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Author’s note

What began as a desire to sketch out a simple “answer key” for the problems in *Understanding Analysis* has inevitably evolved into something a bit more ambitious. As I was generating solutions for the over 350 exercises in the text, I found myself adding regular commentary on common pitfalls and strategies that frequently arise. My sense is that this manual should be a useful supplement to instructors teaching a course or to individuals engaged in an independent study. As with the textbook itself, I tried to write with the introductory student firmly in mind. In my teaching of analysis, I have come to understand the strong correlation between how students learn analysis and how they write it. A final goal I have for these notes is to illustrate by example how the form and grammar of a written argument are intimately connected to the clarity of a proof and, ultimately, to its validity.

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Middlebury, Vermont Stephen Abbott
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Chapter 1

The Real Numbers

1.1 Discussion: The Irrationality of $\sqrt{2}$

1.2 Some Preliminaries

Exercise 1.2.1. (a) Assume, for contradiction, that there exist integers p and q satisfying

$$\left(\frac{p}{q}\right)^2 = 3. \tag{1}$$

Let us also assume that p and q have no common factor. Now, equation (1) implies

$$p^2 = 3q^2. \tag{2}$$

From this, we can see that p^2 is a multiple of 3 and hence p must also be a multiple of 3. This allows us to write $p = 3r$, where r is an integer. After substituting $3r$ for p in equation (2), we get $(3r)^2 = 3q^2$, which can be simplified to $3r^2 = q^2$. This implies q^2 is a multiple of 3 and hence q is also a multiple of 3. Thus we have shown p and q have a common factor, namely 3, when they were originally assumed to have no common factor.

A similar argument will work for $\sqrt{6}$ as well because we get $p^2 = 6q^2$ which implies p is a multiple of 2 and 3. After making the necessary substitutions, we can conclude q is a multiple of 6, and therefore $\sqrt{6}$ must be irrational.

(b) In this case, the fact that p^2 is a multiple of 4 does not imply p is also a multiple of 4. Thus, our proof breaks down at this point.

Exercise 1.2.2. (a) False, as seen in Example 1.2.2.

(b) True. This will follow from upcoming results about compactness in Chapter 3.

(c) False. Consider sets $A = \{1, 2, 3\}, B = \{3, 6, 7\}$ and $C = \{5\}$. Note that $A \cap (B \cup C) = \{3\}$ is not equal to $(A \cap B) \cup C = \{3, 5\}$.

(d) True.
(e) True.

Exercise 1.2.3. (a) If $x \in (A \cap B)^c$ then $x \notin (A \cap B)$. But this implies $x \notin A$ or $x \notin B$. From this we know $x \in A^c$ or $x \in B^c$. Thus, $x \in A^c \cup B^c$ by the definition of union.

(b) To show $A^c \cup B^c \subseteq (A \cap B)^c$, let $x \in A^c \cup B^c$ and show $x \in (A \cap B)^c$. So, if $x \in A^c \cup B^c$ then $x \in A^c$ or $x \in B^c$. From this, we know that $x \notin A$ or $x \notin B$, which implies $x \notin (A \cap B)$. This means $x \in (A \cap B)^c$, which is precisely what we wanted to show.

(c) In order to prove $(A \cup B)^c = A^c \cap B^c$ we have to show,

$$(A \cup B)^c \subseteq A^c \cap B^c \text{ and,} \tag{1}$$

$$A^c \cap B^c \subseteq (A \cup B)^c. \tag{2}$$

To demonstrate part (1) take $x \in (A \cup B)^c$ and show that $x \in (A^c \cap B^c)$. So, if $x \in (A \cup B)^c$ then $x \notin (A \cup B)$. From this, we know that $x \notin$A and $x \notin B$ which implies $x \in A^c$ and $x \in B^c$. This means $x \in (A^c \cap B^c)$.

Similarly, part (2) can be shown by taking $x \in (A^c \cap B^c)$ and showing that $x \in (A \cup B)^c$. So, if $x \in (A^c \cap B^c)$ then $x \in A^c$ and $x \in B^c$. From this, we know that $x \notin A$ and $x \notin B$ which implies $x \notin (A \cup B)$. This means $x \in (A \cup B)^c$. Since we have shown inclusion both ways, we conclude that $(A \cup B)^c = A^c \cap B^c$.

Exercise 1.2.4. (a)When a and b have the same sign, consider the following two cases:

(i) If $a \geq 0$ and $b \geq 0$ then we have $a + b > 0$ which implies $|a + b| = a + b$. Furthermore, because $|a| = a$ and $|b| = b$, we have $|a| + |b| = a + b$. This implies, $|a + b| = |a| + |b|$, which satisfies the triangle inequality.

(ii) If $a \leq 0$ and $b \leq 0$ then we have $a + b \leq 0$ which implies $|a + b| = -a - b$. Furthermore, since we know $|a| = -a$ and $|b| = -b$ we have $|a| + |b| = -a - b$. This implies, $|a + b| = |a| + |b|$, which satisfies the triangle inequality.

(b)If $a \geq 0$, $b < 0$, and $a + b \geq 0$ then we have $|a + b| = a + b = a - (-b) = |a| - |b| < |a| + |b|$. This implies $|a + b| \leq |a| + |b|$ as desired.

Exercise 1.2.5. (a) Observe that $|a - b| = |a + (-b)| \leq |a| + |-b| = |a| + |b|$ which implies $|a - b| \leq |a| + |b|$.

(b) First note that $|a| = |a - b + b| \leq |a - b| + |b|$. Taking $|b|$ to the left side of the inequality we get $|a| - |b| \leq |a - b|$. Reversing the roles of a and b in the previous argument gives $|b| - |a| \leq |b - a|$, and because $|a - b| = |b - a|$ the result follows.

Exercise 1.2.6. (a) $f(A) = [0, 4]$ and $f(B) = [1, 16]$. In this case, $f(A \cap B) = f(A) \cap f(B) = [1, 4]$ and $f(A \cup B) = f(A) \cup f(B) = [0, 16]$.

(b) Take $A = [0, 2]$ and $B = [-2, 0]$ and note that $f(A \cap B) = \{0\}$ but $f(A) \cap f(B) = [0, 4]$.

(c) We have to show $y \in g(A \cap B)$ implies $y \in g(A) \cap g(B)$. If $y \in g(A \cap B)$ then there exists an $x \in A \cap B$ with $g(x) = y$. But this means $x \in A$ and $x \in B$ and hence $g(x) \in g(A)$ and $g(x) \in g(B)$. Therefore, $g(x) = y \in g(A) \cap g(B)$.

(d) Our claim is $g(A \cup B) = g(A) \cup g(B)$. In order to prove it, we have to show,

$$g(A \cup B) \subseteq g(A) \cup g(B) \text{ and,} \tag{1}$$

$$g(A) \cup g(B) \subseteq g(A \cup B). \tag{2}$$

To demonstrate part (1), we let $y \in g(A \cup B)$ and show $y \in g(A) \cup g(B)$. If $y \in g(A \cup B)$ then there exists $x \in A \cup B$ with $g(x) = y$. But this means $x \in A$ or $x \in B$, and hence $g(x) \in g(A)$ or $g(x) \in g(B)$. Therefore, $g(x) = y \in g(A) \cup g(B)$.

To demonstrate the reverse inclusion, we let $y \in g(A) \cup g(B)$ and show $y \in g(A \cup B)$. If $y \in g(A) \cup g(B)$ then $y \in g(A)$ or $y \in g(B)$. This means we have an $x \in A$ or $x \in B$ such that $g(x) = y$. This implies, $x \in A \cup B$, and hence $g(x) \in g(A \cup B)$. Since we have shown parts (1) and (2), we can conclude $g(A \cup B) = g(A) \cup g(B)$.

Exercise 1.2.7. (a) $f^{-1}(A) = [-2, 2]$ and $f^{-1}(B) = [-1, 1]$. In this case, $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B) = [-1, 1]$ and $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B) = [-2, 2]$.

(b) In order to prove $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$, we have to show,

$$g^{-1}(A \cap B) \subseteq g^{-1}(A) \cap g^{-1}(B) \text{ and,} \tag{1}$$

$$g^{-1}(A) \cap g^{-1}(B) \subseteq g^{-1}(A \cap B). \tag{2}$$

To demonstrate part (1), we let $x \in g^{-1}(A \cap B)$ and show $x \in g^{-1}(A) \cap g^{-1}(B)$. So, if $x \in g^{-1}(A \cap B)$ then $g(x) \in (A \cap B)$. But this means $g(x) \in A$ and $g(x) \in B$, and hence $g(x) \in A \cap B$. This implies, $x \in g^{-1}(A) \cap g^{-1}(B)$.

To demonstrate the reverse inclusion, we let $x \in g^{-1}(A) \cap g^{-1}(B)$ and show $x \in g^{-1}(A \cap B)$. So, if $x \in g^{-1}(A) \cap g^{-1}(B)$ then $x \in g^{-1}(A)$ and $x \in g^{-1}(B)$. This implies $g(x) \in A$ and $g(x) \in B$, and hence $g(x) \in A \cap B$. This means, $x \in g^{-1}(A \cap B)$.

Similarly, in order to prove $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$, we have to show,

$$g^{-1}(A \cup B) \subseteq g^{-1}(A) \cup g^{-1}(B) \text{ and,} \tag{1}$$

$$g^{-1}(A) \cup g^{-1}(B) \subseteq g^{-1}(A \cup B). \tag{2}$$

To demonstrate part (1), we let $x \in g^{-1}(A \cup B)$ and show $x \in g^{-1}(A) \cup g^{-1}(B)$. So, if $x \in g^{-1}(A \cup B)$ then $g(x) \in (A \cup B)$. But this means $g(x) \in A$ or $g(x) \in B$, which implies $x \in g^{-1}(A)$ or $x \in g^{-1}(B)$. From this we know $x \in g^{-1}(A) \cup g^{-1}(B)$.

To demonstrate the reverse inclusion, we let $x \in g^{-1}(A) \cup g^{-1}(B)$ and show $x \in g^{-1}(A \cup B)$. So, if $x \in g^{-1}(A) \cap g^{-1}(B)$ then $x \in g^{-1}(A)$ or $x \in g^{-1}(B)$. This implies $g(x) \in A$ or $g(x) \in B$, and hence $g(x) \in A \cup B$. This means, $x \in g^{-1}(A \cup B)$.

Exercise 1.2.8. (a) There exist two real numbers a and b satisfying $a < b$ such that for all $n \in \mathbf{N}$ we have $a + 1/n \geq b$.

(b) There exist two distinct rational numbers with the property that every number in between them is irrational.

(c) There exists a natural number n where $\sqrt{n}$ is rational but not a natural number.

(d) There exists a real number x such that $n \leq x$ for all $n \in \mathbf{N}$.

Exercise 1.2.9. (a) We will use induction to prove $x_n \leq 2$, for every $n \in \mathbf{N}$. For $n = 1$, we can easily see $x_1 = 1 \leq 2$. Now, we want to show that

$$\text{if we have } x_n \leq 2, \text{ then it follows that } x_{n+1} \leq 2.$$

Starting from the induction hypothesis $x_n \leq 2$, we multiply across the inequality by $1/2$ and add 1 to get

$$\frac{1}{2}x_n + 1 \leq \frac{1}{2}2 + 1 = 2,$$

which is precisely the the desired conclusion $x_{n+1} \leq 2$. By induction, the claim is proved for all $n \in \mathbf{N}$.

Exercise 1.2.10. (a) For $n = 1$, we can easily see $y_1 = 1 < 4$, and this proves the base case. Now, we want to show that

$$\text{if we have } y_n < 4, \text{ then it follows that } y_{n+1} < 4.$$

Starting from the induction hypothesis $y_n < 4$, we can multiply across the inequality by $3/4$ and add 1 to get

$$\frac{3}{4}y_n + 1 < \frac{3}{4}4 + 1 = 4,$$

which is the the desired conclusion $y_{n+1} \leq 4$. By induction, the claim is proved for all $n \in \mathbf{N}$.

(b) For $n = 1$, we can easily see $y_1 = 1 < 7/4 = y_2$, proving the base case. Now, we want to show that

$$\text{if we have } y_n \leq y_{n+1}, \text{ then it follows that } y_{n+1} \leq y_{n+2}.$$

Starting from the induction hypothesis $y_n \leq y_{n+1}$, we can multiply across the inequality by $3/4$ and add 1 to get

$$\frac{3}{4}y_n + 1 < \frac{3}{4}y_{n+1} + 1$$

which is the the desired conclusion $y_{n+1} \leq y_{n+1}$. By induction, the claim is proved for all $n \in \mathbf{N}$.

Exercise 1.2.11. We will use induction, this time starting with $n = 0$, to prove the claim. When $n = 0$ then $A = \emptyset$. For this case, the set A has just the empty set as its only subset. Since $2^0 = 1$, the claim is true in this case.

Now we have to show that if sets of size n have 2^n different subsets, then it follows that sets of size $n + 1$ have 2^{n+1} different subsets. Given a set A of size $n + 1$, first remove an arbitrary element $a \in A$. The set $A \backslash \{a\}$ has n elements, and we can use the induction hypothesis to say that there are exactly 2^n subsets of $A \backslash \{a\}$. Said another way, there are precisely 2^n subsets of A that do not contain the particular element a. By adding the element a to each of these we will produce 2^n new subsets of A. Since every subset of A either contains a or does not contain a, we can be sure that we have listed them all. Thus, the total number of subsets of A is given by 2^n (for the subsets without a) plus 2^n (for the subsets that do contain a), and $2^n + 2^n = 2^{n+1}$. By induction, the claim is proved for all $n \in \mathbf{N}$.

Exercise 1.2.12. (a) From Exercise 1.2.3 we know $(A_1 \cup A_2)^c = A_1^c \cap A_2^c$ which proves the base case. Now we want to show that

if we have $(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$, then it follows that

$$(A_1 \cup A_2 \cup \cdots \cup A_{n+1})^c = A_1^c \cap A_2^c \cap \cdots \cap A_{n+1}^c.$$

Since the union of sets obey the associative law,

$$(A_1 \cup A_2 \cup \cdots \cup A_{n+1})^c = ((A_1 \cup A_2 \cup \cdots \cup A_n) \cup A_{n+1})^c$$

which is equal to

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c \cap A_{n+1}^c.$$

Now from our induction hypothesis we know that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$$

which implies that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c \cap A_{n+1}^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c \cap A_{n+1}^c.$$

By induction, the claim is proved for all $n \in \mathbf{N}$.

(b) The point here is to distinguish between asserting that a statement is true for all values of $n \in \mathbf{N}$ and asserting that it is true in the infinite case. Induction cannot be used when we have an infinite number of sets. It is used to prove facts that hold true for each value of $n \in \mathbf{N}$. For instance, in Exercise 1.2.2, we could use induction to show that $\bigcap_{k=1}^{n} A_k$ is infinite for all choices of $n \in N$, but notice that this conclusion is not true for $\bigcap_{k=1}^{\infty} A_n$.

(c) In order to prove $\left(\bigcup_{n=1}^{\infty} A_n\right)^c = \bigcap_{n=1}^{\infty} A_n^c$ we have to show,

$$\left(\bigcup_{n=1}^{\infty} A_n\right)^c \subseteq \bigcap_{n=1}^{\infty} A_n^c \text{ and,} \tag{1}$$

$$\bigcap_{n=1}^{\infty} A_n^c \subseteq \left(\bigcup_{n=1}^{\infty} A_n\right)^c. \tag{2}$$

To demonstrate part (1), we let $x \in \left(\bigcup_{n=1}^{\infty} A_n\right)^c$ and show $x \in \bigcap_{n=1}^{\infty} A_n^c$. So, if $x \in \left(\bigcup_{n=1}^{\infty} A_n\right)^c$ then $x \notin A_n$ for all $n \in \mathbf{N}$. This implies x is in the complement of each A_n and by the definition of intersection $x \in \bigcap_{n=1}^{\infty} A_n^c$.

To demonstrate the reverse inclusion, we let $x \in \bigcap_{n=1}^{\infty} A_n^c$ and show $x \in \left(\bigcup_{n=1}^{\infty} A_n\right)^c$. So, if $x \in \bigcap_{n=1}^{\infty} A_n^c$ then $x \in A_n^c$ for all $n \in \mathbf{N}$ which means $x \notin A_n$ for all $n \in \mathbf{N}$. This implies $x \notin \left(\bigcup_{n=1}^{\infty} A_n\right)$ and we can now conclude $x \in \left(\bigcup_{n=1}^{\infty} A_n\right)^c$.

1.3 The Axiom of Completeness

Exercise 1.3.1. (a) For any $z \in \mathbf{Z}_5$ the additive inverse is $y = 5 - z$.

(b) For $z = 1$ the additive inverse is $x = 1$, for $z = 2$ it is $x = 3$, for $z = 3$ it is $x = 2$,and for $z = 4$ it is $x = 4$.

(c) For any $z \in \mathbf{Z}_4$ the *additive inverse* is $y = 4 - z$. However, the *multiplicative inverse* of 2 does not exist. In general, additive inverses exist in $\mathbf{Z}_n$ for all values of n. Multiplicative inverses exist for prime values of n only.

Exercise 1.3.2. (a) A real number i is the greatest upper bound, or the infimum, for a set $A \subseteq \mathbf{R}$ if it meets the following two criteria:

(i) i is a lower bound for A; i.e., $i \leq a$ for all $a \in A$, and

(ii) if l is any lower bound for A, then $l \leq i$.

(b) Lemma: Assume $i \in \mathbf{R}$ is a lower bound for a set $A \subseteq \mathbf{R}$. Then, $i = \inf A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $i + \epsilon > a$.

(i) To prove this in the forward direction, assume $i = \inf A$ and consider $i + \epsilon$, where $\epsilon > 0$ has been arbitrarily chosen. Because $i + \epsilon > i$, statement (ii) implies $i + \epsilon$ is not a lower bound for A. Since this is the case, there must be some element $a \in A$ for which $i + \epsilon > a$ because otherwise $i + \epsilon$ would be a lower bound.

(ii) For the backward direction, assume i is a lower bound with the property that no matter how $\epsilon > 0$ is chosen, $i + \epsilon$ is no longer a lower bound for A. This implies that if l is any number greater than i then l is no longer a lower bound for A. Because any number greater than i cannot be a lower bound, it follows that if l is some other lower bound for A, then $l \leq i$. This completes the proof of the lemma.

Exercise 1.3.3. (a) Because A is bounded below, B is not empty. Also, for all $a \in A$ and $b \in B$, we have $b \leq a$. The first thing this tells us is that B is bounded above and thus $\alpha = \sup B$ exists by the Axiom of Completeness. It remains to show that $\alpha = \inf A$. The second thing we see is that every element of A is an upper bound for B. By part (ii) of the definition of supremum, $\alpha \leq a$ for all $a \in A$ and we conclude that α is a lower bound for A.

Is it the greatest lower bound? Sure it is. If l is an arbitrary lower bound for A then $l \in B$, and part (i) of the definition of supremum implies $l \leq \alpha$. This completes the proof.

(b) We do not need to assume that greatest lower bounds exist as part of the Axiom of Completeness because we now have a proof that they exist. By demonstrating that the infimum of a set A is always equal to the supremum of a different set, we can use the existence of least upper bounds to assert the assert the existence of greatest lower bounds.

(c) Given a set A, define $-A = \{-a : a \in A\}$. Now if A is bounded below it follows that $-A$ is bounded above and it is not too hard to prove $\inf A = \sup(-A)$ using an argument much like those in Exercise 1.3.5.

Exercise 1.3.4. Observe that all elements of B are contained in A and hence $\sup A \geq b$ for all $b \in B$. By Definition 1.3.2 part (ii), $\sup B$ is less than or equal to any other upper bounds of B. Because $\sup A$ is an upper bound for B, it follows that $\sup B \leq \sup A$.

Exercise 1.3.5. (a) Note that $c + \sup A$ is an upper bound for $c + A$. Now, we have to show if d is any upper bound for $c + A$, then $c + \sup A \leq d$. We know $c + a \leq d$ for all $a \in A$, and thus $a \leq d - c$ for all $a \in A$. This means $d - c$ is an upper bound for A and by part (ii) of Definition 1.3.2, $\sup A \leq d - c$. But this implies $c + \sup A \leq d$ which is precisely what we wanted to show.

(b) In the case $c = 0$, $cA = \{0\}$ and without too much difficulty we can argue that $\sup(cA) = 0 = c \sup A$. So let's focus on the case where $c > 0$. Observe that $c \sup A$ is an upper bound for cA. Now, we have to show if d is any upper bound for cA, then $c \sup A \leq d$. We know $ca \leq d$ for all $a \in A$, and thus $a \leq d/c$ for all $a \in A$. This means d/c is an upper bound for A, and by Definition 1.3.2 $\sup A \leq d/c$. But this implies $c \sup A \leq c(d/c) = d$, which is precisely what we wanted to show.

(c) Assuming the set A is bounded below, we claim $\sup(cA) = c \inf A$ for the case $c < 0$. In order to prove our claim we first show $c \inf A$ is an upper bound for cA. Since $\inf A \leq a$ for all $a \in A$, we multiply both sides of the equation to get $c \inf A \geq ca$ for all $a \in A$. This shows that $c \inf A$ is an upper bound for cA. Now, we have to show if d is any upper bound for cA, then $c \inf A \leq d$. We know $ca \leq d$ for all $a \in A$, and thus $d/c \leq a$ for all $a \in A$. This means d/c is a lower bound for A and from Exercise 1.3.2, $d/c \leq \inf A$. But this implies $c \inf A \leq c(d/c) \leq d$, which is precisely what we wanted to show.

Exercise 1.3.6. (a) The supremum is 3 and the infimum is 1.

(b) The supremum is 1 and the infimum is 0.

(c) The supremum is 1/2 and the infimum is 1/3.

(d) The supremum is 9 and the infimum is 1/9.

Exercise 1.3.7. Since a is an upper bound for A, we just need to verify the second part of the definition of supremum and show that if d is any upper bound then $a \leq d$. By the definition of upper bound $a \leq d$ because a is an element of A. Hence, by Definition 1.3.2, a is the supremum of A.

Exercise 1.3.8. Set $\epsilon = \sup B - \sup A > 0$. By Lemma 1.3.7, there exists an element $b \in B$ satisfying $\sup B - \epsilon < b$, which implies $\sup A < b$. Because $\sup A$ is an upper bound for A, then b is as well.

Exercise 1.3.9. (a) True.

(b) False. If we consider $A = (a, b)$, every element in A is less than b but $\sup A = b$.

(c) False. Consider, the open sets $A = (c, d)$ and $B = (d, f)$. Then $a < b$ for every $a \in A$ and $b \in B$, but $\sup A = d = \inf B$.

(d) True

(e) False. If we take $A = [0, 2]$ and $B = (0, 2)$, we see that $\sup A = \sup B$ but there is no element $b \in B$ that is an upper bound for A.

1.4 Consequences of Completeness

Exercise 1.4.1. We have to show there exists a rational number between a and b when $a < 0$. If $b > 0$, then by Theorem 1.4.3 we know there exists a rational number r satisfying $0 < r < b$ and so $a < r < b$ as well. If $b \leq 0$ then we can use Theorem 1.4.3 to say that there exists $r \in \mathbf{Q}$ satisfying $-b < r < -a$ and it follows that $a < -r < b$. The proof that $-r$ is rational is part of the next exercise.

Exercise 1.4.2. (a) We have to show if $a, b \in \mathbf{Q}$, then ab and $a+b$ are elements of $\mathbf{Q}$. By definition, $\mathbf{Q} = \{p/q : p, q \in \mathbf{Z}, q \neq 0\}$. So take $a = p/q$ and $b = c/d$ where $p, q, c, d \in \mathbf{Z}$ and $q, d \neq 0$. Then, $ab = \frac{pc}{qd}$ where $pc, qd \in \mathbf{Z}$ because $\mathbf{Z}$ is closed under multiplication. This implies $ab \in \mathbf{Q}$. To see that $a + b$ is rational, write

$$\frac{p}{q} + \frac{c}{d} = \frac{pd + qc}{qd},$$

and observe that both $pd + qc$ and qd are integers with $qd \neq 0$.

(b) Assume, for contradiction, that $a + t \in \mathbf{Q}$. Then $t = (t + a) - a$ is the difference of two rational numbers, and by part (a) t must be rational as well. This contradiction implies $a + t \in \mathbf{I}$.

Likewise, if we assume $at \in \mathbf{Q}$, then $t = (at)(1/a)$ would again be rational by the result in (a). This implies $at \in \mathbf{I}$.

(c) The set of irrationals is not closed under addition and multiplication. Given two irrationals s and t, $s + t$ can be either irrational or rational. For instance, if $s = \sqrt{2}$ and $t = -\sqrt{2}$, then $s + t = 0$ which is an element of $\mathbf{Q}$. However, if $s = \sqrt{2}$ and $t = 2\sqrt{2}$ then $s + t = \sqrt{2} + 2\sqrt{2} = 3\sqrt{3}$ which is an element of $\mathbf{I}$. Similarly, st can be either irrational or rational. If $s = \sqrt{2}$ and $t = -\sqrt{2}$, then $st = -1$ which is a rational number. However,if $s = \sqrt{2}$ and $t = \sqrt{3}$ then $st = \sqrt{2}\sqrt{3} = \sqrt{6}$ which is an irrational number.

Exercise 1.4.3. We have to show the existence of an irrational number between any two real numbers a and b. By applying Theorem 1.4.3 on the real numbers $a-\sqrt{2}$ and $b-\sqrt{2}$ we can find a rational number r satisfying $a-\sqrt{2} < r < b-\sqrt{2}$. This implies $a < r + \sqrt{2} < b$. From Exercise 1.4.2(b) we know $r + \sqrt{2}$ is an irrational number between a and b.

Exercise 1.4.4. Observe that $0 < 1/n$ for all $n \in \mathbf{N}$. This implies 0 is a lower bound. Now we have to show that if c is any lower bound then $c \leq 0$. If $c > 0$, then the Archimedean Property of $\mathbf{R}$ states there exists $n \in \mathbf{N}$ such that $1/n < c$. This means that any $c > 0$ is not a lower bound. Thus, if c is a lower bound it must satisfy $c \leq 0$ as desired.

Exercise 1.4.5. Let $x \in \mathbf{R}$ be arbitrary. To prove $\bigcap_{n=1}^{\infty}(0, 1/n) = \emptyset$ it is enough to show that $x \notin (0, 1/n)$ for some $n \in \mathbf{N}$. If $x \leq 0$ then we can take $n = 1$ and observe $x \notin (0, 1)$. If $x > 0$ then by Theorem 1.4.2 we know there exists an $n_0 \in \mathbf{N}$ such that $1/n_0 < x$. This implies $x \notin \bigcap_{n=1}^{\infty}(0, 1/n)$, and our proof is complete.

Exercise 1.4.6. (a) Now, we need to pick n_0 large enough so that

$$\frac{1}{n_0} < \frac{\alpha^2 - 2}{2\alpha} \quad \text{or} \quad \frac{2\alpha}{n_0} < \alpha^2 - 2.$$

With this choice of n_0, we have

$$(\alpha - 1/n_0)^2 > \alpha^2 - 2\alpha/n_0 = \alpha^2 - (\alpha^2 - 2) = 2.$$

This means $(\alpha - 1/n_0)$ is an upper bound for T. But $(\alpha - 1/n_0) < \alpha$ and $\alpha = \sup T$ is supposed to be the least upper bound. This contradiction means that the case $\alpha^2 > 2$ can be ruled out. Because we have already ruled out $\alpha^2 < 2$, we are left with $\alpha^2 = 2$ which implies $\alpha = \sqrt{2}$ exists in $\mathbf{R}$.

(b) Define the set $T = \{t \in \mathbf{R} : t^2 < b$, and let $\alpha = \sup T$ which we know exists because T is non-empty (it contains 0) and bounded above. As before, we'll show $\alpha^2 = b$ by ruling out the possibilities $\alpha^2 < b$ and $\alpha^2 > b$.

First assume $\alpha^2 < b$ and observe,

$$\begin{aligned}\left(\alpha + \frac{1}{n}\right)^2 &= \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} \\ &< \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n} \\ &= \alpha^2 + \frac{2\alpha + 1}{n}.\end{aligned}$$

From Theorem 1.4.2(ii), choose n_0 large enough so that

$$\frac{1}{n_0} < \frac{b - \alpha^2}{2\alpha + 1}.$$

This implies $(2\alpha+1)/n_0 < 2-\alpha^2$, and consequently that

$$\left(\alpha + \frac{1}{n_0}\right)^2 < \alpha^2 + (b-\alpha^2) = b.$$

Thus, $\alpha + 1/n_0 \in T$, contradicting the fact that α is an upper bound for T. We conclude that $\alpha^2 < b$ cannot happen.

Now, let's assume $\alpha^2 > b$. This time, we have

$$\begin{aligned} \left(\alpha - \frac{1}{n}\right)^2 &= \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} \\ &> \alpha^2 - \frac{2\alpha}{n}. \end{aligned}$$

This time pick n_0 large enough so that

$$\frac{1}{n_0} < \frac{\alpha^2 - b}{2\alpha} \quad \text{or} \quad \frac{2\alpha}{n_0} < \alpha^2 - b.$$

With this choice of n_0, we have

$$(\alpha - 1/n_0)^2 > \alpha^2 - 2\alpha/n_0 = \alpha^2 - (\alpha^2 - b) = b.$$

This means $(\alpha - 1/n_0)$ is an upper bound for T. But then $(\alpha - 1/n_0) < \alpha = \sup T$ leads to a contradiction because all upper bounds of T should be greater than or equal to the supremum α. Thus, $\alpha^2 > b$ is not a possibility and we are left with $\alpha^2 = b$ as desired.

Exercise 1.4.7. Next let $n_2 = \min\{n \in \mathbf{N} : f(n) \in A \backslash \{f(n_1)\}\}$ and set $g(2) = f(n_2)$. In general, assume we have defined $g(k)$ for $k < m$, and let $g(m) = f(n_m)$ where $n_m = \min\{n \in \mathbf{N} : f(n) \in A \backslash \{f(n_1) \ldots f(n_{k-1})\}\}$.

To show that $g : N \to A$ is 1–1, observe that $m \neq m'$ implies $n_m \neq n_{m'}$ and it follows that $f(n_m) = g(m) \neq g(m') = f(n_{m'})$ because f is assumed to be 1–1. To show that g is onto, let $a \in A$ be arbitrary. Because f is onto, $a = f(n')$ for some $n' \in \mathbf{N}$. This means $n' \in \{n : f(n) \in A\}$ and as we inductively remove the minimal element, n' must eventually be the minimum by at least the $n' - 1$st step.

Exercise 1.4.8. (a) Because A_1 is countable, there exists a 1–1 and onto function $f : \mathbf{N} \to A_1$.

If $B_2 = \emptyset$, then $A_1 \cup A_2 = A_1$ which we already know to be countable.

If $B_2 = \{b_1, b_2, \ldots, b_m\}$ has m elements then define $h : A_1 \cup B_2$ via

$$h(n) = \begin{cases} b_n & \text{if } n \leq m \\ f(n-m) & \text{if } n > m. \end{cases}$$

The fact that h is a 1–1 and onto follows immediately from the same properties of f.

If B_2 is infinite, then by Theorem 1.4.12 it is countable, and so there exists a 1–1 onto function $g : \mathbf{N} \to B_2$. In this case we define $h : A_1 \cup B_2$ by

$$h(n) = \begin{cases} f((n+1)/2) & \text{if } n \text{ is odd} \\ g(n/2) & \text{if } n \text{ is even.} \end{cases}$$

Again, the proof that h is 1–1 and onto is derived directly from the fact that f and g are both bijections. Graphically, the correspondence takes the form

$$\begin{array}{lccccccc} \mathbf{N}: & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ A_1 \cup B_2: & a_1 & b_1 & a_2 & b_2 & a_3 & b_3 & \cdots \end{array}$$

To prove the more general statement in Theorem 1.4.13, we may use induction. We have just seen that the result holds for two countable sets. Now let's assume that the union of m countable sets is countable, and show that the union of $m+1$ countable sets is countable.

Given $m+1$ countable sets $A_1, A_2, \ldots, A_{m+1}$, we can write

$$A_1 \cup A_2 \cup \cdots \cup A_{m+1} = (A_1 \cup A_2 \cup \cdots \cup A_m) \cup A_{m+1}.$$

Then $C_m = A_1 \cup \cdots \cup A_m$ is countable by the induction hypothesis, and $C_m \cup A_{n+1}$ is just the union of two countable sets which we know to be countable. This completes the proof.

(b) Induction can not be used when we have infinite number of sets. It can only be used to prove facts that hold true for each value of $n \in \mathbf{N}$. See the discussion in Exercise 1.2.12 for more on this.

(c) Let's first consider the case where the sets $\{A_n\}$ are disjoint. In order to achieve 1-1 correspondence between the set $\mathbf{N}$ and $\bigcup_{n=1}^{\infty} A_n$, we first label the elements in each countable set A_n as

$$A_n = \{a_{n1}, a_{n2}, a_{n3}, \ldots\}.$$

Now arrange the elements of $\bigcup_{n=1}^{\infty} A_n$ in an array similar to the one for $\mathbf{N}$ given in the exercise:

$$\begin{array}{llllll} A_1 = a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & \cdots \\ A_2 = a_{21} & a_{22} & a_{23} & a_{24} & \cdots & \\ A_3 = a_{31} & a_{32} & a_{33} & \cdots & & \\ A_4 = a_{41} & a_{42} & \cdots & & & \\ A_5 = a_{51} & \cdots & & & & \\ \quad \vdots & & & & & \end{array}$$

This establishes a 1–1 and onto mapping $g : \mathbf{N} \to \bigcup_{n=1}^{\infty} A_n$ where $g(n)$ corresponds to the element a_{jk} where (j,k) is the row and column location of n in the array for $\mathbf{N}$ given in the exercise.

