c .

Fix $\alpha > 0$, given $\varepsilon > 0$ choose $g \in C_0^\infty(\mathbb{R}^d)$ s.t. $\|f - g\|_{L^1} < \varepsilon$. Already seen

$$\begin{aligned} \lim_{|B| \rightarrow 0, x \in B} \oint_B g &= g(x) \quad \forall x \\ \oint_B f - f(x) &= \oint_B (f - g) + \oint_B g - g(x) + g(x) - f(x) \end{aligned}$$

$$F_\alpha = \{x : M(f - g)(x) > \alpha\}$$

$$G_\alpha = \{x : |f(x) - g(x)| > \alpha\}$$

$E_\alpha \subset F_\alpha \cup G_\alpha$ since $u_1, u_2 > 0$, $u_1 + u_2 > 2\alpha \Rightarrow u_1 > \alpha \vee u_2 > \alpha$.

$$|G_\alpha| \leq \frac{1}{\alpha} \|f - g\|_{L^1} \quad (\text{Chebyshev})$$

$$|F_\alpha| \leq \frac{c_d}{\alpha} \|f - g\|_{L^1} \quad (\text{weak type})$$

$$|E_\alpha| \leq |F_\alpha| + |G_\alpha| \leq \left(\frac{c_d}{\alpha} + \frac{1}{\alpha}\right) \|f - g\|_{L^1} \leq \frac{c'_d \varepsilon}{\alpha}$$

Since $\varepsilon > 0$ was arbitrary $|E_\alpha| = 0$. □

$h \in L^1 \subset L^{1,\infty}$ by Chebyshev: $\infty > \|h\|_{L^1} = \int_{\mathbb{R}^d} |h(y)| dy \geq \int_{h(y) \geq \alpha} |h(y)| dy \geq \alpha |\{h > \alpha\}|$.

Would have been enough to replace $L^1(\mathbb{R}^d)$ by L^1_{loc} .

Sets $E \subset \mathbb{R}^d$ measurable, $x \in \mathbb{R}^d$ (not necc. in E) x is a point of Lebesgue density of E if

$$\lim_{|B| \rightarrow 0, x \in B} \frac{|B \cap E|}{|B|} = 1$$

Corollary. Let $E \subset \mathbb{R}^d$ be measurable. Then

- (i) Almost every $x \in E$ is a point of Lebesgue density of E .
- (ii) Almost every $x \notin E$ is not a point of Lebesgue density.

Functions $f \in L^1_{\text{loc}}(\mathbb{R}^d)$.

$$\text{Leb}(f) := \{x \in \mathbb{R}^d : f(x) < \infty \text{ and } \lim_{|B| \rightarrow 0, x \in B} \int_B |f(y) - f(x)| dy = 0\}$$

f continuous at $\bar{x} \Rightarrow \bar{x} \in \text{Leb}(f) \Rightarrow \int_B f \rightarrow \int_B f(\bar{x})$ (all the inverse implications are wrong)

Corollary. $f \in L^1_{\text{loc}}(\mathbb{R}^d) \Rightarrow$ Almost every point belongs to $\text{Leb}(f)$.

(By checking the proof again?)

These things also works with other sets that "shrink regularly to x than balls". It gets worse however when one takes all parallel rectangles and even worse when arbitrarily oriented rectangles are allowed.

Q.2 Key: bounded variation (BV) $F : [a, b] \rightarrow \mathbb{R}$, $P = \{a = t_0 < t_1 < \dots < t_N = b\}$

$$V_F^P = \sum_{j=0}^N |F(t_j) - F(t_{j+1})|$$

is the variation of f over P . F is of bounded variation if

$$T_F(a, b) = \sup_P V_F^P < \infty$$

$$P \subset \tilde{P} \text{ partitions} \Rightarrow V_F^P \leq V_F^{\tilde{P}}$$

Example. (i) f monotonic (increasing) and bounded, $|f| \leq M \Rightarrow F \in \text{BV}$

$$V_F^P = \sum_{j=1}^N |F(t_j) - F(t_{j-1})| = F(b) - F(a) \leq 2M$$

(ii) F differentiable with F' bounded, $|F'| \leq M$, then by mean value theorem $f \in \text{BV}$. Or F Lipschitz

(iii) F α -Hölder ($\alpha < 1$) $\Rightarrow F \in \text{BV}$. Take $F : [0, 1] \rightarrow \mathbb{R}$, $x \mapsto d(x, C)^\alpha$, where C is the cantor set. 2^{n-1} intervals of length 3^{-n}

$$\alpha > \frac{\log 2}{\log 3} \Rightarrow \sum_{n=1}^{\infty} 2^{n-1} (3^{-n})^\alpha < \infty$$

- *Total variation* of F on $[a, x]$ (where $a \leq x \leq b$) is

$$T_F(a, x) = \sup \sum_{j=0}^N |F(t_j) - F(t_{j-1})|$$

- *Positive variation* of F on $[a, 1]$ is

$$P_F(a, x) = \sup_{(+)} \sum (F(t_j) - F(t_{j-1})) \quad \text{all } j : F(t_j) \geq F(t_{j-1})$$

- *Negative variation* of F on $[a, 1]$ is

$$N_F(a, x) = \sup_{(-)} \sum -(F(t_j) - F(t_{j-1}))$$

Lemma. $f : [a, b] \rightarrow \mathbb{R}$. Then

$$(i) \quad F(x) = F(a) + P_F(a, x) - N_F(a, x)$$

$$(ii) \quad T_F(a, x) = P_F(a, x) + N_F(a, x)$$

($\forall x \in [a, b]$)

recall from measure theory: $f = f^+ - f^-$, $|f| = f^+ + f^-$

Proof. (i) given $\varepsilon > 0$, $\exists P = \{a = t_0 < t_1 < \dots < t_N = x\}$

$$|PVF - \sum_{(+)} (F(t_j) - F(t_{j-1}))| < \varepsilon$$

$$|N_F - \sum_{(-)} -(F(t_j) - F(t_{j-1}))| < \varepsilon$$

Also

$$F(x) - F(a) = \sum_{(+)} F(t_j) - F(t_{j-1}) - \sum_{(-)} -(F(t_j) - F(t_{j-1}))$$

□

Corollary. $F : [a, b] \rightarrow \mathbb{R} \in \text{BV}$ iff F is the difference of two increasing bounded functions

Theorem. $F : [a, b] \rightarrow \mathbb{R} \in \text{BV} \Rightarrow F$ differentiable a.e.

Wlog f monotonically increasing, "Wlog" f continuous

Lemma (of the rising sun). $G : \mathbb{R} \rightarrow \mathbb{R}$ continuous.

$$E = \{x \in \mathbb{R} : \exists h = h_x > 0 \, G(x+h) > G(x)\}$$

Then

$$(i) \quad E \text{ is open } (E = \bigcup_{n=1}^{\infty} (a_n, b_n))$$

$$(ii) \quad g(a_n) = G(b_n), \text{ provided } b_n - a_n < \infty.$$

Proof. Let (a_n, b_n) be a finite interval in the decomposition. $a_k \notin E$ then $g(a_k) \geq G(b_k)$. Assume $G(a_k) > G(b_k)$. $\exists c \in (a_k, b_k)$ $g(c) = \frac{g(a_k) + g(b_k)}{2}$. Choose rightmost such c . $\exists d \in (c, b_k)$ $G(d) > G(c)$. But then by continuity c could not have been chosen rightmost, contradiction. \square

Can replace \mathbb{R} by $[a, b]$, but then only get for $a_0 = a$ that $G(a_0) \leq G(b_0)$

Proof. of theorem

$$\begin{aligned}\Delta_h(F)(x) &= \frac{F(x+h) - F(x)}{h} \\ D^\pm(F)(x) &= \limsup_{h \rightarrow 0, h > < 0} \Delta_h(F)(x) \\ D_\pm(F)(x) &= \liminf_{h \rightarrow 0, h > < 0} \Delta_h(F)(x)\end{aligned}$$

Dini numbers. Upshot: They are all the same and finite. $D_- \leq D^-$, $D_+ \leq D^+$ clear. ETS

(i) $D^+(F)(x) < \infty$ (a.e. x)

(ii) $D^+(F)(x) \leq D_-(F)(x)$ (a.e. x)

(ii) is equivalent to $D^-(F)(x) \leq D_+(F)(x)$ by replacing $F(x)$ by $-F(-x)$ somewhere. Then $D^+ \leq D_- \leq D^- \leq D_+ \leq D^+ < \infty$.

(i) relacc: F increasing, bounded, continuous on $[a, b]$. Fix $\gamma > 0$,

$$E_\gamma := \{x : D^+(F)(x) > \gamma\}$$

- E_γ is measurable
- Apply rising sun to $G(x) = F(x) - \gamma x$

$$E_\gamma \subset E = \{x \in [a, b] : \exists h > 0 \ G(x+h) > G(x)\} = \bigcup_{k=1}^{\infty} (a_k, b_k)$$

The condition in the set is equivalent to

$$\begin{aligned}\Leftrightarrow \exists h > 0 \ F(x+h) - \gamma x - \gamma h > F(x) - \gamma x \\ \Leftrightarrow \exists h > 0 \ \frac{F(x+h) - F(x)}{h} > \gamma \\ \Leftrightarrow D^+(F)(x) > \gamma\end{aligned}$$

$$G(a_k) \leq G(b_k) \Leftrightarrow F(a_k) - \gamma a_k \leq F(b_k) - \gamma b_k \Leftrightarrow \gamma(b_k - a_k) \leq F(b_k) - F(a_k).$$

Therefore

$$|E_\gamma| \leq |E| \leq \sum_{k=1}^{\infty} (b_k - a_k) \leq \frac{1}{\gamma} \sum_{k=1}^{\infty} (F(b_k) - F(a_k)) = \frac{1}{\gamma} (F(b) - F(a))$$

Take $\gamma \rightarrow \infty$, done.

(ii) see Stein-Shakarchi (vol 3)

\square

Corollary. F increasing, continuous $\Rightarrow F'$ exists a.e., measurable, nonnegative and

$$\int_a^b F'(x) dx \leq F(b) - F(a).$$

Proof. Let

$$G_h(x) = \frac{F(x + \frac{1}{n}) - F(x)}{\frac{1}{h}}$$

By the theorem, $G_h(x) \rightarrow F'(x)$ ($h \rightarrow 0$) pointwise a.e. By Fatou

$$\int_{[a,b]} F' \leq \liminf_{n \rightarrow \infty} \int_a^b G_h(x) dx = \liminf_{n \rightarrow \infty} \int_b^{b+\frac{1}{n}} F(x) dx - \int_a^{a+\frac{1}{n}} F(x) dx$$

□

Cannot do better than \leq : For the Devil's staircase the left hand side is 0 while the right hand side is 1.

Why is the sunrise Lemma a covering Lemma?

$$\left. \begin{aligned} f_+^* &= \sup \frac{1}{h} \int_x^{x+h} |f(y)| dy \\ E_\alpha^+ &= \{x \in \mathbb{R} : f_+^*(x) > \alpha\} \end{aligned} \right\} |E_\alpha^+| = \frac{1}{\alpha} \int_{E_\alpha^+} |f|$$

Why? Let

$$G(x) = \int_0^x |f(y)| dy - \alpha x$$

$$x \in E_\alpha^+ \iff f_+^*(x) > \alpha \iff \exists h > 0 \frac{1}{h} \int_x^{x+h} |f(y)| dy > \alpha \iff \exists h > 0 G(x+h) > G(x)$$

$$\{x \in \mathbb{R} : \exists h > 0 G(x+h) > G(x)\} = \bigcup_{k \in \mathbb{N}} (a_k, b_k), \quad G(a_k) = G(b_k)$$

$$|E_\alpha^+| = \sum_k (b_k - a_k) = \frac{1}{\alpha} \sum_k \int_{(a_k, b_k)} |f| = \frac{1}{\alpha} \int_{\bigcup_k (a_k, b_k)} |f| = \frac{1}{\alpha} \int_{|E_\alpha^+|} |f|.$$

Definition. $F : [a, b] \rightarrow \mathbb{R}$ is *absolutely continuous* if

$$\forall \varepsilon > 0 \exists \delta > 0 : \sum_{k=1}^N (B_k - a_k) < \delta \implies \sum_{k=1}^N |F(b_k) - F(a_k)| < \varepsilon$$

intervals (a_k, b_k) disjoint ($k = 1, \dots, N$) \Rightarrow

$$\sum_{k=1}^N |F(b_k) - F(a_k)| < \varepsilon$$

Remark. (i) On a bounded interval $I \subset \mathbb{R}$

$$C^1(I) \subset \text{Lip}(I) \subset AC(I) \subset BV(I)$$

So they are diff. a.e.. All the inclusions are strict.

(ii) abs cont \Rightarrow unif. con. \Rightarrow cont.

(iii) $f \in L^1_{\text{loc}}(\mathbb{R})$ $F(x) = \int_0^x f(t) dt$ Then F is absolutely continuous. $(\forall \varepsilon \exists \delta |E| < \delta \Rightarrow \int_E |f| < \varepsilon)$

Upshot: AC functions are the ones which re diff a.e. and vrfy FTC.