If the sets $\{A_n\}$ are not disjoint then our mapping may not be 1–1. In this case we could again replace A_n with $B_n = A_n \backslash \{A_1 \cup \cdots \cup A_{n-1}\}$. Another approach is to use the previous argument to establish a 1–1 correspondence between $\bigcup_{n=1}^{\infty} A_n$ and an infinite *subset* of $\mathbf{N}$, and then appeal to Theorem 1.4.12.

Exercise 1.4.9. (a) Since $A \sim B$ we know there is 1-1, onto function from A onto B. This means we can define another function $g : B \to A$ that is also 1-1 and onto. More specifically, if $f : A \to B$ is 1-1 and onto then $f^{-1} : B \to A$ exists and is also 1-1 and onto.

(b) We will show there exists a 1-1, onto function $h : A \to C$. Because $A \sim B$, there exists $g : A \to B$ that is 1–1 and onto. Likewise, $B \sim C$ implies that there exists $f : B \to C$ that is also 1-1 and onto. So let's define $h : A \to C$ by the composition $h = f \circ g$.

In order to show $f \circ g$ is 1-1, take $a_1, a_2 \in A$ where $a_1 \neq a_2$ and show $f(g(a_1)) \neq f(g(a_2))$. Well, $a_1 \neq a_2$ implies that $g(a_1) \neq g(a_2)$ because g is 1–1. And $g(a_1) \neq g(a_2)$ implies that $f(g(a_1)) \neq (f(g(a-2))$ because f is 1–1. This shows $f \circ g$ is 1–1.

In order to show $f \circ g$ is onto, we take $c \in C$ and show that there exists an $a \in A$ with $f(g(a)) = c$. If $c \in C$ then there exists $b \in B$ such that $f(b) = c$ because f is onto. But for this same $b \in B$ we have an $a \in A$ such that $g(a) = b$ since g is onto. This implies $f(b) = f(g(a)) = c$ and therefore $f \circ g$ is onto.

Exercise 1.4.10. For each $k \in \mathbf{N}$, let A_k be the set of subsets of $\mathbf{N}$ whose maximal element is k. For example, A_1 is the set containing just the subset $\{1\}$. In A_2 we would have $\{2\}$ and $\{1,2\}$. For A_3 there would be four elements: $\{3\}, \{1,3\}, \{2,3\}$, and $\{1,2,3\}$. There are two key observations to make. The first is that every A_k contains a *finite* number of elements. The second is that every finite subset of $\mathbf{N}$ must appear in exactly one of the sets A_k. Setting $A_0 = \emptyset$, this allows us to assert that the set of *all* finite subsets of $\mathbf{N}$ is equal to $\bigcup_{k=0}^{\infty} A_k$. Now we may proceed as in the proof of Theorem 1.4.11 (i) and argue that the countable union of finite subsets is countable.

Exercise 1.4.11. (a). The function $f(x) = (x, \frac{1}{3})$ is 1–1 from $(0,1)$ to S.

(b) Given $(x, y) \in S$, let's write x and y in their decimal expansions

$$x = .x_1x_2x_3\ldots \quad \text{and} \quad y = .y_1y_2y_3\ldots$$

where we make the convention that we always use the terminating form (or repeated 0s) over the repeating 9s form when the situation arises.

Now define $f : S \to (0,1)$ by

$$f(x,y) = .x_1y_1x_2y_2x_3y_3\ldots$$

In order to show f is 1–1, assume we have two distinct points $(x,y) \neq (w,z)$ from S. Then it must be that either $x \neq w$ or $y \neq z$, and this implies that in at least one decimal place we have $x_i \neq w_i$ or $y_i \neq z_i$. But this is enough to conclude $f(x,y) \neq f(w,z)$.

The function f is not onto. For instance the point $t = .555959595\ldots$ is not in the range of f because the ordered pair (x, y) with $x = .555\ldots$ and $y = .5999\ldots$ would not be allowed due to our convention of using terminating decimals instead of repeated 9s.

Exercise 1.4.12. (a) $\sqrt{2}$ is a root of the polynomial $x^2 - 2$, $\sqrt[3]{2}$ is a root of the polynomials $x^3 - 2$, and $\sqrt{3} + \sqrt{2}$ is a root of $x^4 - 10x^2 + 1$. Since all of these numbers are roots of polynomials with integer coefficients, they are all algebraic.

(b) Fix $n, m \in \mathbf{N}$. The set of polynomials of the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

satisfying $|a_n| + |a_{n-1}| + \cdots + |a_0| \leq m$ is *finite* because there are only a finite number of choices for each of the coefficients (given that they must be integers.) If we let A_{nm} be the set of all the roots of polynomials of this form, then because each one of these polynomials has at most n roots, the set A_{nm} is finite. Thus A_n, the set of algebraic numbers obtained as roots of any polynomial (with integer coefficients) of degree n, can be written as a countable union of finite sets

$$A_n = \bigcup_{m=1}^{\infty} A_{nm}.$$

It follows that A_n is countable.

(c) If A is the set of all algebraic numbers, then $A = \bigcup_{n=1}^{\infty} A_n$. Because each A_n is countable, we may use Theorem 1.4.13 to conclude that A is countable as well.

If T is the set transcendental numbers, then $A \cup T = \mathbf{R}$. Now if T were countable, then $\mathbf{R} = A \cup T$ would also be countable. But this is a contradiction because we know $\mathbf{R}$ is uncountable, and hence the collection of transcendental numbers must also be uncountable.

Exercise 1.4.13. (a) Given $y \in f(X)$, the fact that f is 1–1 implies that the $x \in X$ satisfying $f(x) = y$ must be *unique*. This allows us to define the function f^{-1} from $f(X)$ back to X because now there is no ambiguity about the value of $f^{-1}(y)$. By focusing only on the range $f(X)$ (and not all of Y) we may say that f is a 1–1 and onto function from X to $f(X)$. Its inverse, f^{-1}, from $f(X)$ back to X is then easily seen to be 1–1 and onto as well.

(b) If $x \notin g(Y)$ then $g^{-1}(x)$ is not defined, and the number of elements to the left of x in C_x is 0. Similarly, if $g^{-1}(x) \notin f(X)$, then $f^{-1}(g^{-1}(x))$ is not defined and the chain terminates with just one element to the left of x. In general, we get a finite number of elements to the left of x if some iterate falls outside of either $g(Y)$ or $f(X)$.

(c) Given $x, x' \in X$, assume $C_x \cap C_{x'} \neq \emptyset$. Without loss of generality, let $y \in Y$ satisfy $y \in C_x \cap C_{x'}$. Then either

$$y = f(g(\cdots g(f(x)))) \quad \text{or} \quad y = g^{-1}(f^{-1}(\cdots f^{-1}(g^{-1}(x)))),$$

and a similar statement is true with x' in place of x. Equating the expressions for x and x' and applying the appropriate combination of f, g, f^{-1} and g^{-1}, we can show that $x \in C_{x'}$ and $x' \in C_x$. This is sufficient to conclude $C_x = C_{x'}$.

(d)Let C by a chain in B. Then $C \cap Y$ is not a subset of $f(X)$, so there exists $y \subset Y$ with $y \in C$ but $y \notin f(X)$. Note that C and C_y have a point in common, so they must be equal.

(e) Note that $Y_1 \subseteq f(X)$. To show $f : X_1 \to Y_1$ is onto we pick a $y_1 \in Y_1$ and show there exists $x_1 \in X_1$ with $f(x_1) = y_1$. Well, if $y_1 \in Y_1 \subseteq f(X)$, we know there exists $x_1 \in X$ such that $f(x_1) = y_1$. But we must be sure that $x_1 \in X_1$. However, C_{x_1} contains y_1 which is an element of some chain in A. Since chains that intersect must be identical, $C_{x_1} \subseteq A$, and $x_1 \in X_1$.

Now we have to show $g : Y_2 \to X_2$ is onto. To do this, we pick $x_2 \in X_2$ and show that there exists $y_2 \in Y_2$ with $g(y_2) = x_2$. Since $x_2 \in X_2 \subseteq g(Y)$ we know there exists $y_2 \in Y$ such that $g(y_2) = x_2$. Now we need to show that $y_2 \in B$ because if $y_2 \in Y$ and $y_2 \in B$ then $y_2 \in B \cap Y = Y_2$. We know that C_{x_2} contains y_2 which is an element of some chain B. Since chains that intersect are identical $C_{x_2} \subseteq B$, $y_2 \in B$, and hence $y_2 \in Y_2$.

Finally, to prove $X \sim Y$ define $h : X \to Y$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \in X_1 \\ g^{-1}(x) & \text{if } x \in X_2 \end{cases}$$

Because $X = X_1 \cup X_2$ and f and g^{-1} are 1–1 and have disjoint ranges on these respective spaces, we get that h is 1–1. Because $Y = Y_1 \cup Y_2$ and f and g^{-1} are respectively onto, it follows that h is onto as well.

1.5 Cantor's Theorem

Exercise 1.5.1. The function $f(x) = (x - 1/2)/(x - x^2)$ is a 1–1, onto mapping from $(0, 1)$ to $\mathbf{R}$. This shows $(0, 1) \sim \mathbf{R}$, and the result follows using the ideas in Exercise 1.4.9.

Exercise 1.5.2. (a) The real number $x = .b_1b_2b_3b_4\ldots$ cannot be equal to $f(1) = a_{11}a_{12}a_{13}a_{14}\ldots$ because they differ in the first decimal place; i.e., $b_1 \neq a_{11}$.

(b) The real number x cannot be equal to $f(2)$ because $b_2 \neq a_{22}$ and they differ in the second decimal place. In general, $x \neq f(n)$ because $b_n \neq a_{nn}$.

(c) Since f is onto, every real number $x \in (0, 1)$ should be in the indexed array. However, the specific x we have constructed is not equal to $f(n)$ for any $n \in \mathbf{N}$, and hence not contained in the range of f. This is contradiction to the assumption that f is onto. We conclude that (0,1) must be uncountable.

Exercise 1.5.3. (a) If we imitate the proof to try and show that $\mathbf{Q}$ is uncountable, we can construct a real number x in the same way. This x will again fail to be in the range of our function f, *but there is no reason to expect x to be rational.* The decimal expansions for rational numbers either terminate or repeat, and this will not be true of the constructed x.

(b) By using the digits 2 and 3 in our definition of b_n we eliminate the possibility that the point $x = .b_1b_2b_3\ldots$ has some other possible decimal representation (and thus it cannot exist somewhere in the range of f in a different form.)

Exercise 1.5.4. Our proof will have the same structure as that of Cantor's. So let us assume for contradiction that there exists a function $f : \mathbf{N} \to S$ that is 1-1 and onto. The 1-1 correspondence between $\mathbf{N}$ and S can be represented by the following indexed array

$$\begin{array}{cccccccccccc}
\mathbf{N} & & & & & & & & & & & \\
\hline
1 & \longleftrightarrow & f(1) & = & .\boldsymbol{a_{11}} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & \cdots \\
2 & \longleftrightarrow & f(2) & = & .a_{21} & \boldsymbol{a_{22}} & a_{23} & a_{24} & a_{25} & a_{26} & \cdots \\
3 & \longleftrightarrow & f(3) & = & .a_{31} & a_{32} & \boldsymbol{a_{33}} & a_{34} & a_{35} & a_{36} & \cdots \\
4 & \longleftrightarrow & f(4) & = & .a_{41} & a_{42} & a_{43} & \boldsymbol{a_{44}} & a_{45} & a_{46} & \cdots \\
5 & \longleftrightarrow & f(5) & = & .a_{51} & a_{52} & a_{53} & a_{54} & \boldsymbol{a_{55}} & a_{56} & \cdots \\
6 & \longleftrightarrow & f(6) & = & .a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & \boldsymbol{a_{66}} & \cdots \\
\vdots & & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

where $a_{mn} = 1$ or 0 for $m, n \in \mathbf{N}$. Now let us define a sequence $(x_n) = (x_1, x_2, x_3, \ldots) \in S$ via

$$x_n = \begin{cases} 0 & \text{if } a_{nn} = 1 \\ 1 & \text{if } a_{nn} = 0. \end{cases}$$

From this definition we can see that $f(1)$ is not the sequence (x_n) because a_{11} is not the same as x_1. Similarly, $f(2) \neq (x_n)$ since $a_{22} \neq x_2$. In general, $f(n) \neq (x_n)$ since $a_{nn} \neq x_n$ for all $n \in N$. Because f is onto, all sequences in S should be in the range of f. However, the specific sequence (x_n) we defined above is not equal to $f(n)$ for any $n \in \mathbf{N}$. This contradiction implies that the set S is uncountable.

Exercise 1.5.5. (a) $P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

(b) An induction proof for this fact is given in Exercise 1.2.11. A more combinatoric proof can be obtained by listing the n elements of A. To construct a subset of A, we consider each element and associate either a 'Y' if we decide to include it in our subset or an 'N' if we decide not to include it. Thus, to each subset of A there is an associated sequence of length n of Ys and Ns. This correspondence is 1–1, and the proof is done by observing there are 2^n such sequences.

Exercise 1.5.6. (a) Given set $A = \{a, b, c\}$, A can be mapped in a 1–1 fashion into $P(A)$ in many ways. For example, we could write

(i)

$$\begin{array}{l} a \to \{a\} \\ b \to \{a, c\} \\ c \to \{a, b, c\} \end{array}$$

As another example we might say

(ii)

$$\begin{aligned} &a \to \{b, c\} \\ &b \to \emptyset \\ &c \to \{a, c\}. \end{aligned}$$

(b) An example of 1–1 mapping from B to $P(B)$ is:

$$\begin{aligned} &1 \to \{1\} \\ &2 \to \{2, 3, 4\} \\ &3 \to \{1, 2, 4\} \\ &4 \to \{2, 3\}. \end{aligned}$$

(c) Because $2^n > n$ for every n, the power set $P(A)$ simply has too many elements to be mapped into A in a 1–1 fashion.

Exercise 1.5.7. For the example in (a) (i), the set $B = \{b\}$. For example (ii) we get $B = \{a, b\}$. In part (b) we find $B = \{3, 4\}$. In every case, the set B fails to be in the range of the function that we defined.

Exercise 1.5.8. (a) If $a' \in B$, then by the definition of B we conclude that $a' \notin f(a')$. But $f(a') = B$ which means that $a' \notin B$, a contradiction. Thus we must reject the possibility that $a' \in B$.

(b) But now let's assume that $a' \notin B$. Then by the definition of B, $a' \in f(a')$. Because $f(a') = B$, this implies that $a' \in B$, which is another contradiction. Therefore $a' \notin B$ is equally unacceptable.

Because it is impossible for a' to be in neither B nor B^c, the initial assumption that $B = f(a')$ for some $a' \in A$ must have been false. In other words, such an element a' does not exist, and the function $f : A \to P(A)$ is not onto.

Exercise 1.5.9. (a) The set A of functions from $\{0, 1\}$ to $\mathbf{N}$ is countable. To see this, first observe that A can be put into a 1–1 correspondence with the set of ordered pairs $\{(m, n) : m, n \in \mathbf{N}\}$. To be precise, if $f \in A$, then f is a function from $\{0, 1\}$ to $\mathbf{N}$, and we can match it up with the ordered pair (m, n) where $m = f(0)$ and $n = f(1)$.

To show that $\{(m, n) : m, n \in \mathbf{N}\}$ is countable, we can either use an argument similar to the proof of Theorem 1.4.11 (i) where we showed that $\mathbf{Q}$ is countable. Another approach would be to write

$$\{(m, n) : m, n \in \mathbf{N}\} = \bigcup_{n=1}^{\infty} \{(m, n) : m \in \mathbf{N}\}$$

and use Theorem 1.4.13.

(b) This set is uncountable. A function from $\mathbf{N}$ to $\{0, 1\}$ is in fact just a sequence consisting of 0's and 1's, so the set of such functions is precisely the set S from Exercise 1.5.4.

(c) The set $P(\mathbf{N})$ does contain an uncountable antichain. To construct such an antichain, let's first let $E = \{2, 4, \ldots, 2n, \ldots\}$ be the even natural numbers and $O = \{1, 3, \ldots, 2n-1, \ldots\}$ be the odd natural numbers, enumerated in the standard way. Now consider the set S from Exercise 1.5.4 which we know to be uncountable. For each $s = (s_1, s_2, s_3, \ldots)$ we construct the subset $A_s \subseteq \mathbf{N}$ using the rule that

$2n \in A_s$ if and only if $s_n = 1$ and

$2n - 1 \in A_s$ if and only if $s_n = 0$.

The fact that E and O are disjoint with $\mathbf{N} = E \cup O$ is enough to prove that the collection $\{A_s : s \in S\}$ is an uncountable antichain.

Chapter 2

Sequences and Series

2.1 Discussion: Rearrangements of Infinite Series

2.2 The Limit of a Sequence

Exercise 2.2.1. a) Let $\epsilon > 0$ be arbitrary. We must show that there exists an $N \in \mathbf{N}$ such that $n \geq N$ implies $|\frac{1}{6n^2+1} - 0| < \epsilon$. Well,

$$\left| \frac{1}{6n^2+1} - 0 \right| = \frac{1}{6n^2+1},$$

as this will always be positive. So pick N to satisfy $N > \sqrt{1/6\epsilon}$. It then follows that for $n \geq N$ implies $\frac{1}{6n^2+1} < \epsilon$.

b) Let $\epsilon > 0$ be arbitrary. Now we must produce an N so that $n \geq N$ implies $|\frac{3n+1}{2n+5} - \frac{3}{2}| < \epsilon$. This time notice,

$$\left| \frac{3n+1}{2n+5} - \frac{3}{2} \right| = \frac{3}{2} - \frac{3n+1}{2n+5} = \frac{6n+15-6n-2}{4n+10} = \frac{13}{4n+10}.$$

Now pick N such that $N > \frac{13-10\epsilon}{4\epsilon}$, then for $n \geq N$ it follows that $\frac{13}{4n+10} < \epsilon$.

c) Let $\epsilon > 0$ be arbitrary. We must produce an N so that $n \geq N$ implies $|\frac{2}{\sqrt{n+3}} - 0| < \epsilon$. Well,

$$\left| \frac{2}{\sqrt{n+3}} - 0 \right| = \frac{2}{\sqrt{n+3}}.$$

Pick N to satisfy $N > 4/\epsilon^2 - 3$. It then follows that when $n \geq N$ we get $\frac{2}{\sqrt{n+3}} < \epsilon$ as desired.

Exercise 2.2.2. Consider the sequence $(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \cdots)$ This sequence vercongs to $x = 0$. To see this, note that we only have to produce *a single* $\epsilon > 0$

where the prescribed condition follows, and in this case we can take $\epsilon = 1$. This ϵ works because for all $N \in \mathbf{N}$, it is true that $n \geq N$ implies $|x_n - \frac{1}{2}| < 1$.

This is also an example of a vercongent sequence that is divergent. Notice that the "limit" $x = 0$ is not unique. We could also show this same sequence verconges to $x = 1$ by choosing $\epsilon = 2$.

In general, a vercongent sequence is a bounded sequence. By a bounded sequence, we mean that there exists an $M \geq 0$ satisfying $|x_n| <\leq M$ for all $n \in \mathbf{N}$. In this case we can always take $x = 0$ and $\epsilon = M + 1$. Then $|x_n - x| = |x_n| < \epsilon$, and the sequence (x_n) verconges to 0.

Exercise 2.2.3. a) There exists at least one college in the United States where all students are less than seven feet tall.

b) There exists a college in the United States where all professors gave at least one student a grade of C or less.

c) At every college in the United States, there is a student less than six feet tall.

Exercise 2.2.4. For any ϵ that is greater than 1, there exists a response N. In this case, N can be any natural number.

For any ϵ that is less than or equal to 1, there exists no suitable response. This is because, although the 1s in the sequence occur less and less frequently as we go out the sequence, there is still no point in the sequence where the sequence enters the neighborhood $(-\epsilon, \epsilon)$ and *never leaves.*

Exercise 2.2.5. a) The limit of (a_n) is zero. To show this let $\epsilon > 0$ be arbitrary. We must show that there exists an $N \in \mathbf{N}$ such that $n \geq N$ implies $|[[1/n]] - 0| < \epsilon$. Well, pick $N > 1$. If $n \geq N$ we then have;

$$\left| \left[\left[\frac{1}{n}\right]\right] - 0 \right| = |0 - 0| < \epsilon,$$

because $[[1/n]] = 0$ for all $n > 1$.

b) Again the limit of a_n is zero. Let $\epsilon > 0$ be arbitrary. By picking $N > 10$ we have that for $n \geq N$,

$$\left| \left[\left[\frac{10+n}{2n}\right]\right] - 0 \right| = |0 - 0| < \epsilon,$$

because $[[(10 + n)/2n]] = 0$ for all $n > 10$.

In these exercises, the choice of N does not depend on ϵ in the usual way. In exercise (b) for instance, setting $N = 11$ is a suitable response for every choice of $\epsilon > 0$. Thus, this is a rare example where a smaller $\epsilon > 0$ does not require a larger N in response.

Exercise 2.2.6. (a) Any larger N will also work for the same $\epsilon > 0$.

(b) This same N will also work for any larger value of ϵ.

Exercise 2.2.7. a) A sequence (a_n) "converges to infinity" if, for every positive number a, there exists an $N \in \mathbf{N}$ such that whenever $n \geq N$ it follows that $a_n > a$.

Let $a > 0$ be arbitrary. We must show that there exists an $N \in \mathbf{N}$ such that $n \geq N$ implies that $\sqrt{n} > a$. Well, pick $N > a^2$. Then, $\sqrt{n} > a$ for all $n \geq N$.

b) According to the above definition, this sequence does not converge to infinity.

Exercise 2.2.8. (a) The sequence $(-1)^n$ is *frequently* in the set 1.

(b) Definition (i) is stronger. "Frequently" does not imply "eventually", but "eventually" implies "frequently".

(c) A sequence (a_n) converges to a real number a if, given any ϵ-neighborhood $V_\epsilon(a)$ of a, (a_n) is *eventually* in $V_\epsilon(a)$.

(d) Suppose an infinite number of terms of a sequence (x_n) are equal to 2, then (x_n) is *frequently* in the interval (1.9, 2.1). However, (x_n) is not necessarily *eventually* in the interval (1.9, 2.1). Consider the sequence $(2, 0, 2, 0, 2, \cdots)$, for instance.

2.3 The Algebraic and Order Limit Theorems

Exercise 2.3.1. Let $\epsilon > 0$ be arbitrary. We need to show that there exists an N such that when $n \geq N$, $|a_n - a| < \epsilon$. Well, for all n

$$|a_n - a| = |a - a| = 0 < \epsilon.$$

So we can choose N to be anything we like.

Exercise 2.3.2. (a) Let $\epsilon > 0$ be arbitrary. We must find an N such that $n \geq N$ implies $|\sqrt{x_n} - 0| < \epsilon$. Because $(x_n) \to 0$, there exists $N \in \mathbf{N}$ such that $n \geq N$ implies $|x_n - 0| = x_n < \epsilon^2$. Using this N, we have $\sqrt{(x_n)^2} < \epsilon^2$, which gives $|\sqrt{x_n} - 0| < \epsilon$ for all $n \geq N$, as desired.

(b) Part (a) handles the case $x = 0$, so we may assume $x > 0$. Let $\epsilon > 0$. This time we must find an N such that $n \geq N$ implies $|\sqrt{x_n} - \sqrt{x}| < \epsilon$, for all $n \geq N$. Well,

$$\begin{aligned} |\sqrt{x_n} - \sqrt{x}| &= |\sqrt{x_n} - \sqrt{x}| \left(\frac{\sqrt{x_n} + \sqrt{x}}{\sqrt{x_n} + \sqrt{x}} \right) \\ &= \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \\ &\leq \frac{|x_n - x|}{\sqrt{x}} \end{aligned}$$

Now because $(x_n) \to x$ and $x > 0$, we can choose N such that $|x_n - x| < \epsilon\sqrt{x}$ whenever $n \geq N$. And this implies that for all $n \geq N$,

$$|\sqrt{x_n} - \sqrt{x}| < \frac{\epsilon\sqrt{x}}{\sqrt{x}} = \epsilon$$

as desired.

Exercise 2.3.3. Let $\epsilon > 0$ be arbitrary. We must show that there exists an N such that $n \geq N$ implies $|y_n - l| < \epsilon$. In terms of ϵ-neighborhoods (which are a bit easier to use in this case), we must equivalently show $y_n \in (l - \epsilon, l + \epsilon)$ for all $n \geq N$.

Because $(x_n) \to l$, we can pick an N_1 such that $x_n \in (l - \epsilon, l + \epsilon)$ for all $n \geq N_1$. Similarly, because $(z_n) \to l$ we can pick an N_2 such that $z_n \in (l-\epsilon, l+\epsilon)$ whenever $n \geq N_2$. Now, because $x_n \leq y_n \leq z_n$, if we let $N = \max\{N_1, N_2\}$, then it follows that $y_n \in (l - \epsilon, l + \epsilon)$, for all $n \geq N$. This completes the proof.

Exercise 2.3.4. We can prove this directly from the definition of convergence, or by using the Algebraic Limit Theorem.

(i) Proof using the definition of convergence:

Let $\epsilon > 0$ be arbitrary. Let's show $|l_1 - l_2| < \epsilon$. We know that $\lim a_n = l_1$, so there exists $N_1 \in \mathbf{N}$ such that $n \geq N_1$ implies $|a_n - l_1| < \epsilon/2$. Similarly, since $\lim a_n = l_2$, there exists $N_2 \in \mathbf{N}$ such that $n \geq N_2$ implies $|a_n - l_2| < \epsilon/2$. Setting $N = \max\{N_1, N_2\}$ gives us that for $n \geq N$,

$$
\begin{aligned}
|l_1 - l_2| &= |l_1 - a_n + a_n - l_2| \\
&\leq |l_1 - a_n| + |a_n - l_2| \\
&= |a_n - l_n| + |a_n - l_2| \\
&< \epsilon/2 + \epsilon/2 \\
&= \epsilon.
\end{aligned}
$$

Thus it is clear that $|l_1 - l_2| < \epsilon$. By Theorem 1.2.6, $l_1 = l_2$.

(ii) Proof using the Algebraic Limit Theorem:

First observe that

$$\lim(a_n - a_n) = \lim(a_n) - \lim(a_n) = l_1 - l_2$$

But we also have

$$\lim(a_n - a_n) = \lim 0 = 0,$$

and therefore $l_1 - l_2 = 0$ which implies $l_1 = l_2$.

Exercise 2.3.5. ($\Rightarrow$) Let $\epsilon > 0$ be arbitrary. Let's call the limit that (z_n) converges to L. Then we need to show that there exists an N such that when $n \geq N$, it follows that $|y_n - L| < \epsilon$. Because $(z_n) \to L$, we can pick N so that $|z_n - L| < \epsilon$ for all $n \geq N$. Because $y_n = z_{2N}$ it certainly follows that $|y_n - L| < \epsilon$ whenever $n \geq N$. A similar argument holds for the (x_n) sequence.

($\Leftarrow$) Let $\epsilon > 0$ be arbitrary. Again, let L be the common limit of (x_n) and (y_n). We need to show that there exists an N such that when $n \geq N$ it follows that $|z_n - L| < \epsilon$. Choose N_1 so that $|x_n - L| < \epsilon$ for all $n \geq N_1$, and choose N_2 such that $|y_n - L| < \epsilon$ for all $n \geq N_2$. Finally, let $N = \max\{2N_1, 2N_2\}$, and it follows from the construction of the sequence (z_n) that $|z_n - L| < \epsilon$ whenever $n \geq N$.

Exercise 2.3.6. (a)By the triangle inequality,

$$|b_n| = |b_n - b + b| \leq |b_n - b| + |b|$$

Thus

$$|b_n| - |b| \leq |b_n - b|,$$

and in fact

$$||b_n| - |b|| \leq |b_n - b|.$$

Since $(b_n) \to b$, there exists $N \in \mathbf{N}$ such that $|b_n - b| < \epsilon$ whenever $n \geq N$. Therefore, $||b_n| - |b|| \leq |b_n - b| < \epsilon$ for all $n \geq N$ as well, which proves $|b_n| \to |b|$.

(b) The converse of (a) is false. Consider $b_n = (-1)^n$. We can see that $|b_n| \to 1$, but (b_n) is divergent.

Exercise 2.3.7. a) Because (a_n) is bounded, there exists a K satisfying $|a_n| \leq K$. Let $\epsilon > 0$ be arbitrary. We need to find an N such that when $n \geq N$ it follows that $|a_n b_n - 0| < \epsilon$. Well,

$$|a_n b_n - 0| = |a_n||b_n| \leq K|b_n|.$$

Because $(b_n) \to 0$, we can pick an N such that

$$|b_n| < \frac{\epsilon}{K}.$$

Finally, we can conclude that for this choice of N,

$$|a_n b_n - 0| \leq K|b_n| < K\frac{\epsilon}{K} = \epsilon$$

for all $n \geq N$. Therefore, $(a_n b_n) \to 0$.

We may not use the Algebraic Limit Theorem in this case because the hypothesis of that theorem requires that both (a_n) and (b_n) be convergent. (And this may not be so for (a_n).)

b) No, for instance if $(a_n) = (1, -1, 1, -1, \cdots)$, $(a_n b_n)$ will not converge.

c) All convergent series are bounded. Therefore, if $(a_n) \to a$ and $(b_n) \to 0$, then by part (a), $(a_n b_n) \to 0$.

Exercise 2.3.8. (a) Consider $(x_n) = (-1, 1, -1, 1, \cdots)$, and $(y_n) = (1, -1, 1, -1, \cdots)$. Both sequences diverge but $(x_n + y_n)$ converges.

(b) Such a request is impossible because by the Algebraic Limit Theorem, if $(x_n + y_n)$ converges to l and (x_n) converges to x, then

$$\lim(y_n) = \lim(y_n + x_n - x_n) = \lim(x_n + y_n) - \lim(x_n) = l - x.$$

So (y_n) must also converge.

(c) Consider the sequence $(b_n) = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots)$. To prevent the sequence from "converging to infinity" we could also add alternating negative signs.

(d) Such a request is impossible. By Theorem 2.3.2,(b_n) is bounded. If $(a_n - b_n)$ were bounded, then we could show that

$$(a_n) = (a_n - b_n) + (b_n)$$

would also have to be bounded, which is not the case. Thus, $(a_n - b_n)$ is unbounded.

(e) Take $(a_n) = 1/n$, and $(b_n) = (-1)^n$. Such a request would be impossible if we were given that $\lim a_n \neq 0$.

Exercise 2.3.9. No, the Order Limit Theorem (Theorem 2.3.4) does not remain valid if the inequalities are assumed to be strict (in the conclusion of the theorem). For example, the sequence $(1/n)$ converges to zero although every term is strictly positive.

Exercise 2.3.10. Let $\epsilon > 0$ be arbitrary. We want to produce an N such that for every $n \geq N$, $|b_n - b| < \epsilon$. Because $(a_n) \to 0$, there exists an $N \in \mathbf{N}$ such that for every $n \geq \mathbf{N}$,

$$a_n = |a_n - 0| < \epsilon.$$

Using this same N, we have $|b_n - b| \leq a_n < \epsilon$ whenever $n \geq N$. Therefore $(b_n) \to b$.

Exercise 2.3.11. Let $\epsilon > 0$ be arbitrary. Then we need to find an N such that $n \geq N$ implies $|y_n - L| < \epsilon$. Because $(x_n) \to L$, we know that there exists $M > 0$ such that $|x_n - L| < M$ for all n. Also, there exists an N_1 such that $n \geq N_1$ implies $|x_n - L| < \epsilon/2$. Now for $n \geq N_1$ we can write

$$\begin{aligned}
|y_n - L| &= \left| \frac{x_1 + x_2 + \cdots + x_{N_1} + \cdots + x_n}{n} - \frac{nL}{n} \right| \\
&= \left| \frac{(x_1 - L) + (x_2 - L) + \cdots + (x_{N_1 - 1} - L)}{n} + \frac{(x_{N_1} - L) + \cdots + (x_n - L)}{n} \right| \\
&\leq \left| \frac{(x_1 - L) + (x_2 - L) + \cdots + (x_{N_1 - 1} - L)}{n} \right| + \left| \frac{(x_{N_1} - L) + \cdots + (x_n - L)}{n} \right| \\
&\leq \frac{(N_1 - 1)M}{n} + \frac{\epsilon(n - N_1)}{2n}.
\end{aligned}$$

Because N_1 and M are fixed constants at this point, we may choose N_2 so that $\frac{(N_1-1)M}{n} < \epsilon/2$ for all $n \geq N_2$. Finally, let $N = \max\{N_1, N_2\}$ be the desired N. To see that this works, keep in mind that $\frac{n - N_1}{n} < 1$ and observe

$$|y_n - L| \leq \frac{(N_1 - 1)M}{n} + \frac{\epsilon(n - N_1)}{2n} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $n \geq N$. This completes the proof.

The sequence $(x_n) = (1, -1, 1, -1, \cdots)$ does not converge, but the averages satisfy $(y_n) \to 0$.

Exercise 2.3.12. (a) Intuitively speaking,

$$\lim_{m,n\to\infty} a_{m,n}$$

should be a number that is arbitrarily close to the values of $a_{m,n}$ when m and n are both large. However, with two index variables, this raises the question of whether we insist that the variables be large simultaneously or whether we allow them to "go to infinity" one at a time in an iterated fashion.

The "iterated" limit $\lim_{n\to\infty}\lim_{m\to\infty} a_{m,n}$ is the limit of the sequence of the limits of the columns in the doubly indexed array $a_{m,n}$. To compute this, first fix $n \in \mathbf{N}$ and let

$$b_n = \lim_{m\to\infty} a_{m,n} = \lim_{m\to\infty} \frac{1}{1+n/m} = \frac{1}{1+0} = 1.$$

Thus

$$\lim_{n\to\infty}\lim_{m\to\infty} a_{m,n} = \lim_{n\to\infty} b_n = \lim_{n\to\infty} 1 = 1.$$

In the other order, we first fix m and compute the limit along each row of the $a_{m,n}$ array to get

$$\lim_{m\to\infty}\lim_{n\to\infty} a_{m,n} = \lim_{m\to\infty}\left(\lim_{n\to\infty}\frac{m/n}{(m/n)+1}\right) = \lim_{m\to\infty}\frac{0}{0+1} = 0.$$

From this example we can see that it is possible for

$$\lim_{m\to\infty}\lim_{n\to\infty} a_{m,n} \neq \lim_{n\to\infty}\lim_{m\to\infty} a_{m,n},$$

and so defining doubly indexed limits in this fashion would be problematic to say the least.

(b) A doubly indexed array $(a_{m,n})$ satisfies

$$\lim_{m,n\to\infty} a_{m,n} = l$$

if for every positive number ϵ, there exists an $N \in \mathbf{N}$ such that whenever $n, m \geq N$ it follows that $|a_{m,n} - l| < \epsilon$.

2.4 The Monotone Convergence Theorem and a First Look at Infinite Series

Exercise 2.4.1. We will show that if $\sum_{n=0}^{\infty} 2^n b_{2n}$ diverges then $\sum_{n=1}^{\infty} b_n$ diverges by again exploiting a relationship between the partial sums

$$s_m = b_1 + b_2 + \cdots + b_m, \quad \text{and} \quad t_k = b_1 + 2b_2 + \cdots + 2^k b_{2^k}.$$

Because $\sum_{n=0}^{\infty} 2^n b_{2n}$ diverges, its monotone sequence of partial sums (t_k) must be unbounded. To show that (s_m) is unbounded it is enough to show that for all $k \in \mathbf{N}$, there is term s_m satisfying $s_m \geq t_k/2$. This argument is similar to the one for the forward direction, only to get the inequality to go the other way we group the terms in s_m so that the *last* (and hence smallest) term in each group is of the form b_{2^k}.

Given an arbitrary k, we focus our attention on s_{2^k} and observe that

$$\begin{aligned}
s_{2^k} &= b_1 + b_2 + (b_3 + b_4) + (b_5 + b_6 + b_7 + b_8) + \cdots + (b_{2^{k-1}+1} + \cdots + b_{2^k}) \\
&\geq b_1 + b_2 + (b_4 + b_4) + (b_8 + b_8 + b_8 + b_8) + \cdots + (b_{2^k} + \cdots + b_{2^k}) \\
&= b_1 + b_2 + 2b_4 + 4b_8 + \cdots + 2^{k-1} b_{2^k} \\
&= \frac{1}{2}\left(2b_1 + 2b_2 + 4b_4 + 8b_8 + \cdots + 2^k b_{2^k}\right) \\
&= b_1/2 + t_k/2.
\end{aligned}$$

Because (t_k) is unbounded, the sequence (s_m) must also be unbounded and cannot converge. Therefore, $\sum_{n=1}^{\infty} b_n$ diverges.

Exercise 2.4.2. (a) We will show that this sequence is decreasing and bounded.

First, let's use induction to show that this sequence is decreasing. Observe that $x_1 = 3 > 1 = x_2$. Now, we need to prove that if $x_n > x_{n+1}$, then $x_{n+1} > x_{n+2}$. Well, $x_n > x_{n+1}$ implies that $-x_n < -x_{n+1}$. Adding 4 to both sides of the inequality gives $4 - x_n < 4 - x_{n+1}$. It follows that

$$\frac{1}{4 - x_n} > \frac{1}{4 - x_{n+1}},$$

which is precisely what we need to conclude $x_{n+1} > x_{n+2}$. Thus by induction, (x_n) is decreasing.

The argument above shows that (x_n) is bounded above by 3, so now we'll show that (x_n) is bounded below. Clearly $x_1 > 0$. Now assume $x_n > 0$. Because (x_n) is decreasing, we know that $x_n \leq x_1 = 3$, which implies that $x_{n+1} = \frac{1}{4-x_n}$ is positive. By induction, (x_n) is bounded below by 0 for all $n \in \mathbf{N}$.

Therefore this sequence converges by Monotone Convergence Theorem.

(b) Since the sequence (x_{n+1}) is just the sequence (x_n) shifted by 1 (and without the first term), the two sequences have the same limit.

(c) From (b), we can let $x = \lim(x_n) = \lim(x_{n+1})$. Now the Algebraic Limit Theorem tells us that

$$x = \lim x_{n+1} = \lim \frac{1}{4 - x_n} = \frac{1}{4 - x},$$

and it follows that x must satisfy the equation $x^2 - 4x + 1 = 0$. Solving the equation gives

$$x = \frac{4 \pm \sqrt{16 - 4}}{2} = 2 \pm \sqrt{3},$$

and since $x_1 = 3$ and (x_n) is decreasing, we conclude that $x = 2 - \sqrt{3}$.

Exercise 2.4.3. (a) First, $y_1 = 1 < 7/2 = y_2$. To use induction to prove that (y_n) is increasing we assume $y_n < y_{n+1}$ and show that $y_{n+1} < y_{n+2}$. Starting with the inequality $y_n < y_{n+1}$, we take reciprocals to get $1/y_n > 1/y_{n+1}$. Then multiplying by -1 and adding 4 to each side gives $4 - 1/y_n < 4 - 1/y_{n+1}$ which is precisely the desired statement $y_{n+1} < y_{n+2}$.

Now we know that (y_n) is increasing and bounded below by $y_1 = 1$. Because the terms in (y_n) are all positive, it follows that $y_n < 4$ for all $n \in \mathbf{N}$ and our increasing sequence is bounded above. Thus, by the Monotone Convergence Theorem we may set

$$y = \lim y_n = \lim y_{n+1}.$$

Taking limits across the recursive equation for y_{n+1} gives

$$y = \lim y_{n+1} = \lim(4 - 1/y_n) = 4 - 1/y$$

which implies that y satisfies $y^2 - 4y + 1 = 0$. A little algebra yields $y = \sqrt{3} + 2$.

Exercise 2.4.4. We will show that this sequence is increasing and bounded. First rewrite the sequence in a recursive way: $x_1 = \sqrt{2}$, $x_{n+1} = \sqrt{2x_n}$.

Let's prove that the sequence is increasing by induction. For the base case we observe that

$$x_1 = 2 < \sqrt{2\sqrt{2}} = x_2,$$

so we just need to prove that $x_n < x_{n+1}$ implies $x_{n+1} < x_{n+2}$. But if $x_n < x_{n+1}$ then $\sqrt{x_n} < \sqrt{x_{n+1}}$, and multiplying by $\sqrt{2}$ gives $\sqrt{2x_n} < \sqrt{2x_{n+1}}$. Thus we have $x_{n+1} < x_{n+2}$ and the sequence is increasing.

To show the sequence is bounded above by 2 we first observe that $x_1 < 2$. Now if $x_n < 2$, then $x_{n+1} = \sqrt{2x_n} < \sqrt{2 \cdot 2} = 2$ as well, and (x_n) is bounded.

Therefore this sequence converges by Monotone Convergence Theorem and we can assert that both (x_n) and (x_{n+1}) converge to some real number l. Taking limits across the recursive equation $x_{n+1} = \sqrt{2x_n}$ yields $l = \sqrt{2l}$, which implies $l = 2$.

We should note that the last steps in this problem involved taking the limit inside a square root sign, and this is not a manipulation that is justified by the Algebraic Limit Theorem. Instead we should reference Exercise 2.3.2 to support this part of the argument.

Exercise 2.4.5. (a) We first observe that a simple induction argument shows that x_n is positive for all n. We can also write

$$x_{n+1}^2 - 2 = \left(\frac{1}{2}\left(x_n + \frac{2}{x_n}\right)\right)^2 - 2 = \frac{x_n^2}{4} + \frac{1}{x_n^2} - 1 = \left(\frac{x_n}{2} - \frac{1}{x_n}\right)^2 \geq 0$$

as any number squared is positive. This shows that $x_n^2 \geq 2$ for all choices of n. (It's worth mentioning that this part of the argument, and the next, is not by induction.)

Now let's argue that (x_n) is decreasing. If we write

$$x_n - x_{n+1} = x_n - \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) = \frac{1}{2}x_n - \frac{1}{x_n} = \frac{x_n^2 - 2}{2x_n},$$

then we can see that $x_n - x_{n+1}$ is positive because $x_n^2 \geq 2$. Because we have shown that (x_n) is decreasing and bounded below, we may set $x = \lim x_n = \lim x_{n+1}$. Taking limits across the recursive equation we find

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) = \frac{x}{2} + \frac{1}{x}$$

which implies $x = \sqrt{2}$.

(b) The sequence

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right)$$

converges to $\sqrt{c}$ using a similar argument.

Exercise 2.4.6. (a) For each $n \in \mathbf{N}$, set $A_n = \{a_k : k \geq n\}$ so that $y_n = \sup A_n$. Because $A_{n+1} \subseteq A_n$ it follows (by Exercise 1.3.4) that $y_{n+1} \leq y_n$ and so (y_n) is decreasing. If L is a lower bound for (a_n), then for all $n \in \mathbf{N}$ it must be that $y_n \geq a_n \geq L$. Thus (y_n) is both decreasing and bounded, and it follows from the Monotone Convergence Theorem that (y_n) converges.

(b)Define the *limit inferior* of (a_n) as

$$\liminf a_n = \lim z_n,$$

where $z_n = \inf\{a_k : k \geq n\}$. The sequence (z_n) is increasing (because we are taking the greatest lower bound of a smaller set each time) and bounded above (because (a_n) is bounded.) Thus (z_n) converges by MCT.

(c) For each $n \in \mathbf{N}$ we have $y_n \geq z_n$, so by the Order Limit Theorem (Theorem 2.3.4) $\lim y_n \geq \lim z_n$. This shows $\limsup a_n \geq \liminf a_n$ for every bounded sequence.

The sequence $(a_n) = (1, 0, 1, 0, 1, 0, \cdots)$ has $\limsup a_n = 1$ and $\liminf a_n = 0$. Notice that this sequence is not convergent.

(d) First let's prove that if $\lim y_n = \lim z_n = l$, then $\lim a_n = l$ as well. Let $\epsilon > 0$. There exists an $N \in \mathbf{N}$ such that $y_n \in V_\epsilon(l)$ and $z_n \in V_\epsilon(l)$ for all $n \geq N$. Because $z_n \leq a_n \leq y_n$, it must also be the case that $a_n \in V_\epsilon(l)$ for all $n \geq N$. Therefore $\lim a_n$ exists and is equal to l.

Next, let's show that if $\lim a_n = l$, then $\lim y_n = l$. (The proof that $\lim z_n = l$ is similar.) Let $\epsilon > 0$ be arbitrary. Because $\lim a_n = l$, there exists an $N \in \mathbf{N}$ such that $n \geq N$ implies $a_n \in V_\epsilon(l)$. This means that $l-\epsilon$ and $l+\epsilon$ are lower and upper bounds for the set $\{a_n, a_{n+1}, a_{n+2}, \cdots\}$. It follows that $l - \epsilon \leq y_n \leq l + \epsilon$ for all $n \geq N$. Keeping in mind that we already know $y = \lim y_n$ exists, we can use the Order Limit Theorem to assert that $l - \epsilon \leq y \leq l + \epsilon$, and because ϵ is arbitrary we must have $y = l$. (Theorem 1.2.6 could be referenced in this last step.)

2.5 Subsequences and the Bolzano–Weierstrass Theorem

Exercise 2.5.1. Assume $(a_n) \to L$ and let (a_{n_j}) be a subsequence of (a_n). We must show $(a_{n_j}) \to L$ as well. Let $\epsilon > 0$ be arbitrary. We need to produce an $N \in \mathbf{N}$ such that $j \geq N$ implies $|a_{n_j} - L| < \epsilon$. Because $(a_n) \to L$ we know there exists an N such that $|a_n - L| < \epsilon$ for $n \geq N$. But $n_j \geq j$, so this same N works for the subsequence as well. To be precise, $j \geq N$ implies $n_j \geq N$, and so $|a_{n_j} - L| < \epsilon$ as desired.

Exercise 2.5.2. (a) Letting $s_n = a_1 + a_2 + \cdots + s_n$, we are given that $\lim s_n = L$. For the regrouped series, let's write

$$\begin{aligned} b_1 &= a_1 + a_2 + \cdots + a_{n_1}, \\ b_2 &= a_{n_1+1} + a_{n_1+2} + \cdots + a_{n_2}, \\ &\vdots \\ b_m &= a_{n_{m-1}+1} + \cdots + a_{n_m}, \end{aligned}$$

and the claim is that the series $\sum_{m=1}^{\infty} b_m$ converges to L as well.

To prove this, just observe that if (t_m) is the sequence of partial sums for the regrouped series, then

$$\begin{aligned} t_m &= b_1 + b_2 + \cdots + b_m \\ &= (a_1 + \cdots + a_{n_1}) + \cdots + (a_{n_{m-1}+1} + \cdots + a_{n_m}) = s_{n_m}. \end{aligned}$$

which means that (t_m) is a subsequence of (s_n) and therefore converges to L by Theorem 2.5.2.

Exercise 2.5.3. (a) $(1/2, 1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 4/5, \cdots, 1/n, (n-1)/n, \cdots)$

(b) Impossible. This convergent subsequence would then be bounded; however, this would imply that the original sequence was also bounded. Because the original sequence is monotone, we know it cannot be bounded because we are told it diverges.

(c) The sequence

$$\left(1, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, 1, \cdots\right)$$

has this property. Notice that there is also a subsequence converging to 0. We shall see that this is unavoidable.

(d) $(1, 1, 2, 1, 3, 1, 4, 1, 5, 1, \cdots)$

(e) Impossible. Theorem 2.5.5 guarantees us that all bounded sequences have convergent subsequences.

Exercise 2.5.4. Let's assume, for contradiction, that (a_n) does not converge to a. Paying close attention to the quantifiers in the definition of convergence for a sequence, what this means is that there exists an $\epsilon_0 > 0$ such that for every $N \in \mathbf{N}$ we can find an $n \geq N$ for which $|a - a_n| \geq \epsilon_0$. Using this, we can build a subsequence of (a_n) that never enters the ϵ–neighborhood $V_{\epsilon_0}(a)$. To see how, first pick n_1 so that $|a - a_{n_1}| \geq \epsilon$. Next choose $n_2 > n_1$ so that $|a - a_{n_2}| \geq \epsilon_0$. Because our negated definition says that "...for every $N \in \mathbf{N}$, we can find an $n \geq N$..." we can be sure that having chosen n_j, we may pick $n_{j+1} > n_j$ so that $|a - a_{n_{j+1}}| \geq \epsilon_0$.

Because (a_n) is bounded, the resulting subsequence (a_{n_j}) must be bounded as well. Now apply the Bolzano–Weierstrass Theorem to (a_{n_j}) to say that there exists a convergent subsequence (of (a_{n_j}) and hence also of (a_n)) which we will write as $(a_{n_{j_k}})$. By hypothesis, this convergent subsequence must converge to a, but therein lies the contradiction. Because $(a_{n_{j_k}})$ is a subsequence of (a_{n_j}), it never enters the neighborhood $V_{\epsilon_0}(a)$ and it cannot converge to a. This completes the proof.

Exercise 2.5.5. From Example 2.5.3 we know that this is true for $0 < b < 1$. If $b = 0$ we get the constant sequence $(0, 0, 0, \ldots)$, so let's focus on the case $-1 < b < 0$. Let $\epsilon > 0$ be arbitrary and set $a = |b|$. Because we know $(a^n) \to 0$ (by Example 2.5.3), we may choose N so that $n \geq N$ implies $|a^n - 0| < \epsilon$. But this N will also work for the sequence (b^n) because

$$|b^n - 0| = |b^n| = |a^n| < \epsilon$$

whenever $n \geq N$.

Exercise 2.5.6. Because (a_n) is bounded, the set S is not empty and bounded above. By AoC, we know there exists an $s \in \mathbf{R}$ satisfying $s = \sup S$. For a fixed $k \in \mathbf{N}$ consider $s - 1/k$. Because s is the *least* upper bound, $s - 1/k$ is not an upper bound and there exists a point $s' \in S$ satisfying $s - 1/k < s'$. A quick look at the definition of S then shows that, in fact, $s - 1/k \in S$ and consequently there exist an infinite number of terms a_n satisfying $s - 1/k < a_n$.

Because s is an upper bound for S we can be sure that $s + 1/k \notin S$ from which we can conclude that there are only a finite number of terms a_n satisfying $s + 1/k < a_n$. Taken together, these observations show that for all $k \in \mathbf{N}$, there are an infinite number of terms a_n satisfying

$$s - \frac{1}{k} < a_n \leq s + \frac{1}{k}.$$

To inductively build our convergent subsequence (a_{n_k}) first pick a_{n_1} to satisfy $s - 1 < a_{n_1} \leq s + 1$. Now given that we have constructed a_{n_k}, choose $n_{k+1} > n_k$ so that

$$s - \frac{1}{k+1} < a_{n_{k+1}} \leq s + \frac{1}{k+1}.$$

(Here we are using the fact that this inequality is satisfied by an infinite number of terms a_n and so there is certainly one where $n > n_k$.) To show $(a_{n_k}) \to s$, we let $\epsilon > 0$ be arbitrary and choose $K > 1/\epsilon$. If $k \geq K$ then $1/k < \epsilon$ which implies $s - \epsilon < a_{n_k} < s + \epsilon$, and the proof is complete.

2.6 The Cauchy Criterion

Exercise 2.6.1. (a) $(1, -1/2, 1/3, -1/4, 1/5, -1/6, \ldots)$

(b) $(1, 2, 3, 4, 5, 6, \ldots)$

(c) Impossible, if a sequence is Cauchy then Theorem 2.6.4 tells us that it converges and Theorem 2.5.2 says that subsequences of convergent sequences converge.

(d) $(1, 1, 1/2, 2, 1/3, 3, 1/4, 4, 1/5, 5, \ldots)$

Exercise 2.6.2. Assume (x_n) converges to x, and let $\epsilon > 0$ be arbitary. Becuase $(x_n) \to x$, there exists $N \in \mathbf{N}$ such that $n, m \geq N$ implies $|x_n - x| < \epsilon/2$ and $|x_m - x| < \epsilon/2$. By the triangle inequality,

$$\begin{aligned} |x_n - x_m| &= |x_n - x + x - x_m| \\ &\leq |x_n - x| + |x_m - x| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Therefore, $|x_n - x_m| < \epsilon$ whenever $n, m \geq N$, and (x_n) is a Cauchy sequence.

Exercise 2.6.3. (a) The difference is that this definition only requires that the difference between consecutive elements become arbitrarily small, whereas the real Cauchy property requires that any two elements beyond a certain point in the sequence differ by an arbitrarily small amount.

(b) $(1), (1 + 1/2), (1 + 1/2 + 1/3), (1 + 1/2 + 1/3 + 1/4), \ldots$
This is the sequence of partial sums for the harmonic series $\sum 1/n$ which we have seen diverges even though $s_{n+1} - s_n = 1/n$ goes to zero.

Exercise 2.6.4. Let $\epsilon > 0$ be arbitrary. We know that there exists an $N_1 \in \mathbf{N}$ such that $n, m \geq N$ implies $|a_n - a_m| < \epsilon/2$. Also, we know there exists an $N_2 \in \mathbf{N}$ such that $n, m \geq N_2$ implies $|b_n - b_m| < \epsilon/2$. Set $N = \max\{N_1, N_2\}$. By the triangle inequality and its variation in Exercise 1.2.5 (b),

$$\begin{aligned} |c_n - c_m| &= ||a_n - b_n| - |a_m - b_m|| \\ &\leq |(a_n - b_n) - (a_m - b_m)| \\ &= |(a_n - a_m) + (b_m - b_n)| \\ &\leq |a_n - a_m| + |b_m - b_n| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

whenever $n, m \geq N$. Therefore (c_n) is a Cauchy sequence.

Exercise 2.6.5. (a) Let $\epsilon > 0$ be arbitrary. We need to find an N so that $n, m \geq N$ implies $|(x_n + y_n) - (x_m + y_m)| < \epsilon$. Because (x_n) and (y_n) are Cauchy we can pick N so that when $n, m \geq N$ it follows that $|x_n - x_m| < \epsilon/2$ and $|y_n - y_m| < \epsilon/2$. Now write,

$$|(x_n + y_n) - (x_m + y_m)| \leq |x_n - x_m| + |y_n - y_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

(b) Let $\epsilon > 0$ be arbitrary. We must produce an N such that $n, m \geq N$ implies $|x_n y_n - x_m y_m| < \epsilon$. Note that

$$\begin{aligned} |x_n y_n - x_m y_m| &= |x_n y_n - x_n y_m + x_n y_m - x_m y_m| \\ &\leq |x_n y_n - x_n y_m| + |x_n y_m - x_m y_m| \\ &= |x_n||y_n - y_m| + |y_m||x_n - x_m|. \end{aligned}$$

Because (x_n) and (y_n) are Cauchy, we know by Lemma 2.6.3 that they are bounded. So let $K \geq |x_n|$ and $L \geq |y_m|$ for all m, n. We also know that we can pick N_1 such that $m, n \geq N_1$ implies $|x_n - x_m| < \frac{\epsilon}{2L}$. Similarly, pick N_2 so that $m, n \geq N_2$ implies $|y_n - y_m| < \frac{\epsilon}{2K}$. Now let $N = \max\{N_1, N_2\}$. Then for $m, n \geq N$ it follows that

$$|x_n y_n - x_m y_m| \leq |x_n||y_n - y_m| + |y_m||x_n - x_m| < K\frac{\epsilon}{2K} + L\frac{\epsilon}{2L} = \epsilon.$$

Exercise 2.6.6. (a) Let A be a non-empty set that is bounded above, and let b_1 be an upper bound for A. Next choose a real number a_1 that is *not* an upper bound for A. Necessarily, $a_1 < b_1$ and we can let I_1 be the closed interval $I_1 = [a_1, b_1]$.

Our goal in this proof is to show that the set A has a least upper bound, and our major tool for getting there is the Nested Interval Property. With an eye toward using NIP, bisect the interval I_1 and let c_1 be the midpoint. If c_1 is an upper bound for A, then set $b_2 = c_1$ and $a_2 = a_1$. If c_1 is not an upper bound for A, then set $b_2 = b_1$ and $a_2 = c_1$. Letting $I_2 = [a_2, b_2]$, we see that in either case described the left endpoint a_2 is not an upper bound for A while the right endpoint b_2 is an upper bound for A.

We now continue this process inductively. Given that we have constructed $I_n = [a_n, b_n]$, we let c_n be the midpoint. If c_n is an upper bound for A, we let $b_{n+1} = c_n$ and $a_{n+1} = a_n$. If c_n is not an upper bound then it becomes the left endpoint; i.e., $a_{n+1} = c_n$ and $b_{n+1} = b_n$. The resulting collection $I_n = [a_n, b_n]$ of nested intervals has the property that, for every $n \in \mathbf{N}$, the point a_n fails to be an upper bound while b_n is an upper bound.

By the Nested Interval Property, we know there exists a real number

$$s \in \bigcap_{n=1}^{\infty} I_n.$$

Setting $M = b_1 - a_1$, we can also see that the length of I_n is $M/2^{n-1}$ which tends to zero. From this fact, we can easily prove that

$$s = \lim a_n \quad \text{and} \quad s = \lim b_n.$$

We now claim that $s = \sup A$. To show that s is an upper bound for A, we let $a \in A$ be arbitrary. Because each b_n is an upper bound, we observe that $a \leq b_n$ for all n. By the Order Limit Theorem, $a \leq s$ as well, and we conclude that s is an upper bound for A.

To show that s is the least upper bound, we let l be some arbitrary upper bound and observe that $a_n < l$ for all n. Again using the Order Limit Theorem, we may conclude $s \leq l$, and this completes the argument.

(b) Let $I_n = [a_n, b_n]$ be a nested collection of closed intervals. To prove NIP, we must produce an $x \in \mathbf{R}$ satisfying $a_m \leq x \leq b_m$ for all $m \in \mathbf{N}$.

Because the intervals are nested, the sequence (a_n) is increasing and bounded above (by b_1 for instance.) By MCT, we know there exists a real number x satisfying $x = \lim a_n$. Now fix $m \in \mathbf{N}$. A short contradiction argument shows $a_m \leq x$. The nested property of the intervals also gives us that $a_n \leq b_m$ for all $n \in \mathbf{N}$, and the Order Limit Theorem then implies $x \leq b_m$, as desired.

(c) Just as in (b), we start with a nested collection of closed intervals $I_n = [a_n, b_n]$ and argue that there is a real number x common to all of them. Focusing on the sequence (a_n) of left-hand endpoints, we may not assert (because MCT is off limits) that it converges, but it is certainly bounded. By the Bolzano–Weierstrass Theorem, there exists a convergent subsequence (a_{n_k}), and we can set $x = \lim a_{n_k}$.

Now fix $m \in \mathbf{N}$. Because $a_{n_k} \leq b_m$ for all $k \in \mathbf{N}$, the Order Limit Theorem implies, just as before, that $x \leq b_m$. Also, choosing a particular term $n_K \geq m$ we can argue that $a_m \leq a_{n_K} \leq x$ must be true. Thus $x \in I_m$ for all m, and $\bigcap_{n=1}^{\infty} I_m$ is not empty.

(d) Let (a_n) be a bounded sequence so that there exists $M > 0$ satisfying $|a_n| \leq M$ for all n. Our goal is to use the Cauchy Criterion to produce a convergent subsequence.

First construct the sequence of closed intervals and the subsequence with $a_{n_k} \in I_k$ according to the method described in the proof of the Bolzano-Weierstrass Theorem in the text. Rather than using NIP to produce a candidate for the limit of this subsequence, we can argue that (a_{n_k}) is convergent by appealing to the Cauchy Criterion.

Let $\epsilon > 0$. By construction, the length of I_k is $M(1/2)^{k-1}$ which converges to zero. Choose N so that $k \geq N$ implies that the length of I_k is less than ϵ. So for any $s, t \geq N$, because a_{n_s} and a_{n_t} are in I_k, it follows that $|a_{n_s} - a_{n_t}| < \epsilon$. Having shown (a_{n_k}) is a Cauchy sequence, we know it converges.

2.7 Properties of Infinite Series

Exercise 2.7.1. (a) Here we show that the sequence of partial sums (s_n) converges by showing that it is a Cauchy sequence. Let $\epsilon > 0$ be arbitrary. We need to find an N such that $n > m \geq N$ implies $|s_n - s_m| < \epsilon$. First recall,

$$|s_n - s_m| = |a_{m+1} - a_{m+2} + a_{m+3} - \cdots \pm a_n|.$$

Because (a_n) is decreasing and the terms are positive, an induction argument shows that for all $n > m$ we have

$$|a_{m+1} - a_{m+2} + a_{m+3} - \cdots \pm a_n| \leq |a_{m+1}|.$$

So, by virtue of the fact that $(a_n) \to 0$, we can choose N so that $m \geq N$ implies $|a_m| \leq \epsilon$. But this implies

$$|s_n - s_m| = |a_{m+1} - a_{m+2} + \cdots \pm a_n| \leq |a_{m+1}| < \epsilon$$

whenever $n > m \geq N$, as desired.

(b) Let I_1 be the closed interval $[0, s_1]$. Then let I_2 be the closed interval $[s_2, s_1]$, which must be contained in I_1 as (a_n) is decreasing. Continuing in this fashion, we can construct a nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots .$$

By the Nested Interval Property there exists at least one point S satisfying $S \in I_n$ for every $n \in \mathbf{N}$. We now have a candidate for the limit, and it remains to show that $(s_n) \to S$.

Let $\epsilon > 0$ be arbitrary. We need to demonstrate that there exists an N such that $|s_n - S| < \epsilon$ whenever $n \geq N$. By construction, the length of I_n is $|s_n - s_{n-1}| = a_n$. Because $(a_n) \to 0$ we can choose N such that $a_n < \epsilon$ whenever $n \geq N$. Thus,

$$|s_n - S| \leq a_n < \epsilon$$

because both $s_n, S \in I_n$.

(c) The subsequence (s_{2n}) is increasing and bounded above (by a_1 for instance.) The Monotone Convergence Theorem allows us to assert that there exists an $S \in \mathbf{R}$ satisfying $S = \lim(s_{2n})$. One way to prove that the other subsequence (s_{2n+1}) converges to the same value is to use the Algebraic Limit Theorem and the fact that $(a_n) \to 0$ to write

$$\lim(s_{2n+1}) = \lim(s_{2n} + a_{2n+1}) = S + \lim(a_{2n+1}) = S + 0 = S.$$

The fact that both (s_{2n}) and (s_{2n+1}) converge to S implies that $(s_n) \to S$ as well. (See Exercise 2.3.5.)

Exercise 2.7.2. (a) (i) Assume $\sum_{k=1}^{\infty} b_k$ converges. Thus, given $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that whenever $n > m \geq N$ it follows that $|b_{m+1} + b_{m+2} + \cdots + b_n| < \epsilon$. Since $0 \leq a_k \leq b_k$ for all $k \in \mathbf{N}$, we have

$$|a_{m+1} + a_{m+2} + \cdots + a_n| < |b_{m+1} + b_{m+2} + \cdots + b_n| < \epsilon$$

whenever $n > m \geq N$, and $\sum_{k=1}^{\infty} a_k$ converges as well.

(ii) Rather than trying to work with a negated version of the Cauchy Criterion, we can argue by contradiction. This is actually an example of a contrapositive proof. Rather than proving "If P, then Q," we can argue that "Not Q implies not P." In the context of this particular problem, "Not Q implies not P" is just the statement "$\sum_{k=1}^{\infty} b_k$ converges implies that $\sum_{k=1}^{\infty} a_k$ converges." But this is exactly what we showed in (i).

(b)(i) Let $s_n = a_1 + \cdots + a_n$ be the partial sums for $\sum_{k=1}^{\infty} a_k$, and let $t_n = b_1 + \cdots + b_n$ be the partial sums for $\sum_{k=1}^{\infty}$. Because $0 \leq a_k \leq b_k$ for all

$k \in \mathbf{N}$, both (s_n) and (t_n) are increasing, and in addition we have $s_n \leq t_n$ for all $n \in \mathbf{N}$. Because $\sum_{k=1}^{\infty} b_k$ converges, (t_n) is bounded and thus (s_n) is also bounded. By MCT, $\sum_{k=1}^{\infty} a_k$ converges.

(ii) As mentioned previously, this is just the contrapositive version of the statement in (i).

Exercise 2.7.3. (a) The key observation is that $a_n = p_n + q_n$. If both $\sum p_n$ and $\sum q_n$ converge, then by the Algebraic Limit Theorem, $\sum a_n$ would also converge, and this is not the case.

(b) In addition to $a_n = p_n + q_n$, we also have $|a_n| = p_n - q_n$. Because we are given that $\sum |a_n|$ diverges, it must be (for the reasons similar to those in (a)) that at least one of $\sum p_n$ or $\sum q_n$ diverges. So let's assume (without loss of generality) that $\sum p_n$ diverges. If $\sum q_n$ were to converge, then we could write $p_n = a_n - q_n$. Keeping in mind that we are assuming $\sum a_n$ converges, the Algebraic Limit Theorem would imply that $\sum p_n$ should also converge. This contradiction implies that $\sum q_n$ must, in fact, diverge.

Exercise 2.7.4. One example would be

$$x_n = (1, 0, 1, 0, 1, 0, \ldots) \quad \text{and} \quad y_n = (0, 1, 0, 1, 0, 1, \ldots).$$

Another would be to set $x_n = y_n = 1/n$ for all $n \in \mathbf{N}$.

Exercise 2.7.5. (a) By definition of absolute convergence, $\sum |a_n|$ must converge. Theorem 2.7.3 tells us that there must be an N such that $n \geq N$ implies $|a_n| < 1$. Now, $a_n^2 < |a_n|$ for $n \geq N$. Thus, by Theorem 2.7.4, $\sum_{n=N}^{\infty} a_n^2$ converges and therefore so does $\sum_{n=1}^{\infty} a_n^2$ as there are only a finite number of terms before N. Because $a_n^2 \geq 0$, the convergence is absolute.

This result does not hold without absolute convergence. Consider $\sum (-1)^{n+1}/\sqrt{n}$ which converges conditionally; however, $\sum 1/n$ diverges.

(b) This is not a true statement. Consider $\sum 1/n^2$ which converges, however, $\sum 1/n$ does not converge.