Theorem. $F \in AC(a, b) \Rightarrow F'$ exists a.e., $F' = 0$ a.e. $\Rightarrow F$ constant

- Existence of F' clear \checkmark
- $F' = 0$ a.e. $\Rightarrow F$ constant: refinement of Vitali

Definition. A collection $\mathcal{B} = \{B\}$ of (open) balls on \mathbb{R}^d . is a *Vitali covering* of a set E if

$$\forall x \in E \forall \eta > 0 \exists B \in \mathcal{B} : x \in B, |B| < \eta$$

Lemma. $E \subset \mathbb{R}^d$ meas. $|E| < \infty$, \mathcal{B} Vitali covering of E , $\delta > 0$. Then there exist finitely many disjoint balls $B_1, \dots, B_N \in \mathcal{B}$

$$\sum_{j=1}^N |B_j| \geq |E| - \delta$$

Recall elementary Vitali: $\mathcal{B} = \{B_1, \dots, B_N\}$ finite collection of pen balls in $\mathbb{R}^d \Rightarrow \exists$ disjoint subcollection B_{i_1}, \dots, B_{i_k} with

$$|\bigcup_{j=1}^B B_j| \leq 3^d \sum_{j=1}^k |B_{i_j}|$$

Proof of Lemma. wlog $\delta > |E|$. Vitali $\Rightarrow \exists$ disjoint subcollection $B_1, \dots, B_N \in \mathcal{B}$

$$\sum_{i=1}^{N_1} |B_i| \geq 3^{-d} \delta$$

Sequence of balls B_1, \dots, B_N . question: Is $\sum_{j=1}^{N_1} |B_j| \geq |E| - \delta$? Yes: done with $N = N_1$. No: work harder.

$$\sum_{j=1}^{N_1} |B_j| < |E| - \delta$$

$$E_2 = E \setminus \bigcup_{j=1}^{N_1} B_j$$

$$|E_2| \geq |E| - \sum_{j=1}^{N_1} |B_j| > |E| - (|E| - \delta) = \delta$$

\mathcal{B} Vitali covering \Rightarrow balls in \mathcal{B} disjoint from $\bigcup_{i=1}^{N_1} B_i$ still covers E_2 . Vitali $\Rightarrow \exists$ finite disjoint subcollection of these balls $B_{N_1+1}, \dots, B_{N_2}$

$$\sum_{N_1 < j < N_2} |B_j| \geq 3^{-d} \delta.$$

After k steps, $B_1, \dots, B_{N_1}, \dots, B_{N_k}$ with

$$\sum_{j=1}^{N_k} |B_j| \geq k 3^{-d} \delta \geq |E| - \delta$$

iff $k \geq 3^d \frac{|E| - \delta}{\delta}$, stop. □

need to approximate with compact from inside somewhere and with open from outside somewhere else

Corollary. The balls can be arranged in such a way that

$$|E \cup \bigcup_{i=1}^N B_i| < 2\delta$$

Proof. Choose open $O \supset E$: $|O \setminus E| < \delta$. \mathcal{B} Vitali covering \Rightarrow wlog all balls in \mathcal{B} are contained in O .

$$(E \cup \bigcup_{i=1}^N B_i) \cup \bigcup_{i=1}^N NB_i \subset O. : |E \cup \bigcup_{i=1}^N B_i| \leq |O| - \sum_{i=1}^N |B_i| \leq |E| + \delta - (|E| - \delta) = 2\delta$$

□

$F \in AC$ Back to the real line: Goal: $F' = 0$ a.e. $\Rightarrow F$ constant. ETS $F(a) = F(b)$

$$E = \{x \in (a, b) : F'(x) \text{ exists and } = 0\} \quad |E| = b - a$$

Fix $\varepsilon > 0$. For $x \in E$, $\lim_{h \rightarrow 0} \left| \frac{F(x+h) - F(x)}{h} \right| = 0$. $\forall \eta > 0 \exists$ open interval $I = (a_x, b_x) \subset [a, b]$ containing x . $|F(b_x) - F(a_x)| \leq \varepsilon(b_x - a_x)$ and $b_x - a_x < \eta$. The collection of these intervals (over all $\eta > 0$) forms a Vitali covering of E . Lemma \Rightarrow Given $\delta > 0$ can select finitely many, disjoint $I_j = (a_j, b_j)_{j=1}^N$ such that

$$\sum_{j=1}^N |I_j| \geq |E| - \delta = b - a - \delta$$

But

$$\sum_{j=1}^N |F(b_j) - F(a_j)| \leq \varepsilon \sum_{j=1}^N (b_j - a_j) \leq \varepsilon(b - a)$$

$$[a, b] \supset \bigcup_{j=1}^N I_j = \bigcup_{k=1}^M [\alpha_k, \beta_k]$$

with total length $\leq \delta$. $\therefore F \in AC$

$$\sum_{k=1}^M |F(\beta_k) - F(\alpha_k)| < \varepsilon$$

$$|F(b) - F(a)| \leq \sum |F(b_j) - F(a_j)| + \sum |F(\beta_k) - F(\alpha_k)| \leq \varepsilon(b - a) + \varepsilon,$$

done. □

Theorem. $F \in AC(a, b)$. Then

(i) F' exists a.e. and is integrable

(ii)

$$F(x) - F(a) = \int_a^x F'(t) dt \quad (\forall a \leq x \leq b)$$

Conversely, if $f \in L^1(a, b)$, then there exists $F \in AC(a, b) : F' = f$ a.e..

Proof. \Rightarrow

(i) seen last lecture.

(ii)

$$G(x) := \int_a^x F'(t) dt$$

$\therefore G \in AC \therefore F - G \in AC$. Lebesgue diff. $\Rightarrow G'(x) = F'(x)$ (a.e. x) $\therefore (F - G)' = 0$ a.e..
Therefore $(F - G)(x) = (F - G)(a)$, $F(x) - G(x) = F(a)$, equivalent to (*)

\Leftarrow

$$F(x) = \int_a^x f(t) dt$$

AC \checkmark Leb. diff $\Rightarrow F' = f$ a.e. □

Next: Monotone functions which are not nec. continuous. Wlog. F increasing, bounded on $[a, b]$.

$$F(x^-)F \lim_{y \rightarrow x, y < x} F(y) \quad F(x^+) := \dots$$

$F(x^-) \leq F(x) \leq F(x^+)$, F cont. at x if $F(x^-) = F(x^+)$. Otherwise F has a jump discontinuity at x .

Obs: A (bounded) increasing function F on $[a, b]$ has at most countable many jumps. There exists an injective map $\text{Disc}(F) \rightarrow \mathbb{Q}$

$\therefore \text{Disc}(F) = \{x_n\}_{n=1}^\infty$ $\alpha_n = F(x_n^+) - F(x_n^-) = \text{jump of } F \text{ at } x_n$. $F(x_n^+) = F(x_n^-) + \alpha_n$
 $F(x_n) = F(x_n^-) + \theta_n \alpha_n$, $\theta_n \in [0, 1]$. $F(x) = \mu((-\infty, x])$. Corresponds to singular + abs. cont measures.

$$j_n(x) = \begin{cases} 0 & x < x_n \\ \phi_n & x = x_n \\ 1 & x > x_n \end{cases}$$

Jump function associated to F is

$$J_F(x) = \sum_{n=1}^\infty j_n(x)$$

$$\sum_{n=1}^\infty \alpha_n = \sum_{n=1}^\infty F(x_n^+) - F(x_n^-) \leq F(b) - F(a) < \infty$$

because F incr, and F bounded.

Lemma. F increasing, bounded on $[a, b]$, $\text{Disc}(F) = \{x_n\}_{n=1}^\infty$

(i) $J_F(x)$ is discontinuous precisely at $\{x_n\}_{n=1}^\infty$, has a jump at x_n equal to that of F .

(ii) The function $F - J_F$ is increasing and continuous.

Proof. (i) $x \neq x_n (\forall n) \Rightarrow$ each j_n is continuous at $x \Rightarrow J_F$ is continuous at x because of uniform convergence. $x = x_N (\exists N) \Rightarrow J_F = \sum_{n=1}^N \alpha_n j_n(x) + \sum_{n>N} \alpha_n j_n(x)$. First sum has jump discontinuity x_N of size α_N

(ii) $F - J_F$ is continuous

$$F(x) - J_F(x) \leq F(y) - J_F(y) \iff J_F(y) - J_F(x) \leq F(y) - F(x)$$

, where

$$J_F(y) = \sum_{x < x_n \leq y} \alpha_n = \sum_{x < x_n \leq y} F(x_n^+) - F(x_n^-) \leq F(y) - F(x)$$

□

Since $F = (F - J_F) + J_F$ ETS J_F is diff a.e.. This was essential step of

$$\mu = \mu_{AC} + \mu_S + \mu_{PP}$$

$z(t) = (x(t), y(t))$. curve γ . $x, y : [a, b] \rightarrow \mathbb{R}$ continuous.

γ rectifiable if length

$$L(\gamma) = \sup \sum_{j=1}^N |z(t_j) - z(t_{j-1})| < \infty$$

sup over all partitions $P = \{a = t_0 < t_1 < \dots < t_N = b\}$ of $[a, b]$

When is

$$L(\gamma) = \int_a^b |z'(t)| dt?$$

Lemma. γ is rectifiable iff x, y are of bounded variation (and cont.).

see $F = x + iy$

Assume γ rectifiable, let $L(A, B)$ length of $\gamma(A, B)$, ($a \leq A \leq B \leq b$)

(i) $L(A, B) = T_F(A, B)$ (where $F(t) = z(t)$)

(ii) $L(A, C) + L(C, B) = L(A, B)$ ($A \leq C \leq B$)

(iii) $A \mapsto L(A, B)$ (fix B) is continuous

$B \mapsto L(A, B)$ (fix A)

seen: $F \in \text{BV}(a, b)$, cont. $\Rightarrow T_F$ cont.

Warning: $[0, 1] \ni t \mapsto (F(t), F(t))$, F Cantor. F cont. incr. $F(0) = 0$, $F(1) = 1$, $F' = 0$ a.e.

Theorem. $z : [a, b] \rightarrow \mathbb{R}^2, t \mapsto (x(t), y(t)) \sim \text{curve } \gamma$. $x, y \in \text{AC}(a, b)$. $\Rightarrow \gamma$ rectifiable and

$$L(\gamma) = \int_a^b |z'(t)| dt$$

why? $F : [a, b] \rightarrow \mathbb{C}$ is $\text{AC}(a, b)$

$$\Rightarrow T_F(a, b) = \int_a^b |F'(t)| dt.$$

$$\sum_{j=1}^N |F(t_j) - F(t_{j-1})| = \sum_{j=1}^N \left| \int_{t_{j-1}}^{t_j} F'(t) dt \right| \leq \sum_{j=1}^N \int_{t_{j-1}}^{t_j} |F'(t)| dt = \int_a^b |F'(t)| dt$$

First inequality by FTC. For \geq , write $F' = g + h$, g step function, h small in L^1 $G, H = \text{def.}$ integrals of g, h . Check $T_F \geq T_G, T_H$, T_H small, $T_G \geq \int_a^b |g(t)| dt$.

Minkowski content of a curve simple, simple closed, quasi-simple curves.

trace of γ : $\Gamma = \{z(t) \in \mathbb{R}^2 : t \in [a, b]\}$.

Given $K \in \mathbb{R}^2$ and $\delta > 0$ define

$$K^\delta = \{x \in \mathbb{R}^2 : d(x, K) < \delta\}$$

where $d(x, K) = \inf_{k \in K} d(x, k)$

Definition. The set K has (1D) Minkowski content if

$$\lim_{\delta \rightarrow 0} \frac{|K^\delta|}{2\delta}$$

exists (in \mathbb{R}), denoted $M(K)$.

Theorem. Let $\Gamma = \{z(t) : a \leq t \leq b\}$ be (the trace of) a quasi-simple curve γ . Then Γ has Minkowski content iff γ is rectifiable (in which case $M(\Gamma) = L(\gamma)$).

Upper Mink: content.

$$\limsup_{\delta \rightarrow 0^+} \frac{|K^\delta|}{2\delta} =: M^*(K)$$

lower

$$\liminf_{\delta \rightarrow 0^+} \frac{|K^\delta|}{2\delta} =: M_*(K).$$

Proposition. $\Gamma = \{z(t) : a \leq t \leq b\}$ quasi simple. If $M_*(\Gamma) < \infty$, then γ is rectifiable and $L(\gamma) \leq M_*(\Gamma)$.

Proposition. $\Gamma = \{z(t) : a \leq t \leq b\}$ rectifiable γ . Then $M^*(\Gamma) \leq L(\gamma)$.

Proof of Prop. 1, for simple curves. Obs: $\Gamma = \{z(t) : a \leq t \leq b\}$ any curve. $\Delta = |z(b) - z(a)|$. $|\Gamma^\delta| \geq 2\delta\Delta$.

Take any partition P of $[a, b]$. $L_P = \sum_{j=1}^N |z(t_j) - z(t_{j-1})|$. Given $\varepsilon > 0$, $\exists N$ proper closed subintervals $I_j = [a_j, b_j] \subset (t_{j-1}, t_j)$:

$$\sum_{j=1}^N |z(b_j) - z(a_j)| \leq L_P - \varepsilon$$

I_1, \dots, I_N disjoint $\Rightarrow \Gamma_1, \dots, \Gamma_N$ disjoint because Γ is simple. $\Leftrightarrow \Gamma_1^\delta, \dots, \Gamma_N^\delta$ disjoint, provided $\delta > 0$ small enough.

$$\bigcup_{j=1}^N \Gamma_j^\delta \subset \Gamma^\delta$$

$$|\Gamma^\delta| \geq \sum_{j=1}^N |\Gamma_j^\delta| \geq 2\delta \sum_{j=1}^N |z(t_j) - z(t_{j-1})| \geq 2\delta(L_P - \varepsilon)$$

□

Isoperimetric inequality (soft) $\gamma : [a, b] \rightarrow \mathbb{R}^2$, $\gamma \in C^1(a, b) : \gamma'(s) \neq 0 \forall s$, $\gamma(a) = \gamma(b)$.
 Arclength parametrization: $\gamma : [0, L] \rightarrow \mathbb{R}^2$, $|\gamma'(s)| = 1 \forall s$.

Theorem. $\Gamma \subset \mathbb{R}^2$ simple closed C^2 curve of length L . A area of the region enclosed by Γ .

$$A = \frac{1}{2} \left| \int_{\Gamma} (x dy - y dx) \right| = \frac{1}{2} \left| \int_0^L (x(s)y'(s) - x'(s)y(s)) ds \right|.$$

Then $4\pi A \leq L^2$. Equality iff Γ is a circle.

Proof. wlog (rescale) $L = 2\pi$: WTS $A \leq \pi$, equality iff Γ is a circle of radius 1.