Exercise 2.7.6. (a) Because (y_n) is bounded, there exits $M \geq 0$ such that $|y_n| \leq M$. Now we are given that $\sum |x_n|$ converges, and the Algebraic Limit Theorem tells us that $\sum M|x_n|$ also converges. Because $|x_n y_n| \leq M|x_n|$, we may use the Comparison Test to assert that $\sum |x_n y_n|$ converges. Finally, the Absolute Convergence Test implies $\sum x_n y_n$ converges.

(b) Set $x_n = (-1)^n/n$, and let $y_n = (-1)^n$ which is certainly bounded. Then $\sum x_n$ converges conditionally, but $\sum x_n y_n = \sum 1/n$ diverges.

Exercise 2.7.7. By the Cauchy Condensation Test (Theorem 2.4.6) $\sum 1/n^p$ converges if and only if $\sum 2^n \,(1/2^n)^p$ converges. But notice that

$$\sum 2^n \left(\frac{1}{2^n}\right)^p = \sum \left(\frac{1}{2^n}\right)^{p-1} = \sum \left(\frac{1}{2^{p-1}}\right)^n.$$

By the Geometric Series Test (Example 2.7.5), this series converges if and only if $|\frac{1}{2^{p-1}}| < 1$. Solving for p we find that p must satisfy $p > 1$.

Exercise 2.7.8. In order to show that $\sum_{k=1}^{\infty}(a_k+b_k)=A+B$, we must argue that the sequence of partial sums

$$r_m=(a_1+b_1)+(a_2+b_2)+(a_3+b_3)+\cdots+(a_m+b_m)$$

converges to $A+B$. We are given that $\sum_{k=1}^{\infty}a_k=A$ and $\sum_{k=1}^{\infty}b_k=B$, meaning that the partial sums

$$s_m=a_1+a_2+a_3+\cdots+a_m$$

converge to A and

$$t_m=b_1+b_2+b_3+\cdots+b_m$$

converge to B. Because $s_m+t_m=r_m$, applying the Algebraic Limit Theorem for sequences (Theorem 2.3.3) yields $(r_m)\to A+B$, as desired.

Exercise 2.7.9. (a) This r' must exist because $\mathbf{R}$ is dense in itself.

First, pick an ϵ-neighborhood around r of size $\epsilon_0=|r-r'|$. Because $\lim\left|\frac{a_{n+1}}{a_n}\right|=r$, there exists an N such that $n\geq N$ implies $\left|\frac{a_{n+1}}{a_n}\right|\in V_{\epsilon_0}(r)$. It follows that $\left|\frac{a_{n+1}}{a_n}\right|\leq r'$ for all $n\geq N$, and this implies the statement in (a)

(b) Having chosen N, $|a_N|$ is now a fixed number. Also, $\sum(r')^n$ is a geometric series with $|r'|<1$, so it converges. Therefore, by the Algebraic Limit Theorem $|a_N|\sum(r')^n$ converges.

(c) From (a) we know that there exists an N such that $|a_{N+1}|\leq|a_N|r'$. Extending this we find $|a_{N+2}|\leq|a_{N+1}|r'\leq|a_N|(r')^2$, and using induction we can say that

$$|a_k|\leq|a_N|(r')^{k-N}\quad\text{for all } k\geq N.$$

Thus, $\sum_{k=N}^{\infty}|a_k|$ converges by the Comparison Test and part (b). Because

$$\sum_{k=1}^{\infty}|a_k|=\sum_{k=1}^{N-1}|a_k|+\sum_{k=N}^{\infty}|a_k|$$

and $\sum_{k=1}^{N-1}|a_k|$ is just a finite sum, the series $\sum_{k=1}^{\infty}|a_k|$ converges.

Exercise 2.7.10. (a) The idea here is that eventually the terms a_n "look like" a non-zero constant times $1/n$, and we know that any series of this form diverges. To make this precise, let $\epsilon_0=l/2>0$. Because $(na_n)\to l$, there exists $N\in\mathbf{N}$ such that $na_n\in V_{\epsilon_0}(l)$ for all $n\geq N$. A little algebra shows that this implies we must have $na_n>l/2$, or

$$a_n>(l/2)\,(1/n)\quad\text{for all } n\geq N.$$

Because this inequality is true for all but some finite number of terms, we may still appeal to the Comparison Test to assert that $\sum a_n$ diverges.

(b) Assume that $\lim(n^2a_n)\to L\geq 0$. The definition of convergence (with $\epsilon_0=1$) tells us that there exists an N such that $n^2a_n<L+1$ for all $n\geq N$.

This means that eventually $a_n < (L+1)/n^2$. We know that the series $\sum 1/n^2$ converges, and by the Algebraic Limit Theorem for series (Theorem 2.7.1), $\sum (L+1)/n^2$ converges as well. Thus, by the Comparison Test $\sum a_n$ must converge.

Exercise 2.7.11. A preliminary example would be to let

$$(a_n) = (1, 0, 1, 0, 1, \dots) \quad \text{and} \quad (b_n) = (0, 1, 0, 1, 0, \dots).$$

To handle the more challenging version, we shall construct two positive decreasing sequences (a_n) and (b_n) with $\min\{a_n, b_n\} = 1/n^2$ where $\sum a_n$ and $\sum b_n$ each diverge. First set $a_1 = b_1 = 1$. For $2 \le n \le 5$, let $a_n = 1/4$ and let $b_n = 1/n^2$. By holding $a_n = 1/4$ constant over 4 terms, we have added 1 to the partial sums of $\sum a_n$. For $6 \le n \le 6 + 24$, let $a_n = 1/n^2$ and hold $b_n = 1/25$ constant. This will add one to the partial sums of $\sum b_n$. Now we switch again and hold $a_n = 1/30^2$ for the next 30^2 terms while letting $b_n = 1/n^2$. Continuing this process will ensure that the partial sums of $\sum a_n$ and $\sum b_n$ are unbounded while $\sum \min\{a_n, b_n\} = \sum 1/n^2$ converges.

Exercise 2.7.12. First write

$$\begin{aligned}
\sum_{j=m+1}^{n} x_j y_j &= \sum_{j=m+1}^{n} (s_j - s_{j-1}) y_j \\
&= \sum_{j=m+1}^{n} s_j y_j - \sum_{j=m+1}^{n} s_{j-1} y_j.
\end{aligned}$$

Then, focusing on the second sum in the above expression, we have

$$\sum_{j=m+1}^{n} s_{j-1} y_j = \sum_{j=m}^{n-1} s_j y_{j+1} = s_m y_{m+1} - s_n y_{n+1} + \sum_{j=m+1}^{n} s_j y_{j+1}.$$

Substituting this back into our first equation gives the result.

Exercise 2.7.13. (a) Let $M > 0$ be an upper bound for the partial sums, s_n, of $\sum x_n$. Making use of Exercise 2.7.12 and overestimating the partial sums of $\sum x_n$ with M, we find

$$\begin{aligned}
\left| \sum_{j=m+1}^{n} x_j y_j \right| &= \left| s_n y_{n+1} - s_m y_{m+1} + \sum_{j=m+1}^{n} s_j (y_j - y_{j+1}) \right| \\
&\le M y_{n+1} + M y_{m+1} + \sum_{j=m+1}^{n} M (y_j - y_{j+1}) \\
&= M y_{n+1} + M y_{m+1} + M (y_{m+1} - y_{n+1}) \\
&= 2M y_{m+1}.
\end{aligned}$$

(b) In order to show that the series converges we will use the Cauchy Criterion for Series. Let $\epsilon > 0$ be arbitrary. We must show that there exists an N such that whenever $n > m \geq N$ it follows that $|x_{m+1}y_{m+1}+x_{m+2}y_{m+2}+\cdots+x_ny_n| < \epsilon$. By part (a),

$$|x_{m+1}y_{m+1} + x_{m+2}y_{m+2} + \cdots + x_ny_n| = \left|\sum_{j=m+1}^{n} x_jy_j\right| \leq 2My_{m+1}.$$

Because $(y_n) \to 0$, we can pick N such that $m \geq N$ implies $y_m < \epsilon/(2M)$. Using this N, we find that

$$|x_{m+1}y_{m+1} + x_{m+2}y_{m+2} + \cdots + x_ny_n| \leq 2My_{m+1} < 2M\frac{\epsilon}{2M} = \epsilon$$

whenever $n > m \geq N$ as desired.

(c) The Alternating Series Test is the special case where $x_n = (-1)^{n+1}$. The partial sums of $\sum x_n$ in this case look like $(1, 0, 1, 0, 1, \ldots)$ which is a bounded sequence.

Exercise 2.7.14. (a) Abel's Test differs from Dirichlet's Test in that we assume more about $\sum x_n$ but less about (y_n). Specifically, we now assume that $\sum x_n$ converges; however, (y_n) may converge to a limit greater than zero.

(b) Let $A > 0$ be an upper bound for the partial sums, s_n, of $\sum a_n$. By making use of Exercise 2.7.12 and replacing the partial sums of $\sum a_n$ with A, we find

$$\begin{aligned}
\left|\sum_{j=1}^{n} a_jb_j\right| &= \left|s_nb_{n+1} - s_mb_{m+1} + \sum_{j=m+1}^{n} s_j(b_j - b_{j+1})\right| \\
&\leq Ab_{n+1} + Ab_{m+1} + \left|\sum_{j=m+1}^{n} A(b_j - b_{j+1})\right| \\
&= Ab_{n+1} + Ab_{m+1} + A(b_{m+1} - b_{n+1}) \\
&= 2Ab_{m+1} \leq 2Ab_1.
\end{aligned}$$

(c) In order to show that the series converges we will use the Cauchy Criterion. Let $\epsilon > 0$ be arbitrary. We must show that there exists an N such that whenever $n > m \geq N$ it follows that

$$|x_{m+1}y_{m+1} + x_{m+2}y_{m+2} + \cdots + x_ny_n| = \left|\sum_{j=m+1}^{n} x_jy_j\right| < \epsilon.$$

Thinking of m as fixed for the moment, let $a_n = x_{m+n}$ and $b_n = y_{m+n}$, and apply part (b) to get

$$\left|\sum_{j=m+1}^{n} x_jy_j\right| = \left|\sum_{j=1}^{n-m} a_jb_j\right| \leq 2A_1b_1$$

where A_1 is an upper bound on the partial sums of $\sum_{j=m+1}^{\infty} x_j$. But this is the crucial point. Because $\sum x_n$ converges, the Cauchy Criterion tells us that its "tail" can be made arbitrarily small. That is, we can pick N such that $n > m \geq N$ implies

$$\sum_{j=m+1}^{n} x_j < \frac{\epsilon}{2y_1}.$$

Looking again at what the constant A_1 represents, it now follows that if $m \geq N$ then

$$A_1 \leq \left| \sum_{j=m+1}^{n} x_j \right| < \frac{\epsilon}{2y_1}.$$

Putting this altogether and noting that $b_1 = y_{1+n} \leq y_1$, we find

$$|x_{m+1}y_{m+1} + x_{m+2}y_{m+2} + \cdots + x_n y_n| \leq 2A_1 b_1 < 2y_1 \frac{\epsilon}{2y_1} = \epsilon$$

whenever $n > m \geq N$ as desired.

2.8 Double Summations and Products of Infinite Series

Exercise 2.8.1. Examining the sum over squares we get $s_{11} = -1$, $s_{22} = -3/2$, $s_{33} = -7/4$, and in general

$$s_{nn} = -2 + \frac{1}{2^{n-1}}.$$

Now taking the limit we find $(s_{nn}) \to -2$. This value corresponds to the value previously computed by fixing j and summing down each column.

Exercise 2.8.2. In order to show that the iterated series

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$$

converges we must first show that for each fixed $i \in \mathbf{N}$ the series $\sum_{j=1}^{\infty} a_{ij}$ converges to some real number r_i. Then we need to show that the series $\sum_{i=1}^{\infty} r_i$ converges.

Fix $i \in R$. By our hypothesis, $\sum_{j=1}^{\infty} |a_{ij}|$ converges. Thus, the Absolute Convergence Test tells us $\sum_{j=1}^{\infty} a_{ij}$ converges to some real number r_i. By looking at the partial sums, we can use the Order Limit Theorem to assert that $|r_i| \leq b_i$, where $b_i = \sum_{j=1}^{\infty} |a_{ij}|$. Because $\sum_{i=1}^{\infty} b_i$ converges, $\sum_{i=1}^{\infty} |r_i|$ converges by the Comparison Test, and then $\sum r_i$ must converge as well.

Exercise 2.8.3. (a) As we have been doing, let $b_i = \sum_{j=1}^{\infty} |a_{ij}|$ for all $i \in \mathbf{N}$. Our hypothesis tells us that there exists $L \geq 0$ satisfying $\sum_{i=1}^{\infty} b_i = L$. Because we are adding all non-negative terms, it follows that

$$t_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}| \leq \sum_{i=1}^{m}\sum_{j=1}^{\infty} |a_{ij}| \leq \sum_{i=1}^{m} b_i \leq L.$$

Thus, t_{mn} is bounded. We can now conclude that (t_{nn}) converges by the Monotone Convergence Theorem, as it is both increasing and bounded.

(b) Let $\epsilon > 0$ be arbitrary. We need to find an N such that $n > m \geq N$ implies $|s_{nn} - s_{mm}| < \epsilon$. Now the expression $s_{nn} - s_{mm}$ is really a sum over a finite collection of a_{ij} terms. If each a_{ij} included in the sum is replaced with $|a_{ij}|$, the sum only gets larger (this is just the triangle inequality), and the result is that

$$|s_{nn} - s_{mm}| = \left| \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} - \sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij} \right| \leq |t_{nn} - t_{mm}|.$$

We know that (t_{nn}) converges, so pick N so that $n > m \geq N$ implies $|t_{nn} - t_{mm}| < \epsilon$. It follows that (s_{nn}) is Cauchy and must converge.

Exercise 2.8.4. (a) The fact that t_{mn} is a sum of non-negative terms implies that if $m_1 \geq m$ and $n_1 \geq n$ then $t_{m_1 n_1} \geq t_{mn}$. So let $N_1 = \max\{m_0, n_0\}$. Then it follows that

$$B - \frac{\epsilon}{2} < t_{m_0,n_0} \leq t_{mn} \leq B$$

for all $m, n \geq N_1$.

(b) Without loss of generality, let $n > m \geq N$. Then,

$$\begin{aligned}
|s_{mn} - S| &= |s_{mn} - s_{mm} + s_{mm} - S| \\
&\leq |s_{mn} - s_{mm}| + |s_{mm} - S| \\
&= \left| \sum_{i=1}^{m}\sum_{j=m+1}^{n} a_{ij} \right| + |s_{mm} - S| \\
&\leq |t_{mn} - t_{mm}| + |s_{mm} - S|.
\end{aligned}$$

We have already chosen N_1 such that

$$|t_{mn} - t_{mm}| < \frac{\epsilon}{2} \quad \text{whenever } n > m \geq N_1.$$

Because $(s_{nn}) \to S$, we can pick N_2 so that

$$|s_{mm} - S| < \frac{\epsilon}{2} \quad \text{whenever } m \geq N_2.$$

Setting $N = \max\{N_1, N_2\}$, we can conclude that $|s_{mn} - S| < \epsilon/2 + \epsilon/2 = \epsilon$ for all $n > m \geq N$.

Exercise 2.8.5. Thinking of m as fixed and n as the limiting variable, the Algebraic Limit Theorem can be applied to the finite number of components of

$$s_{mn} = \sum_{j=1}^{n} a_{1j} + \sum_{j=1}^{n} a_{2j} + \cdots + \sum_{j=1}^{n} a_{mj}$$

to conclude that

$$\lim_{n\to\infty} s_{mn} = r_1 + r_2 + \cdots + r_m.$$

If, in addition, we insist that $m \geq N$ (where N is the one constructed in the previous exercise), then we have that

$$-\epsilon < s_{mn} - S < \epsilon$$

is eventually true once n is larger than N. Applying the Order Limit Theorem we find

$$-\epsilon \leq (r_1 + r_2 + \cdots + r_m) - S \leq \epsilon$$

for all $m \geq N$.

This last statement is extremely close to what we need to conclude that $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$ converges to S. Given an arbitrary $\epsilon > 0$, we have produced an N such that

$$|(r_1 + r_2 + \cdots + r_m) - S| \leq \epsilon \quad \text{for all } m \geq N$$

The only distraction is that our definition of convergence requires a strict inequality, and we have a "less than or equal to ϵ" result. This, however, is not a problem. Because ϵ is arbitrary, we could just as easily have chosen to let $\epsilon' < \epsilon$ at the beginning and constructed our argument using ϵ' throughout the proof. On a more general note, while we strive at the introductory level to adhere to the exact wording of our definitions, there comes a point in epsilon–style arguments where it becomes more convenient to simply make quantities less than something that we know can be made arbitrarily small.

Exercise 2.8.6. As the exercise explains, the same argument can be used to prove $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$ converges to S once we show that for each $j \in \mathbf{N}$ the sum $\sum_{i=1}^{\infty} a_{ij}$ converges to some real number c_j.

To show $\sum_{i=1}^{\infty} a_{ij}$ converges for each $j \in \mathbf{N}$, it suffices to prove that the absolute series $\sum_{i=1}^{\infty} |a_{ij}|$ converges. Recall that $b_i = \sum_{j=1}^{\infty} |a_{ij}|$, so it is certainly the case that $b_i \geq |a_{ij}|$ for all $i, j \in \mathbf{N}$. But our hypothesis says that $\sum_{i=1}^{\infty} b_i$ converges, and so by the Comparison Test, $\sum_{i=1}^{\infty} |a_{ij}|$ converges for all values of j.

Exercise 2.8.7. (a) In order to prove absolute convergence, let

$$u_n = |d_2| + |d_3| + |d_4| + \cdots + |d_n| = \sum_{k=2}^{n} |d_k|.$$

We must now show that (u_n) converges. Well,

$$u_n = \sum_{k=2}^{n} |d_k| \le \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}| = t_{nn}.$$

Because, $u_n \le t_{nn}$ for all n and (t_{nn}) converges, we know that u_n converges by the Comparison Test.

(b) Let $\epsilon > 0$ be arbitrary. We need to find N such that $n \ge N$ implies $|\sum_{k=2}^{n} d_k - S| < \epsilon$. By hypothesis, $(s_{nn}) \to S$, so choose N_1 so that

$$|s_{nn} - S| < \frac{\epsilon}{2} \quad \text{for all } n \ge N_1.$$

We are also given that (t_{nn}) converges (this is the absolute convergence hypothesis), an so there exists N_2 such that

$$|t_{nn} - t_{mm}| < \frac{\epsilon}{2} \quad \text{for all } n > m \ge N_2.$$

In essence, this says that once we get far enough out into the array (a_{ij}) in any direction, the absolute values of the terms do not add up to anything significant. To take advantage of this we set $N = \max\{N_1, 2N_2\}$. Then, for $n \ge N$

$$\begin{aligned}\left|\sum_{k=2}^{n} d_k - S\right| &= \left|\sum_{k=2}^{n} d_k - s_{nn} + s_{nn} - S\right| \\ &\le \left|\sum_{k=2}^{n} d_k - s_{nn}\right| + |s_{nn} - S| \\ &< \left|\sum_{k=2}^{n} d_k - s_{nn}\right| + \frac{\epsilon}{2}\end{aligned}$$

Because $n \ge 2N_2$, the partial sum $\sum_{k=2}^{n} d_k$ along diagonals contains every term in the "square" sum $s_{N_2N_2}$. It follows that

$$\left|s_{nn} - \sum_{k=2}^{n} d_k\right| \le (t_{nn} - t_{N_2N_2}) < \frac{\epsilon}{2}.$$

Putting it altogether, we have

$$\left|\sum_{k=2}^{n} d_k - S\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{for all } n \ge N,$$

and we conclude that $\sum_{k=2}^{\infty} d_k = S$.

Exercise 2.8.8. (a) It is possible, as suggested, to prove that $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_i b_j|$ converges by first proving that it is bounded and then taking advantage of the fact that it is monotone. However, a method similar to proving that it is bounded can be used to directly prove that it converges. We will use this method. Let

$$\sum_{i=1}^{\infty}|a_i| = L \quad \text{and} \quad \sum_{j=1}^{\infty}|b_j| = M.$$

For each fixed $i \in \mathbf{N}$, the Algebraic Limit Theorem allows us write $\sum_{j=1}^{\infty}|a_ib_j| = |a_i|\sum_{j=1}^{\infty}|b_j|$. Continuing this process, we see

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_ib_j| = \sum_{i=1}^{\infty}|a_i|\sum_{j=1}^{\infty}|b_j| = \sum_{i=1}^{\infty}|a_i|M = M\sum_{i=1}^{\infty}|a_i| = ML,$$

and therefore $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_ib_j|$ converges to ML.

(b) Again, fix $i \in R$. Now we can write

$$\lim_{n\to\infty} s_{nn} = \lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{n}a_ib_j = \lim_{n\to\infty}\left(\sum_{i=1}^{n}a_i\right)\left(\sum_{j=1}^{n}b_j\right).$$

Applying the Algebraic Limit Theorem to the limits of these partial sums we find that $\lim_{n\to\infty} s_{nn} = AB$. From part (a) we know that $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_ib_j|$ converges, so we can use Theorem 2.8.1 and Exercise 2.8.7 to conclude that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_ib_j = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}a_ib_j = \sum_{k=2}^{\infty}d_k = \lim_{n\to\infty}s_{nn} = AB.$$

Chapter 3

Basic Topology of R

3.1 Discussion: The Cantor Set

3.2 Open and Closed Sets

Exercise 3.2.1. (a) We cannot always take minimums of infinite sets. Therefore the step where we let $\epsilon = \min\{\epsilon_1, \epsilon_2, \ldots, \epsilon_N\} > 0$ requres that we are working with a finite collection of open sets. You can, however, take the infimum of an infinite set, but the infimum of the set could be 0.

(b) Let $O_n = (\frac{-1}{n}, \frac{1}{n})$. Then $\bigcap_{n=1}^{\infty} O_n = \emptyset$.

Exercise 3.2.2. (a) $\{-1, 1\}$

(b) B is not a closed set because it does not contain its limit points.

(c) B is not an open set. Given any point of B, it is impossible to find an ϵ-neighborhood contained in B.

(d) All points in B are isolated points.

(e) $\overline{B} = B \cup \{-1, 1\}$

Exercise 3.2.3. (a) Neither. Given any point in $\mathbf{Q}$, there is no ϵ-neighborhood contained in $\mathbf{Q}$. The set of limit points not contained in $\mathbf{Q}$ is $\mathbf{I}$.

(b) Closed. Given any point in $\mathbf{N}$, there is no ϵ-neighborhood of that point contained in the set.

(c) Open. The limit point 0 is not contained in the set $\{x \in \mathbf{R} : x > 0\}$.

(d) Neither. There is no ϵ-neighborhood of 1 contained in $(0, 1]$. The limit point 0 is not contained in the set.

(e) Neither. There is no ϵ-neighborhood of any point in the set contained in $\{1 + 1/4 + 1/9 + \cdots + 1/n^2 : n \in \mathbf{N}\}$. Without the square on the n in this set, we would have no limit point. However, since $\sum_{n=1}^{\infty} 1/n^2$ converges, the limit of the partial sums is a limit point for this set. This limit point is not an element of the set.

Exercise 3.2.4. Let $x = \lim a_n$ for some sequence (a_n) contained in A, and assume that $a_n \neq x$ for all n in $\mathbf{N}$. We want to show that x is a limit point of A.

The sequence (a_n) converges to x, so by Definition 2.2.3B, every ϵ-neighborhood $V_\epsilon(x)$ contains all but a finite number of the terms of (a_n). Since (a_n) is contained in A, this means that $V_\epsilon(x) \cap A$ is non-empty and contains elements other than x. Hence, x is a limit point of A.

Exercise 3.2.5. ($\Rightarrow$) Assume a is an isolated point of A. By Definition 3.2.5, a is not a limit point. Therefore there exists an ϵ-neighborhood $V_\epsilon(a)$ such that $V_\epsilon(a) \cap A = \emptyset$ or $V_\epsilon(a) \cap A = \{a\}$. Since a is an element of A the former cannot be true. Therefore, $V_\epsilon(a) \cap A = \{a\}$.

($\Leftarrow$) Assume that there exists an ϵ-neighborhood $V_\epsilon(a)$ such that $V_\epsilon(a) \cap A = \{a\}$. It follows from Definition 3.2.4 that a is not a limit point of A, and hence it is isolated.

Exercise 3.2.6. ($\Rightarrow$) Assume that the set $F \subseteq \mathbf{R}$ is closed. Then F contains its limit points. We will show that that every Cauchy sequence (a_n) contained in F has its limit in F by showing that the limit of (a_n) is either a limit point or possibly an isolated point of F. Because (a_n) is Cauchy, we know $x = \lim a_n$ exists. If $a_n \neq x$ for all x, then it follows from Theorem 3.2.5 that x is a limit point of F. Now consider a Cauchy sequence a_n where $a_n = x$ for some n. Because $(a_n) \subseteq F$ it follows that $x \in F$ as well. (Note that if a_n is eventually equal to x, then it may not be true that x is a limit point of F.)

($\Leftarrow$) Assume that every Cauchy sequence contained in F has a limit that is also an element of F. To show that F is closed we want to show that it contains its limit points. Let x be a limit point of F. By Theorem 3.2.5, $x = \lim a_n$ for some sequence (a_n). Because (a_n) converges, it must be a Cauchy sequence. So x is contained in F, and therefore F is closed.

Exercise 3.2.7. Let $x \in O$, where O is an open set. Let $x = \lim x_n$. It follows from Definition 3.2.1 that there exists an ϵ-neighborhood $V_\epsilon(x)$ of x such that $V_\epsilon(x) \in O$. Because (x_n) is a convergent sequence, by Definition 2.2.3B every ϵ-neighborhood $V_\epsilon(x)$ of x contains all but a finite number of the terms of (x_n). Therefore all but a finite number of terms of (x_n) are contained in O.

Exercise 3.2.8. (a) Let L be the set of limit points of A, and suppose that x is a limit point of L. We want to show that x is an element of L; in other words, that x is a limit point of A. Let $V_\epsilon(x)$ be arbitrary. By the definition of a limit point, $V_\epsilon(x)$ intersects L at a point $l \in L$, where $l \neq x$. Now choose $\epsilon' > 0$ small enough so that $V_{\epsilon'}(l) \subseteq V_\epsilon(x)$ and $x \notin V_{\epsilon'}(l)$. Since $l \in L$, l is a limit point of A and so $V_{\epsilon'}(l)$ intersects A. This implies $V_\epsilon(x)$ intersects A at a point different than x, and therefore x is a limit point of A and thus an element of L.

(b) Assume x is a limit point of $A \cup L$ and consider the ϵ-neighborhood $V_\epsilon(x)$ for an arbitrary $\epsilon > 0$. We know $V_\epsilon(x)$ must intersect $A \cup L$ and we would like to argue that it in fact intersects A. If $V_\epsilon(x)$ intersects A at a point different than x we are done, so let's assume that there exists an $l \in L$ with $l \in V_\epsilon(x)$. Using the same argument employed in (a), we take $\epsilon' > 0$ small enough so that $V_{\epsilon'}(l) \subseteq V_\epsilon(x)$, and $x \notin V_{\epsilon'}(l)$. Because l is a limit point of A we have that there exists an $a \in V_{\epsilon'}(l) \subseteq V_\epsilon(x)$ and thus $V_\epsilon(x)$ intersects A at some point other than x, as desired.

Exercise 3.2.9. (a) Let y be a limit point of $A \cup B$. By Theorem 3.2.5, there exists a sequence (c_n) contained in $A \cup B$ satisfying $y = \lim c_n$ with $y \neq c_n$ for all $n \in \mathbf{N}$. Because (c_n) is contained in $A \cup B$ it must be that either A or B (or both) contains an infinite number of terms of (c_n). This subsequence contained entirely in one set or the other will also converge to y, and we are done with another nod to Theorem 3.2.5.

(b) Clearly $A \subseteq A \cup B$, and any limit point of A will by definition be a limit point of $A \cup B$. Thus $\overline{A} \subseteq \overline{A \cup B}$. Similarly, $\overline{B} \subseteq \overline{A \cup B}$. It follows that $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. We also have that $A \cup B \subseteq \overline{A} \cup \overline{B}$, and so $\overline{A \cup B} \subseteq \overline{\overline{A} \cup \overline{B}}$. But by Theorem 3.2.14, $\overline{A} \cup \overline{B}$ is closed, so $\overline{\overline{A} \cup \overline{B}} = \overline{A} \cup \overline{B}$. Hence, $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$, and so $\overline{A} \cup \overline{B} = \overline{A \cup B}$.

(c) No. Take $A_n = \{1/n\}$. Then $\bigcup_{n=1}^{\infty} \overline{A_n} = \{1/n : n \in \mathbf{N}\}$. But $\overline{\bigcup_{n=1}^{\infty} A_n} = \{1/n : n \in \mathbf{N}\} \cup \{0\}$.

Exercise 3.2.10. (a) Let $x \in \left(\bigcup_{\lambda \in \Lambda} E_\lambda\right)^c$. Then x is not an element of E_λ for all λ. Hence $x \in E_\lambda^c$ for all λ. So $x \in \bigcap_{\lambda \in \Lambda} E_\lambda^c$. We have just shown that $\left(\bigcup_{\lambda \in \Lambda} E_\lambda\right)^c \subseteq \bigcap_{\lambda \in \Lambda} E_\lambda^c$. Now we will show that $\bigcap_{\lambda \in \Lambda} E_\lambda^c \subseteq \left(\bigcup_{\lambda \in \Lambda} E_\lambda\right)^c$. Let $x \in \bigcap_{\lambda \in \Lambda} E_\lambda^c$. Then for all λ, $x \notin E_\lambda$. So $x \notin \bigcup_{\lambda \in \Lambda} E_\lambda$, and hence $x \in \left(\bigcup_{\lambda \in \Lambda} E_\lambda\right)^c$. Therefore

$$\left(\bigcup_{\lambda \in \Lambda} E_\lambda\right)^c = \bigcap_{\lambda \in \Lambda} E_\lambda^c.$$

Secondly, we want to show that

$$\left(\bigcap_{\lambda \Lambda} E_\lambda\right)^c = \bigcup_{\lambda \in \Lambda} E_\lambda^c.$$

Let $x \in \left(\bigcap_{\lambda \in \Lambda} E_\lambda\right)^c$. Then there exists a $\lambda' \in \Lambda$ for which x is not an element of $E_{\lambda'}$. Therefore $x \in E_{\lambda'}^c$. So $x \in \bigcup_{\lambda \in \Lambda} E_\lambda^c$, and we have $\left(\bigcap_{\lambda \in \Lambda} E_\lambda\right)^c \subseteq \bigcup_{\lambda \in \Lambda} E_\lambda^c$. Now assume $x \in \bigcup_{\lambda \in \Lambda} E_\lambda^c$. Then there exists a $\lambda' \in \Lambda$ such that $x \notin E_{\lambda'}$. Therefore $x \notin \bigcap_{\lambda \in \Lambda} E_\lambda$, so $x \in \left(\bigcap_{\lambda \in \Lambda} E_\lambda\right)^c$. So it is also true that $\bigcup_{\lambda \in \Lambda} E_\lambda^c \subseteq \left(\bigcap_{\lambda \in \Lambda} E_\lambda\right)^c$ and we have reached our desired conclusion.

(b) (i)Suppose that E_λ is a finite collection of closed sets. Then their complements, E_λ^c are a finite collection of open sets. We know by Theorem 3.2.3 that the intersection of a finite collection of open sets is open. In symbols,

$$\bigcap_{\lambda \in \Lambda} E_\lambda^c = \left(\bigcup_{\lambda \in \Lambda} E_\lambda\right)^c$$

is an open set. Therefore the union of a finite collection of closed sets, $\bigcup_{\lambda \in \lambda} E_\lambda$ is closed.

(ii) Now suppose that E_λ is an arbitrary collection of closed sets. Then $\bigcup_{\lambda \in \Lambda} E_\lambda^c$ is open by Theorem 3.2.3. By De Morgan's Laws,

$$\bigcup_{\lambda \in \Lambda} E_\lambda^c = \left(\bigcap_{\lambda \in \lambda} E_\lambda\right)^c.$$

It then follows from Theorem 3.2.13 that the intersection of an arbitrary collection of closed sets is closed.

Exercise 3.2.11. Let A be bounded above and let $s = \sup A$. Then for $\epsilon > 0$, there exists an $a \in A$ such that $s - \epsilon < a$. Hence a falls in the ϵ-neighborhood $V_\epsilon(s)$ of s. So $V_\epsilon(s)$ intersects A at a point other than s, and hence s is a limit point of A. Therefore $s \in \overline{A}$.

Exercise 3.2.12. (a) True. By Theorem 3.2.12 $\overline{A}$ is closed. It then follows from Theorem 3.2.13 that $\overline{A}^c$ is open.

(b) True. If $a \in A$ is an isolated point, then there exists an $\epsilon_0 > 0$ satisfying $V_{\epsilon_0}(a) \cap A = \{a\}$. It follows that for any $0 < \epsilon \leq \epsilon_0$ we would again have $V_\epsilon(a) \cap A = \{a\}$. However, for A to be open, it would have to be that $V_\epsilon(a) \subseteq A$ for some $0 < \epsilon \leq \epsilon_0$, and this is impossible.

(c) True. Throughout the proof, let's let L be the set of limit points for A.

($\Rightarrow$) Suppose that A is closed. Then A includes its limit points, so $A = A \cup L = \overline{A}$. ($\Leftarrow$) Let $A = \overline{A}$. Then $A = A \cup L$, hence A contains its limit points and therefore it is closed.