$\gamma : [0, 2\pi] \rightarrow \mathbb{R}^2$, $s \mapsto \gamma(s) = (x(s), y(s))$ arclength par. $x'(s)^2 + y'(s)^2 = 1 \forall s$:

$$\frac{1}{2\pi} \int_0^{2\pi} (x'(s)^2 + y'(s)^2) ds = 1$$

Γ closed $\Rightarrow x(s), y(s)$ 2π -periodic.

$$x'(s) = \sum_n a_n i n e^{i n s}$$

$$y'(s) = \sum_n b_n i n e^{i n s}$$

Parseval \Rightarrow

$$\sum_{n \in \mathbb{Z}} n^2 (|a_n|^2 + |b_n|^2) = 1$$

$$A = \frac{1}{2} \int_0^{2\pi} (x(s)y'(s) - x'(s)y(s)) ds = \pi \left| \sum_{n \in \mathbb{Z}} n (a_n \bar{b}_n - b_n \bar{a}_n) \right|$$

by bilinear Parseval

$$|a_n \bar{b}_n - b_n \bar{a}_n| \leq 2|a_n||b_n| \leq |a_n|^2 + |b_n|^2$$

$$A \leq \pi \sum_{n \in \mathbb{Z}} n^2 (|a_n|^2 + |b_n|^2) \leq \pi$$

□

Cases of equality: $A = \pi \Rightarrow$

$$x(s) = a_{-1}e^{-is} + a_0 + a_1e^{is}$$

$$y(s) = b_{-1}e^{-is} + b_0 + b_1e^{is}$$

x, y real-valued $\Rightarrow a_1 = \bar{a}_{-1}$, $b_1 = \bar{b}_{-1}$. $(**) \Rightarrow 2(|a_1|^2 + |b_1|^2) = 1$.

$(***) \Rightarrow$

$$|a_1| = |b_1| = \frac{1}{2} : a_1 = \frac{1}{2}e^{i\alpha} \quad b_1 = \frac{1}{2}e^{i\beta} \quad (\alpha, \beta \in \mathbb{R})$$

$$1 = 2|a_1 \bar{b}_1 - \bar{a}_1 b_1| = \sin(\alpha - \beta) : \alpha - \beta = \frac{k\pi}{2} \quad (\text{odd } k)$$

$$x(s) = a_0 + \cos(s + \alpha)$$

$$y(s) = b_0 \pm \sin(s + \alpha)$$

\pm dep. on parity of $\frac{k-1}{2}$.

Isoperimetric inequality (hard) $\Omega \subset \mathbb{R}^2$ bounded, open, $\partial\Omega = \bar{\Omega} - \Omega =: \Gamma$ rectifiable curve (not nec. simple) with length $l(\Gamma)$.

Theorem.

$$4\pi|\Omega| \leq l(\Omega)^2$$

Proof. inner:

$$\Omega_-^\delta = \{x \in \mathbb{R}^2 : d(x, \mathbb{R}^2 \setminus \Omega) \geq \delta\}$$

outer:

$$\Omega_+^\delta = \{x \in \mathbb{R}^2 : d(x, \bar{\Omega}) < \delta\}$$

$$\Gamma^\delta = \{x : d(x, \Gamma) < \delta\}$$

$$\Omega_+^\delta = \Omega_-^\delta \dot{\cup} \Gamma^\delta > \delta$$

$A, B \subset \mathbb{R}^d$, $A + B = \{a + b : a \in A, b \in B\}$ Note: $\Omega + B_\delta \subset \Omega_+^\delta$, $\Omega_-^\delta + B_\delta \subset \Omega$.

Brunn-Minkowski: $A, B \subset \mathbb{R}^2$ meas., $A + B$ meas.

$$|A + B|^{\frac{1}{2}} \geq |A|^{\frac{1}{2}} + |B|^{\frac{1}{2}}$$

$$|\Omega_-^\delta| \geq (|\Omega|^{\frac{1}{2}} + |B_\delta|^{\frac{1}{2}})^2 \geq |\Omega| + 2|\Omega|^{\frac{1}{2}} \underbrace{|B_\delta|^{\frac{1}{2}}}_{=(\pi\delta^2)^{\frac{1}{2}}}$$

$$|\Omega| \geq (|\Omega_-^\delta|^{\frac{1}{2}} + |B_\delta|^{\frac{1}{2}})^2 \geq |\Omega_-^\delta| + 2|\Omega_-^\delta|^{\frac{1}{2}}|B_\delta|^{\frac{1}{2}}$$

$$|\Gamma^\delta| \geq |\Omega| + 2|\Omega|^{\frac{1}{2}}\sqrt{\pi} - |\Omega_-^\delta| + |\Omega_-^\delta|^{\frac{1}{2}}\sqrt{\pi}$$

$$\limsup_{\delta \rightarrow 0^+} \frac{|\Gamma^\delta|}{2\delta} \geq 2|\Omega|^{\frac{1}{2}}\sqrt{\pi}$$

$$4\pi|\Omega| \leq M^*(\Gamma)^2 \leq l(\Gamma)^2$$

Note, that only in the very last inequality did we use the rectifiability of Γ . □

Brunn-Minkowski ineq. (\mathbb{R}^d) $A, B \subset \mathbb{R}^d$ measurable. $A + B = \{a + b : a \in A, b \in B\}$. $\lambda A = \{\lambda a : a \in A\}$ ($\lambda > 0$).

Q.: Can $|A + B|$ be controlled in terms of $|A|, |B|$? No! There exist sets A, B $|A| = |B| = 0$ with $|A + B| > 0$. Example $[0, 1] \times [0, 1]$. Another example $A = B = C \subset [0, 1]$ Cantor set. Then $A + B = [0, 2]$.

Q.: Can $|A + B|^\alpha \geq c_\alpha(|A|^\alpha + |B|^\alpha)$ hold? (for some $\alpha > 0$ with $c_\alpha < \infty$, indep of A, B) Best possible $c_\alpha = 1$.

What about α ? Convex sets play a role. A = convex, $B = \lambda A$. $|B| = |\lambda A| = \lambda^d |A|$. $|A + B| = |A + \lambda A| = |(1 + \lambda)A| = (1 + \lambda)^d |A|$ because A is convex.

($\lambda_1 A + \lambda_2 A = (\lambda_1 + \lambda_2)A$ iff A is convex.)

$|A + B|^\alpha \geq |A|^\alpha + |B|^\alpha$ iff $(1 + \lambda)^{d\alpha} \geq 1 + \lambda^{d\alpha} \Rightarrow \alpha \geq \frac{1}{d}$.

$(a + b)^\gamma \geq a^\gamma + b^\gamma \forall a, b \geq 0, \gamma \geq 1$.

Candidate inequality:

$$|A + B|^{\frac{1}{d}} \geq |A|^{\frac{1}{d}} + |B|^{\frac{1}{d}}$$

(BM)

A, B measurable $\Rightarrow A + B$ measurable. Take $[0, 1] \times \text{nonmeasurable}$.

(i) A, B closed $\Rightarrow A + B$ measurable

(ii) A, B compact $\Rightarrow A + B$ compact

(iii) A, B open $\Rightarrow A + B$ open

Theorem. (BM) holds if $A, B, A + B$ measurable.

(i) A, B rectangles with sidelengths $\{a_j\}_{j=1}^\infty, \{b_j\}_{j=1}^\infty$

(ii) A, B unions of finitely many rectangles with disjoint interiors.

(iii) A, B open sets of finite measure

(iv) A, B compact

(v) $A, B, A + B$ measurable.

Proof. (i) (BM) becomes

$$\prod_{j=1}^d (a_j + b_j)^{\frac{1}{d}} \geq \prod_{j=1}^d a_j^{\frac{1}{d}} + \prod_{j=1}^d b_j^{\frac{1}{d}}$$

$a_j \rightarrow \lambda_1 a_j, b_j \rightarrow \lambda_1 b_j$. Both sides are multiplied by $(\lambda_1 \lambda_2 \dots \lambda_d)^{\frac{1}{d}}$: wlog can assume $a_j + b_j = 1 \forall j$ (Choose $\lambda_j = a_j + b_j$)

AMGM:

$$\prod_{j=1}^d a_j^{\frac{1}{d}} \leq \frac{1}{d} \sum_{j=1}^d a_j$$

$$\prod_{j=1}^d b_j^{\frac{1}{d}} \leq \frac{1}{d} \sum_{j=1}^d b_j$$

$$\prod_{j=1}^d a_j^{\frac{1}{d}} + \prod_{j=1}^d b_j^{\frac{1}{d}} \leq \frac{1}{d} \sum_{j=1}^d (a_j + b_j) = 1$$

(ii) Induction on n = number of rectangles in A and B . Choose pair of disjoint rectangles R_1, R_2 in A . Can rotate s.t. R_1 and R_2 are separated by hyperplane $\{x_j = 0\}$. R_1 lies in $A_+ = A \cap \{x_j \geq 0\}$, $A_- = A \cap \{x_j \leq 0\}$.

Rem.: Both A_+, A_- contain at least one less rectangle than A , $A = A_+ \cup A_-$ and $A_B \cap A_-$ has measure zero.

Now: translate B s.t. B_- and B_+ satisfy

$$\frac{|B_\pm|}{|B|} = \frac{|A_\pm|}{|A|}$$

$(A_+ + B_+) \cup (A_- + B_-) \subset A + B$ Number of rectangles in A_+ and B_+ , number of rectangles in A_- and B_- is $< n$.

$$\begin{aligned} |A + B| &\geq |A_+ + B_+| + |A_- + B_-| \geq (|A_+|^{\frac{1}{d}} + |B_+|^{\frac{1}{d}})^d + (|A_-|^{\frac{1}{d}} + |B_-|^{\frac{1}{d}})^d \\ &= (|A_+|(1 + (\frac{|B_+|}{|A_+|})^{\frac{1}{d}}))^d + |A_-|(1 + (\frac{|B_-|}{|A_-|})^{\frac{1}{d}})^d = (|A_+| + |A_-|)(1 + (\frac{|B|}{|A|})^{\frac{1}{d}})^d \\ &= (|A|^{\frac{1}{d}} + |B|^{\frac{1}{d}})^d. \end{aligned}$$

- (iii) Open sets of finite measure A, B . $\forall \varepsilon > 0 \exists A_\varepsilon, B_\varepsilon$ finet unions of parallel rectangles with disjoint interiors. $A_\varepsilon \subset A, B_\varepsilon \subset B, |A| \leq |A_\varepsilon| + \varepsilon, |B| \leq |B_\varepsilon| + \varepsilon$.
 $|A + B| \geq |A_\varepsilon + B_\varepsilon| \geq (|A_\varepsilon|^{\frac{1}{d}} + |B_\varepsilon|^{\frac{1}{d}})^d \geq ((|A| - \varepsilon)^{\frac{1}{d}} + (|B| - \varepsilon)^{\frac{1}{d}})^d$. Let $\varepsilon \rightarrow 0^+$, done.
- (iv) A, B compact. Let $A^\varepsilon = \{x : d(x, A) < \varepsilon\}$. $A + B \subset A^\varepsilon + B^\varepsilon \subset (A + B)^{2\varepsilon}$
- (v) $A, B, A + B$ measurable: use inner regularity of Lebesgue measure.

□

Remark. A, B open sets of finite positive measure. Equality in (BM) iff A, B convex and similar.
 $\exists \delta > 0 \exists h \in \mathbb{R}^d : A = \delta B + h$ (A convex iff $\lambda_1 A + \lambda_2 A = (\lambda_1 + \lambda_2)A$)

Consequences for isoperimetric inequality $A \subset \mathbb{R}^d$ bounded open with smooth boundary.
 $(\partial A, B \subset \mathbb{R}^d \text{ ball } |B| = |A|)$

$$|\partial A| = \lim_{\varepsilon \rightarrow 0^+} \frac{|A + \varepsilon B| - |A|}{\varepsilon}$$

Isoper ineq.: $|\partial A| \geq |\partial B|$.

Proof.

$$\frac{|A + \varepsilon B| - |A|}{\varepsilon} \geq \frac{(|A|^{\frac{1}{d}} + |\varepsilon B|^{\frac{1}{d}})^d - |A|}{\varepsilon} = \frac{(1 + \varepsilon)^d - 1}{\varepsilon} |B| \rightarrow d|B| = |\partial B|$$

for $\varepsilon \rightarrow 0$.

□

Better: $A \subset \mathbb{R}^d$ has finite perimeter ($\iff 1_A \in \text{BV}(U), U \subset \mathbb{R}^d$ bdd open)

$$\frac{\mathcal{H}^{d-1}(\partial A)}{|A|^{\frac{d-1}{d}}} \geq \frac{\mathcal{H}^{d-1}(S^{d-1})}{|B^d(0, 1)|^{\frac{d-1}{d}}}$$

Hausdorff measure Q: How does a set replicate under scaling? $E \rightarrow nE = E_1 \cup \dots \cup E_m$ disjoint congruent copies of E . Examples: line $m = n^1$, square $m = n^2$, cube $m = n^3$, Cantor set $3C = C_1 \cup C_2$ $2 = 3^\alpha \iff \alpha = \frac{\log 2}{\log 3}$

$\#(\varepsilon)$ = least $\#$ of segments that arise from such polygonal lines. Γ rectifiable iff $\#(\varepsilon) \sim \varepsilon^{-1}$ as $\varepsilon \rightarrow 0^+$. If $\#(\varepsilon) \sim \varepsilon^{-\alpha}$ ($\alpha > 1$) In this case, say " Γ has dim α ". Snowflake has $\alpha = \frac{\log 4}{\log 3} > 1$.

Upshot: E $\alpha > 1$. $m_\alpha(E)$ = α -dimensional mass of E among sets of "dimension" α .

- $\alpha > \dim(E) \Rightarrow m_\alpha(E) = 0$
- $\alpha < \dim(E) \Rightarrow m_\alpha(E) = \infty$
- $\alpha = \dim(E)$ interesting

R. Gardner Bulletin AMS more about Brunn-Minkowski, geometrically, including more proofs, e.g. with induction of the dimension.

Hausdorff measure $E \subset \mathbb{R}^d$ any subset.

$$m_\alpha^*(E) := \lim_{\delta \rightarrow 0^+} \inf \left\{ \underbrace{\sum_k (\text{diam} F_k)^\alpha : E \subset \bigcup_{k=1}^{\infty} F_k, \text{diam}(F_k) \leq \delta}_{H_\alpha^\delta(E)} \right\}$$

exterior/outer α -dim Hausdorff measure.

Remark. $H_\alpha^\delta(E) \leq H_\alpha^\delta(E) \leq m_\alpha^*(E) (\forall \delta > 0)$. $H_\alpha^\delta(E)$ increases when δ decreases. $\therefore m_\alpha^*(E) = \lim_{\delta \rightarrow 0^+} H_\alpha^\delta(E)$ exists

Remark. Coverings must be by sets of arb. small measure. (If we allowed the δ to be arbitrary then two parallel lines would get the same 1d-measure as one of them.)