(d) True. See Exercise 3.2.11.

(e) True. If $A = \{a_1, a_2, \cdots, a_n\}$ is a finite set, then A has no limit points. (To prove this, let $x \in \mathbf{R}$ be arbitrary and let $\epsilon_0 = \min\{|x - a_n| : a_n \neq x\}$. Then $V_{\epsilon_0}(x)$ cannot intersect A at a point other than x, and therefore x is not a limit point.) By default, A contains its empty set of limit points and thus is closed.

(f) False. $(-\infty, \sqrt{2}) \cup (\sqrt{2}, \infty)$ is a counterexample. For a more interesting example, see Exercise 3.4.10.

Exercise 3.2.13. For contradiction, assume that there exists a nonempty set A that is both open and closed. Because $A \neq \mathbf{R}$, $B = A^c$ is also non-empty, and B is open and closed as well. Pick a point $a_1 \in A$ and $b_1 \in B$. We can assume, without loss of generality, that $a_1 < b_1$. Bisect the interval $[a_1, b_1]$ at $c = (b_1 - a_1)/2$. Now $c \in A$ or $c \in B$. If $c \in A$, let $a_2 = c$ and let $b_2 = b_1$. If $c \in B$, let $b_2 = c$ and let $a_2 = a_1$. Continuing this process yields a sequence of nested intervals $I_n = [a_n, b_n]$, where $a_n \in A$ and $b_n \in B$. By the Nested Interval Property, there exists an $x \in \bigcap_{n=1}^\infty I_n$. Because the lengths $(b_n - a_n) \to 0$, we can show $\lim a_n = x$ which implies that $x \in A$ because A is closed. However, it is also true that $\lim b_n = x$ and thus $x \in B$ because B is closed. Thus we have shown $x \in A$ and $x \in A^c$. This contradiction implies that no such A exists, and we conclude that $\mathbf{R}$ are $\emptyset$ are the only two sets that are both open and closed. (This argument is closely related to the discussion of connected sets in the next section.)

Exercise 3.2.14. (a) $[a, b] = \bigcap_{n=1}^\infty (a - 1/n, b + 1/n)$.

(b) $(a, b] = \bigcap_{n=1}^\infty (a, b + 1/n)$; $(a, b] = \bigcup_{n=1}^\infty [a + 1/n, b]$

(c) Because $\mathbf{Q}$ is countable, we can write $\mathbf{Q} = \{r_1, r_2, r_3, \ldots\}$. Note that each singleton set $\{r_n\}$ is closed and the complement $\{r_n\}^c$ is open. Then $\mathbf{Q} = \bigcup_{n=1}^\infty \{r_n\}$ shows that $\mathbf{Q}$ is an F_σ set, and $\mathbf{I} = \mathbf{Q}^c = \bigcap_{n=1}^\infty \{r_n\}^c$ shows that $\mathbf{I}$ is a G_δ set.

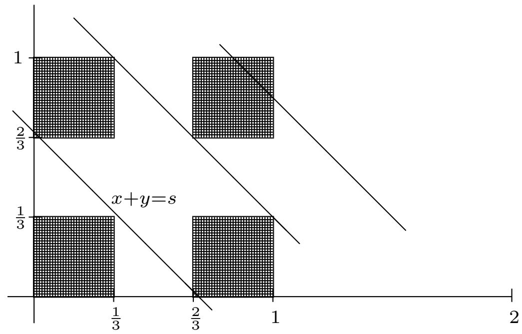


Figure 3.1: $x + y = s$ MUST INTERSECT $C_1 \times C_1$.

3.3 Compact Sets

Exercise 3.3.1. Let K be compact. Then by Theorem 3.3.4, K is closed and bounded. By the Axiom of Completeness, $\sup K$ exists, and by Exercise 3.2.11 we know $\sup K \in \overline{K}$. Because K is closed, $K = \overline{K}$ and hence $\sup K \in K$. A similar argument shows $\inf K \in K$.

Exercise 3.3.2. Let $K \subseteq \mathbf{R}$ be closed and bounded. Since K is bounded, the Balzano-Weierstrass Theorem guarantees that for any sequence (a_n) contained in K, we can find a convergent subsequence (a_{n_k}). Because the set is closed, the limit of this subsequence is also in K. Hence K is compact.

Exercise 3.3.3. We will show that the Cantor set is closed and bounded. Recall that $C = \bigcap_{n=0}^{\infty} C_n$. Each C_n is closed because it is a finite union of closed intervals. Now since C is an intersection of closed sets, C itself is closed by Theorem 3.2.14. By construction, the Cantor set is bounded above by 1 and below by 0. Hence, C is a compact set.

Exercise 3.3.4. Let K be compact and let F be closed. Then $K \cap F$ is closed by Theorem 3.2.14. Because K is bounded, $K \cap F$ must be bounded as well. Thus $K \cap F$ is closed and bounded, and hence compact.

Exercise 3.3.5. (a) Not compact. Let (a_n) be a sequence of rational numbers converging to $\sqrt{2}$.

(b) Not compact. Let (a_n) be a sequence of rational numbers converging to an irrational number in the interval $(0, 1)$.

(c) Not compact. Let $a_n = n$.

(d) Compact.

(e) Not compact. Let $a_n = 1/n$. The sequence (a_n) converges to 0 (and thus so does every subsequence), which is not an element of the set.

(f) Compact.

Exercise 3.3.6. (a) Fix $s \in [0,2]$. We want to find an $x_1, y_1 \in C_1$ such that $x_1 + y_1 = s$. We know that $C_1 = [0,1/3] \cup [2/3,1]$. Then we have that:

$$[0,1/3] + [0,1/3] = [0,2/3]$$

$$[0,1/3] + [2/3,1] = [2/3,4/3]$$

$$[2/3,1] + [2/3,1] = [4/3,1].$$

Hence $C_1 + C_1 = [0,2/3] \cup [2/3,4/3] \cup [4/3,2] = [0,2]$, so for any $s \in [0,2]$, we can find an $x_1, y_1 \in C_1$ such that $x_1 + y_1 = s$.

A convenient way to visualize this result in the (x,y)–plane is to shade in the four squares corresponding to the components of $C_1 \times C_1$ (see Figure 3.1) and observe that, for each $s \in [0,2]$, the line $x + y = s$ must intersect at least one of the squares. For each n we can draw a similar picture (with increasing numbers of smaller squares), and our job is to argue that the line $x + y = s$ continues to intersect at least one of the smaller squares

To argue by induction, suppose that we can find $x_n, y_n \in C_n$ such that $x_n + y_n = s$. To show that this must hold for $n+1$, let's focus attention on a square from the nth stage where $x_n + y_n = s$ holds (i.e., where $x + y = s$ intersects an nth stage square). Moving to the $n+1$th stage means removing the open middle third of this shaded region. But this results in a situation precisely like the one in Figure 3.1, implying that the line $x + y = s$ must intersect a $(n+1)$st stage square. This shows that there exist $x_{n+1}, y_{n+1} \in C_{n+1}$ where $x_{n+1} + y_{n+1} = s$.

(b) We have (x_n) and (y_n) with $x_n, y_n \in C_n$ and $x_n + y_n = s$ for all n. The sequence (x_n) doesn't converge, but (x_n) is bounded so by the Bolzano-Weierstrass Theorem there exists a convergent subsequence (x_{n_k}). Set $x = \lim x_{n_k}$. Now look at the corresponding subsequence $(y_{n_k}) = s - x_{n_k}$. Using the Algebraic Limit Theorem, we see that this subsequence converges to $y = \lim(x - x_{n_k}) = s - x$. This shows $x + y = s$. We now need to argue that $x, y \in C$.

One temptation is to say that because C is closed, $x = \lim(x_{n_k})$ must be in C. However, we don't know (and it probably isn't true) that (x_{n_k}) is in C. We *can* say that (x_{n_k}) is in C_1, and because C_1 is closed we may conclude $x \in C_1$. In fact, given any fixed n_0, we can argue that $x \in C_{n_0}$ because x_{n_k} is (with the exception of some finite number of terms) contained in C_{n_0}. This implies $x \in \bigcap_{n=1}^{\infty} C_n = C$ as desired, and a similar argument works for y.

Exercise 3.3.7. (a) True. By Theorem 3.2.14, an arbitrary intersection of closed sets is closed. Boundedness is also preserved by intersections; therefore, the arbitrary intersection of compact sets will be compact.

(b) False. Let K be a closed interval and let A be an open set such that $A \subseteq K$. Then $A \cap K$ is not closed, and hence it is not compact.

(c) False. Let $F_n = [n, \infty)$. Then F_n is closed for all n, but the intersection of these sets is empty.

(d) True. A finite set is clearly bounded, and by a previous exercise we know that a finite set is closed.

(e) False. The rational numbers are countable but they are not compact.

Exercise 3.3.8. (a) If $A_1 \cap K$ and $B_1 \cap K$ both had finite subcovers consisting of the form $\{O_\lambda : \lambda \in \Lambda\}$, then there would exist a finite subcover for K. But we assumed that such a finite subcover did not exist for K. Hence either $A_1 \cap K$ or $B_1 \cap K$ (or both) has no finite subcover.

(b) Let I_1 be a half of I_0 whose intersection with K does not have a finite subcover, so that $I_1 \cap K$ cannot be finitely covered and $I_1 \subseteq I_0$. Then bisect I_1 into two closed intervals, A_2 and B_2 and again let $I_2 = A_2$ if $A_2 \cap K$ does not have a finite subcover. Otherwise, let $I_2 = B_2$ Continuing this process of bisecting the interval I_n, we get the desired sequence I_n with $\lim |I_n| = 0$.

(c) Because K is compact, $K \cap I_n$ is also compact for each $n \in \mathbf{N}$. By Theorem 3.3.5, $\bigcap_{n=1}^{\infty} I_n \cap K$ is non-empty, and there exists an $x \in K \cap I_n$ for all n.

(d) Let $x \in K$ and let O_{λ_0} be an open set that contains x. Because O_{λ_0} is open, there exists $\epsilon_0 > 0$ such that $V_{\epsilon_0}(x) \subseteq O_{\lambda_0}$. Now choose n_0 such that $|I_{n_0}| < \epsilon_0$. Then I_{n_0} is contained in the single open set O_{λ_0} and thus it has a finite subcover. This contradiction implies that K must have originally had a finite subcover.

Exercise 3.3.9. (a) Let $O_\lambda = (\lambda - 1, \lambda + 1)$ where $\lambda \in \mathbf{N}$.

(b) Let α be a fixed irrational number in the interval $(0, 1)$. For each $n \in \mathbf{N}$ set $O_n = (-1, \alpha - 1/n) \cup (\alpha + 1/n, 2)$. The union over n of all these sets gives $(-1, \alpha) \cup (\alpha, 2)$ which contains $\mathbf{Q} \cap [0, 1]$. This cover has no finite subcover.

(c) Let $O_\lambda = (\lambda - 1, \lambda + 1)$ where $\lambda \in \mathbf{N}$.

(e) Let $O_n = (1/n, 2)$ for each $n \in \mathbf{N}$. The union gives $(0, 2)$ and there is no finite subcover.

Exercise 3.3.10. If A is a finite set then it clearly clompact. Conversely, assume A is clompact. Because a singleton set is a closed set, the collection of singleton sets consisting of the elements of A is a closed cover. This cover must have a finite subcover, and it follows that A is a finite set. To summarize, a set is "clompact" if and only if it is finite.

3.4 Perfect Sets and Connected Sets

Exercise 3.4.1. Let P be a perfect set and let K be compact. Consider the set $P \cap K$. This set is closed by Theorem 3.2.14. Since K is bounded, $P \cap K$ will be bounded as well, and thus the intersection of the two sets is compact. However, $P \cap K$ is not necessarily perfect. For example, let K be a singleton set contained in P. Then $P \cap K$ is a singleton set and is not perfect.

Exercise 3.4.2. No. A non-empty perfect set must be uncountable and subsets of $\mathbf{Q}$ are all countable sets.

Exercise 3.4.3. (a) We are given an arbitrary $x \in C$. Because $x \in C_1 \subseteq C$, x must fall in one of the two intervals that make up C_1. The key idea to remember is that C contains at least the endpoints of these two intervals. Thus, if $0 \leq x < 1/3$, let $x_1 = 1/3$. If $x = 1/3$, then take $x = 0$. We can do a similar

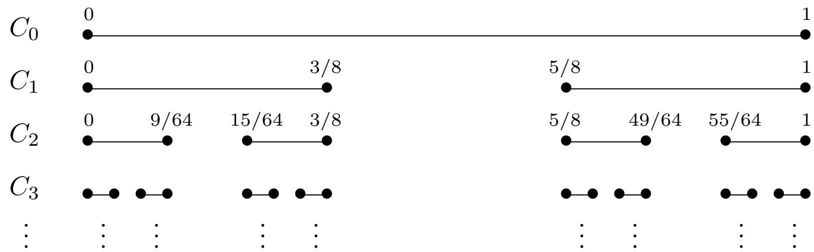


Figure 3.2: THE "OPEN MIDDLE–FOURTH" CANTOR SET.

thing if x falls in the other interval. This is, if $2/3 \leq x < 1$, then let $x_1 = 1$, and if $x = 1$ then set $x_1 = 2/3$. In all of these cases we have $x_1 \in C$ with $|x - x_1| \leq 1/3$.

(b) For each $n \in \mathbf{N}$, the length of each interval that makes up C_n is $1/3^n$. It is also true that the endpoints of these intervals are always elements of C. For every n, let x_n be an endpoint of the interval that contains x. If x happens to be an endpoint of a C_n interval, then let x_n be the opposite endpoint of this interval. Thus we have $x_n \in C$ with $x_n \neq x$ such that $|x - x_n| \leq 1/3^n$. Because $1/3^n \to 0$, it follows that $x_n \to x$. This means that $x \in C$ is not an isolated point. Having already seen that C is closed, we conclude that C is perfect.

Exercise 3.4.4. (a) This set is compact and perfect, and the arguments proceed exactly as they do for the original Cantor set. (See Figure 3.2.)

(b) The length of this set is equal to 1 minus the lengths of the missing pieces:

$$\begin{aligned}
\textit{Length} &= 1 - \left(\frac{1}{4} + 2(\frac{3}{32}) + 4(\frac{9}{256}) + \cdots\right) \\
&= 1 - \left(\frac{1}{4} + \frac{3}{16} + \frac{9}{64} + \cdots\right) \\
&= 1 - \left(\frac{1/4}{1 - 3/4}\right) \\
&= 1 - 1 = 0.
\end{aligned}$$

To find the dimension of this set, magnify the set by $\frac{8}{3}$. Then $C_0 = [0, 8/3]$ and $C_1 = [0, 1] \cup [5/3, 8/3]$. Thus we obtain two copies of the set. If x is the dimension of the set, then x should satisfy $2 = (\frac{8}{3})^x$, or $x = \frac{\ln 2}{\ln(8/3)} \approx .707$.

Exercise 3.4.5. Let U and V be disjoint, open sets with $A \subseteq U$ and $B \subseteq V$. We claim that $\overline{U} \cap V = \emptyset$ and $U \cap \overline{V} = \emptyset$. To see why this is true, note that because U and V are disjoint we have $U \subseteq V^c$. Now V^c is closed (because V is open) and thus $\overline{U}$ must also satisfy $\overline{U} \subseteq V^c$ by Theorem 3.2.12. This proves $\overline{U} \cap V = \emptyset$, and the other statement has a similar proof.

Since $A \subseteq U$, limit points of A will also be limit points of U and we get $\overline{A} \subseteq \overline{U}$. Hence $\overline{A} \cap V = \emptyset$ and therefore $\overline{A} \cap B = \emptyset$. Similarly, $\overline{B} \subseteq \overline{V}$, so $A \cap \overline{B} = \emptyset$. Therefore, A and B are separated.

Exercise 3.4.6. ($\Rightarrow$) Let E be a connected set. Assume $E = A \cup B$ where A, B are disjoint, non-empty sets. Since E is connected, A and B are not separated. So either $\overline{A} \cap B$ or $A \cap \overline{B}$ is not empty. Without loss of generality, assume $x \in \overline{A} \cap B$. Then $x \in B$ and $x \in \overline{A}$, but $x \notin A$ because A and B were assumed to be disjoint. Therefore x is a limit point of A. Then by Theorem 3.2.5 there exists a convergent sequence (x_n) contained in A that converges to x.

($\Leftarrow$) We will prove this direction by proving the contrapositive. Assume $E \subseteq \mathbf{R}$ is disconnected. We want to find two non-empty, disjoint sets A, B satisfying $E = A \cup B$ such that there never exists a convergent sequence $(x_n) \to x$ with (x_n) contained in one of A or B, and x an element of the other. Because E is disconnected, there exist separated sets A and B satisfying $E = A \cup B$. Now suppose (x_n) is contained in A and $(x_n) \to x$. Then either $x \in A$ or x is a limit point of A, and in either case $x \in \overline{A}$. Because $\overline{A} \cup B = \emptyset$, we know $x \notin B$. If we assume (x_n) is convergent sequence in B, a similar argument shows that its limit cannot be in A. This completes the proof.

Exercise 3.4.7. (a) Consider $A = Q \cap (0, 5)$. Then A is disconnected, for we can write $A = (0, \sqrt{2}) \cup (\sqrt{2}, 5)$. But $\overline{A} = [0, 5]$, which is connected.

(b) If A is connected, $\overline{A}$ is connected as well. This follows directly from Theorem 3.4.7. If A is perfect then A is closed and $A = \overline{A}$. Hence, $\overline{A}$ is perfect as well.

Exercise 3.4.8. (a) Given any $x, y \in \mathbf{Q}$, choose $z \in \mathbf{I}$ such that $x < z < y$. We know that such a z exists because the irrational numbers are dense. Then let $\mathbf{Q} = A \cup B$, where $A = \mathbf{Q} \cap (-\infty, z)$ and $B = \mathbf{Q} \cap (z, \infty)$. The sets A and B are separated (see Example 3.4.5(ii)), and $x \in A$ and $y \in B$.

(b) The set of irrational numbers is totally disconnected because the rational numbers are also dense in $\mathbf{R}$. Thus we can follow the same argument as in part (a) by letting $x, y \in \mathbf{I}$ and choosing $z \in \mathbf{Q}$.

Exercise 3.4.9. (a) The length of each interval in C_n is $1/3^n$. If we choose an N so that $1/3^N < \epsilon$, then x, y cannot belong to the same interval.

(b) Let x and y be on separate intervals of C_N, where N is chosen as in (a). Then there exists an open interval between x and y that is not contained in C. Choose z in this interval. Then $x < z < y$ and $z \notin C$.

If (a, b) were an open interval satisfying $(a, b) \subseteq C$, then we could find $x, y \in C$ with $a < x < y < b$, and it would follow that $[x, y] \subseteq C$. However, we have now shown that for all such x and y there exists a point $z \in (x, y)$ with $z \notin C$. Thus, C contains no intervals (open or closed).

(c) Informally speaking, totally disconnected sets in $\mathbf{R}$ and sets that do not contain any intervals. This is the content of part (b). To say it again, we know that given any $x, y \in C$ with $x < y$, there exists a $z \notin C$ satisfying $x < z < y$. Take $A = C \cap [0, z)$ and $B = C \cap (z, 1]$. Then A, B are separated with $x \in A$ and $y \in B$, and $C = A \cup B$.

Exercise 3.4.10. (a) Since $O = \bigcup_{n=1}^{\infty} V_{\epsilon_n}(r_n)$ is a union of open sets, O is open. Therefore F is closed. Every rational number is contained in O, so F must contain only irrationals. Now we must argue that F is non-empty. To informally see this, look at the "length" of O. Since O is the union of open sets of length $1/2^{n-1}$, the length of O must be no greater than $\sum_{n=1}^{\infty} 1/2^n n - 1 = 2$. Therefore the entire real line cannot be covered by O, and hence F is non-empty.

A way to avoid applying the concept of "length" to sets that are not *finite* unions of intervals would be to assume, for contradiction, that $F = \emptyset$. Then $O = \mathbf{R}$ and, in particular, the compact set $[0, 3]$ is covered by $\{V_{\epsilon_n}(r_n) : n \in \mathbf{N}\}$. Now let $\{V_{\epsilon_{n_1}}(r_{n_1}), V_{\epsilon_{n_2}}(r_{n_2}), \ldots V_{\epsilon_{n_m}}(r_{n_m})\}$ be a finite subcover for $[0, 3]$. The lengths of this finite collection of open intervals must sum to a total less than 2, and therefore they cannot cover the set $[0, 3]$.

(b) No, the set F does not contain any non-empty open intervals. Every non-trivial interval contains a rational number and this rational is not an element of F. Hence F contains no such intervals. This proves that F is totally disconnected. Given arbitrary $a, b \in F$ with $a < b$, we can find a rational number c with $a < c < b$. Then writing $F = A \cup B$ where $A = F \cap (-\infty, c)$ and $B = F \cap (c, \infty)$ finishes the argument.

(c) It is not possible to know whether F is perfect as it is possible for F to contain isolated points.

There does exist a non-empty perfect set of irrational numbers. To modify the construction, we again write $\mathbf{Q} = \{r_1, r_2, r_3, \ldots\}$, but this time we define ϵ_n inductively. Set $\epsilon_1 = \sqrt{2}/2$ and, as a convention, let $V_\epsilon(x) = \emptyset$ whenever $\epsilon = 0$. For $n \geq 2$, let $\epsilon_n = \min\{\sqrt{2}/2^n, d_n/2\}$ where

$$d_n = \inf\{|x - r_n| : x \in \bigcup_{k=1}^{n-1} V_{\epsilon_k}(r_k)\}.$$

Geometrically, d_n is the distance from r_n to the set $O_{n-1} = \bigcup_{k=1}^{n-1} V_{\epsilon_k}(r_k)$. The idea is to inductively build the open set O as a disjoint union of positively spaced neighborhoods of the form $V_{\epsilon_n}(r_n)$. If $d_n = 0$, then because ϵ_n is always irrational whenever it is non-zero, we may conclude $r_n \in O_{n-1}$. If $d_n > 0$, then the definition of ϵ_n ensures that

$$\text{(1)} \qquad \overline{V_{\epsilon_n}(r_n)} \cap \overline{V_{\epsilon_m}(r_m)} = \emptyset \quad \text{for all } 1 \leq m < n.$$

Now $O = \bigcup_{n=1}^{\infty} V_{\epsilon_n}(r_n)$ is open and contains $\mathbf{Q}$, so $F = O^c$ is again a closed set inside the irrationals. It remains to show that it contains no isolated points.

Let $x \in F$ be arbitrary and assume, for contradiction, that x is isolated. Thus there exits $\epsilon_0 > 0$ such that $(x - \epsilon_0, x)$ and $(x, x + \epsilon_0)$ are both be contained in O. Because of the way we constructed O it now follows that there must exist n' and m' such that

$$(x - \epsilon_0, x) \subseteq V_{\epsilon_{n'}}(r_{n'}) \quad \text{and} \quad (x, x + \epsilon_0) \subseteq V_{\epsilon_{m'}}(r_{m'}).$$

But this contradicts statement (1) above because the point x is a limit point of each of these two neighborhoods. This contradiction proves x is not isolated and the proof is complete.

3.5 Baire's Theorem

Exercise 3.5.1. ($\Rightarrow$) Let A be a G_δ set. We want to show that this implies that A^c is an F_σ set. By the definition of a G_δ set, A can be written as the countable intersection of open sets. In symbols, $A = \bigcap_{n=1}^{\infty} O_n$ where O_n is open for each $n \in \mathbf{N}$. Then by De Morgan's Law, $A^c = \bigcup_{n=1}^{\infty} O_n^c$. Because O_n is open, O_n^c is closed. Hence, A^c is the countable union of closed closed sets, and therefore it is an F_σ set.

($\Leftarrow$) Now let B be an F_σ set. Then we know that $B = \bigcup_{n=1}^{\infty} F_n$, where F_n is closed for each $n \in \mathbf{N}$. It then follows from De Morgan's Law that $B^c = \bigcap_{n=1}^{\infty} F_n^c$. Therefore, B^c is the countable intersection of open sets, which makes it a G_δ set.

Exercise 3.5.2. (a) countable.
(b) finite.
(c) finite.
(d) countable.

Exercise 3.5.3. See Exercise 3.2.14.

Exercise 3.5.4. (a) Pick a point $x_1 \in G_1$. Since G_1 is open, there exists an $\epsilon_1 > 0$ such that $V_{\epsilon_1}(x_1) \subseteq G_1$. Now take $\epsilon_1' < \epsilon_1$, and let $I_1 = \overline{V_{\epsilon_1'}(x_1)}$. The significant point to make here is that I_1 is a closed interval but we still have the containment $I_1 \subseteq V_{\epsilon_1}(x_1) \subseteq G_1$.

Because G_2 is dense, there exists an $x_2 \in V_{\epsilon_1'}(x_1) \subseteq G_1$. Now $G_2 \cap V_{\epsilon_1'}(x_1)$ is open, so there exists an $\epsilon_2 > 0$ such that $V_{\epsilon_2}(x_2) \subseteq G_2 \cap V_{\epsilon_1'}(x_1)$. If we again choose a smaller $\epsilon_2' < \epsilon_2$, then as before the closed interval $I_2 = \overline{V_{\epsilon_2'}(x_2)}$ satisfies $I_2 \subseteq G_2$ as well as $I_2 \subseteq I_1$. We may continue this process to create a nested sequence of closed intervals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ satisfying $I_n \subseteq G_n$ for all $n \in \mathbf{N}$.

(b) By the Nested Interval Property, there exists an $x \in \bigcap_{n=1}^{\infty} I_n$. Because $I_n \subseteq G_n$ it follows that $x \in G_n$ for all n. Hence $\bigcap_{n=1}^{\infty} G_n$ is not empty.

Exercise 3.5.5. Let F be a closed set containing no non-empty open intervals. Then F^c is open and we claim that it must also be *dense* in $\mathbf{R}$. To see why, assume $x, y \in R$ satisfy $x < y$. By hypothesis, the open interval (x, y) is *not* contained in F which means there exists a point $z \in F^c$ satisfying $x < z < y$. This proves F^c is dense.

Turning to the statement in the exercise, assume for contradiction that $\mathbf{R} = \bigcup_{n=1}^{\infty} F_n$ where each F_n is a closed set containing no non-empty open intervals. Taking complements we get $\emptyset = \bigcap_{n=1}^{\infty} F_n^c$, and we have just seen that each F_n^c is a dense open set in $\mathbf{R}$. But this is a contradiction, because the intersection of dense open sets is not empty.

Exercise 3.5.6. Assume, for contradiction, that $\mathbf{I}$ is an F_σ set. Then we can write $\mathbf{I} = \bigcup_{n=1}^{\infty} F_n$, where each F_n is a closed set. Because each F_n is a subset of $\mathbf{I}$, we can also assert that F_n fails to contain any open intervals. Now $\mathbf{Q}$ is the countable union of singleton sets, and each singleton set certainly qualifies as a closed set containing no open intervals. But this implies that we can write $\mathbf{R}$

as the countable union of closed sets, none of which contain any open intervals. In the previous exercise we showed that this is impossible, hence **I** cannot be an F_σ set.

If **Q** were a G_δ set, then by Exercise 3.5.1, we could show that **I** was an F_σ set, which we have just shown to be impossible.

Exercise 3.5.7. The set $(\mathbf{I} \cap (-\infty, 0]) \cup (\mathbf{Q} \cap [0, \infty))$ is neither an F_σ set nor a G_δ set.

Exercise 3.5.8. ($\Rightarrow$) Assume that E is nowhere dense in **R**. Then $\overline{E}$ contains no nonempty open intervals. Given any $x, y \in \mathbf{R}$ with $x < y$, we know (x, y) is not a subset of $\overline{E}$. So there exists a $z \in \overline{E}^c$ satisfying $x < z < y$. We also have that $\overline{E}^c$ is open because $\overline{E}$ is closed. This proves $\overline{E}^c$ is dense.

($\Leftarrow$) Assume that $\overline{E}^c$ is dense. Then for any $x, y \in \mathbf{R}$ with $x < y$, we can find a $z \in \overline{E}^c$ satisfying $x < z < y$. Therefore $\overline{E}$ cannot contain any nonempty open intervals. It then follows from the definition that E is nowhere-dense.

Exercise 3.5.9. (a) Somewhere in between.

(b) Nowhere dense.

(c) Dense.

(d) Nowhere dense.

Exercise 3.5.10. Assume, for contradiction, that $\mathbf{R} = \bigcup_{n=1}^\infty E_n$. Then certainly $\mathbf{R} = \bigcup_{n=1}^\infty \overline{E_n}$. By De Morgan's Law this implies that $\emptyset = \bigcap_{n=1}^\infty \overline{E_n}^c$. Because E_n is nowhere dense, $\overline{E_n}^c$ is dense. We also know that $\overline{E_n}^c$ is open. Then we have reached a contradiction, since by Theorem 3.5.2 the countable intersection of dense, open sets is not empty.

Chapter 4

Functional Limits and Continuity

4.1 Discussion: Examples of Dirichlet and Thomae

4.2 Functional Limits

Exercise 4.2.1. (a) Let $\epsilon > 0$. Notice that

$$|f(x) - 8| = |(2x + 4) - 8| = |2x - 4| = 2|x - 2|.$$

Choose $\delta = \epsilon/2$. Then $0 < |x - 2| < \delta = \epsilon/2$ implies that

$$|f(x) - 8| = 2|x - 2| < 2\left(\frac{\epsilon}{2}\right) = \epsilon.$$

(b) Let $\epsilon > 0$. Choose $\delta = \epsilon^{\frac{1}{3}}$. Then $0 < |x| < \delta = \epsilon^{\frac{1}{3}}$ implies that

$$|f(x) - 0| = |x^3| < (\epsilon^{\frac{1}{3}})^3 = \epsilon.$$

(c) Given an arbitrary $\epsilon > 0$, our goal is to make $|x^3 - 8| < \epsilon$ by restricting $|x - 2|$ to be smaller than some carefully chosen δ. Note that

$$|x^3 - 8| = |(x^2 + 2x + 4)(x - 2)| = |(x^2 + 2x + 4)||x - 2|.$$

By insisting that $\delta \leq 1$, we can restrict x to fall in the interval $(1, 3)$. This implies $|(x^2 + 2x + 4)| \leq 9 + 6 + 4 = 19$.

Now choose $\delta = \min\{1, \epsilon/19\}$. If $0 < |x - 2| < \delta$, then it follows that

$$|x^3 - 8| = |(x^2 + 2x + 4)|\,|x - 2| \leq 19\left(\frac{\epsilon}{19}\right) = \epsilon$$

as desired.

(d) For arbitrary $\epsilon > 0$, choose $\delta = 1/10$. Then, $0 < |x - \pi| < \delta = 1/10$ implies $3 < x < 4$ and hence $[[x]] = 3$. Thus, $|[[x]] - 3| = |3 - 3| = 0 < \epsilon$ as desired.

Although in most cases smaller values of ϵ require smaller values of δ in response, this is a non-standard situation where δ can be chosen independently of the value of ϵ.

Exercise 4.2.2. Then any *smaller* δ will also suffice.

Exercise 4.2.3. (a) If $x_n = -1/n$ and $y_n = 1/n$ for $n \in N$, then $\lim(x_n) = \lim(y_n) = 0$. However,

$$\frac{|x_n|}{x_n} = \frac{|-1/n|}{-1/n} = -1 \quad \text{and} \quad \frac{|y_n|}{y_n} = \frac{|1/n|}{1/n} = 1.$$

Thus,

$$\lim \frac{|x_n|}{x_n} \neq \lim \frac{|y_n|}{y_n},$$

and so by Corollary 4.2.5, $\lim_{x\to 0} |x|/x$ does not exist.

(b) Let $x_n = \frac{n+1}{n}$ and $y_n = \sqrt{\frac{n+1}{n}}$ for $n \in N$. Then $\lim(x_n) = \lim(y_n) = 0$. We also have $x_n \in \mathbf{Q}$ and $y_n \in \mathbf{I}$ for all $n \in \mathbf{N}$, so that

$$\lim g(x_n) = \lim 1 = 1 \quad \text{while} \quad \lim g(y_n) = \lim 0 = 0.$$

By Corollary 4.2.5, $\lim_{x\to 1} g(x)$ does not exist.

Exercise 4.2.4. (a) Let $x_n = (n+1)/n$, $y_n = \sqrt{(n+1)/n}$ and $z_n = (2n+1)/2n$. Note that $\lim(x_n) = \lim(y_n) = \lim(z_n) = 1$.

(b) For (x_n) we get $t(x_n) = 1/n$ which converges to 0.

For (y_n) we get $t(y_n) = 0$ which converges to 0.

For (z_n) we get $t(z_n) = 1/2n$ which converges to 0.

(c) The point to make is that the closer a rational number is to 1, the larger its denominator has to be, and thus the smaller the value of $t(x)$. Because $t(x) = 0$ for all irrational numbers, the conjecture is that $\lim_{x\to 1} t(x) = 0$.

In order to prove our claim, we have to show that given $\epsilon > 0$, there exists a δ neighborhood around 1 such that $x \in V_\delta(1)$ implies $t(x) \in V_\epsilon(0)$. If we set $T = \{x \in R : t(x) \geq \epsilon\}$, then notice that $x \in T$ if and only if x is a rational number of the form $x = m/n$ where $n \leq 1/\epsilon$. If we focus on some finite interval such as $[0, 2]$ then the restriction on the size of n implies that the set $T \cap [0, 2]$ is *finite*. With finite sets, we are allowed to take minimums and so let

$$\delta = \min\{y : y \in T \cap [0, 2]\} > 0.$$

To see that this choice of δ "works", we note that if $x \in V_\delta(1)$ then $x \notin T$ and thus $t(x) \in V_\epsilon(0)$.