Remark (Scaling). "The measure of a set should scale like its dimension". E.g.: $\Gamma \subset \mathbb{R}^d$ smooth curve of length L *sim* $\lambda\Gamma$ has length λL . $Q \subset \mathbb{R}^d$ cube *sum* λQ has measure $\lambda^d |Q|$. $|F|$ scaled by $\lambda \Rightarrow (\text{diam} F)^\alpha$ scaled by λ^α

Properties

- (i) $E_1 \subset E_2 \Rightarrow m_\alpha^*(E_1) \leq m_\alpha^*(E_2)$
- (ii) $\{E_j\} \subset \mathbb{R}^d$ countable family of sets $\Rightarrow m_\alpha^*(\bigcup_{j=1}^{\infty} E_j) \leq \sum_{j=1}^{\infty} m_\alpha^*(E_j)$
- (iii) (Finite additivity) $\inf_{x \in E_1, y \in E_2} |x - y| = d(E_1, E_2) > 0 \Rightarrow m_\alpha^*(E_1 \cup E_2) = m_\alpha^*(E_1) + m_\alpha^*(E_2)$

Proof. ETS \geq . Fix $0 < \varepsilon < d(E_1, E_2)$. Given any cover of $E_1 \cup E_2$ with sets F_1, F_2, \dots of $\text{diam} \leq \delta < \varepsilon$, let $F'_j = F_j \cap E_1$, $F''_j = F_j \cap E_2$.

$$\sum (\text{diam}_j F'_j)^\alpha + \sum (\text{diam}_j F''_j)^\alpha \leq \sum_k (\text{diam}(F_k))^\alpha$$

Take inf over all covers, let $\delta \rightarrow 0^+$, done. □

m_α^* satisfies all properties of a Caratheodory outer measure $\therefore m_\alpha^*$ is a countably additive measure when restricted to Borel sets, call it $m_\alpha = \alpha$ -dim Hausdorff measure.

- (iv) $\{E_j\}$ countable family of disjoint Borel sets \Rightarrow

$$m_\alpha(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} m_\alpha(E_j)$$

- (v) Hausdorff measure is invariant under translation and rotations. It scales like:

$$m_\alpha(\lambda E) = \lambda^\alpha m_\alpha(E)$$

- (vi) $m_0(E) = \#E$, $m_1(E) = |E|$ (=1D Lebesgue measure of E), $E \subset \mathbb{R}$ Borel.

- (vii) $E \subset \mathbb{R}^d$ Borel, $m_\alpha(E) \simeq |E|$

Proof. (i) Isodiametric inequality: $|E| \leq v_d \left(\frac{\text{diam} E}{2}\right)^d$, v_d volume of the unit ball in \mathbb{R}^d .
Prove first for sets $E = -E$ and then something hard.

- (ii) Covering argument: Given $\varepsilon, \delta > 0$, there exists a covering of E by balls $\{B_j\}$:
 $\text{diam} B_j < \delta$, $\sum_j |B_j| \leq |E| + \varepsilon$

$$H_d^\delta(E) \leq \sum_j (\text{diam} B_j)^d = c_d \sum_j |B_j| \leq c_d(|E| + \varepsilon),$$

let $\delta, \varepsilon \rightarrow 0^+$, get one of the inequalities.

□

- (viii) if $m_\alpha^*(E) < \infty$ and $\beta > \alpha$, then $m_\beta^*(E) = 0$. If $m_\alpha^*(E) > 0$ and $\beta < \alpha$, then $m_\beta^*(E) = \infty$.

Proof. $\text{diam} F < \delta, \beta > \alpha \Rightarrow (\text{diam} F)^\beta = (\text{diam} F)^{\beta-\alpha} (\text{diam} F)^\alpha < \delta^{\beta-\alpha} (\text{diam} F)^\alpha$ □

Consequence: Given $E \subset \mathbb{R}^d$ Borel, $\exists! \alpha$ such that

$$m_\beta(E) = \begin{cases} \infty & \beta < \alpha \\ 0 & \beta > \alpha \end{cases}$$

$$\alpha = \sup\{\beta : m_\beta(E) = \infty\} = \inf\{\beta : m_\beta(E) = 0\} := \text{Hausdorff dimension of } E = \dim E$$

At the critical value $\alpha = \dim E$ $0 \leq m_\alpha(E) \leq \infty$. If E is bounded and the inequalities are strict, we say that E has strict Hausdorff dimension α .

Theorem. The Cantor set $C \subset [0, 1]$ has strict Hausdorff dimension $\frac{\log 2}{\log 3}$.

ETS: $0 < m_\alpha(C) \leq 1$

Proof. $m_\alpha(C) \leq 1$: $C = \bigcap C_k$ where each C_k is a finite union of 2^k intervals of length 3^{-k} . Given $\delta > 0$ choose k large enough such that $3^{-k} < \delta$. C_k covers C and consists of 2^k intervals of diameter $3^{-k} < \delta$. $H_\alpha^\delta(C) \leq 2^k (3^{-k})^\alpha = 1$, let $\delta \rightarrow 0^+$, done.

$$m_\alpha(C) > 0:$$

Lemma. $E \subseteq \mathbb{R}^d$ compact, $f : E \rightarrow \mathbb{R}$ γ -Hölder,

$$|f(x) - f(y)| \leq m|x - y|^\gamma \quad (\forall x, y \in E) \quad 0 < \gamma \leq 1$$

Then

$$(i) \quad m_\beta(f(E)) \leq M^\beta m_\alpha(E) \text{ if } \beta = \frac{\alpha}{\gamma}.$$

$$(ii) \quad \dim f(E) \leq \frac{1}{\gamma} \dim(E)$$

Proof. $\{F_k\}$ countable family of sets that covers E : $\{f(F_k \cap E)\}$ covers $f(E)$. $\text{diam} f(F_k \cap E) \leq M(\text{diam} F_k)^\gamma$.

$$\sum_k (\text{diam} f(F_k \cap E))^{\frac{\alpha}{\gamma}} \leq M^{\frac{\alpha}{\gamma}} \sum_k (\text{diam} F_k)^\alpha,$$

done. and 1 implies 2. □

Lemma. The Cantor-Lebesgue function $F : C \rightarrow [0, 1]$ is $\gamma = \frac{\log 2}{\log 3}$ -Hölder.

Proof. Goal: $|F(x) - F(y)| \leq c|x - y|^\gamma \forall x, y \in C$.

F_n increases at most 2^{-n} on an interval of length 3^{-n} . \therefore slope $\leq (\frac{3}{2})^n \therefore |F_n(x) - F_n(y)| \leq (\frac{3}{2})^n |x - y|$. $|F_n(x) - F(x)| \leq 2^{-n}$. Given x, y chose n : $3^n |x - y| \sim 1$, $3^\gamma = 2$.

$$|F(x) - F(y)| \leq |F_n(x) - F_n(y)| + |F_n(x) - F(x)| + |F_n(y) - F(y)| \leq (\frac{3}{2})^n |x - y| + 2 \cdot 2^{-n} \leq c 2^{-n} = c(3^{-n})^\gamma \leq c' |x - y|^\gamma$$

□

Apply Lemma 1 with $E = C$, $f = F$, $\gamma = \frac{\log 2}{\log 3} \Rightarrow 1 = m_1([0, 1]) \leq M m_\alpha(C)$, $\dim C = \frac{\log 2}{\log 3}$

□

Rectifiable curves

Theorem. $\gamma : [a, b] \rightarrow \mathbb{R}^d$ continuous and simple. Then γ is rectifiable iff $\Gamma = \{\gamma(t) : a \leq t \leq b\}$ has strict Hausdorff dimension equal to 1. $m_1(\Gamma) = l(\gamma)$.

Proof. \Rightarrow : Let γ be rectifiable of length L . Consider arclength parametrization $\tilde{\gamma}$. $\Gamma = \{\tilde{\gamma}(s) : 0 \leq s \leq L\}$.

$$|\tilde{\gamma}(s_1) - \tilde{\gamma}(s_2)| \leq |s_1 - s_2|$$

By Lemma 1 (i) $m_1(\Gamma) \leq L$. Wh $m_1(\Gamma) \geq L$?

$$\Gamma_j = \{\gamma(t) : t_j \leq t \leq t_{j+1}\}$$

$$\Gamma = \bigcup_{j=1}^{N-1} \Gamma_j \quad m_1(\Gamma) = \sum_{j=0}^{N-1} m_1(\Gamma_j)$$

Claim: $m_1(\Gamma_j) \geq l_j := |\gamma(t_j) - \gamma(t_{j+1})|$

Proof. $\pi : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2$ $(x, y) \mapsto x$ Lipschitz, $\pi(\Gamma_j) \subset [0, l_j]$ Lemma 1 (i) implies the claim.

□

$$\therefore m_1(\Gamma) \geq \sum m_1(\Gamma_j) \geq \sum l_j, \quad L := \sup_p \sum l_j : m_1(\Gamma) \geq L, \text{ done.}$$

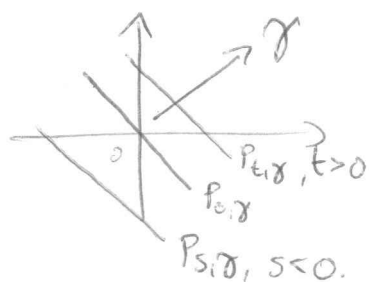
□

Radon transforms

$$R(f)(t, \gamma) = \int_{P_{t, \gamma}} f. \quad \text{where ...}$$

$$f: \mathbb{R}^d \rightarrow \mathbb{R}, \quad t \in \mathbb{R}, \quad \gamma \in S^{d-1} \subseteq \mathbb{R}^d$$

$$P_{t, \gamma} = \{x \in \mathbb{R}^d : \underbrace{x \cdot \gamma}_{\text{inner pr.}} = t\} \quad \text{hyperplane}$$



$P_{t, \gamma}$ equipped with natural $(d-1)$ -dim. Leb. measure, denoted by m_{d-1} (coincides with $(d-1)$ -Hausdorff measure)

Remarks (i) $f \in C_0^\infty(\mathbb{R}^d) \Rightarrow f$ integrable on every $P_{t, \gamma} \Rightarrow R(f)(t, \gamma)$ defined for every (t, γ) .
($R(f)$ cont. fct. of (t, γ) ,
cpctly spp. in t)

(ii) $f \in L^1(\mathbb{R}^d) \Rightarrow f$ may fail to be measurable/integrable on some $P_{t, \gamma}$ ($\rightarrow R(f)(t, \gamma)$ not defined.)

(iii) $f = \chi_E$ ($E \subset \mathbb{R}^d$ mb.) $\Rightarrow R(f)(t, \gamma) = m_{d-1}(E_{t, \gamma})$
if $E_{t, \gamma}$ measurable.
 $E_{t, \gamma} = E \cap P_{t, \gamma}$

Look instead at maximal Radon transform:

$$R^*(f)(\gamma) = \sup_{t \in \mathbb{R}} |R(f)(t, \gamma)|.$$

\rightarrow Want to study L^p -mapping properties of R in order to study regularity of subsets of \mathbb{R}^d .

Thm. 1 $f \in C^0(\mathbb{R}^d)$, $n \geq 3$:

(*) $\int_{S^{d-1}} R^*(f)(\gamma) d\sigma_\gamma \leq C(\|f\|_{L^1(\mathbb{R}^d)} + \|f\|_{L^2(\mathbb{R}^d)}).$

~~the f needs~~

Rem. i) necessary conditions:

•) $f \in L^1$: $f(x) = (1+|x|^{d-1})^{-1} \in (L^2 \setminus L^1)(\mathbb{R}^d)$ if $d \geq 3$
 f is not integrable in any plane $P_{t,\gamma}$.
($\rightarrow f \in L^1$ gives global control.)

•) $f \in L^2$: $f_\varepsilon(x) = (|x| + \varepsilon)^{-d+\delta}$ if $|x| \leq 1$, $\delta \in (0,1)$ fixed.
Let $\varepsilon \rightarrow 0^+$ to see that (*) fails if $\|\cdot\|_{L^2}$
on the RHS is not there
($\rightarrow f \in L^2$ gives local control.)

Key: Interplay between Radon and Fourier transform.
 $t \mapsto \lambda \in \mathbb{R}$ dual variable.

Fourier transform:

$$\hat{R}(f)(\lambda, \gamma) = \int_{-\infty}^{\infty} R(f)(t, \gamma) e^{-2\pi i \lambda t} dt$$

Lemma 1 $f \in C^0(\mathbb{R}^d)$, $\gamma \in S^{d-1}$:

$$\hat{R}(f)(\lambda, \gamma) = \hat{f}(\lambda \gamma).$$

Proof $\hat{f}(\lambda \gamma) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot (\lambda \gamma)} dx$

$$= \int_{-\infty}^{\infty} \left(\int_{\mathbb{R}^{d-1}} f(u, t) du \right) e^{-2\pi i \lambda t} dt = \int_{-\infty}^{\infty} \left(\int_{P_{t,\gamma}} f \right) e^{-2\pi i \lambda t} dt.$$

Choose coordinates

$$x = (u, t), \quad t = x \cdot \gamma = x_d \in \mathbb{R},$$

$$u = (x_1, \dots, x_{d-1}) \in \mathbb{R}^{d-1}.$$

□

Lemma 2 $f \in C_0^\infty(\mathbb{R}^d)$: (2)

$$\int_{S^{d-1}} \left(\int_{-\infty}^{\infty} |\hat{F}(f)(\lambda, y)|^2 |\lambda|^{d-1} d\lambda \right) d\sigma_y = 2 \int_{\mathbb{R}^d} |f(x)|^2 dx$$

Proof $2 \int_{\mathbb{R}^d} |f(x)|^2 dx = 2 \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 d\xi$
 $= 2 \int_{S^{d-1}} \left(\int_0^\infty |\hat{f}(\lambda y)|^2 \lambda^{d-1} d\lambda \right) d\sigma_y$
 $= 2 \int_{S^{d-1}} \left(\int_{-\infty}^\infty |\hat{f}(\lambda y)|^2 |\lambda|^{d-1} d\lambda \right) d\sigma_y$
 done by Lemma 2. \square

Lemma 3 Suppose $F(t) = \int_{-\infty}^{\infty} \hat{F}(\lambda) e^{2\pi i \lambda t} d\lambda$
 where $\sup_{\lambda \in \mathbb{R}} |\hat{F}(\lambda)| \leq A$ and $\int_{-\infty}^{\infty} |\hat{F}(\lambda)|^2 |\lambda|^{d-1} d\lambda \leq B^2$.

Then $\sup_{t \in \mathbb{R}} |F(t)| \leq C(A+B)$ (This is essential in the proof of Thm. 1)

Moreover, if $0 < \alpha < \frac{1}{2}$, then $|F(t_1) - F(t_2)| \leq C_\alpha |t_1 - t_2|^\alpha \cdot (A+B)$

($\rightarrow \alpha$ -Hölder)

Proof.

$$F(t) = \underbrace{\int_{|\lambda| \leq 1} \hat{F}(\lambda) e^{2\pi i \lambda t} d\lambda}_{\leq CA} + \underbrace{\int_{|\lambda| > 1} \hat{F}(\lambda) e^{2\pi i \lambda t} d\lambda}_{\leq \int_{|\lambda| > 1} \hat{F}(\lambda) d\lambda}$$

$$\stackrel{CS}{\leq} \left(\int_{|\lambda| > 1} |\hat{F}(\lambda)|^2 |\lambda|^{d-1} d\lambda \right)^{1/2} \cdot \underbrace{\left(\int_{|\lambda| > 1} |\lambda|^{-d+1} d\lambda \right)^{1/2}}_{< \infty \text{ if } -d+1 < -1 \Rightarrow d \geq 3. \checkmark} \leq B$$

\Rightarrow first estimate.

$$\begin{aligned}
|F(t_1) - F(t_2)| &= \int_{-\infty}^{\infty} \hat{F}(\lambda) \underbrace{(e^{2\pi i \lambda t_1} - e^{2\pi i \lambda t_2})}_{1 - |e^{2\pi i \lambda(t_1 - t_2)}|} d\lambda \\
&\leq \int_{|\lambda| \leq 1} \hat{F}(\lambda) d\lambda + \int_{|\lambda| > 1} |\hat{F}(\lambda)| |\lambda|^{-\alpha} d\lambda \\
&\leq C_\alpha A |t_1 - t_2|^\alpha \quad (\text{Lipschitz}) \\
&\quad + \int_{|\lambda| > 1} |\hat{F}(\lambda)| |\lambda|^{-\alpha} d\lambda \\
&\leq \left(\int |\hat{F}(\lambda)|^2 |\lambda|^{-d+1} d\lambda \right)^{1/2} \cdot \left(\int_{|\lambda| > 1} |\lambda|^{-d+1+2\alpha} d\lambda \right)^{1/2} \\
&\leq \infty \text{ if } \alpha < \frac{1}{2} \text{ for } d \leq 3.
\end{aligned}$$

□

Proof of Thm. For each $\gamma \in S^{d-1}$, let $F(t) = R(f)(t, \gamma)$

$$\Rightarrow \sup_{t \in \mathbb{R}} |F(t)| = R^*(f)(\gamma).$$

$$\text{Let } A(\gamma) = \sup_{\lambda \in \mathbb{R}} |\hat{F}(\lambda)|, \quad B^2(\gamma) = \int_{-\infty}^{\infty} |\hat{F}(\lambda)|^2 |\lambda|^{d-1} d\lambda$$

$$\text{Lemma 3 } \stackrel{\text{check assumptions!}}{\Rightarrow} \sup_{t \in \mathbb{R}} |F(t)| \leq C(A(\gamma) + B(\gamma))$$

$$\text{Lemma 1 } \Rightarrow \hat{F}(\lambda) = \hat{R}(f)(\lambda, \gamma) = \hat{f}(\lambda \gamma) \Rightarrow A(\gamma) \leq \|f\|_{L^1(\mathbb{R}^d)}$$

$$\text{Lemma 2 } \Rightarrow \int_{S^{d-1}} B^2(\gamma) d\sigma_\gamma = 2 \|f\|_{L^2(\mathbb{R}^d)}^2.$$

$$\text{We have } \sup_{t \in \mathbb{R}} |F(t)|^2 \leq C^2 (A^2(\gamma) + B^2(\gamma))$$

Integrate both sides:

$$\int_{S^{d-1}} R^*(f)(\gamma)^2 d\sigma_\gamma \lesssim \underbrace{\int A^2(\gamma) d\sigma_\gamma}_{\approx \|f\|_1^2} + \underbrace{\int B^2(\gamma) d\sigma_\gamma}_{\approx \|f\|_{L^2}^2}$$

Use Hölder, because on cpt. space:

$$\lesssim (\|f\|_{L^1}^2 + \|f\|_{L^2}^2)^2$$

$$\|R^*(f)\|_{L^1}^2 \lesssim \int_{S^{d-1}} R^*(f)(\gamma)^2 d\sigma_\gamma.$$

□

Regularity of sets when $d \geq 3$.

(3)

$E \subset \mathbb{R}^d$ meas. $E_{t,\gamma} = E \cap P_{t,\gamma}$ (t varies, γ fixed)

Fubini $\Rightarrow E_{t,\gamma}$ is m_{d-1} -measurable for a.e. t

$t \mapsto m_{d-1}(E_{t,\gamma})$ is a measurable fctn of t .

Thm. 2 $E \subset \mathbb{R}^d$ ($d \geq 3$) of finite measure.

Then for a.e. $\gamma \in S^{d-1}$:

(i) $E_{t,\gamma}$ is m_{d-1} -measurable for every t .

(ii) $t \mapsto m_{d-1}(E_{t,\gamma})$ is a cont. fct. of t .

Moreover, this fctn is α -Hölder $\forall \alpha \in (0, \frac{1}{2})$.

Cor. $d \geq 3$, $E \subset \mathbb{R}^d$ of Lebesgue measure zero.

Then, for a.e. $\gamma \in S^{d-1}$, the slice $E_{t,\gamma}$ has zero measure for every $t \in \mathbb{R}$.

Prop. $d \geq 3$, $f \in (L^1 \cap L^2)(\mathbb{R}^d)$. Then for a.e.

$\gamma \in S^{d-1}$:

i) f is meas. and int. on the plane for every $t \in \mathbb{R}$.

ii) $t \mapsto \mathcal{R}(f)(t,\gamma)$ is cont. and α -Hölder if $\alpha < \frac{1}{2}$.

Moreover, estimate (*) from Thm. 1 holds for f .

Rem. Prop. implies Thm. 2 by taking char. fctn of E .

$$\mathcal{R}(\chi_E)(t,\gamma) = m_{d-1}(E_{t,\gamma}).$$

We skip the proof of Prop. (follows from Thm. 1 using some delicate measure theory)

What about $d=2$?

Given $f \in L^1(\mathbb{R}^2)$, define

$$\mathcal{R}_\delta(f)(t,\gamma) = \frac{1}{2\delta} \int_{t-\delta}^{t+\delta} \mathcal{R}(f)(s,\gamma) ds. \quad (\text{averaged version of } \mathcal{R})$$

(integration over thickened line = hypplane.)

$$= \frac{1}{2\delta} \int_{\{t-\delta \leq x \cdot \gamma \leq t+\delta\}} f(x) dx.$$

Thm. 3 $f \in C_0^\infty(\mathbb{R}^2)$, $0 < \delta \leq 1/2$.

$$\int_{S^1} R_\delta^*(f)(\gamma) d\sigma_\gamma \lesssim (\log \frac{1}{\delta})^{1/2} \cdot (\|f\|_{L^1(\mathbb{R}^2)} + \|f\|_{L^2(\mathbb{R}^2)})$$

Theorem. $f \in C_0^0(\mathbb{R}^2)$, $0 < \delta \leq \frac{1}{2}$. Then

$$\int_{S^1} R_\delta^*(f)(\gamma) d\sigma_\gamma \lesssim (\log \frac{1}{\delta})^{\frac{1}{2}} (\|f\|_{L^1(\mathbb{R}^2)} + \|f\|_{L^2(\mathbb{R}^2)})$$

Proof of Theorem. Modified version of lemma 3: Setting

$$F_\delta(t) = \int_{-\infty}^{\infty} \hat{F}(\lambda) \left(\frac{e^{2\pi i(t+\delta)\lambda} - e^{2\pi i(t-\delta)\lambda}}{2\pi i\lambda(2\delta)} \right) d\lambda$$

Suppose $\sup_{\lambda \in \mathbb{R}} |\hat{F}(\lambda)| \leq A$ and $\int_{-\infty}^{\infty} |\hat{F}(\lambda)|^2 |\lambda|^{d-1} d\lambda \leq B^2$.

Claim:

$$\sup_t |F_\delta(t)| \lesssim (\log \frac{1}{\delta})^{\frac{1}{2}} (A + B)$$

$$F_\delta(t) = \int_{-\infty}^{\infty} \hat{F}(\lambda) \left(\frac{e^{2\pi i(t+\delta)\lambda} - e^{2\pi i(t-\delta)\lambda}}{2\pi i\lambda(2\delta)} \right) d\lambda = I + II$$

CS:

$$I \lesssim \left(\int_{\mathbb{R}} |\hat{F}(\lambda)|^2 |\lambda| d\lambda \right)^{\frac{1}{2}} \left(\int_{1 < |\lambda| \leq \frac{1}{\delta}} |\lambda|^{-1} d\lambda \right)^{\frac{1}{2}} \leq B (\log \frac{1}{\delta})^{\frac{1}{2}}$$

$$II \lesssim \frac{c}{\delta} \left(\int_{\mathbb{R}^2} |\hat{F}(\lambda)|^2 |\lambda| d\lambda \right)^{\frac{1}{2}} \left(\int_{|\lambda| > \frac{1}{\delta}} |\lambda|^{-3} d\lambda \right)^{\frac{1}{2}} \lesssim B$$

□

Theorem. There exists a subset $K \subset \mathbb{R}^2$ such that

- (i) K is compact
- (ii) K has Lebesgue measure zero
- (iii) K contains a translate of every unit line segment

Theorem. Suppose F is any set that satisfies conditions (i) and (iii) from Theorem 1. Then F has Hausdorff dimension 2.

Proof of Theorem 2. Let F be a Kakeya set. Fix $0 < \alpha < 2$. Let $F \subset \bigcup_{i=1}^{\infty} B_i$ be a covering with balls B_i of diameter $\leq \delta$. It is enough to show

$$\sum (\text{diam} B_i)^\alpha \geq c_\alpha > 0$$

for $\alpha < 2$.

Case 1: Assume $\text{diam} B_1 = \delta \leq \frac{1}{2}$ and let $N < \infty$ be the number of balls in the covering. WTS $N\delta^\alpha \geq c_\alpha$. B_i^* = double of B_i . $F^* = \bigcup_i B_i^*$. $|F^*| \leq \sum |B_i^*| = cN\delta^2$. F Kakeya $\Rightarrow \forall \gamma \in S^1 \exists s_\gamma \perp \gamma$ unit line segment: $s_\gamma \subset F$. $s_\gamma^\delta \subset F^*$. $\therefore R_\delta^*(\chi_{F^*})(\gamma) \geq 1$ ($\forall \gamma \in S^1$). Take $f = \chi_{F^*}$ in (*). Since $L^2 \subset L^1$,

$$\|\chi_{F^*}\|_{L^1} \lesssim \|\chi_{F^*}\|_{L^2} = |F^*|^{\frac{1}{2}} \lesssim N^{\frac{1}{2}} \delta.$$

(*) $\Rightarrow 0 < c \leq (\log \frac{1}{\delta})^{\frac{1}{2}} N^{\frac{1}{2}} \delta$. This implies $N\delta^\alpha \geq c_\alpha > 0$.

Case 2: General case. $F \subset \bigcup_{i=1}^{\infty} B_i$ with each ball B_i of diameter ≤ 1 . For each $k \in \mathbb{N}$, let N_k be the number of balls in $\{B_i\}$ with diameter $B_k \sim 2^{-k}$, i.e. $\in [2^{-k-1}, 2^{-k}]$. WTS

$$\sum_{k=0}^{\infty} N_k 2^{-k\alpha} \geq c_{\alpha} > 0.$$

ETS $\exists k' : N_{k'} 2^{-k'\alpha} \geq c_{\alpha}$.

$$F_k = F \cap \left(\bigcup_{\text{diam } B_i \sim 2^{-k}} B_i \right)$$

$$F_k^* = \bigcup_{\text{diam } B_v \sim 2^{-k}} B_v^*$$

$$|F_k^*| \leq c N_k 2^{-2k} \quad \forall k$$

F Keakeya $\Rightarrow \forall \gamma \in S^2 \exists s_{\gamma} \perp \gamma : s_{\gamma} \subset F$ (in particular $m_1(s_{\gamma} \cap F) = 1$).

Key: For some k , a large proportion of s_{γ} belongs to F_k . Pick $\{a_k\}_{k=0}^{\infty}$ such that $0 \leq a_k < 1$, $\sum a_k = 1$, (a_k) does not tend to 0 too quickly, e.g. $a_k = c_{\varepsilon} 2^{-k\varepsilon}$ (for sufficiently small ε).

Claim:

$$\exists k : m_1(s_{\gamma} \cap F_k) \geq a_k.$$

Otherwise $m_1(s_{\gamma} \cap F) \leq \sum_k m_1(s_{\gamma} \cap F_k) < \sum a_k = 1$, contradicts (**)

For this value of k ,

$$R_{2^{-k}}^*(\chi_{F_k^*})(\gamma) \geq a_k.$$

Since this choice of k depends on γ , let

$$E_k = \{\gamma \in S^1 : R_{2^{-k}}^*(\chi_{F_k^*})(\gamma) \geq a_k\}.$$

$S^1 = \bigcup_{k=1}^{\infty} E_k$. Therefore $\exists k' : |E_{k'}| \geq 2\pi a_{k'}$.

$$2\pi a_{k'}^2 = 2\pi a_{k'} a_{k'} \leq \int_{E_{k'}} a_{k'} d\sigma \leq_{S^1} \int_{E_{k'}} R_{2^{-k'}}^*(\chi_{F_{k'}^*})(\gamma) d\sigma_{\gamma}$$

$$2^{-2k\varepsilon} \sim a_{k'}^2 \leq c(\log 2^{k'})^{\frac{1}{2}} |F_{k'}^*|^{\frac{1}{2}} \leq c(\log 2^{k'})^{\frac{1}{2}} N_{k'}^{\frac{1}{2}} 2^{-k'}$$

$\Rightarrow N_{k'} 2^{-\alpha k'} \geq c_{\alpha}$, provided $4\varepsilon < 2 - \alpha$. □

Construction of a Keakeya set I (Stein-Shakarchi, III)

Thinner Cantor set, always taking away the half.