Exercise 4.2.5. (a) Showing $\lim_{x \to c}[f(x)+g(x)] = L+M$ is equivalent to showing $f(x_n) + g(x_n) \to L + M$ whenever $x_n \to c$. Since we are given $f(x_n) \to L$ and $g(x_n) \to M$, we can use Theorem 2.3.3 part (ii) to conclude $f(x_n) + g(x_n) \to L + M$.

(b) Let $\epsilon > 0$ be arbitrary. We need to show, there exists δ such that $0 < |x - c| < \delta$ implies $|(f(x) + g(x)) - (L + M)| < \epsilon$. Note that,

$$|(f(x)+g(x)) - (L+M)| = |(f(x) - L) + (g(x) - M)| \leq |f(x) - L| + |g(x) - M|.$$

Since $\lim_{x \to c} f(x) = L$, there exists δ_1 such that $0 < |x - c| < \delta_1$ implies $|f(x) - L| < \epsilon/2$. In addition, because $\lim_{x \to c} g(x) = M$, there exists δ_2 such that $0 < |x - c| < \delta_2$ implies $|g(x) - M| < \epsilon/2$. Now if we pick $\delta = \min\{\delta_1, \delta_2\}$ then $0 < |x - c| < \delta$ implies that

$$\begin{aligned} |(f(x) + g(x)) - (L + M)| &\leq |f(x) - L| + |g(x) - M| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

as desired.

(c) Showing $\lim_{x \to c}[f(x)g(x)] = LM$ is equivalent to showing $f(x_n)g(x_n) \to LM$ whenever $x_n \to c$. Since we are given $f(x_n) \to L$ and $g(x_n) \to M$, we can use Theorem 2.3.3 part (iii) to conclude $f(x_n)g(x_n) \to LM$.

Now let's write another proof of the corollary based on Definition 4.2.1. Note that,

$$\begin{aligned} |f(x)g(x) - (LM)| &= |f(x)g(x) - f(x)M + f(x)M - (LM)| \\ &\leq |f(x)(g(x) - M)| + |M(f(x) - L)| \\ &= |f(x)||g(x) - M| + |M||f(x) - L|. \end{aligned}$$

Since $\lim_{x \to c} f(x) = L$, there exists δ_1 such that $0 < |x - c| < \delta_1$ implies $|f(x) - L| < \epsilon/(2M)$.

Next we need a lemma that says $f(x)$ is bounded. Although this may not be the case over the whole domain A, it is certainly true in some neighborhood around $x = c$. Given $\epsilon_0 = 1$, for instance, we know there exists $\delta_2 > 0$ such that $0 < |x - c| < \delta_2$ implies $|f(x) - L| < 1$, and in this case we then have $|f(x)| < |L| + 1$.

We now use the fact that $\lim_{x \to c} g(x) = M$ to assert that there exists $\delta_3 > 0$ such that $0 < |x - c| < \delta_3$ implies $|g(x) - M| < \epsilon/(2(|L| + 1))$. Finally, if we pick $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, then

$$\begin{aligned} |f(x)g(x) - (LM)| &\leq |f(x)||g(x) - M| + |M||f(x) - L| \\ &< (|L| + 1)\left(\frac{\epsilon}{2(|L| + 1)}\right) + M\left(\frac{\epsilon}{2M}\right) = \epsilon \end{aligned}$$

whenever $0 < |x - c| < \delta$.

Finally, we point out that this proof assumes $M \neq 0$. The case $M = 0$ is a little easier in fact and can be handled as a corollary of the next exercise.

Exercise 4.2.6. We are given that there exists an $M > 0$ such that $|f(x)| \leq M$ for all $x \in A$. Let $\epsilon > 0$ be arbitrary. Because we know $\lim_{x\to c} g(x) = 0$, there exists $\delta > 0$ such that $0 < |x - c| < \delta$ implies $|g(x) - 0| = |g(x)| < \epsilon/M$. It follows that

$$|g(x)f(x) - 0| = |g(x)||f(x)| < \left(\frac{\epsilon}{M}\right) M = \epsilon$$

whenever $0 < |x - c| < \delta$. Therefore, $\lim_{x\to c} g(x)f(x) = 0$.

Exercise 4.2.7. (a) We say $\lim_{x\to c} f(x) = \infty$ if for every arbitrarily large M, there exists $\delta > 0$, such that whenever $0 < |x-c| < \delta$ it follows that $f(x) > M$.

Let $M > 0$ be arbitrary. To prove $\lim_{x\to c} 1/x^2 = \infty$, we can choose $\delta = \sqrt{\frac{1}{M}}$. Then $0 < |x| < \delta = \sqrt{\frac{1}{M}}$ implies $x^2 < \frac{1}{M}$ from which it follows that $1/x^2 > M$, as desired.

(b) We say $\lim_{x\to\infty} f(x) = L$ if for every $\epsilon > 0$, there exists $K > 0$, such that whenever $x > K$ it follows that $|f(x) - L| < \epsilon$.

Let $\epsilon > 0$. To prove $\lim_{x\to\infty} 1/x = 0$, choose $K = 1/\epsilon$. If $x > K = 1/\epsilon$, then $1/x < \epsilon$ as desired.

(c) We say $\lim_{x\to\infty} f(x) = \infty$ if for every $M > 0$ there exists $K > 0$, such that whenever $x > K$ it follows that $f(x) > M$. An example of function with such a limit would be $f(x) = \sqrt{x}$. Given an arbitrary $M > 0$, choose $K = M^2$. If $x > K = M^2$, then it follows that $\sqrt{x} > M$ as desired.

Exercise 4.2.8. Let $\lim_{x\to c} f(x) = L$ and let $\lim_{x\to c} g(x) = M$. We are asked to show $L \geq M$. This result is in the same spirit as the Order Limit Theorem (Theorem 2.3.4), and using the Sequential Criterion for Functional Limits we can in fact derive this result from OLT.

Let (x_n) be a sequence in A satisfying $(x_n) \to c$ with $x_n \neq c$ for all n. We are given than $f(x_n) \geq g(x_n)$, and thus the Order Limit Theorem tells us $\lim f(x_n) \geq \lim g(x_n)$. (This requires that we know the limits exist, a hypothesis not included in early editions of this problem.) By the Sequential Criterion for Functional Limits, $L = \lim f(x_n)$ and $M = \lim g(x_n)$, and thus $L \geq M$ as desired.

Exercise 4.2.9. This is another situation where we could use the analogous statement for sequences (Exercise 2.3.3) to prove the functional limit version. (We could also apply the previous exercise to each inequality.) Instead, we shall give a proof in terms of the ϵ–δ definition of functional limits.

Let $\epsilon > 0$. Because $\lim_{x\to c} f(x) = L$, there exists $\delta_1 > 0$ such that $0 < |x - c| < \delta_1$ implies $L - \epsilon < f(x) < L + \epsilon$. Likewise, there exists $\delta_2 > 0$ such that $0 < |x - c| < \delta_2$ implies $L - \epsilon < h(x) < L + \epsilon$. Choosing $\delta = \min\{\delta_1, \delta_2\}$, we see that

$$L - \epsilon < f(x) \leq g(x) \leq h(x) < L + \epsilon$$

whenever $0 < |x - c| < \delta$, which implies $|g(x) - L| < \epsilon$ as desired.

4.3 Combinations of Continuous Functions

Exercise 4.3.1. (a) Let $\epsilon > 0$. Note that $|g(x) - c| = |\sqrt[3]{x} - 0| = |\sqrt[3]{x}|$ where $c = 0$. Now if we set $\delta = \epsilon^3$, then $|x - 0| < \delta = \epsilon^3$ implies $|\sqrt[3]{x}| < \epsilon$. This shows $g(x)$ is continuous at $c = 0$.

(b) For $c \neq 0$ write,

$$\begin{aligned} |g(x) - g(c)| &= |\sqrt[3]{x} - \sqrt[3]{c}| = |\sqrt[3]{x} - \sqrt[3]{c}| \left(\frac{\sqrt[3]{x^2} + \sqrt[3]{xc} + \sqrt[3]{c^2}}{\sqrt[3]{x^2} + \sqrt[3]{xc} + \sqrt[3]{c^2}} \right) \\ &= \frac{|x - c|}{\sqrt[3]{x^2} + \sqrt[3]{xc} + \sqrt[3]{c^2}} \leq \frac{|x - c|}{\sqrt[3]{c^2}}. \end{aligned}$$

Therefore, if we pick $\delta = \epsilon\sqrt[3]{c^2}$, then $|x - c| < \delta = \epsilon\sqrt[3]{c^2}$ implies

$$|g(x) - g(c)| = |\sqrt[3]{x} - \sqrt[3]{c}| \leq \frac{|x - c|}{\sqrt[3]{c^2}} < \frac{\epsilon\sqrt[3]{c^2}}{\sqrt[3]{c^2}} = \epsilon.$$

Exercise 4.3.2. (a) Let $\epsilon > 0$. Because g is continuous at $f(c) \in B$, for every $\epsilon > 0$, there exists an $\alpha > 0$ such that $|g(y) - g(f(c))| < \epsilon$ whenever y satisfies $|y - f(c)| < \alpha$. Now, because f is continuous at $c \in A$, for this value of α, we can find a $\delta > 0$ such that $|x - c| < \delta$ implies that $|f(x) - f(c)| < \delta$. Combining the two statements, we see that for $\epsilon > 0$, there exists $\delta > 0$ such that $|x - c| < \delta$ implies $|g(f(x)) - g(f(c))| < \epsilon$. Therefore, $g \circ f$ is continuous at c.

(b) Let's now prove Theorem 4.3.9 using the sequential characterization of continuity in Theorem 4.3.2 (iv).

Assume $(x_n) \to c$ (with $c \in A$). Our goal is to show $g(f(x_n)) \to g(f(c))$. Because f is continuous at c, we know $f(x_n) \to f(c)$. Then, because g is continuous at $f(c)$, we know that $g(f(x_n)) \to g(f(c))$. This completes the proof.

Exercise 4.3.3. Let $\epsilon > 0$. We need to argue that $|f(x) - f(c)|$ can be made less than ϵ for all values of x in some δ neighborhood around an arbitrary c. For the case where $a \neq 0$, write

$$|f(x) - f(c)| = |(ax + b) - (ac + b)| = |ax - ac| = |a||x - c|.$$

So if we pick $\delta = \epsilon/|a|$, then $|x - c| < \delta = \epsilon/|a|$ implies

$$|f(x) - f(c)| = |a||x - c| < |a|\frac{\epsilon}{|a|} = \epsilon.$$

Therefore, $f(x)$ is continuous.

If $a = 0$, then $|f(x) - f(c)| = 0$, and we may choose $\delta = 1$ regardless of how ϵ is chosen.

Exercise 4.3.4. (a) Let $\epsilon > 0$ and fix $n \in \mathbf{Z}$. If we set $\delta = 1$, then the point $x = n$ will be the only element of the domain that lies in the $V_\delta(n)$

neighborhood. It follows trivially that $f(x) \in V_\epsilon(f(n))$ for the point $x = n$, and we may conclude that f is continuous at n by Theorem 4.3.2 (iii).

(b) Let $\epsilon > 0$. If c is an isolated point of A, then there exists a neighborhood $V_\delta(c)$ that intersects the set A only at c. Because $x \in V_\delta(c) \cap A$ implies that $x = c$, we see $f(x) = f(c) \in V_\epsilon(f(c))$. Thus $f(x)$ is continuous at the isolated point c using the criterion in Theorem 4.3.2 (iii).

Exercise 4.3.5. Set $\epsilon_0 = |g(c)|$ which we are assuming to be greater than zero. Because g is continuous, we know there exists an open neighborhood $V_\delta(c)$ with the property that $g(x) \in V_{\epsilon_0}(g(c))$ provided $x \in V_\delta(c)$. But notice that $V_{\epsilon_0}(g(c))$ does not contain zero and so we can be sure that $g(x) \neq 0$ whenever $x \in V_\delta(c)$. This guarantees $f(x)/g(x)$ is defined on $V_\delta(c)$ as long as $x \in A$. (To properly answer this question as written we need the additional assumption that c be an interior point to the common domain A.)

Exercise 4.3.6. (a) We are asked to show Dirichlet's function $g(x)$ is nowhere-continuous on $\mathbf{R}$. First consider an arbitrary $r \in \mathbf{Q}$. Because $\mathbf{I}$ is dense in $\mathbf{R}$ there exists a sequence $(x_n) \subseteq \mathbf{I}$ with $(x_n) \to r$. Then, $g(x_n) = 0$ for all $n \in N$ while $g(r) = 1$. Since $\lim g(x_n) = 0 \neq g(r)$ we can use Corollary 4.3.3 to conclude $g(x)$ is not continuous at $r \in \mathbf{Q}$.

Now let's consider an arbitrary $i \in \mathbf{I}$. Because $\mathbf{Q}$ is dense in $\mathbf{R}$ we can find a sequence $(y_n) \subseteq \mathbf{Q}$ with $(y_n) \to i$. This time $g(y_n) = 1$ for all $n \in N$ while $g(i) = 0$. Because $\lim g(y_n) = 1 \neq g(i)$ we can conclude that g is not continuous at i. Combining the two results, we can conclude that Dirichlet's function is indeed nowhere continuous on $\mathbf{R}$.

(b) Consider an arbitrary rational number $r \in \mathbf{Q}$ and observe that $t(r) \neq 0$. Because $\mathbf{I}$ is dense, there exists a sequence $(x_n) \subseteq \mathbf{I}$ with $(x_n) \to r$. Then, $t(x_n) = 0$ for all $n \in N$ while $t(r) \neq 0$. Thus, $\lim t(x_n) \neq t(r)$ and $t(x)$ is not continuous at r.

(c) Consider an arbitrary $c \in \mathbf{I}$. Given $\epsilon > 0$, set $T = \{x \in R : t(x) \geq \epsilon\}$. If $x \in T$, then x is a rational number of the form $x = m/n$ with $m, n \in \mathbf{Z}$ where n satisfies $|n| \leq 1/\epsilon$. By focusing our attention on the interval $[c-1, c+1]$ around the point c, we see that the restriction on the size of n implies that the set $T \cap [c-1, c+1]$ is *finite*. In a finite set, all points are isolated so we can pick a neighborhood $V_\delta(c)$ around c such that all $x \in V_\delta(c)$ implies $x \notin T$. But if $x \notin T$ then $t(x) < \epsilon$ or $t(x) \in V_\epsilon(t(c))$. By Theorem 4.3.2 (iii), we conclude $t(x)$ is continuous at c.

Exercise 4.3.7. We will prove the set K is closed by showing that it contains all its limit points. Let c be a limit point of K. By Theorem 3.2.5 there is a sequence $(x_n) \subseteq K$ with $(x_n) \to c$. Because h is continuous on $\mathbf{R}$, $\lim h(x_n) = h(c)$. But notice $x_n \in K$, implies $h(x_n) = 0$, and thus $\lim h(x_n) = 0$. We conclude $h(c) = 0$, which implies $c \in K$, as desired.

Exercise 4.3.8. (a) Consider an arbitrary $c \in \mathbf{I}$. Because $\mathbf{Q}$ is dense in $\mathbf{R}$ we can find a sequence $(r_n) \subseteq \mathbf{Q}$ such that $(r_n) \to c$. Using the continuity of f, we see $\lim f(x_n) = f(c)$. But we are given that $x_n \subseteq \mathbf{Q}$ implies $f(x_n) = 0$, and so $f(c) = \lim f(x_n) = 0$.

(b) First define a new function $h(x) = f(x) - g(x)$. By Theorem 4.2.4, $h(x)$ is continuous. Because $f(r) = g(r)$ at every $r \in \mathbf{Q}$, we have $h(r) = 0$ on $\mathbf{Q}$ and part(a) implies $h(x) = 0$ on all of $\mathbf{R}$. This shows f and g are the same function. (The hypothesis that f and g are continuous was not included in early editions.)

Exercise 4.3.9. Geometrically speaking, the condition on f described in this problem says that if f is applied to any two points x and y, then the image values $f(x)$ and $f(y)$ are closer together (in a uniform way) than x and y. This is the reason for the term "contraction."

(a) Let $\epsilon > 0$ and fix $y \in \mathbf{R}$. To show f is continuous at y, choose $\delta = \epsilon/c$, and observe that $|x - y| < \delta = \epsilon/c$ implies

$$|f(x) - f(y)| \leq c|x - y| < c\left(\frac{\epsilon}{c}\right) = \epsilon.$$

Because y is arbitrary, $f(x)$ must be continuous on $\mathbf{R}$.

(b) Observe that for any fixed $n \in \mathbf{N}$,

$$|y_{m+1} - y_{m+2}| = |f(y_m) - f(y_{m+1})| \leq c|y_m - y_{m+1}|.$$

This idea can be extended inductively to conclude that

$$\begin{aligned} |y_{m+1} - y_{m+2}| &\leq c|y_m - y_{m+1}| \\ &\leq c^2|y_{m-1} - y_m| \\ &\leq \cdots \leq c^m|y_1 - y_2|. \end{aligned}$$

The fact that $0 < c < 1$ means $\sum_{n=1}^{\infty} c^n$ converges, and this will enable us to conclude that (y_n) is a Cauchy sequence. To see how, first note that for $m < n$ we have

$$\begin{aligned} |y_m - y_n| &\leq |y_m - y_{m+1}| + |y_{m+1} - y_{m+2}| + \cdots + |y_{n-1} - y_n| \\ &\leq c^{m-1}|y_1 - y_2| + c^m|y_1 - y_2| + \cdots + c^{n-2}|y_1 - y_2| \\ &= c^{m-1}|y_1 - y_2|(1 + c + \cdots + c^{n-m-1}) \\ &< c^{m-1}|y_1 - y_2|\left(\frac{1}{1-c}\right). \end{aligned}$$

Let $\epsilon > 0$, and choose $N \in \mathbf{N}$ large enough so that $c^{N-1} < \epsilon(1-c)/|y_1 - y_2|$. Then the previous calculation shows that $n > m \geq \mathbf{N}$ implies $|y_m - y_n| < \epsilon$. We conclude that (y_n) is Cauchy.

(c) Set $y = \lim y_n$. Because f is continuous, $f(y) = \lim f(y_n)$. But $f(y_n) = y_{n+1}$, and so $f(y) = \lim y_{n+1}$. Because $\lim y_{n+1} = \lim y_n = y$, it follows that $f(y) = y$ and y is a "fixed point."

(d) The argument in (b) and (c) applies to any sequence of iterates. Thus, given an arbitrary x, we may assert that $(x, f(x), f(f(x)), \ldots)$ converges to a limit x' and that x' is a fixed point of f. But y is also a fixed point and so

$$|f(x') - f(y)| = |x' - y|.$$

However,

$$|f(x') - f(y)| \leq c|x' - y|,$$

must also be true, and because $0 < c < 1$ we conclude that $x' = y$.

In summary, if f is a contraction on $\mathbf{R}$, then f has a unique fixed point, and every sequence of iterates converges to this unique point.

Exercise 4.3.10. (a) Note that $f(0) = f(0+0) = f(0) + f(0)$ which implies $f(0) = 0$. For any $x \in \mathbf{R}$, $f(0) = f(x - x) = f(x) + f(-x) = 0$. This implies $f(-x) = -f(x)$.

(b) Fix $c \in \mathbf{R}$ and let $(x_n) \to c$. To prove that f is continuous at c it is enough to show $\lim f(x_n) = f(c)$.

Now $(c - x_n) \to 0$. Because we are given that f is continuous at 0, it follows that

$$\lim f(x_n - c) = f(0) = 0.$$

Combining the additive condition on f with the Algebraic Limit Theorem then gives

$$0 = \lim f(c - x_n) = \lim(f(c) - f(x_n)) = f(c) - \lim f(x_n),$$

and we get $f(c) = \lim f(x_n)$ as desired.

(c) For any $n \in \mathbf{N}$,

$$f(n) = f(1 + 1 + \ldots + 1) = f(1) + f(1) + \ldots + f(1) = nf(1) = nk.$$

For $z \in \mathbf{Z}$, the case $z < 0$ is all that remains to do. In (a) we saw $f(-x) = -f(x)$. Observing that $z = -|z|$ and $|z| \in \mathbf{N}$, we can write

$$f(z) = f(-|z|) = -f(|z|) = -|z|k = zk. \tag{1}$$

Before taking on an arbitrary rational number, let's consider $1/n$ where $n \in \mathbf{N}$. In this case,

$$\begin{aligned} k = f(1) &= f\left(\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}\right) \\ &= nf\left(\frac{1}{n}\right), \end{aligned}$$

which gives $f(1/n) = k/n$. For $m, n \in \mathbf{N}$ we then get

$$\begin{aligned} f(m/n) &= f\left(\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}\right) \\ &= mf\left(\frac{1}{n}\right) = k(m/n). \end{aligned}$$

Finally, for any $r \in \mathbf{Q}$ satisfying $r < 0$, an argument similar to equation (1) above gives the result.

(d) Fix $x \in \mathbf{R}$. Because $\mathbf{Q}$ is dense in $\mathbf{R}$, there exists a sequence $(r_n) \subseteq \mathbf{Q}$ with $(r_n) \to x$. By our work in part (c) we know that $f(r_n) = kr_n$ for all n. Then, because f is continuous at x, we have

$$f(x) = \lim f(r_n) = \lim kr_n = kx.$$

This completes the proof.

Exercise 4.3.11. (a) The greatest integer function, $h(x) = [[x]]$ from Example 4.3.7 is a suitable example.

(b) Let

$$k(x) = \begin{cases} x(1-x) & \text{if } 0 < x < 1 \text{ with } x \in \mathbf{Q} \\ 0 & \text{if } 0 < x < 1 \text{ with } x \notin \mathbf{Q} \\ 0 & \text{if } x \geq 1 \text{ or } x \leq 0. \end{cases}$$

Because $x(1-x)$ tends to zero as x approaches 0 and 1, it is possible to show that k is continuous at these points.

(c) This time let

$$l(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \text{ with } x \in \mathbf{Q} \\ 0 & \text{if } 0 \leq x \leq 1 \text{ with } x \notin \mathbf{Q} \\ 0 & \text{if } x > 1 \text{ or } x > 0. \end{cases}$$

which fails to be continuous at 0 and 1, as requested.

(d) The function

$$g(x) = \begin{cases} 1/n & \text{if } x = 1/n \text{ for some } n \in \mathbf{N} \\ 0 & \text{otherwise} \end{cases}$$

is not continuous on A, but observe that it is continuous at 0. (Setting $g(x) = 1$ when $x \in A$ would not work, instance.)

Exercise 4.3.12. (a) Fix $c \in C$ so that $g(c) = 1$. The standard way to proceed is to find a sequence (x_n) in the complement of C with $(x_n) \to c$. Then $\lim g(x_n) \neq g(c)$ would show g is not continuous at c. Finding this sequence amounts to arguing that the Cantor set does not contain any intervals, and this is the content of Exercise 3.4.9.

A more concise approach might be the following. Let $\epsilon_0 = 1/2$. Then for every $\delta > 0$, the neighborhood $V_\delta(c)$ is *not* a subset of C (because C contains no intervals). Thus there exists a point $x \in V_\delta(c)$ with $x \notin C$, and consequently $g(x) = 0 \notin V_{\epsilon_0}(g(c))$. By the criterion in Theorem 4.3.2 (iii), g is not continuous at c.

(b) Now fix $c \notin C$, and let $\epsilon > 0$ be arbitrary. Because C is closed, C^c is open. This means that there exists a $\delta > 0$ with $V_\delta(c) \subseteq C^c$. Now, if we consider any $x \in V_\delta(c)$, then $x \in C^c$ implies $g(x) = 0$. Looking again at the criterion for continuity in Theorem 4.3.2 (iii), we see that $x \in V_\delta(c)$ implies $g(x) \in V_\epsilon(g(c))$, and thus $g(x)$ is continuous at every $c \notin C$.

4.4 Continuous Functions on Compact Sets

Exercise 4.4.1. (a) Fix $c \in \mathbf{R}$ and write

$$|f(x) - f(c)| = |x^3 - c^3| = |x - c||x^2 + xc + c^2|.$$

Insisting that $\delta \leq 1$ means that x will fall in the interval $(c-1, c+1)$ and thus

$$|x^2 + xc + c^2| < (c+1)^2 + (c+1)^2 + c^2 < 3(c+1)^2.$$

Now pick $\delta = \min\{1, \epsilon/(3(c+1)^2)\}$. Then $|x - c| < \delta$ implies

$$|f(x) - f(c)| < \left(\frac{\epsilon}{3(c+1)^2}\right) 3(c+1)^2 = \epsilon.$$

(b) The dependence of ϵ on the point c is evident in the previous formula with larger choices of c resulting in smaller values of δ. This means that the sequences (x_n) and (y_n) we seek are necessarily going to tend to infinity.

Set $x_n = n$ and $y_n = n + 1/n$. Then $|x_n - y_n| = 1/n$ tends to zero as required, while

$$|f(x_n) - f(y_n)| = |n^3 - \left(n + \frac{1}{n}\right)^3| = 3n + \frac{3}{n} + \frac{1}{n^3} \geq 3,$$

stays $e_0 = 3$ units apart for all $n \in \mathbf{N}$. This proves f is not uniformly continuous on $\mathbf{R}$.

(c) Let A be bounded by M. If $x, c \in A$ then $|x^2 + xc + c^2| \leq 3M^2$. Given $\epsilon > 0$ we can now choose $\delta = \epsilon/(3M^2)$, which is independent of c. If $|x - c| < \delta$, it follow that

$$|f(x) - f(c)| \leq \left(\frac{\epsilon}{3M^2}\right) 3M^2 = \epsilon,$$

and f is uniformly continuous on A.

Exercise 4.4.2. For $f(x) = 1/x^2$ we see

$$|f(x) - f(y)| = \left|\frac{1}{x^2} - \frac{1}{y^2}\right| = \left|\frac{y^2 - x^2}{x^2 y^2}\right| = |y - x| \left(\frac{y + x}{x^2 y^2}\right).$$

If we restrict our attention to $x, y \geq 1$, then we can estimate

$$\frac{y + x}{x^2 y^2} = \frac{1}{x^2 y} + \frac{1}{x y^2} \leq 1 + 1 = 2.$$

Given $\epsilon > 0$, we may then choose $\delta = \epsilon/2$ (independent of x and y), and it follows that $|f(x) - f(y)| < (\epsilon/2)2 = \epsilon$ whenever $|x - y| < \delta$. This shows f is uniformly continuous on $[1, \infty)$.

If x and y are allowed to be arbitrarily close to zero, then the expression $(x+y)/(x^2y^2)$ is unbounded and we get into trouble. To see this more explicitly, set $x_n = 1/\sqrt{n}$ and $y_n = 1/\sqrt{n+1}$. Then $|x_n - y_n| \to 0$ while

$$|f(x_n) - f(y_n)| = |n - (n+1)| = 1.$$

By the criterion in Theorem 4.4.6, we conclude that f is not uniformly continuous on $(0, 1]$.

Exercise 4.4.3. Because compactness is preserved by continuous functions, the set $f(K)$ is compact. By Exercise 3.3.1, $y_1 = \sup f(K)$ exists and $y_1 \in f(K)$. Because $y_1 \in f(K)$, there must exist (at least one point) $x_1 \in K$ satisfying $f(x_1) = y_1$, and it follows immediately from the definition of the supremum that $f(x) \leq f(x_1)$ for all $x \in K$.

A similar argument using the infimum yields x_0.

Exercise 4.4.4. Because $[a, b]$ is a compact set, it follows from the Extreme Value Theorem that f attains a minimum. That is, there exists a point $x_0 \in [a, b]$ where $f(x_0) \leq f(x)$ for all $x \in [a, b]$. Then, because $f(x_0) > 0$, we may write

$$\frac{1}{f(x)} \leq \frac{1}{f(x_0)},$$

and we see that $1/f$ is bounded.

Exercise 4.4.5. Negating the definition of uniform continuity gives the following: A function $f : A \to \mathbf{R}$ fails to be uniformly continuous on A if there exists $\epsilon_0 > 0$ such that for all $\delta > 0$ we can find two points x and y satisfying $|x - y| < \delta$ but with $|f(x) - f(y)| \geq \epsilon_0$.

The fact that no δ "works" means that if we were to try $\delta = 1$, we would be able to find points x_1 and y_1 where $|x_1 - y_1| < 1$ but $|f(x_1) - f(y_1)| \geq \epsilon_0$. In a similar way, if we try $\delta = 1/n$ where $n \in \mathbf{N}$, it follows that there exist points x_n and y_n with $|x_n - y_n| < 1/n$ but where $|f(x_n) - f(y_n)| \geq \epsilon_0$. The sequences (x_n) and (y_n) are precisely the ones described in Theorem 4.4.6.

Exercise 4.4.6. (a) Let $f(x) = 1/x$ and set $x_n = 1/n$. Then $f(x_n) = n$ which is not a Cauchy sequence.

(b) This is impossible. A Cauchy sequence (x_n) in $[0, 1]$ must have a limit in $[0, 1]$ because this is a closed set. If $x = \lim x_n$, then by continuity $f(x) = \lim f(x_n)$. Because $f(x_n)$ converges, it is a Cauchy sequence as well.

(c) This is also impossible for the same reasons as in (b). Note that we did not use the compactness of $[0, 1]$ but only the fact that it was closed.

(d) The function $f(x) = x(1 - x)$ has this property.

Exercise 4.4.7. Let $\epsilon > 0$ be arbitrary. Because f is uniformly continuous on $(a, b]$, there exists $\delta_1 > 0$ such that $|f(x) - f(y)| < \epsilon/2$ whenever $x, y \in (a, b]$ satisfy $|x - y| < \delta_1$. Likewise, there exists $\delta_2 > 0$ such that $|f(x) - f(y)| < \epsilon/2$ whenever $x, y \in [b, c)$ satisfy $|x - y| < \delta_2$.

Now set $\delta = \min\{\delta_1, \delta_2\}$ and assume we have x and y satisfying $|x-y| < \delta$. If both x and y fall in $(a, b]$, or if they both fall in $[b, c)$, then we get $|f(x)-f(y)| < \epsilon/2 < \epsilon$. In the case where $x < b$ and $y > b$ we may write

$$|f(x) - f(y)| \leq |f(x) - f(b)| + |f(b) - f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Because δ_1 and δ_2 are both independent of x and y, δ is as well and we conclude that f is uniformly continuous on (a, c).

Exercise 4.4.8. (a) We are given that f is uniformly continuous on $[b, \infty)$. The set $[0, b]$ is compact, and so by Theorem 4.4.8, f is also uniformly continuous on $[0, b]$. By argument precisely like the one in Exercise 4.4.7, we can show f is uniformly continuous on $[0, \infty)$.

(b) Let's first focus our attention on the domain $[1, \infty)$. If $x, y \geq 1$, it follows that

$$|\sqrt{x} - \sqrt{y}| = \left| \frac{x - y}{\sqrt{x} + \sqrt{y}} \right| \leq |x - y| \frac{1}{2}.$$

So, given $\epsilon > 0$ we can choose $\delta = 2\epsilon$, and it follows that $f(x) = \sqrt{x}$ is uniformly continuous on $[1, \infty)$. By the observation in part (a), we get that f is uniformly continuous on $[0, \infty)$.

Exercise 4.4.9. (a) First write the Lipschitz condition in the form

$$|f(x) - f(y)| \leq M|x - y| \quad \text{for all } x, y \in A.$$

Given $\epsilon > 0$, we choose $\delta = \epsilon / M$. Then $|x - y| < \delta$ implies

$$|f(x) - f(y)| < M \frac{\epsilon}{M} = \epsilon.$$

This proves f is uniformly continuous.

(b) No, all uniformly continuous functions are not Lipschitz. Consider $f(x) = \sqrt{x}$ on $[0, 1]$. A continuous function on a compact set is uniformly continuous. However, if we set $y = 0$ and consider $x > 0$, then we get

$$\left| \frac{f(x) - f(y)}{x - y} \right| = \left| \frac{\sqrt{x}}{x} \right| = \frac{1}{\sqrt{x}},$$

which is *not* bounded for x values arbitrarily close to zero.

Exercise 4.4.10. Yes, uniformly continuous functions map bounded sets to bounded sets.

Given $\epsilon_0 = 1$, there exists $\delta_0 > 0$ such that $|f(x) - f(y)| < 1$ as long as $|x - y| < \delta_0$. Now the fact that A is bounded means that we can find a *finite* collection of points $\{x_1, x_2, \ldots, x_n\}$ where the δ_0 neighborhoods

$$\{V_{\delta_0}(x_1), V_{\delta_0}(x_2), \ldots, V_{\delta_0}(x_n)\}$$

cover A. For each $0 \leq i \leq n$, the image set $f(V_{\delta_0}(x_i) \cap A)$ is bounded because $|f(x) - f(y)| \leq 1$ whenever $x, y \in V_{\delta_0}(x_i) \cap A$. Because $f(A)$ is covered by the finite collection of bounded sets $\{f(V_{\delta_0}(x_i) \cap A) : 0 \leq i \leq n\}$, it follows that $f(A)$ is bounded as well.

Exercise 4.4.11. ($\Rightarrow$) Assume g is continuous on $\mathbf{R}$ and let $O \subseteq \mathbf{R}$ be open. We want to prove $g^{-1}(O)$ is open. To do this, we fix $c \in g^{-1}(O)$ and show that there is a δ–neighborhood of c satisfying $V_\delta(c) \subseteq g^{-1}(O)$.

Because $c \in g^{-1}(O)$, we know $g(c) \in O$. Now O is open, so there exists an $\epsilon > 0$ such that $V_\epsilon(g(c)) \subseteq O$. Given this particular ϵ, the continuity of g at

c allows us to assert that there exists a neighborhood $V_\delta(c)$ with the property that $x \in V_\delta(c)$ implies $g(x) \in V_\epsilon(g(c)) \subseteq O$. But this implies $V_\delta(c) \subseteq g^{-1}(O)$, which proves that $g^{-1}(O)$ is open.

($\Leftarrow$) Conversely, we assume $g^{-1}(O)$ is open whenever O is open, and show that g is continuous at an arbitrary point $c \in \mathbf{R}$.

Let $\epsilon > 0$, and set $O = V_\epsilon(g(c))$. Certainly O is open, so our hypothesis gives us that $g^{-1}(O)$ is open. Because $c \in g^{-1}(O)$, there exists a $\delta > 0$ with $V_\delta(c) \subseteq g^{-1}(O)$. But this means that whenever $x \in V_\delta(c)$ we get $g(x) \in O = V_\epsilon(g(c))$, and we conclude that g is continuous at c by the criterion in Theorem 4.3.2 (iii).

Exercise 4.4.12. Assume f is continuous on a compact set K. We must show f is uniformly continuous.

Let $\epsilon > 0$. Then for each $x \in K$, the continuity of f tells us that there exists a $\delta_x > 0$ (depending on x) with the property that

$$|y - x| < \delta_x \quad \text{implies} \quad |f(y) - f(x)| < \epsilon/2.$$

Now consider the open cover of K consisting of the neighborhoods of the form

$$\{V_{\frac{1}{2}\delta_x}(x) : x \in K\}.$$

Because K is compact, there exists a finite subcover corresponding to a finite set of points $\{x_1, x_2, \ldots, x_n\}$ in K. That is,

$$K \subseteq V_{\frac{1}{2}\delta_{x_1}}(x_1) \cup V_{\frac{1}{2}\delta_{x_2}}(x_2) \cup \cdots \cup V_{\frac{1}{2}\delta_{x_n}}(x_n).$$

Because we have a finite cover, we may now let

$$\delta = \min\{\frac{1}{2}\delta_{x_1}, \frac{1}{2}\delta_{x_2}, \ldots, \frac{1}{2}\delta_{x_n}\},$$

and be confident that $\delta > 0$.

Now assume $|x - y| < \delta$. Because we have a cover for K, there must exist x_i for some $0 \leq i \leq n$ where $|x_i - x| < \frac{1}{2}\delta_{x_i} < \delta_{x_i}$. It follows that $|f(x) - f(x_i)| < \epsilon/2$. Also,

$$|y - x_i| \leq |y - x| + |x - x_i| < \delta + \frac{1}{2}\delta_{x_i} < \delta_{x_i},$$

and so we get $|f(y) - f(x_i)| < \epsilon/2$ as well. Finally,

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f(x_i)| + |f(x_i) - f(y)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Because δ is chosen independently of x, this shows f is uniformly continuous on K.

Exercise 4.4.13. (a) We want to show that $f(x_n)$ is a Cauchy sequence, so let $\epsilon > 0$ be arbitrary. Because f is uniformly continuous, there exists $\delta > 0$

such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$. Given this δ, we use the fact that (x_n) is a Cauchy sequence to say that there exists an $N \in \mathbf{N}$ such that $|x_n - y_n| < \delta$ whenever $m, n \geq N$. Combining the last two statements we see that $|f(x_n) - f(y_n)| < \epsilon$ whenever $m, n \geq N$, which shows that $f(x_n)$ is Cauchy.

(b) ($\Rightarrow \infty$) Let's first assume f is uniformly continuous on (a, b). Now fix a sequence (x_n) in (a, b) with $(x_n) \to a$. It follows from (a) that $g(x_n)$ converges, so let's *define* the value of $g(a)$ by asserting that $g(a) = \lim g(x_n)$.

Proving that g is continuous at a amounts to showing that if we now take an *arbitrary* sequence (y_n) that converges to a, then it follows that $g(a) = \lim g(y_n)$ as well. This is equivalent to showing that

$$\lim[g(y_n) - g(x_n)] = 0.$$

Given $\epsilon > 0$, there exists a $\delta > 0$ such that $|g(y) - g(x)| < \epsilon$ whenever $|x - y| < \delta$. Because (x_n) and (y_n) each converge to a, we see that $(y_n - x_n) \to 0$. Thus, there exists an $N \in N$ such that $|y_n - x_n| < \delta$ for all $n \geq N$. But this implies

$$|g(y_n) - g(x_n)| < \epsilon \quad \text{for all } n \geq N,$$

and we conclude $\lim[g(y_n) - g(x_n)] = 0$. Because this implies $g(a) = \lim g(y_n)$, we see that g is continuous at a.

A similar argument can be used for the point b.

($\Leftarrow$) Given that g can be continuously extended to the domain $[a, b]$, we immediately get that g is uniformly continuous because $[a, b]$ is a compact set. Thus g is certainly uniformly continuous on the smaller set (a, b).

4.5 The Intermediate Value Theorem

Exercise 4.5.1. The set $[a, b]$ is connected, and so by Theorem 4.5.2, the image set $f([a, b])$ is also connected. Because $f(a)$ and $f(b)$ are both elements of $f([a, b])$, we see that $L \in f([a, b])$ as well by Theorem 3.4.6. But this implies that there exists a point $c \in (a, b)$ satisfying $L = f(c)$, as desired.

Exercise 4.5.2. (a) False. The function $f(x) = 1/x$ takes the bounded interval $(0, 1)$ to the unbounded interval $(1, \infty)$.

(b) False. The function $f(x) = x(1 - x)$ takes the open interval $(0, 1)$ to the set $(0, 1/4]$, which is clearly not open.

(c) True. By the Preservation of Compactness result, a continuous function maps a bounded closed set (i.e., a compact set) to another compact set. Then, by the Preservation of Connectedness result we may conclude that this compact set is, in fact, an interval.

Exercise 4.5.3. No, because $\mathbf{Q}$ is not connected. If such a function were to contain 1 and 2 in its range, then by the Intermediate Value Theorem, its range would also have to contain $\sqrt{2}$ (and many other irrational points).

Exercise 4.5.4. Assume $f : [a, b] \to \infty$ is increasing and satisfies the intermediate value property stated in Definition 4.5.3. Let's fix $c \in (a, b)$ (the case where c is an endpoint is similar), and let $\epsilon > 0$. Our task is to produce a $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$.

We know $f(a) \leq f(c)$. If $f(c) - \epsilon/2 < f(a)$, then set $x_1 = a$. If $f(a) \leq f(c) - \epsilon/2$, then the intermediate value property for f implies that there exists $x_1 < c$ where $f(x_1) = f(c) - \epsilon/2$. Because f is increasing, we see that in either case $x \in (x_1, c]$ implies

$$f(c) - \frac{\epsilon}{2} = f(x_1) \leq f(x) \leq f(c).$$

We can follow a similar process on the other side to get that there exists a point $x_2 > c$ with the property that

$$f(c) \leq f(x) \leq f(x_2) = f(c) + \frac{\epsilon}{2},$$

whenever $x \in [c, x_2)$. Finally, we set $\delta = \min\{c - x_1, x_2 - c\}$, and it follows that

$$f(c) - \frac{\epsilon}{2} \leq f(x) \leq f(c) + \frac{\epsilon}{2} \quad \text{provided } |x - c| < \delta.$$

This completes the proof.

Exercise 4.5.5. Assume, for contradiction, that $f(c) > 0$. If we set $\epsilon_0 = f(c)$, then the continuity of f implies that there exists a $\delta_0 > 0$ with the property that $x \in V_{\delta_0}(c)$ implies $f(x) \in V_{\epsilon_0}(f(c))$. But this implies that $f(x) > 0$ and thus $x \notin K$ for all $x \in V_{\delta_0}(c)$. What this means is that if c is an upper bound on K, then $c - \delta_0$ is a smaller upper bound, violating the definition of the supremum. We conclude that $f(x) > 0$ is not allowed.

Now assume that $f(c) < 0$. This time, the continuity of f allows us to produce a neighborhood $V_{\delta_1}(c)$ where $x \in V_{\delta_1}(c)$ implies $f(x) < 0$. But this implies that a point such as $c + \delta_1/2$ is an element of K, violating the fact that c is an upper bound for K.

It follows that $f(c) < 0$ is also impossible, and we conclude that $f(c) = 0$ as desired.

This proves the Intermediate Value Theorem for the special case where $L = 0$. To prove the more general version, we consider the auxiliary function $h(x) = f(x) - L$ which is certainly continuous. From the special case just considered we know $h(c) = 0$ for some point $c \in (a, b)$ from which it follows that $f(c) = L$.

Exercise 4.5.6. By repeating the construction started in the text, we get a nested sequence of intervals $I_n = [a_n, b_n]$ where $f(a_n) < 0$ and $f(b_n) \geq 0$ for all $n \in \mathbf{N}$. By the Nested Interval Property, there exists a point $c \in \bigcap_{n=1}^{\infty} I_n$. The fact that the lengths of the intervals are tending to zero means that the two sequences (a_n) and (b_n) each converge to c.

Because f is continuous at c, we get $f(c) = \lim f(a_n)$ where $f(a_n) < 0$ for all n. Then the Order Limit Theorem implies $f(c) \leq 0$. Because we also have $f(c) = \lim f(b_n)$ with $f(b_n) \geq 0$, it must be that $f(c) \geq 0$. We conclude that $f(c) = 0$.

Exercise 4.5.7. The trick here is to apply the Intermediate Value Theorem to the function $g(x) = f(x) - x$. Because the range of f is contained in the interval $[0, 1]$ we see that

$$g(0) = f(0) \geq 0 \quad \text{and} \quad g(1) = f(1) - 1 \leq 0.$$

It follows from IVT that we must have $g(c) = 0$ for some point $c \in [0, 1]$, and this is equivalent to $f(c) = c$.

Exercise 4.5.8. No. Let (m_1, h_1) represent the position of the minute and hour hands respectively on "clock 1", where the variables take values in the interval $[0, 12]$ (with zero identified with 12). Let (m_2, h_2) be the same for "clock 2."

Assume that clock 1 is set at 12:00 and clock 2 is set at H:00, where $H \in \{1, 2, \ldots 11\}$. If we set $x = m_2 = h_1$, then we may consider $m_1 = m_1(x)$ to be a continuous function of x with $m_1(0) = 0$ and $m_1(1) = 12$. Likewise, $h_2 = h_2(x)$ is also a continuous function of x with $h_2(0) = H$ and $h_2(1) = H + (1/12)$. Now what happens to the function $d(x) = h_2(x) - m_1(x)$ as x ranges over the domain $[0, 1]$? Well, $d(0) = H > 0$ and $d(1) = H + (1/12) - 12 < 0$, and so by IVT there must exist a point $c \in (0, 1)$ where $d(c) = 0$. For this value of c, the two times corresponding to

$$m_1 = m_1(c),\ h_1 = c \quad \text{and} \quad m_2 = c,\ h_2 = h_2(c),$$

are indistinguishable if the hands on the two clocks are identical. This happens 11 times (once for each value of H) in the course of a twelve hour span of time.

(Note: Refinements in this solution have admittedly made the use of IVT a bit artificial in this problem. We could explicitly write $h_2(x) = H + (x/12)$ and $m_1(x) = 12x$, and then solve to get $c = (12H)/143$. As an example, let's set $H = 1$. Then for clock 1 we have (144/143,12/143) which is approximately 12:05:02, and for clock 2 we have (12/143,144/143) which is approximately 1:00:25.)

4.6 Sets of Discontinuity

Exercise 4.6.1. This problem is contained in Exercise 4.3.11

Exercise 4.6.2. We say that $\lim_{x \to c^-} f(x) = L$ if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < c - x < \delta$.

Exercise 4.6.3. ($\Rightarrow$) Let's assume $\lim_{x \to c} f(x) = L$. Then given $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$. This δ then satisfies the required condition to prove the existence of the left and right limits.

($\Leftarrow$) In the other direction, if we are given $\epsilon > 0$ then we know that there exists a $\delta_1 > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < x - c < \delta_1$. We also know there exists a $\delta_2 > 0$ such that $|f(x) - L| < \delta_2$ whenever $0 < c - x < \delta_2$. If we set $\delta = \min\{\delta_1, \delta_2\}$, then it follows that $|f(x) - l| < \epsilon$ for all $0 < |x - c| < \delta$. We conclude $\lim_{x \to c} f(x) = L$.

Exercise 4.6.4. This argument is very similar in spirit to the proof of the Monotone Convergence Theorem.

Given $c \in \mathbf{R}$, let's prove that $\lim_{x\to c^-} f(x)$ exists for an increasing function f. Our first task is to produce a candidate for the value of the limit. To this end, set

$$A = \{f(x) : x < c\}.$$

Because f is increasing, A is bounded above by $f(c)$. By AoC, we can set $L = \sup A$. The claim is that $\lim_{x\to c^-} f(x) = L$.

Let $\epsilon > 0$. By the least upper bound property of the supremum, we know that there exists an $x_0 < c$ satisfying

$$L - \epsilon < f(x_0) \leq L.$$

If we set $\delta = c - x_0$, then the fact that f is increasing implies that

$$L - \epsilon < f(x_0) \leq f(x) \leq L$$

whenever $0 < c - x < \delta$. This proves the claim.

For the right-hand limit we can fashion a similar argument to show that

$$\lim_{x\to c^+} f(x) = L',$$

where $L' = \inf\{f(x) : x > c\}$. A final consequence of this argument is that the value of the function at c must satisfy

$$L \leq f(c) \leq L'.$$

If $L = L'$ then f is continuous at c, and if $L < L'$ then we have a jump discontinuity. There are no other possibilities.

Exercise 4.6.5. Let c be a point of discontinuity for an increasing function f. If we set

$$\lim_{x\to c^-} f(x) = L_c \quad \text{and} \quad \lim_{x\to c^+} f(x) = L'_c,$$

then we know $L_c < L'_c$. Because $\mathbf{Q}$ is dense in $\mathbf{R}$, there exists a rational number r_c satisfying $L_c < r_c < L'_c$. It is also true that $c_1 < c_2$ implies $r_{c_1} < r_{c_2}$ which implies that the mapping $\phi(c) = r_c$ defined on the set of discontinuities of f must be 1–1. Because the range of ϕ is a subset of $\mathbf{Q}$, it follows that the set of discontinuities of f is either countable or finite.

Exercise 4.6.6. For Dirichlet's function we see $\mathbf{R}$ is closed.

For the modified Dirichlet function, we set $A_n = (-\infty, -1/n] \cup [1/n, \infty)$ which is closed for each $n \in \mathbf{N}$. Then $\mathbf{R}\backslash\{0\} = \bigcup_{n=1}^{\infty} A_n$ is an F_σ set.

For Thomae's function we observe that $\mathbf{Q}$ is the countable union of singleton sets, and a singleton set is closed.

For the interval $(0, 1]$ write $(0, 1] = \bigcup_{n=1}^{\infty}[1/n, 1]$.

Exercise 4.6.7. Before getting started on this proof, let's observe that the statement $c \in D_\alpha$ is equivalent to saying that *for all* $\delta > 0$ there exist points $y, z \in V_\delta(c)$ satisfying $|f(y) - f(z)| \geq \alpha$.

To prove D_α is closed we let c be a limit point of D_α and argue that $c \in D_\alpha$. So let $\delta > 0$ be arbitrary. Because c is a limit point, there must exist $x' \in D_\alpha$ satisfying $x' \in V_{\delta/2}(c)$. But this means that there exist points $y, z \in V_{\delta/2}(x')$ where $|f(y) - f(z)| \geq \alpha$. Because $V_{\delta/2}(x') \subseteq V_\delta(c)$, the points y, z provide us with exactly what we need to conclude that $c \in D_\alpha$.

(An alternate proof showing D_α^c is open is also a productive way to attack this problem.)

Exercise 4.6.8. Assume $\alpha_1 < \alpha_2$ and let $c \in D_{\alpha_2}$. Given $\delta > 0$, the statement $c \in D_{\alpha_2}$ implies that there exist $y, z \in V_\delta(c)$ satisfying

$$|f(y) - f(z)| \geq \alpha_2 > \alpha_1.$$

Thus $c \in D_{\alpha_1}$ as well.

Exercise 4.6.9. Assume f is continuous at x. Then given our fixed $\alpha > 0$, we know there exists a $\delta > 0$ such that

$$|f(y) - f(x)| < \frac{\alpha}{2} \quad \text{provided } y \in V_\delta(x).$$

Thus, if $y, z \in V_\delta(x)$ we then get

$$\begin{aligned} |f(y) - f(z)| &\leq |f(y) - f(x)| + |f(x) - f(z)| \\ &< \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha, \end{aligned}$$

and we conclude that f is α-continuous at x. The contrapositive of this conclusion is that if f is not α-continuous at x, then it certainly cannot be continuous at x. This is precisely what it means to say $D_\alpha \subseteq D_f$.

Exercise 4.6.10. Assume f is not continuous at x. Negating the ϵ–δ definition of continuity we get that there exists an $\epsilon_0 > 0$ with the property that for all $\delta > 0$ there exists a point $y \in V_\delta(x)$ where $|f(y) - f(x)| \geq \epsilon_0$. Noting simply that both $x, y \in V_\delta(x)$, we conclude that f is not α-continuous for $\alpha = \epsilon_0$ (or anything smaller.)

To prove $D_f = \bigcup_{n=1}^\infty D_{1/n}$ we argue for inclusion each way. If $x \in D_f$, then we have just shown that $x \in D_{\epsilon_0}$ for some $\epsilon_0 > 0$. Choosing $n_0 \in \mathbf{N}$ small enough so that $1/n_0 \leq \epsilon_0$, it follows that $x \in D_{1/n_0}$. This proves $D_f \subseteq \bigcup_{n=1}^\infty D_{1/n}$.

For the reverse inclusion we observe that Exercise 4.6.9 implies $D_{1/n} \subseteq D_f$ for all $n \in \mathbf{N}$, and the result follows.

Chapter 5

The Derivative

5.1 Discussion: Are Derivatives Continuous?

5.2 Derivatives and the Intermediate Value Property

Exercise 5.2.1. (i) First we rewrite the difference quotient as

$$\begin{aligned}\frac{(f+g)(x)-(f+g)(c)}{x-c} &= \frac{f(x)+g(x)-f(c)-g(c)}{x-c} \\ &= \frac{f(x)-f(c)}{x-c}+\frac{g(x)-g(c)}{x-c}.\end{aligned}$$

The fact that f and g are differentiable at c together with the functional-limit version of the Algebraic Limit Theorem (Theorem 4.2.4) justifies the conclusion

$$(f+g)'(c) = f'(c) + g'(c).$$

(ii) This time we rewrite the difference quotient as

$$\begin{aligned}\frac{(kf)(x)-(kf)(c)}{x-c} &= \frac{kf(x)-kf(c)}{x-c} \\ &= k\left(\frac{f(x)-f(c)}{x-c}\right)\end{aligned}$$

Because f is differentiable at c, it follows from the functional-limit version of the Algebraic Limit Theorem that

$$(kf)'(c) = kf'(c).$$

Exercise 5.2.2. (a) For $c \neq 0$, the derivative of f at c is given by the formula

$$f'(c) = \lim_{x\to c}\frac{1/x-1/c}{x-c} = \lim_{x\to c}\frac{(c-x)/xc}{x-c} = \lim_{x\to c}\frac{-1}{xc} = \frac{-1}{c^2}.$$

(b) To avoid confusion with the notation in Theorem 5.2.4, let's set $h(x) = 1/x$. By the Chain Rule,

$$\left(\frac{1}{g(x)}\right)' = (h \circ g)'(x) = \frac{-g'(x)}{[g(x)]^2}.$$

Then using the product rule (Theorem 5.2.4 (iii)), we have

$$\begin{aligned}\left(\frac{f}{g}\right)'(x) = [f(x)(h \circ g)(x)]' &= f'(x)(h \circ g)(x) + f(x)(h \circ g)'(x) \\ &= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{[g(x)]^2} \\ &= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}\end{aligned}$$

provided that $g(c) \neq 0$.

(c) Rewrite the difference quotient as

$$\begin{aligned}\frac{(f/g)(x) - (f/g)(c)}{x - c} &= \frac{1}{x-c}\left(\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}\right) \\ &= \frac{1}{x-c}\left(\frac{f(x)g(c) - f(c)g(x)}{g(x)g(c)}\right) \\ &= \frac{1}{x-c}\left(\frac{f(x)g(c) - f(c)g(c) + f(c)g(c) - f(c)g(x)}{g(x)g(c)}\right) \\ &= \frac{1}{g(x)g(c)}\left(g(c)\frac{f(x)-f(c)}{x-c} - f(c)\frac{g(x)-g(c)}{x-c}\right).\end{aligned}$$

Applying the Algebraic Limit Theorem for functional limits gives

$$\left(\frac{f}{g}\right)'(c) = \frac{1}{[g(c)]^2}\left(g(c)f'(c) - f(c)g'(c)\right),$$

which gives the result.

Exercise 5.2.3. Consider

$$h(x) = \begin{cases} x^2 & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

For points different from zero this function is not continuous and thus not differentiable either. At zero, we have

$$h'(0) = \lim_{x \to 0} \frac{h(x)}{x}.$$

Given $\epsilon > 0$, choose $\delta = \epsilon$. Because $|h(x)/x| \leq x$, we see that $|h(x)/x| < \epsilon$ whenever $0 < |x| < \delta$ and it follows that h is differentiable at zero with $h'(0) = 0$.

Exercise 5.2.4. (a) From the left side of zero we have $\lim_{x\to 0^-} f(x) = 0$, so we require that $\lim_{x\to 0^+} x^a = 0$ as well. This occurs if and only if $a > 0$.

(b) From (a) we know $f_a(0) = 0$. For $f_a'(0)$ we again begin by considering the limit from the left and see that

$$\lim_{x\to 0^-} \frac{f_a(x) - f_a(0)}{x - 0} = \lim_{x\to 0^-} \frac{0}{x} = 0.$$

Thus, we require that

$$\lim_{x\to 0^+} \frac{x^a}{x} = \lim_{x\to 0^+} x^{a-1} = 0$$

as well. This occurs if and only if $a > 1$. The derivative formula $(x^a)' = ax^{a-1}$ (which we have not justified for $a \notin \mathbf{N}$) shows that $f_a'(x)$ is continuous in this case.

(c) Because we continue to get zero on the left, for the second derivative to exist we must have

$$\lim_{x\to 0^+} \frac{(x^a)' - 0}{x - 0} = \lim_{x\to 0^+} \frac{ax^{a-1}}{x} = \lim_{x\to 0^+} ax^{a-2} = 0.$$

This occurs whenever $a > 2$.

Exercise 5.2.5. (a) With regards to the existence of $g_a'(x)$ at $x = 0$ we see that

$$g_a'(0) = \lim_{x\to 0} \frac{x^a \sin(1/x)}{x} = \lim_{x\to 0} x^{a-1} \sin(1/x) = 0,$$

as long as $a > 1$. For $x \neq 0$, $g_a'(x)$ always exists and using the standard rules of differentiation we get

$$g_a'(x) = -x^{a-2}\cos(1/x) + ax^{a-1}\sin(1/x).$$

Setting $1 < a < 2$ makes $x^{a-2}\cos(1/x)$ unbounded near zero and yields the desired function.

(b) For $g_a'(x)$ to be continuous we need

$$\lim_{x\to 0} g_a'(x) = g_a'(0) = 0$$

and, looking at the above expression for $g_a'(x)$, we see that this happens as long as $a > 2$. For the second derivative $g_a''(0)$ we consider the limit

$$\begin{aligned} g_a''(0) = \lim_{x\to 0} \frac{g_a'(x)}{x} &= \lim_{x\to 0} \left(\frac{1}{x}\right)\left(-x^{a-2}\cos(1/x) + ax^{a-1}\sin(1/x)\right) \\ &= \lim_{x\to 0} \left(-x^{a-3}\cos(1/x) + ax^{a-2}\sin(1/x)\right) \end{aligned}$$

which exists if and only if $a > 3$. Thus setting $2 < a \leq 3$ gives the desired function.

(c) From (b) we see that choosing $a > 3$ makes g'_a differentiable at zero. Away from zero we get

$$g''_a(x) = -x^{a-4}\sin(1/x) - (2a-2)x^{a-3}\cos(1/x) + a(a-1)x^{a-2}\sin(1/x),$$

which fails to be continuous at zero when $a \leq 4$. Setting $3 < a \leq 4$ gives the desired function.

Exercise 5.2.6. (a) First let's prove that there exists $x \in (a, b)$ where $g(x) < g(a)$. Let (x_n) be a sequence in (a, b) satisfying $(x_n) \to a$. Then we have

$$g'(a) = \lim_{n\to\infty} \frac{g(x_n) - g(a)}{x_n - a} < 0.$$

The denominator is always positive. If the numerator were always positive then the Order Limit Theorem would imply $g'(a) \geq 0$. Because we know this is not the case, we may conclude that the numerator is eventually negative and thus $g(x) < g(a)$ for some x near a.

The proof that there exists $y \in (a, b)$ where $g(y) < g(b)$ is similar.

(b) We must show that $g'(c) = 0$ for some $c \in (a, b)$. Because g is differentiable on the compact set $[a, b]$ it must also be continuous here, and so by Extreme Value Theorem (Theorem 4.4.3), g attains a minimum at a point $c \in [a, b]$. From our work in (a) we know that the minimum of g is neither $g(a)$ nor $g(b)$, and therefore $c \in (a, b)$. Finally, the Interior Extremum Theorem (Theorem 5.2.6) allows us to conclude $g'(c) = 0$.

To prove the general result stated in the theorem we just observe that $g'(c) = 0$ is equivalent to the conclusion $f'(c) = \alpha$.

Exercise 5.2.7. (a) A function $f : A \to \mathbf{R}$ is *uniformly differentiable* on A with derivative $f'(t)$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|x - t| < \delta$ implies

$$\left|\frac{f(x) - f(t)}{x - t} - f'(t)\right| < \epsilon.$$

(b) Consider $f(x) = x^2$, which has derivative $f'(x) = 2x$, and observe

$$\left|\frac{x^2 - t^2}{x - t} - 2t\right| = \left|\frac{(x - t)(x + t)}{x - t} - 2t\right| = |x - t|.$$

Given $\epsilon > 0$, we can choose $\delta = \epsilon$. Then $|x - t| < \delta = \epsilon$ implies

$$\left|\frac{x^2 - t^2}{x - t} - 2t\right| = |x - t| < \epsilon$$

as desired.

(c) Not necessarily. Consider $g_2(x)$ in Section 5.1. It is differentiable on $[0, 1]$, but not uniformly differentiable on $[0.1]$. Given a fixed $\epsilon > 0$, the value of the response δ gets progressively smaller as we try to compute $g'_2(t)$ at points closer and closer to zero. To see this explicitly, set $t_n = 1/(2n\pi)$ and $x_n = 0$. Then observe that $|x_n - t_n| \to 0$ while

$$\begin{aligned}\left|\frac{g_2(x_n) - g_2(t_n)}{x_n - t_n} - g_2'(t_n)\right| &= |t_n \sin(1/t_n) + \cos(1/t_n) - 2t_n \sin(1/t_n)| \\ &= |\cos(1/t_n) - t_n \sin(1/t_n)| = 1\end{aligned}$$

for all $n \in \mathbf{N}$. In the spirit of the criterion for non-uniform continuity described in Theorem 4.4.6, we see that $g_2(x)$ is not uniformly differentiable.

Exercise 5.2.8. (a) True. Although the derivative function need not be continuous, it does satisfy the intermediate value property. Thus, if the derivative of a function takes on two distinct values then it attains every value–rational and irrational–in between these two.

(b) False. Consider

$$f(x) = \begin{cases} x/2 + x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

At zero we can show that $f'(0) = 1/2$. Away from zero we get

$$f'(x) = 1/2 - \cos(1/x) + 2x \sin(1/x),$$

which takes on negative values in every δ-neighborhood of zero.

(c)True. Assume, for contradiction, that $L \neq f'(0)$ and choose $\epsilon_0 > 0$ so that $\epsilon_0 < |f'(0) - L|$. From the hypothesis that $\lim_{x\to 0} f'(x) = L$ we know there exists a $\delta > 0$ such that $0 < |x| < \delta$ implies that $|f'(x) - L| < \epsilon_0$. Now our choice of ϵ_0 guarantees that there exists a point α between $f'(0)$ and L but outside $V_{\epsilon_0}(L)$. However, by Darboux's Theorem, there exists a point $x \in V_\delta(0)$ such that $f'(x) = \alpha$. This suggests that $\alpha \in V_{\epsilon_0}(L)$, which is a contradiction. Therefore $L = f'(0)$.

(d) True. More to come...

5.3 The Mean Value Theorem

Exercise 5.3.1. Because f' is continuous on the compact set $[a, b]$, we know that it is bounded. Thus, there exists $M > 0$ such that $|f'(x)| \leq M$ for all $x \in [a, b]$.

Now, given $x < y$ in the interval $[a, b]$, the Mean Value Theorem says that there exists a point $c \in (a, b)$ for which

$$\frac{f(x) - f(y)}{x - y} = f'(c).$$

Because $|f'(c)| \leq M$ (regardless of the value of c), it follows that

$$\left|\frac{f(x) - f(y)}{x - y}\right| \leq M.$$

Exercise 5.3.2. Because f' is continuous on a compact set—let's call it I—the Extreme Value Theorem can be used to conclude that $f'(c)$ attains a maximum and a minimum value on I. Thinking in terms of absolute value, this means that there exists a point $x_0 \in I$ where $|f'(x)| \leq |f'(x_0)|$ for all $x \in I$. Setting $s = |f'(x_0)|$, we see from our hypothesis that $0 \leq s < 1$.

Now, given $x < y$ in I, the Mean Value Theorem tells us that there exists a point $c \in I$ where

$$\left|\frac{f(x) - f(y)}{x - y}\right| = |f'(c)| \leq |f'(x_0)| = s.$$

It follows that

$$|f(x) - f(y)| \leq s|x - y|,$$

and f is contractive on I.

Exercise 5.3.3. (a) Set $g(x) = x - h(x)$. Because $g(1) = -1$ and $g(3) = 1$, by the Intermediate Value Theorem (Theorem 4.5.1), there must exist a $d \in [0, 3]$ where $g(d) = 0$. In terms of h, we note that this implies $h(d) = d$, as desired.

(b) Applying the Mean Value Theorem to h on the interval $[0, 3]$ implies that there exists a point $c \in (0, 3)$ where

$$h'(c) = \frac{h(3) - h(0)}{3 - 0} = \frac{2 - 1}{3} = \frac{1}{3}.$$

(c) Applying Rolle's Theorem to h on the interval $[1, 3]$, we see that there must exist a point $a' \in (1, 3)$ where $h'(a) = 0$. In (b), we found a point where $h'(c) = 1/3$. Because $1/4$ falls between 0 and $1/3$, we can appeal to Darboux's Theorem to assert that $h'(x) = 1/4$ at some point between c and a.

Exercise 5.3.4. (a) Let

$$h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x).$$

From the many "algebraic limit" theorems we know that h is continuous on $[a, b]$ and differentiable on (a, b). We also have $h(a) = g(a)f(b) - f(a)g(b) = h(b)$. Thus by Rolle's Theorem, there exists a $c \in (a, b)$ where $h'(c) = 0$. Because

$$h'(x) = [f(b) - f(a)]g'(x) - [g(b) - g(a)]f'(x),$$

we see that

$$[f(b) - f(a)]g'(c) - [g(b) - g(a)]f'(c) = 0,$$

and the result follows.

(b) Set $x = g(t)$ and $y = f(t)$ and consider the parametric curve in the x–y plane drawn as t ranges over the interval $[a, b]$. The quantity $(f(b)-f(a))/(g(b)-g(a))$ corresponds to the slope of the segment joining the endpoints of this curve, while $f'(c)/g'(c)$ gives the slope of the line tangent to the curve at the point $(g(c), f(c))$. In this context, the Generalized Mean Value Theorem says that if g' is never zero, then at some point along the parametric curve, the tangent line must be parallel to the segment joining the two endpoints.

Exercise 5.3.5. Assume, for contradiction, that f has two distinct fixed points x_1 and x_2. Noting that $f(x_1) = x_1$ and $f(x_2) = x_2$, the Mean Value Theorem implies that there exists c where

$$f'(c) = \frac{f(x_1) - f(x_2)}{x_1 - x_2} = \frac{x_1 - x_2}{x_1 - x_2} = 1.$$

Because this is impossible, we conclude that f can have at most one fixed point.

Exercise 5.3.6. ($\Rightarrow$) First let's show that if $g(d) = d$ for some $d \in (0,1)$, then $g'(1) > 1$. Applying the Mean Value Theorem to g on $[d,1]$, we can see that

$$g'(c) = \frac{g(1) - g(d)}{1-d} = \frac{1-d}{1-d} = 1$$

for some $c \in (d,1)$. Now we apply the Mean Value Theorem to g' on $[c,1]$ to assert that

$$g''(a) = \frac{g'(1) - g'(c)}{1-c}$$

for some $a \in (c,1)$. Because $g''(a) > 0$, the numerator in the previous expression must be strictly positive and it follows that $g'(1) > g'(c) = 1$.