Take two of them, E_0, E_1 , where E_1 has twice the length. Put E_0 on $y = 1$ and E_1 on $y = 0$. Let F be the union of all line segments that join a point in E_0 with one in E_1 .

Construction of an ε -Keakeya set (Stein)

Theorem. Given $\varepsilon > 0$, $\exists N = N_{\varepsilon}$ and 2^N rectangles R_1, \dots, R_{2^N} with side lengths 1×2^{-N} such that

(i)

$$\left| \bigcup_{j=1}^{2^N} R_j \right| < \varepsilon$$

(ii) the reaches \tilde{R}_j are mutually disjoint, i.e.

$$|\bigcup_{j=1}^{2^N} \tilde{R}_j| = 1$$

Proof. Fix $\alpha \in (\frac{1}{2}, 1)$. Symmetric triangle ABC with M opposite C . Push the right part into the left part and call the resulting body $\Phi(T)$. It consists of heart $\Phi_h(T)$ and arms $\Phi_a(T)$. Then

$$|\Phi_h(T)| = \alpha^2 |T|$$

$$|\Phi_a(T)| = 2(1 - \alpha)^2 |T|$$

Conclusion

$$|\Phi(T)| = (\alpha^2 + 2(1 - \alpha)^2) |T|$$

n -fold iteration (Peron trees): Split not into two but 2^n parts and do everything pairwise. Key: right side of $\Phi_h(A_0 A_2 C)$ // left side of $\Phi_n(A_2 A_4 C)$ // CA_2

Then look at heart/arms again.

$$|\text{arms of } \Psi_1(ABC)| \leq 2(1 - \alpha)^2 |T|.$$

$$|\text{heart of } \Psi_1(ABC)| = \alpha^2 |T|$$

$$\therefore |\Psi_1(ABC)| = (\alpha^2 + 2(1 - \alpha)^2) |T|.$$

Iterate: Carry out this process on the heart of $\Psi_1(ABC)$ with n replaced by $n - 1$, given are the union of 2^{n-1} triangles.

Then retranslate all 2^n original triangles to obtain figure $\Psi_2(ABC)$.

$$|\text{heart of } \Psi_2(ABC)| = \alpha^2 \alpha^2 |T|$$

$$|\text{additional arms of } \Psi_2(ABC)| \leq 2(1 - \alpha)^2 \alpha^2 |T|$$

$$\begin{aligned} |\Psi_n(ABC)| &\leq (\alpha^{2n} + 2(1 - \alpha)^2 + 2(1 - \alpha)^2 \alpha^2 + \dots + 2(1 - \alpha)^2 \alpha^{2n-2}) \\ &\leq \alpha^{2n} + 2(1 - \alpha)^2 \underbrace{\sum_{n=0}^{\infty} \alpha^{2n}}_{= \frac{1}{1 - \alpha^2}} \\ &\leq \alpha^{2n} + 2(1 - \alpha) \end{aligned}$$

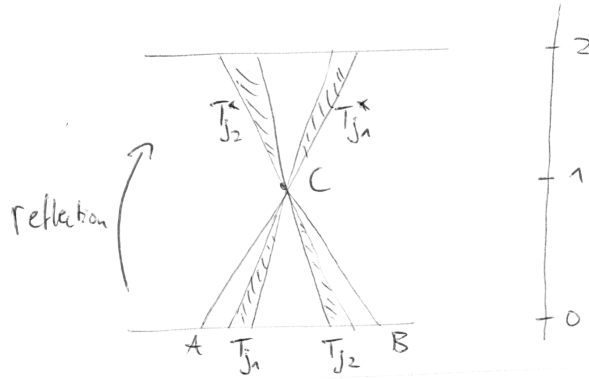


Figure 1: Obtaining mutually disjoint reaches by reflecting in C .

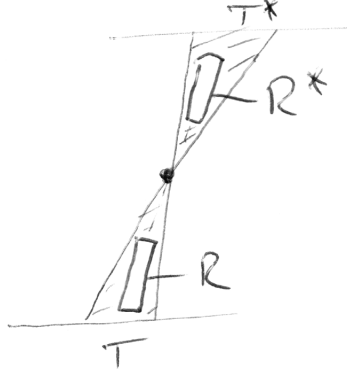


Figure 2: Go from triangles to rectangles by placing rectangles into the triangles with half the length.

□

Application Maximal functions and counterexamples. Q: Given a collection $\mathcal{C} = \{C\}$ of sets, for which class of functions do we have

$$\lim_{\text{diam}(C) \rightarrow 0, c \in \mathcal{C}} \frac{1}{|C|} \int_C f(x-y) dy = f(x) \quad x - \text{a.e.}?$$

Seen:

$$(M_{\mathcal{C}} f)(x) = \sup_{c \in \mathcal{C}} \frac{1}{|C|} \int_C |f(x-y)| dy$$

$\mathcal{C} = \{\text{balls}\}$, weak-type (1,1) inequality for $M_{\mathcal{C}} \Rightarrow$ a.e. convergence of averages. A converse also holds!

$\{\mu_j\}_{j=1}^{\infty}$ collection of finite, nonnegative measures on $\mathbb{R}^d : \text{supp}(\mu_j) \subset K \Subset \mathbb{R}^d$. Define the maximal operator

$$(Mf)(x) = \sup_j |f * \mu_j|(x).$$

Proposition. $1 \leq p < \infty$. Assume for each $f \in L^p(\mathbb{R}^d)$ that $(Mf)(x) < \infty$ for some set of x having positive measure. Then $f \mapsto Mf$ is of weak-type (p, p) , i.e.

$$\exists A < \infty : |\{x \in \mathbb{R}^d : (Mf)(x) > \alpha\}| \leq \frac{A}{\alpha^p} \|f\|_{L^p}^p \quad (\forall \alpha > 0)$$

Lemma. $\{E_j\}$ collection of subsets of a fixed compact set:

$$\sum_{j=1}^{\infty} |E_j| = \infty.$$

Then there exists a sequence of translates $F_j = E_j + x_j$:

$$\limsup F_j = \bigcap_{k=1}^{\infty} \left(\bigcup_{j=k}^{\infty} F_j \right) = \mathbb{R}^n \quad (\text{a.e.})$$

The above set equals $\{x \in \mathbb{R}^d : x \in F_j \text{ infinitely often}\}$.

$$\liminf F_j = \bigcup_{k=1}^{\infty} \left(\bigcap_{j=k}^{\infty} F_j \right)$$

is a subset.

Proof of Lemma. $Q \subset \mathbb{R}^d$ unit cube. $A_1, A_2 \subset Q$. Then $\exists h \in \mathbb{R}^d : |A_1 \cap (A_2 - h)| \geq 2^{-d} |A_1| |A_2|$. Why?

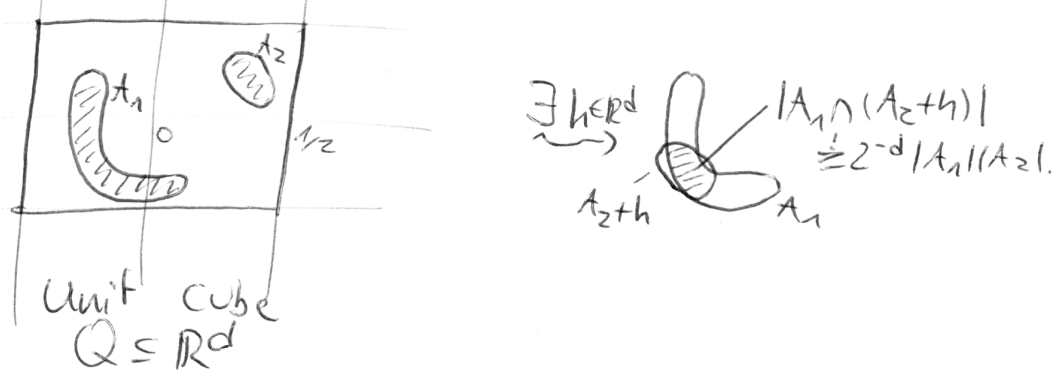


Figure 3: Translation of subsets A_1 and A_2 of a unit cube Q .

$$\eta(x) = \int_{\mathbb{R}^d} \chi_{A_1}(y) \chi_{A_2}(x+y) dy \sim \chi_{A_1} * \chi_{A_2}(x)$$

$$\int_{\mathbb{R}^d} |A_1| |A_2|$$

$$\text{supp}(\eta) \subset Q^*$$

$$|Q^*| = 2^d.$$

$$\exists h \in Q^* : \eta(h) \geq \text{avg}_{Q^*}(\eta) = \frac{1}{|Q^*|} \int_{\mathbb{R}^d} \eta = \frac{|A_1| |A_2|}{2^d}$$

Wlog $\text{supp}(\eta) \subset Q$.

Step 2: There exist translates $F_j = E_j + x_j$ that cover Q at least once.

$$Q \subset \bigcup_j F_j$$

Why? $F_1 = E_1$. Suppose (inductively) that F_1, \dots, F_{j-1} have been constructed. Let $A_1 = Q \cap (F_1 \cup \dots \cup F_{j-1})^c$ and $A_2 = E_j$. Step 1 $\Rightarrow \exists h : |A_1 \cap (A_2 - h)| \geq 2^{-d} |A_1| |A_2|$. Set $F_j = A_2 - h = E_j - h$. Let $p_j = |Q \cap (F_1 \cup \dots \cup F_j)|$. Then

$$p_j = p_{j-1} + \underbrace{|Q \cap (F_1 \cup \dots \cup F_{j-1})^c|}_{A_1} \underbrace{|E_j|}_{A_2-h} = p_{j-1} + |A_1 \cap (A_2 - h)|$$

$$\geq p_{j-1} + 2^{-d} |A_1| |E_j| = p_{j-1} + 2^{-d} (1 - p_{j-1}) |E_j|$$

$$\therefore p_j - p_{j-1} \geq 2^{-d} (1 - p_{j-1}) |E_j|$$

$$\sum_{j=2}^{\infty} (p_j - p_{j-1}) = \lim_{j \rightarrow \infty} p_j - p_1 \div \lim_j p_j = 1$$

Step 3: Decompose (twice) $\{E_j\}$ into a countable infinite number of subcollections so that on each subcollection the sum of the measures diverges. \square

Proof of Proposition. Take a ball B such that $B \supset Q + K$. $\text{supp}(F) \subset Q \Rightarrow \text{supp}(F * \mu_j) \subset \text{supp}(Mf) \subset B$. Key: Estimate (*) (the violation of the weak type estimate) holds if $\text{supp}(f) \subset Q$. For each $k, \exists \alpha_k > 0 \exists g_k \in L^p : \text{supp}(g_k) \subset Q$ such that

$$|\{x \in B : Mg_k(x) > \alpha_k\}| \geq \frac{2^k}{\alpha_k^p} \|g_k\|_{L^p}^p$$

Replace g_k by $\tilde{g}_k = \frac{k}{\alpha_k} g_k$.

$$\frac{2^k}{k^p} \leq \frac{|\{x \in B : M\tilde{g}_k(x) > k\}|}{\|\tilde{g}_k\|_{L^p}^p} \rightarrow \infty \quad \text{as } k \rightarrow \infty$$

\therefore There exists a sequence $\{f_k\} \subset L^p$ and a sequence of constants $R_k \rightarrow \infty$ such that with $E_k = \{x \in B : Mf_k(x) > R_k\}$ we get

$$\sum_k |E_k| = \infty \quad \sum_k \|f_k\|_{L^p}^p < \infty.$$

Remark. $d\mu_j \geq 0$ wlog $f_k \geq 0$.

By the lemma $\exists \{x_k\}$ such that $F_k = E_k + x_k$ satisfy $\limsup F_k = \mathbb{R}^d$ (a.e). Let

$$\tilde{f}_k(x) = f_k(x + x_k), \quad F(x) = \sup_k \tilde{f}_k(x)$$

Then

$$M(F) = \sup_j |F * \mu_j| = \sup_j |(\sup_k \tilde{f}_k) * \mu_j| \geq \sup_k \sup_j |\tilde{f}_k * \mu_j| = \sup_k M(\tilde{f}_k)$$

Also $M(\tilde{f}_k) > R_k$ on F_k . $M(F) = \infty$ a.e. Check $f \in L^p$:

$$|F|^p = |\sup_k \tilde{f}_k|^p \leq \sum_p |\tilde{f}_k|^p$$

$$\|F\|_{L^p}^p \leq \sum_k \|f_k\|_{L^p}^p < \infty$$

Full conclusion

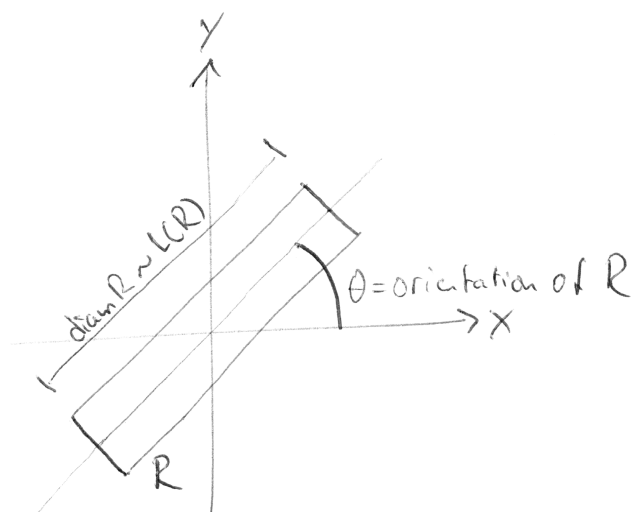
$$f = \sum f \chi_{Q_j} =: \sum f_j$$

$$M(f) \leq \sum_j M(f_j)$$

\square

Example. Rectangles with arbitrary orientation

$$\mathcal{C} = \mathcal{R} = \{\text{all rectangles in } \mathbb{R}^2 \text{ centered at } 0\}$$



Corollary. Given $1 \leq p < \infty$, $\exists f \in L^p(\mathbb{R}^n)$ such that

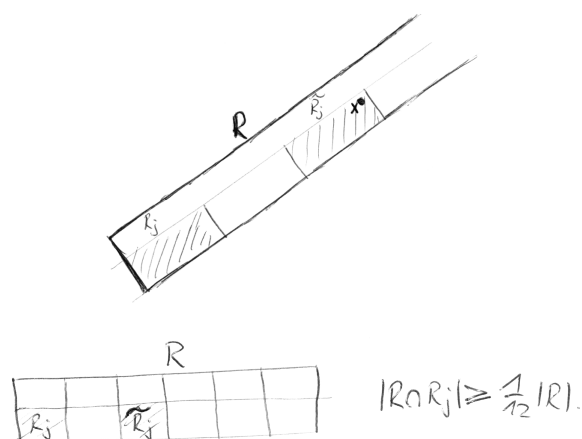
$$\limsup_{\text{diam}(R) \rightarrow 0, R \in \mathcal{R}} \frac{1}{|R|} \int_R f(x-y) dy = \infty \quad (x - \text{a.e.})$$

Idea: Use the ε -Kakeya set to show that M is not weak (p, p)

$$(Mf)(x) = \sup_{\text{diam}(R) < 8} \frac{1}{|R|} \left| \int_R f(x-y) dy \right|$$

Let $E = \bigcup_{j=1}^{2^N} R_j$ as before. $\|\chi_E\|_{L^p}^p = |E| < \varepsilon$. If $x \in \tilde{R}_j$, then \exists rectangle R such that

- R is centered at x
- $\text{diam}(R) \leq 8$
- $|R \cap R_j| \geq \frac{1}{12}|R|$



$$y \in x - E = -(E - x), \quad y - x \in -E, \quad x - y \in E$$

$$M(\chi_E)(x) \geq \int_{R-x} \chi_E(x-y) dy = \frac{|(R-x) \cap (E-x)|}{|R|} \geq \frac{1}{12}$$

Conclusion: $M\chi_E \geq \frac{1}{12}$ on the set $\bigcup_{j=1}^{2^N} \tilde{R}_j$ (of measure 1)

$$\forall A > 0 \exists \text{ set } E : |\{x \in \mathbb{R}^d : M\chi_E > A\}| \leq A\alpha^{-p} \|\chi_E\|_{L^p}^p$$

does not hold! $\therefore M$ is not of weak type (p, p) .

Note, that this is not the complete proof. Therefore still have to replace 8 by δ .

Bochner-Ries summability Q: In which way does Fourier inversion hold? (In $L^p(\mathbb{R}^d)$)

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i x \xi} d\xi = \lim_{R \rightarrow \infty} \underbrace{\int_{|\xi| < R} \hat{f}(\xi) \left(1 - \frac{|\xi|^2}{R^2}\right)^\delta e^{2\pi i x \xi} d\xi}_{f * k_\delta}?$$

for suitable $\delta \geq 0$.

$$\delta > \frac{d-1}{2} \Rightarrow k_\delta \in L^1(\mathbb{R}^d)$$

$$\delta \leq \frac{d-1}{2}$$

$$\delta = 0 \text{ today}$$

$d = 1$: The second equality holds in L^p -norm ($1 < p < \infty$) \iff L^p boundedness of Hilbert transform

It also holds a.e. ($p = 2$ Carleson '66 $1 < p < \infty$ Hunt '68)

$d \geq 2$ Let us only consider norm convergence

$$f \mapsto (S^\delta f)(x) = \int_{|\xi| \leq 1} \hat{f}(\xi) (1 - |\xi|^2)^\delta e^{2\pi i x \xi} d\xi$$

$$Sf(x) = \int_{|\xi| \leq 1} \hat{f}(\xi) e^{2\pi i x \xi} d\xi$$

$$\widehat{Sf} = 1_{|\xi| \leq 1} \hat{f}$$

$$(\widehat{Hf} = i\pi \operatorname{sgn} \xi \hat{f})$$

$$\|Sf\|_{L^2(\mathbb{R}^d)} = \|\widehat{Sf}\|_{L^2} = \|1_{|\xi| \leq 1} \hat{f}\|_{L^2} \leq \|\hat{f}\|_{L^2} = \|f\|_{L^2}$$

He said something about every bounded operator can be written like this or so? Fourier multiplier S is bounded iff multiplier function is bounded.

Theorem (C.Fefferman '71 – Annals of mathematics "The multiplier problem for the ball").
Suppose $q \geq 2$ and $p \neq 2$. Then the operator S (initially defined on $L^p \cap L^2$) is not extendable to a bounded operator from $L^p(\mathbb{R}^d)$ to itself.

Proof. Let $B \subset \mathbb{R}^d$ ball, let S_B be the multiplier operator associated to B :

$$\widehat{S_B f} = 1_B \hat{f}.$$

Given $u \in S^{d-1} \subset \mathbb{R}^d$, let S^u be the multiplier operator associated to the half-space with normal u :

$$\widehat{S^u f} = 1_{\{\xi u > 0\}} \hat{f} \quad (S^u f)(x) = \int_{\xi u > 0} \hat{f}(\xi) e^{2\pi i x \xi} d\xi$$

Upshot: L^p -bound for S implies an L^p vector-valued inequality for S_B s and S_u s.

Lemma (Y. Méyer). Suppose

$$\|Sf\|_{L^p} \leq A_p \|f\|_{L^p}$$

$f \in (L^2 \cap L^p)(\mathbb{R}^d)$ holds for some $p \in [1, \infty]$. Suppose $f_1, \dots, f_M \in L^2 \cap L^p$, $u_1, \dots, u_M \in S^{d-1} \subset \mathbb{R}^d$. Then

$$\left\| \left(\sum_{j=1}^M |S^{u_j}(f_j)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^d)} \leq A_p \left\| \left(\sum_{j=1}^M |f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^d)} \quad (4)$$

where A_p is the same constant as above.

Proof of Lemma. Step 1: $B = B_R$ = ball of radius R centered at 0. Then

$$\|S_B(f)\|_{L^p} \leq A_p \|f\|_{L^p} \quad (g \in L^2 \cap L^p) \quad (5)$$

Why? Scaling:

$$\delta_R(g)(x) = g\left(\frac{x}{R}\right)$$

Check $\delta_{R^{-1}} \circ S \circ \delta_R = S_{B_R}$ since $S = S_{B_1}$. $\hat{\delta}_\rho(g)(\xi) = R^d \hat{g}(R\xi)$.

Step 2: M balls. ($p < \infty$) $f = (f_1, \dots, f_M)$ given M -tuple of functions. $T(f) := (Tf_1, \dots, Tf_M)$. Given a unit vector $\omega = (\omega_1, \dots, \omega_M) \in \mathbb{C}^M$, let

$$S_\omega(f) = \sum_{j=1}^M \bar{\omega}_j S_B(f_j) = S_B\left(\sum_j \bar{\omega}_j f_j\right) = S_B(f_\omega) \quad f_\omega = \sum_{j=1}^M \bar{\omega}_j f_j$$

$$(5) \Rightarrow \int_{\mathbb{R}^d} |S_\omega f(x)|^p dx \leq A_p^k \int_{\mathbb{R}^d} |f_\omega(x)|^p dx \quad (6)$$

$$(S_\omega f)(x) = ?. \quad x, y \in \mathbb{C}^M, \quad \langle x, y \rangle = \sum_{i=1}^M x_i \bar{y}_i$$

$$\begin{aligned} S_\omega(f)(x) &= S_B(f_\omega)(x) = S_B\left(\sum_{j=1}^M \bar{\omega}_j f_j\right)(x) = \sum_{j=1}^M \bar{\omega}_j S_B(f_j)(x) = |\langle S_B(f)(x), \omega \rangle| \\ &= |S_B(f)(x)| \left| \left\langle \frac{S_B(f)(x)}{|S_B(f)(x)|}, \omega \right\rangle \right| = \left(\sum_{j=1}^M |S_B(f_j)(x)|^2 \right)^{\frac{1}{2}} |\varphi(\omega, S_B(f)(x))| \end{aligned}$$

Integrate both sides of (6) with respect to ω (before integrating in x).

$$\begin{aligned} \text{LHS} &= \int_{\mathbb{R}^d} \left(\int_{|\omega|=1} |S_\omega(f)(x)|^p d\omega \right) dx \\ &= \int_{\mathbb{R}^d} \left(\sum_{j=1}^M |S_B(f_j)(x)|^2 \right)^{\frac{p}{2}} \underbrace{\left(\int_{|\omega|=1} |\Phi(\omega, S_B(f)(x))|^p d\omega \right)}_{\gamma_p} dx \end{aligned}$$

$$0 \neq \gamma_p = \int_{|\omega|=1} |\Phi(\omega, 1)|^p d\omega$$

For fixed $\nu \in S^{d-1}$ $\int_{S^{d-1}} |\langle \omega, \nu \rangle|^p d\sigma_\omega = \omega_{d-2} \int_{-1}^1 (1-t^2)^{\frac{d-3}{2}} dt$.

Similarly,

$$\text{RHS} = \int_{\mathbb{R}^d} \left(\sum_{j=1}^M |f_j(x)|^2 \right)^{\frac{p}{2}} dx \gamma_p$$

$$(6) \Rightarrow \left\| \left(\sum_{j=1}^M |S_B(f_j)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^d)} \leq A_p \left\| \left(\sum_{j=1}^M |f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^d)}$$

Step 3: From balls to half-spaces

B_R^u = ball of radius R centered at Ru . Upshot: $B_R^u \rightarrow \{\xi u > 0\}$ as $R \rightarrow \infty$.

Translation

$$(T_y f)(x) = f(x - y)$$

$$\widehat{T_y f}(\xi) = e^{i\xi y} \hat{f}(\xi)$$

$$S_{B_R^u}(f)(x) = e^{2\pi i u R x} S_{B_R}(f e^{-2\pi i u R x})$$

(4) implies

$$\left\| \left(\sum |S_{B_R^{u_j}}(f_j)|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \leq A_p \left\| \left(\sum |f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^p}$$

Let $R \rightarrow \infty$ to finish:

$$\therefore S_{B_R^{u_j}}(f_j) \rightarrow S^{u_j}(f_j) \quad R \rightarrow \infty \text{ (in } L^2)$$

\therefore there exists an almost everywhere converging subsequence, done. \square

$\widehat{Sf} = 1_{B(0,1)} \hat{f}$. S is not bounded in $L^p(\mathbb{R}^d)$ unless $d = 1$ or $p = 2$. Focus on multiplier operator for the half-space (S^u) , $d = 1$.

$$(S^+ f)(x) = \int_0^\infty \hat{f}(\xi) e^{2\pi i x \xi} d\xi \quad (f \in L^2)$$

$$|(S^+ f)(x)| \geq \frac{c}{|x|} \quad \text{if } |x| \geq \frac{1}{2} \tag{7}$$

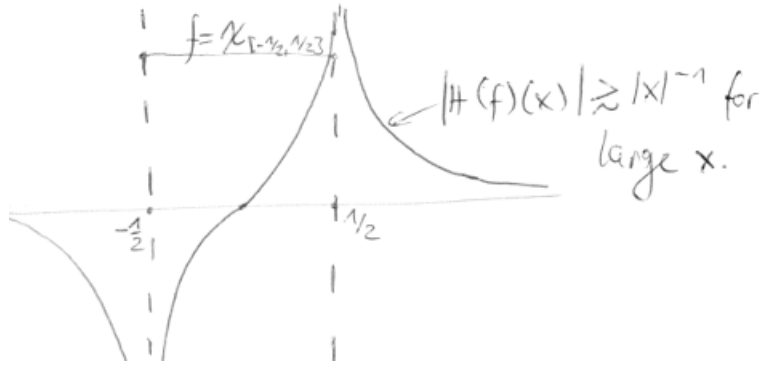
Proof.

$$(S^+ f)(x) = \lim_{\varepsilon \rightarrow 0} \int_0^\infty \hat{f}(\xi) e^{2\pi i (x+i\varepsilon)\xi} d\xi \quad \text{in } L^2$$

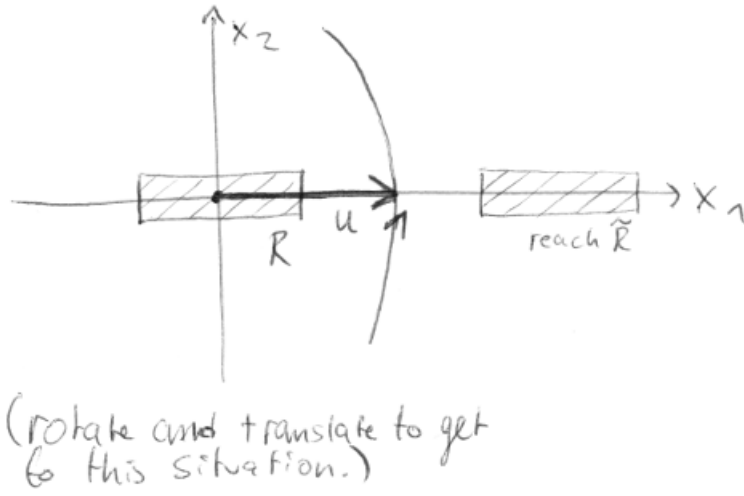
$$\int_{-\infty}^\infty \left(\int_0^\infty e^{-2\pi i y \xi} e^{2\pi i (x+i\varepsilon)\xi} d\xi \right) f(y) dy = \frac{1}{2\pi i} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{y - x - i\varepsilon} dy$$

This has absolute value $\lesssim c|x|$ if $|x| \geq \frac{1}{2}$.

Alternative proof: $|H(f)(x)| \geq |x|^{-1}$ for large x , $S^+ = \frac{1}{2}(I + iH)$.



□



For $R = (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2^{N+1}}, \frac{1}{2^{N+1}})$

$$1_R = 1_{(-\frac{1}{2}, \frac{1}{2})} \otimes 1_{(-2^{-(N+1)}, 2^{-(N+1)})}$$

If u points in the direction of x_1 then

$$(S^u 1_R)(x_1, x_2) = (S^+ 1_{(-\frac{1}{2}, \frac{1}{2})})(x_1) 1_{(-2^{-(N+1)}, 2^{-(N+1)})}(x_2)$$

$$(7) \Rightarrow |S^u(1_R)| \geq c' 1_{\tilde{R}}$$

Similarly for any 1×2^{-N} rectangle R_j . $u_j \in S^1$ in the positive direction of the longest side of R_j . Rotate and translate, then we get

$$|S^{u_j}(1_{R_j})| \geq c' 1_{\tilde{R}_j} \quad (8)$$

Take R_1, \dots, R_{2^N} to be the collection given by ε -Kakeya construction, plug that into result from lemma, get a contradiction.

Key: $p < 2$ and $d = 2$.

Lemma with $f_j = 1_{R_j}$ and $M = 2^N$

$$\begin{aligned} c' &\leq \left\| \left(\sum_{j=1}^{2^N} |S^{u_j}(1_{R_j})|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)} \quad (8), \quad \left| \bigcup \tilde{R}_j \right| = 1 \\ &\leq A_p \left\| \left(\sum_{j=1}^{2^N} |1_{R_j}|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)} = A_p \underbrace{\left(\int_E \left(\sum_{j=1}^{2^N} |1_{R_j}|^2 \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}}}_I \end{aligned}$$

$$\begin{aligned} I &\leq |E|^{\frac{1}{q}} \left(\int \left(\sum |1_{R_j}|^2 \right)^{\frac{p}{2} \frac{2}{p}} dx \right)^{\frac{1}{p} \frac{p}{2}} \quad \text{Hölder} \\ &= |E|^{\frac{1}{q}} \sum_{j=1}^{2^N} |R_j| = |E|^{\frac{1}{q}} \end{aligned}$$

$$E = \bigcup_{j=1}^{2^N} R_j, \quad |E| < \varepsilon, \quad \frac{1}{q} + \frac{1}{p/2} = 1, \quad \frac{1}{q} = 1 - \frac{p}{2}, \quad \frac{1}{pq} = \frac{1}{p} \left(1 - \frac{p}{2}\right) > 0$$

In the end we get

$$c' \leq A_p \varepsilon^{\frac{1}{pq}}.$$

Let $\varepsilon \rightarrow 0^+$ to finish.

$d > 2$:

$$f_j \left(\underset{\in \mathbb{R}^d}{x} \right) = f_j(x_1, x_2, \underset{\in \mathbb{R}^{d-2}}{x'}) = 1_{R_j}(x_1, x_2) f(x') \quad f \in S(\mathbb{R}^{d-2})$$

$p > 2$:

$$\langle Sf, g \rangle = \langle \widehat{Sf}, \hat{g} \rangle = \langle 1_B \hat{f}, \hat{g} \rangle = \langle \hat{f}, 1_B \hat{g} \rangle = \langle f, Sg \rangle \quad S = S^*$$

□

Oscillatory integrals in harmonic analysis Stein (VIII,IX), Stein-Shakarchi (Chapter 8), Sogge

- averaging operators
- restriction theory
- Bochner-Riesz summability

Motivation (in \mathbb{R}^3)

$$(Af)(x) = \int_{\mathbb{S}^2} f(x-y) d\sigma_y = \frac{1}{4\pi} f * \sigma(x)$$

σ surface measure in \mathbb{S}^2 .

Smoothing properties:

$$\left\| \frac{\partial}{\partial x_j} A(f) \right\|_{L^2(\mathbb{R}^3)} \lesssim \|f\|_{L^2(\mathbb{R}^3)} \quad (j = 1, 2, 3) \quad (9)$$

$f \in L^2 \Rightarrow$

$$\|f * \sigma\|_{L^2} \leq \|f\|_{L^2} \underbrace{|\sigma|(\mathbb{R}^3)}_{< \infty}$$

(use Minkowski integral inequality) $\therefore f * \sigma \in L^2$ if $f \in L^2$.

Idea $\widehat{(f * \sigma)} = \widehat{f} \widehat{\sigma}$

$$\widehat{\left(\frac{\partial}{\partial x_j} A(f)\right)}(\xi) = \xi_j \widehat{f}(\xi) \widehat{\sigma}(\xi),$$

so we should study $\widehat{\sigma}$.

without loss of generality $\xi = (0, 0, |\xi|)$ because the integral is spherically symmetric.

$$\widehat{\sigma}(\xi) = \int_{\mathbb{S}^2} e^{-2\pi i \omega \xi} d\sigma_\omega = 2\pi \int_0^\pi e^{-2\pi i |\xi| \cos \theta} \sin \theta d\theta = 2\pi \int_{-1}^1 e^{2\pi i |\xi| t} dt = \frac{2 \sin(2\pi |\xi|)}{|\xi|}$$

$$t = \cos \theta \quad dt = -\sin \theta d\theta$$

$$\therefore |\widehat{\sigma}(\xi)| \lesssim (1 + |\xi|)^{-1}$$

(9) follows from this and Plancherel.

This also generalizes to higher dimensions. There we get a factor $(1-t^2)^{\frac{d-3}{2}}$ in the last integral.

Oscillatory integrals

$$I(\lambda) = \int_{\mathbb{R}^d} e^{i\lambda \phi(x)} \psi(x) dx$$

where $\lambda \in \mathbb{R}$ is the oscillatory parameter, $\phi \in C^\infty(\mathbb{R}^d, \mathbb{R})$ the phase and $\psi(x) \in C$ the amplitude.

Q.: How does $I(\lambda)$ behave for large $|\lambda|$? General principle: Main contribution comes from the critical points of the phase, $x_0 : \nabla \phi(x_0) = 0$.

Principle of non-stationary phase $\phi \in C^\infty, \psi \in C_0^\infty : |\nabla \phi(x)| > 0 (\forall x \in \text{supp } \psi)$. Then for any $N \in \mathbb{N}$

$$|I(\lambda)| \leq c_N |\lambda|^{-N}$$

Proof. involves integration by parts □

$d = 1$:

$$I_1(\lambda) = \int_a^b e^{i\lambda \phi(x)} dx$$

$$0 < a < b < \infty \quad \psi(x) = \chi_{[a,b]}(x) \quad \text{which is rough!}$$

This means we will not get such a fast decay

Lemma (van der Corput (I)). $\phi \in C^2, \phi'$ monotonic, $|\phi'(x)| \geq 1 (\forall x \in [a, b])$. Then

$$|I_1(\lambda)| \leq \frac{3}{|\lambda|} \quad (\forall \lambda > 0)$$

Remark. (i) 3 is neither important nor sharp; independence of a, b, ϕ is the key!

(ii) Order of decrease in λ is sharp ($\phi(x) = x : I_1(\lambda) = \frac{e^{i\lambda b} - e^{i\lambda a}}{i\lambda}$)

(iii) monotonicity of ϕ' is essential

Proof. Integrate by parts(...) □

What if critical points are present? ($d = 1$)
 $x_0 : \phi'(x_0) = 0$ (critical point) and $\phi''(x_0) \neq 0$ (non degenerate), e.g. $\phi(x) = x^2$, $x_0 = 0$. In this case

$$\int_{\mathbb{R}} e^{i\lambda x^2} \psi(x) dx = c_0 \lambda^{-\frac{1}{2}} + \mathcal{O}(|\lambda|^{-\frac{3}{2}}) = \sum_{k=0}^N a_k \lambda^{-\frac{1}{2}-k} + \mathcal{O}(|\lambda|^{-\frac{3}{2}-N}) \quad (\forall N, \lambda \rightarrow \infty)$$

Lemma (van der Corput (II)). $\phi \in C^2[a, b]$, $|\phi''(x)| \geq 1$ ($\forall x \in [a, b]$). Then

$$|I_1(\lambda)| \leq \frac{8}{\lambda^{\frac{1}{2}}} \quad (\forall \lambda > 0)$$

Remark. More generally: $\mathcal{O}(|\lambda|^{\frac{1}{k}})$ if $|\phi^{(k)}| \geq 1$.

Proof. Integration by parts not needed. Instead split up region in small area around critical point with properly chosen size, and rest, and then use results from above. \square

Corollary. Same assumptions as van der Corput (II). $\psi \in C^1[a, b]$.

$$|\int e^{i\lambda \phi(x)} \psi(x) dx| \leq c_{\psi} \lambda^{-\frac{1}{2}}$$

Application: Asymptotics of Bessel functions

$$J_m(r) = \frac{1}{2\pi} \int_0^{2\pi} e^{ir \sin x} e^{-imx} dx \quad \phi(x) = \sin x, \quad \psi(x) = e^{-imx} \quad (m \in \mathbb{Z})$$

Corollary.

$$|J_m(r)| \leq cr^{-\frac{1}{2}} \quad r \rightarrow \infty$$

Recall: Averaging operator in \mathbb{R}^d ($d > 1$) is

$$(Af)(x) = (f * \sigma)(x) \quad \sigma \text{ surface measure on } \mathbb{S}^{d-1}$$

Theorem. $f \mapsto A(f)$ is bounded from $L^2(\mathbb{R}^d)$ to $L_k^2(\mathbb{R}^d)$ with $= \frac{d-1}{2}$.

Proof.

$$\begin{aligned} \hat{\sigma}(\xi) &= 2\pi |\xi|^{-\frac{d}{2}+1} \underbrace{J_{\frac{d}{2}-1}(2\pi|\xi|)}_{=\mathcal{O}(|\xi|^{-\frac{1}{2}}), |\xi| \rightarrow \infty} \\ \therefore |\hat{\sigma}(\xi)| &= \mathcal{O}(|\xi|^{-\frac{d-1}{2}}) \quad |\xi| \rightarrow \infty \end{aligned}$$

\square

"What is van der Corput's lemma in higher dimension?" (Carbery-Wright, 2000)

$$I(\lambda) = \int_{\mathbb{R}^d} e^{i\lambda \phi(x)} \psi(x) dx$$

(ϕ smooth, ψ smooth, compactly supported)
 nondegeneracy hypothesis

$$\det(\nabla^2 \phi)(x) \neq 0 \quad \forall x \in \text{supp}(\psi)$$

Theorem. Under above assumptions

$$|I(\lambda)| = \mathcal{O}(|\lambda|^{-\frac{1}{2}}) \quad \lambda \rightarrow \infty$$

Remark. (i) Decay rate is sharp

(ii) Proof uses TT^* method: $|I(\lambda)|^2 = I(\lambda)\overline{I(\lambda)}$

(iii) variant: $\text{rk}(\nabla^2\phi) \geq m$ for some $0 < m \leq d$ on $\text{supp}(\psi)$. Then

$$|I(\lambda)| = \mathcal{O}(|\lambda|^{-\frac{m}{2}})$$

Application: Fourier transform of surface-carried measures

Recall: $(\mathbb{S}^{d-1}, \sigma)$

$$|\hat{\sigma}(\xi)| \lesssim (1 + |\xi|)^{-\frac{d-1}{2}}$$

(not a Bessel coincidence)

(local) C^∞ -hypersurface M . After translation and rotation $x_0 = 0$, $T_{x_0}M = \{x_d = 0\}$. M can be represented as

$$M = \{(x', x_d) \in B \subset \mathbb{R}^d : x_d = \varphi(x')\}$$

Can arrange $\varphi(0) = 0 = (\nabla_{x'}\varphi)(x')|_{x'=0}$.

$$\varphi(x') = \frac{1}{2} \sum_{k,j=1}^{d-1} \underbrace{\frac{\partial^2 \varphi}{\partial x_k \partial x_j}}_{(a_{jk})} x_k x_j + \mathcal{O}(|x'|^3) = \frac{1}{2} \sum_{j=0}^{d-1} k_j x_j^2 + \mathcal{O}(|x'|^3)$$

(a_{jk}) $(d-1) \times (d-1)$ \mathbb{R} -valued symmetric matrix \therefore diagonalizable. k_j principal curvatures of M at x_0 . $k := \prod_{j=1}^{d-1} k_j$ is the Gaussian curvature of M at x_0 ($k = \det(\nabla^2 \varphi)$)

E.g.

(i) $\mathbb{S}^{d-1} \subset \mathbb{R}^d$. $k_j = 1$ ($\forall j$) $\therefore k = 1$

(ii) $\{x_3 = \underbrace{x_1^2 - x_2^2}_{\varphi(x_1, x_2)}\} \subset \mathbb{R}^3$, $\frac{1}{2}\nabla^2 \varphi(x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(iii) $\{x_1^2 = |x'|^2 : x \neq 0\}$, $x' \in \mathbb{R}^{d-1}$. $d-2$ identical nonvanishing principal curvatures $x_d^{-2} + 1$ vanishing principal curvature.

surface measure σ

$$\int_M f d\sigma = \int_{\mathbb{R}^{d-1}} f(x', \varphi(x')) \underbrace{\frac{\sqrt{1 + |\nabla_{x'} \varphi(x')|^2} dx'}{d\sigma \text{ in our coordinate sys.}}}_{d\sigma \text{ in our coordinate sys.}}$$

$$d\mu = \psi d\sigma, \quad \psi \in C_0^\infty(M, \sigma)$$

is a surface carried measure.

$$\hat{\mu}(\xi) = \int_M e^{-2\pi i x \xi} d\mu(x) = \int_M e^{-2\pi i x \xi} \psi(x) d\sigma_x$$

is bounded on \mathbb{R}^d because $|\mu|(\mathbb{R}^d) < \infty$.

Theorem. *Hypersurface $M \subset \mathbb{R}^d$ with nonvanishing Gauss curvature at each point of $\text{supp}(\psi)$. Then*

$$|\hat{\mu}(\xi)| = \mathcal{O}(|\xi|^{-\frac{d-1}{2}}), \quad |\xi| \rightarrow \infty$$

Corollary. *If M has at least m non vanishing principal curvatures (at each point of $\text{supp}(\psi)$), then*

$$|\hat{\mu}(\xi)| = \mathcal{O}(|\xi|^{-\frac{m}{2}}), \quad |\xi| \rightarrow \infty$$