($\Leftarrow$) Now let's show that if $g'(1) > 1$, then $g(d) = d$ for some $d \in (0,1)$. As we often do in arguments about fixed points, define the auxiliary function $f(x) = g(x) - x$ and observe that $f(0) = g(0) > 0$. If we could find a point $x \in (0,1)$ where $f(x) < 0$, then we could use the Intermediate Value Theorem to conclude $f(d) = 0$ for some d.

At $x = 1$ we have $f(1) = g(1) - 1 = 0$ and $f'(1) = g'(1) - 1 > 0$. If $(x_n) \subseteq (0,1)$ satisfies $(x_n) \to 1$, then

$$f'(1) = \lim_{n\to\infty} \frac{f(1) - f(x_n)}{1 - x_n} = \lim_{n\to\infty} \frac{-f(x_n)}{1-x_n} > 0.$$

If $f(x_n) \geq 0$ for all $n \in \mathbf{N}$, then the Order Limit Theorem would imply $f'(1) \leq 0$. Because this is not the case, it follows that $f(x_n) < 0$ must be true for some values of n. Because $f(0) > 0$, we know that there must exist a point d where $f(d) = 0$. Finally, this implies $g(d) = d$.

Exercise 5.3.7. (a) ($\Rightarrow$) If f is increasing on (a,b), then

$$\frac{f(x) - f(c)}{x - c} \geq 0$$

for every $x, c \in (a,b)$. It follows from the Order Limit Theorem (or an analogous version for functional limits) that

$$f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c} \geq 0$$

for all $c \in (a,b)$.

($\Leftarrow$) For the other direction we use the Mean Value Theorem. Here we are assuming $f'(x) \geq 0$ on (a, b) and we are asked to prove that f is increasing. Given $x < y$, it follows from MVT that

$$f'(c) = \frac{f(y) - f(x)}{y - x}$$

for some point $c \in (a, b)$. Because $f'(c) \geq 0$ and $y - x > 0$, we conclude that $f(x) \leq f(y)$ and f is increasing.

(b) First observe that

$$g'(0) = \lim_{x \to 0} \frac{x/2 + x^2 \sin(1/x)}{x} = \lim_{x \to 0} \frac{1}{2} + x \sin(1/x) = \frac{1}{2}.$$

Thus the derivative is strictly positive at zero. Away from zero, however, we get

$$g'(x) = 1/2 - \cos(1/x) + 2x \sin(1/x).$$

If we set $x_n = 1/(2n\pi)$, then $g'(x_n) < 0$ for all $n \in \mathbf{N}$. Because $(x_n) \to 0$, there is no neighborhood around zero in which $g'(x) \geq 0$, and so by part (a), the function is not increasing in any neighborhood of zero.

The moral here is that knowing that the derivative is positive at a *point* does not imply that the function is increasing near this point.

Exercise 5.3.8. Let's consider the case where $L = g'(c) > 0$. Set $\epsilon_0 = L$. Because

$$L = \lim_{x \to c} \frac{g(x) - g(c)}{x - c},$$

there exists a neighborhood $V_\delta(c)$ with the property that

$$\frac{g(x) - g(c)}{x - c} \in V_{\epsilon_0}(L)$$

whenever $x \in V_\delta(c)$. But notice that $V_{\epsilon_0}(L)$ contains only positive numbers. This means that if $x > c$ then $g(x) > g(c)$, and if $x < c$ then $g(x) < g(c)$.

Exercise 5.3.7 reminds us that a positive derivative at a single point does not imply that the function is increasing in a neighborhood of this point. What this exercise shows is that then we can say something weaker. If $g'(c) > 0$ then it does follow that $x > c$ implies $g(x) > g(c)$ and $x < c$ implies $g(x) < g(c)$. Very roughly speaking, we might say that "g is increasing at the point c."

Exercise 5.3.9. Let $M > 0$ be arbitrary. We need to produce a δ such that $0 < |x - c| < \delta$ implies that $|f(x)/g(x)| \geq M$. Choose δ_1 so that $0 < |x - c| < \delta_1$ implies $|f(x) - L| < |L|/2$. This guarantees that $f(x)$ is not too close to zero, and in particular we have $|f(x)| \geq |L|/2$. Because $\lim_{x \to c} g(x) = 0$, we can choose δ_2 such that $|g(x)| < |L|/2M$ provided $0 < |x - c| < \delta_2$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Then we have

$$\left| \frac{f(x)}{g(x)} \right| \geq \frac{|L|/2}{|L|/2M} = M$$

whenever $0 < |x - c| < \delta$, and the result is proved.

Exercise 5.3.10. The fact that f is bounded means that there exists $M > 0$ satisfying $|f(x)| \leq M$ for all x in the domain.

Let $\epsilon > 0$. Because $\lim_{x\to c} g(x) = \infty$, there exists a $\delta > 0$ such that $0 < |x - c| < \delta$ implies $|g(x)| \geq M/\epsilon$. It then follows that

$$\left|\frac{f(x)}{g(x)}\right| < \frac{M}{M/\epsilon} = \epsilon,$$

provided $0 < |x - c| < \delta$, and the proof is complete.

Exercise 5.3.11. Let $\epsilon > 0$. Because $L = \lim_{x\to a} f'(x)/g'(x)$, we know that there exists a $\delta > 0$ such that

$$\left|\frac{f'(t)}{g'(t)} - L\right| < \epsilon \quad \text{provided } 0 < |t - a| < \delta.$$

This δ is going to suffice to prove $L = \lim_{x\to a} f(x)/g(x)$ as well. To see why, pick $x \in V_\delta(a)$ with $a < x$ (the case $x < a$ is similar) and apply GMVT to f and g on the interval $[a, x]$. In this case we get a point $c \in (a, x)$ where

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f(x)}{g(x)}.$$

Because c must satisfy $0 < |c - a| < \delta$, it follows that

$$\left|\frac{f(x)}{g(x)} - L\right| = \left|\frac{f'(c)}{g'(c)} - L\right| < \epsilon$$

whenever $0 < |x - a| < \delta$. This completes the proof.

Exercise 5.3.12. For all $x \neq a$ we can write

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{(f(x) - f(a))/(x - a)}{(g(x) - g(a))/(x - a)}.$$

Because f and g are differentiable at a, we may use the Algebraic Limit Theorem to conclude

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Finally, the continuity of f' and g' at a implies

$$L = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)},$$

and the result follows.

(Note that this argument also assumes $g'(a) \neq 0$.)

Exercise 5.3.13. Because f and g are continuous on the interval containing a, we can conclude that $f(a) = \lim_{x\to a} f(x) = 0$ and $g(a) = \lim_{x\to a} g(x) = 0$. Now we have the same hypothesis as Theorem 5.3.6, and the rest of the proof will be the same.

5.4 A Continuous Nowhere-Differentiable Function

Exercise 5.4.1. The graph of $h_1(x)$ is similar to the sawtooth function $h(x)$ except that the maximum height is now $1/2$ and the length of the period is 1. For each n, the maximum height of $h_n(x)$ is $1/2^n$ and the period is $1/2^{n-1}$. Note that the slopes of the segments that make up $h_n(x)$ continue to be ± 1 for all values of n.

Exercise 5.4.2. The key observation is that $h_(x) \leq 1$ so that for every n we have

$$0 \leq \frac{1}{2^n} h_n(2^n x) \leq \frac{1}{2^n}.$$

Because the geometric series $\sum_{n=0}^{\infty} 1/2^n$ converges, the Comparison Test implies that our series for $g(x)$ converges for every choice of x. Because all the terms are positive, the convergence is absolute.

Exercise 5.4.3. For each n, the linear function $l(x) = 2^n x$ is certainly continuous. Then the Composition of Continuous Functions Theorem (Theorem 4.3.9) implies $h(2^n x)$ is continuous. The Algebraic Continuity Theorem (Theorem 4.3.4) part (i) implies $\frac{1}{2^n} h(2^n x)$ is continuous. Finally, part (ii) of the same theorem says

$$g_m(x) = h(x) + \frac{1}{2} h(2x) + \cdots + \frac{1}{2^m} h(2^m x)$$

is continuous as long as the sum is finite.

Exercise 5.4.4. For $g'(0)$ to exist, the sequential criterion for limits requires that

$$g'(0) = \lim_{m \to \infty} \frac{g(x_m) - g(0)}{x_m - 0}$$

exist for any sequence $(x_m) \to 0$. Fix $m \in \mathbf{N}$ and consider $x_m = 1/2^m$. Then

$$g(x_m) = \sum_{n=0}^{\infty} \frac{1}{2^n} h(2^{n-m}).$$

If $n > m$ then $h(2^{n-m}) = 0$ because the sawtooth function is zero at any multiple of 2. If $n \leq m$ then we are on the part of the graph where $h(x) = x$ and we get

$$\frac{1}{2^n} h(2^{n-m}) = \frac{1}{2^n} 2^{n-m} = \frac{1}{2_m}.$$

It follows that $g(x_m)$ can be represented with the finite sum

$$g(x_m) = \sum_{n=0}^{m} \frac{1}{2^m}.$$

Turning our attention to the difference quotient, we get

$$\frac{g(x_m) - g(0)}{x_m - 0} = \frac{\sum_{n=0}^{m} 1/2^m}{1/2^m} = \sum_{n=0}^{m} 1 = m + 1.$$

Because this quantity increases without bound, it is impossible for $\lim_{m\to\infty} g(x_m)/x_m$ to exist. It follows that g is not differentiable at zero.

Exercise 5.4.5. (a) To show that $g'(1)$ does not exist we continue to let $x_m = 1/2^m$ and consider

$$g(1 + x_m) = \sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n(1 + 1/2^m)) = \sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n + 2^{n-m}).$$

If $n > m$ then, as before,

$$\frac{1}{2^n} h(2^n + 2^{n-m}) = 0.$$

If $1 \leq n \leq m$, then

$$\frac{1}{2^n} h(2^n + 2^{n-m}) = \frac{1}{2^n} h(2^{n-m}) = \frac{1}{2^n}\, 2^{n-m} = \frac{1}{2^m}.$$

If $n = 0$, then

$$\frac{1}{2^n} h(2^n + 2^{n-m}) = h(1 + 1/2^m) = h(1) - 1/2^m = g(1) - 1/2^m.$$

If we write down the difference quotient for the interval $[1, x_m]$ we get

$$\begin{aligned} \frac{g(1 + x_m) - g(1)}{x_m} &= \frac{\sum_{n=0}^{m} 1/2^n h(2^n + 2^{n-m}) - g(1)}{1/2^m} \\ &= \frac{[\sum_{n=1}^{m} 1/2^m] + (g(1) - 1/2^m) - g(1)}{1/2^m} = m - 1. \end{aligned}$$

Because this is (again) unbounded as $m \to \infty$, it must be that $g'(1)$ does not exist.

(b) Now let $x = p/2^k$ and consider

$$g(x + x_m) = \sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n(x + 1/2^m)) = \sum_{n=0}^{\infty} \frac{1}{2^n} h(p2^{n-k} + 2^{n-m}).$$

Because we are ultimately interested in what happens as $m \to \infty$, let's compute $g(x + x_m)$ assuming $m > k$.

If $n > m$ then because we are at a multiple of 2 on the graph of $h(x)$ it follows that

$$\frac{1}{2^n} h(p2^{n-k} + 2^{n-m}) = 0.$$

If $k < n \leq m$, then the periodicity of h allows us to write

$$\frac{1}{2^n}h(p2^{n-k} + 2^{n-m}) = \frac{1}{2^n}h(2^{n-m}) = \frac{1}{2^n}\,2^{n-m} = \frac{1}{2^m}.$$

Finally, if $0 \leq n \leq k$, then $2^n x$ and 2^{n-m} fall on the same linear segment of $h(x)$ and we get

$$\frac{1}{2^n}h(p2^{n-k} + 2^{n-m}) = \frac{1}{2^n}\left[h(p2^{n-k}) \pm 2^{n-m}\right] = \frac{1}{2^n}h(2^n x) \pm 1/2^m,$$

where the choice of $+$ or $-$ depends on the value of p. Observing that $g(x) = \sum_{n=0}^{k}(1/2^n)h(2^n x)$, it follows that

$$\begin{aligned}
\frac{g(x+x_m) - g(x)}{x_m} &= \frac{\sum_{n=0}^{m} 1/2^n h(p2^{n-k} + 2^{n-m}) - g(x)}{1/2^m} \\
&= \frac{\left[\sum_{n=k+1}^{m} 1/2^m\right] + \left[\sum_{n=0}^{k}(1/2^n)h(2^n x) \pm 1/2^m\right] - g(x)}{1/2^m} \\
&= (m-k-1) + \sum_{n=0}^{k} \pm 1 \geq m - 2k - 1.
\end{aligned}$$

Because this is unbounded as $m \to \infty$, it must be that $g'(x)$ does not exist for any dyadic rational point on the graph. The fact that we get ∞ from the right for all of these limits is reflected in the graph of g by the downward cusps that appear at every dyadic rational point.

Exercise 5.4.6. (i) Because each h_i is differentiable at all nondyadic points, Theorem 5.2.4 implies that the finite sum g_m is differentiable at nondyadic points as well. This same theorem also allows us to say

$$|g'_{m+1}(x) - g'_m(x)| = |h'_{m+1}(x)|$$

and $h'_{m+1}(x) = \pm 1$ because it is a piecewise linear function consisting of segments of slope $\pm$ 1.

(b) The partial sum g_m is a piecewise linear function and $g'_m(x)$ is the slope of the piece containing the nondyadic point $x \in [x_m, y_m]$. The first important observation is that because $h_n(x_m) = h_n(y_m) = 0$ for all $n > m$, it follows that $g_m(x_m) = g(x_m)$ and $g_m(y_m) = g(y_m)$. Focusing on the graphs over the interval $[x_m, y_m]$, what we see is that g_m is the line segment connecting the points $(x_m, g(x_m))$ and $(y_m, g(y_m)$ and thus

$$g'_m(x) = \frac{g(y_m) - g(x_m)}{y_m - x_m}.$$

The other important observation is that because $h_n(x) \geq 0$ for all n, we get that $g(x) > g_m(x)$. Put another way, the segment of g_m over the interval $[x_m, y_m]$ lies *under* the graph of g (and is equal to g at the endpoints). It follows that

$$\frac{g(y_m) - g(x)}{y_m - x} < \frac{g(y_m) - g(x_m)}{y_m - x_m} < \frac{g(x) - g(x_m)}{x - x_m},$$

and the result follows.

(c) If $g'(x)$ did exist, then the sequential criterion for functional limits would imply that

$$g'(x) = \lim_{m\to\infty} \frac{g(x_m) - g(x)}{x_m - x} = \lim_{m\to\infty} \frac{g(y_m) - g(x)}{y_m - x}.$$

Then we could use a squeeze theorem argument to conclude that

$$g'(x) = \lim_{m\to\infty} g'_m(x).$$

The problem is that $\lim_{m\to\infty} g'_m(x)$ does *not* exist. From our work in (a) we see that $g'_m(x)$ is not a Cauchy sequence and so it cannot converge. We conclude that $g'(x)$ does not exist.

Exercise 5.4.7. If we set $g(x) = \sum_{n=0}^{\infty}(1/2^n)h(3^n x)$, then we have

$$|g'_{m+1}(x) - g'_m(x)| = |h'_{m+1}(x)| = (3/2)^{m+1}.$$

Because this does not tend to zero, the sequence $g'_m(x)$ again fails to be a Cauchy sequence and we can conclude that $g'(x)$ does not exist. To set up a parallel with Hardy's result, set $a = 1/2$ and $b = 3$ and notice that $ab = 3/2 \geq 1$.

What happens when $ab < 1$? Letting $a = 1/3$ and $b = 2$ corresponds to the function $g(x) = \sum_{n=0}^{\infty}(1/3^n)h(2^n x)$. In this case we have

$$|g'_{m+1}(x) - g'_m(x)| = |h'_{m+1}(x)| = (2/3)^{m+1},$$

which *does* tend to zero as $m \to \infty$. Thus, our argument no longer works and, in fact, it turns out that $g(x)$ is differentiable at every nondyadic point in its domain (see Theorem 6.4.3.)

Chapter 6

Sequences and Series of Functions

6.1 Discussion: Branching Processes

6.2 Uniform Convergence of a Sequence of Functions

Exercise 6.2.1. (a) By dividing the numerator and denominator by n, we can compute

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1+nx^2} = \lim_{n\to\infty} \frac{x}{1/n + x^2} = \frac{1}{x}.$$

Therefore, the pointwise limit of $f_n(x)$ is $f(x) = 1/x$.

(b) The convergence of $(f_n(x))$ is not uniform on $(0, \infty)$. To see this write

$$|f_n(x) - f(x)| = \left| \frac{nx}{1+nx^2} - \frac{1}{x} \right| = \frac{1}{x + nx^3}.$$

In order to make $|f_n(x) - f(x)| < \epsilon$ we must choose

$$N \geq \frac{1 - \epsilon x}{\epsilon x^3}.$$

For a fixed $\epsilon > 0$, the expression $(1 - \epsilon x)/(\epsilon x^3)$ grows without bound as x tends to zero, and thus there is no way to pick a value of N that will work for every value of x in $(0, \infty)$.

(c) The convergence is not uniform on $(0, 1)$ either. As seen in (b), the problem arises when x tends to zero and this is equally relevant over the domain $(0, 1)$.

(d) The convergence is uniform on the interval $(1, \infty)$. If $x > 1$ then it follows that

$$|f_n(x) - f(x)| = \frac{1}{x + nx^3} \leq \frac{1}{1+n}.$$

Given $\epsilon > 0$, choose N large enough so that $1/(1+n) < \epsilon$ whenever $n \geq N$. It follows that $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$ and thus $(f_n) \to f$ uniformly on $(1, \infty)$.

Exercise 6.2.2. To compute the pointwise limit write

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \left(\frac{x}{2} + \frac{1}{2n}\sin(nx) \right) = \frac{x}{2}.$$

Setting $g(x) = x/2$, we see that

$$|g_n(x) - g(x)| = \left| \frac{1}{2n}\sin(nx) \right| \leq \frac{1}{2n}.$$

Given $\epsilon > 0$, choose $N > 1/(2\epsilon)$ and observe that this is independent of x. Then $n \geq N$ implies

$$|g_n(x) - g(x)| \leq \frac{1}{2n} < \epsilon \qquad \text{for all } n \geq N.$$

It follows that $g_n \to g$ uniformly on $\mathbf{R}$, and thus on any subset of $\mathbf{R}$ as well.

Exercise 6.2.3. (a) The pointwise limit of (h_n) on $[0, \infty)$ is

$$h(x) = \lim_{n\to\infty} \frac{x}{1+x^n} = \begin{cases} x & \text{if } 0 \leq x < 1 \\ 1/2 & \text{if } x = 1 \\ 0 & \text{if } x > 1 \end{cases}$$

(b) Theorem 6.2.6 tells us that if the convergence were uniform then $h(x)$ would be continuous. However, $h(x)$ is not continuous at $x = 1$ and so the convergence cannot be uniform on any domain containing this point. In fact, the convergence is not uniform over any domain that has $x = 1$ as a limit point.

(c) Consider the set $[2, \infty)$. If $x \geq 2$ then

$$|h_n(x) - h(x)| = \left| \frac{x}{1+x^n} - 0 \right| < \frac{x}{x^n} \leq \frac{1}{2^{n-1}}.$$

Given $\epsilon > 0$, pick N so that $n \geq N$ implies $1/2^{n-1} < \epsilon$. Then $|h_n(x) - h(x)| < \epsilon$ for all $n \geq N$, and we conclude that $h_n \to h$ uniformly on $[2, \infty)$.

Exercise 6.2.4. Taking the derivative we find

$$f_n'(x) = \frac{1 - x^2 n}{(x^2 n + 1)^2}$$

which yields critical points $\pm 1/\sqrt{n}$. Using the standard techniques from calculus we can determine that the maximum of f occurs at $1/\sqrt{n}$ and the minimum at $-1/\sqrt{n}$. Because $f_n(1/\sqrt{n}) = |f_n(-1/\sqrt{n})| = 1/(2\sqrt{n})$, we see that

$$|f_n(x)| \leq \frac{1}{2\sqrt{n}} \qquad \text{for all } x \in \mathbf{R}.$$

To show $(f_n) \to 0$ uniformly on $\mathbf{R}$, we let $\epsilon > 0$ and choose N large enough so that $n \geq N$ implies $1/(2\sqrt{n}) < \epsilon$. It follows that $|f_n(x) - 0| < \epsilon$ whenever $n \geq N$, as desired.

Exercise 6.2.5. (a) Taking the limit for each fixed value of x we find that $f_n(x)$ converges pointwise to

$$f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Each of the functions f_n is continuous, but the limit function f is not. Therefore, Theorem 6.2.6 tells us that the convergence cannot be uniform.

(b) We can imitate the construction in (a) but use an unbounded function like $1/x^2$ in place of the constant function 1. Specifically, let

$$f_n(x) = \begin{cases} 1/x^2 & \text{if } |x| \geq 1/n \\ n^3|x| & \text{if } |x| < 1/n \end{cases}$$

Then each f_n is continuous and the pointwise limit is $f(x) = \lim f_n(x) = 1/x^2$, except at zero where we get a limit of $f(0) = 0$.

Exercise 6.2.6. ($\Rightarrow$) This is the easier of the two directions. Let $\epsilon > 0$ be arbitrary. Given that (f_n) converges uniformly on A, our job is to produce an N such that $|f_n(x) - f_m(x)| < \epsilon$ for all $m, n \geq N$ and $x \in A$.

Because we are given that (f_n) converges uniformly, we may let $f(x) = \lim_{n\to\infty} f_n(x)$. By the definition of uniform convergence, there exists an N with the property that

$$|f_n(x) - f(x)| < \frac{\epsilon}{2} \quad \text{for all } n \geq N \text{ and } x \in A.$$

Now given $m, n \geq N$, it follows that

$$\begin{aligned} |f_n(x) - f_m(x)| &= |f_n(x) - f(x) + f(x) - f_m(x)| \\ &\leq |f_n(x) - f(x)| + |f(x) - f_m(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

for all $x \in A$. This completes the proof in the forward direction.

($\Leftarrow$) In this direction we assume that, given $\epsilon > 0$, there exists an N such that $|f_n(x) - f_m(x)| < \epsilon$ for all $m, n \geq N$ and $x \in A$. Our goal is to prove that $f_n(x)$ converges uniformly.

To produce a candidate for the limit, notice that for each $x \in A$ our hypothesis tells us that the sequence $(f_n(x))$ is a Cauchy sequence. Because Cauchy sequences converge, it makes sense to define the limit function

$$f(x) = \lim_{n\to\infty} f_n(x).$$

It is important to realize that, because we are applying the Cauchy Criterion to sequences generated at each point $x \in A$, all we have proved thus far is that $f_n(x) \to f(x)$ pointwise on A.

Let $\epsilon > 0$. Using our hypothesis again (in its full strength this time), we know that there exists an N such that

$$-\epsilon < f_n(x) - f_m(x) < \epsilon \quad \text{for all } m, n \geq N \text{ and } x \in A.$$

The Algebraic Limit Theorem says that

$$\lim_{m\to\infty}(f_n(x) - f_m(x)) = f_n(x) - f(x),$$

and the Order Limit Theorem then implies

$$-\epsilon \leq f_n(x) - f(x) \leq \epsilon \quad \text{for all } n \geq N \text{ and } x \in A.$$

This is sufficient to conclude that $f_n \to f$ uniformly on A.

Exercise 6.2.7. This argument really amounts to adopting the proof of Theorem 6.2.6 to this stronger set of assumptions.

Let $\epsilon > 0$ be arbitrary. We need to show that there exists a $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$ for all $x, y \in A$. First choose N so that

$$|f_N(x) - f(x)| < \frac{\epsilon}{3} \quad \text{for all } x \in A.$$

Because f_N is uniformly continuous on A, there exists a $\delta > 0$ such that

$$|f_N(x) - f_N(y)| < \epsilon/3 \quad \text{whenever } |x - y| < \delta.$$

But this implies

$$\begin{aligned}
|f(x) - f(y)| &= |f(x) - f_N(x) + f_N(x) - f_N(y) + f_N(y) - f(y)| \\
&\leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon
\end{aligned}$$

We conclude that f is uniformly continuous on A.

Exercise 6.2.8. (a) False. Consider Example 6.2.2 (ii).

(b) True. Let $\epsilon > 0$ be arbitrary and assume $|g(x)| \leq M$. We need to show that there exists an N such that $n \geq N$ implies $|f_n g - fg| < \epsilon$. Because $f_n \to f$ uniformly, we know there exists an N such that $|f_n - f| < \epsilon/M$ for all $n \geq N$. It follows that

$$\begin{aligned}
|f_n g - fg| &= |g||f_n - f| \\
&\leq M|f_n - f| \\
&< M(\epsilon/M) = \epsilon
\end{aligned}$$

for all $n \geq N$, as desired.

(c) True. Pick N such that

$$|f_N(x) - f(x)| \leq 1 \quad \text{for all } x \in A.$$

If $|f_N(x)| \leq M$ for all $x \in A$, then it follows that $|f(x)| \leq M + 1$ on A, and hence f is bounded.

(d) True. Let $\epsilon > 0$ be arbitrary. Because $f_n \to f$ uniformly on A we can pick N_1 such that $n \geq N_1$ implies $|f_n - f| < \epsilon$ for all $x \in A$. Similarly, because

$f_n \to f$ uniformly on B we can pick N_2 so that $n \geq N_2$ implies $|f_n - f| < \epsilon$ for all $x \in B$. Now let $N = \max\{N_1, N_2\}$. Then $n \geq N$ implies $|f_n - f| < \epsilon$ for all $x \in A \cup B$.

(e) True. Let $x < y$ be arbitrary points in the domain. We are given that $f_n(x) \leq f_n(y)$ for all n. Because $f_n(x) \to f(x)$ and $f_n(y) \to f(y)$, we may use the Order Limit Theorem to conclude $f(x) \leq f(y)$. This proves f is increasing.

(f) True. The proof in (e) does not require uniform convergence.

Exercise 6.2.9. Let $\epsilon > 0$ be arbitrary. We need to show that there exists an $N \in \mathbf{N}$ such that when $n \geq N$ it follows that $|f_n/g - f/g| < \epsilon$. First write

$$\left|\frac{f_n}{g} - \frac{f}{g}\right| = \left|\frac{1}{g}\right| |f_n - f|.$$

Because g is continuous and never zero, $1/g$ is also continuous on K. The fact that K is compact implies $1/g$ is bounded, so let $M > 0$ satisfy $|1/g| \leq M$. Because $(f_n) \to f$ uniformly, we can pick N such that

$$|f_n - f| < \frac{\epsilon}{M} \quad \text{whenever } n \geq N.$$

It follows that

$$\left|\frac{f_n}{g} - \frac{f}{g}\right| < M\frac{\epsilon}{M} = \epsilon$$

for all $n \geq N$, as desired.

Exercise 6.2.10. Let $\epsilon > 0$ be arbitrary. We need to show that there exists an N such that $n \geq N$ implies $|f_n(x) - f(x)| < \epsilon$.

Because f is uniformly continuous on all of $\mathbf{R}$, we can pick δ so that

$$|f(x) - f(y)| < \epsilon \quad \text{whenever } |x - y| < \delta.$$

Now choose $N > 1/\delta$. If $n \geq N$ then $|(x + 1/n) - x| < \delta$ and it follows that

$$|f_n(x) - f(x)| = |f(x + 1/n) - f(x)| < \epsilon,$$

as desired.

This proposition fails if f is not uniformly continuous. Consider $f(x) = x^2$ which is continuous but not uniformly continuous on all of $\mathbf{R}$. In this case we see

$$|f_n(x) - f(x)| = |(x + 1/n)^2 - x^2| = |2x/n + 1/n^2|.$$

Although for each $x \in R$, this expression tends to zero as $n \to \infty$, we see that larger values of x require larger values of n and the convergence is not uniform.

Exercise 6.2.11. Without the limit functions mentioned, it is a little smoother to argue in terms of the Cauchy Criterion. Let $\epsilon > 0$. Choose N_1 so that

$$|f_n - f_m| < \epsilon/2 \quad \text{for all } n, m \geq N_1,$$

and choose N_2 so that

$$|g_n - g_m| < \epsilon/2 \quad \text{for all } m, n \geq N_2.$$

Letting $N = \max\{N_1, N_2\}$ we see that

$$\begin{aligned} |(f_n + g_n) - (f_m + g_m)| &\leq |f_n - f_m| + |g_n - g_m| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

for all $m, n \geq N$. It follows from the Cauchy Criterion for Uniform Convergence that $(f_n + g_n)$ converges uniformly.

(b) Looking ahead to (c), we see that problems can arise when at least one of the limit functions is unbounded. For example, let $f_n(x) = x + 1/n$ and $g_n(x) = 1/n$. Then $f_n(x) \to x$ uniformly on $\mathbf{R}$ and $g_n(x) \to 0$ uniformly on $\mathbf{R}$. However $f_n(x)g_n(x) = x/n + 1/n^2$. Although $f_n g_n \to 0$ pointwise on $\mathbf{R}$, the convergence is not uniform.

(c) The first step is to write

$$\begin{aligned} |f_n g_n - f_m g_m| &= |f_n g_n - f_n g_m + f_n g_m - f_m g_m| \\ &\leq |f_n||g_n - g_m| + |g_m||f_n - f_m|. \end{aligned}$$

Given $\epsilon > 0$, choose N_1 so that

$$|f_n - f_m| < \frac{\epsilon}{2M} \quad \text{for all } n, m \geq N_1.$$

Also, choose N_2 so that

$$|g_n - g_m| < \frac{\epsilon}{2M} \quad \text{for all } m, n \geq N_2.$$

Letting $N = \max\{N_1, N_2\}$ we see that

$$\begin{aligned} |f_n g_n - f_m g_m| &\leq |f_n||g_n - g_m| + |g_m||f_n - f_m| \\ &< M\frac{\epsilon}{2M} + M\frac{\epsilon}{2M} = \epsilon \end{aligned}$$

whenever $n \geq N$, as desired.

Exercise 6.2.12. (a) Setting $g_n = f - f_n$ we see that

(i) g_n is continuous for each $n \in N$,

(ii) $g_n(x)$ is decreasing for all $x \in K$, and

(iii) $g_n(x) \to 0$ for all $x \in K$.

(b) We first prove that each K_n is closed. To see this, assume (x_m) is a convergent sequence in K_n. If $x = \lim x_m$, then $x \in K$ (because K is closed) and the fact that g_n is continuous on all of K allows us to write $g_n(x) = \lim_{m\to\infty} g_n(x_m)$. Because $g_n(x_m) \geq \epsilon$ for all m, it follows that the limit $g_n(x)$

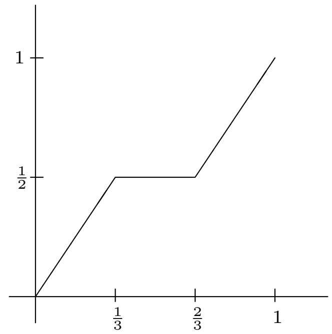


Figure 6.1: SKETCH OF f_1 FOR THE CANTOR FUNCTION.

also satisfies $g_n(x) \geq \epsilon$. But this implies $x \in K_n$ and we see that K_n contains its limit points and thus is closed.

Each K_n is also bounded because it is a subset of the bounded set K, and it follows that K_n is compact.

The nested property $K_n \supseteq K_{n+1}$ is a direct consequence of our assumption that $g_n(x) \geq g_{n+1}(x)$ for all $x \in K$, and so we are prepared to use Theorem 3.3.5. Assume, for contradiction, that K_n is nonempty for every $n \in \mathbf{N}$. Then Theorem 3.3.5 implies there exists a point x satisfying $x \in K_n$ for every n. But this means $g_n(x) \geq \epsilon$ for every n, contradicting our assumption that $g_n(x) \to 0$. We conclude that there must exist an N for which $K_n = \emptyset$ for all $n \geq N$, and this is equivalent to asserting that

$$|g_n(x)| < \epsilon \quad \text{for all } n \geq N \text{ and } x \in K.$$

We conclude that $g_n \to 0$ uniformly, and thus $f_n \to f$ uniformly as well.

Exercise 6.2.13. (a) A sketch of f_1 is given in Figure 6.1.

(b) Looking at f_1 for the moment, notice that for every $n \in \mathbf{N}$ we have $|f_1(x) - f_n(x)| = 0$ if $x \in [1/3, 2/3]$. Off of this middle set, the fact that every f_n is increasing means we still have the estimate

$$|f_1(x) - f_n(x)| \leq \frac{1}{2}.$$

In general, given $m < n$ we see

$$|f_m(x) - f_n(x)| \leq \frac{1}{2^m}.$$

By the Cauchy Criterion for Uniform Convergence (Theorem 6.2.5), we conclude that (f_n) converges uniformly.

(c) The limit function f is continuous by Theorem 6.2.6. Exercise 6.2.8 (e) gives an argument that f is increasing, and $f(0) = 0$ and $f(1) = 1$ follows quickly from the fact that 0 and 1 are fixed by every f_n. Finally, if x is a point in $[0,1]\backslash C$, then x must fall in the complement of some C_m. Notice that $f(x) = f_m(x)$ and the recursive way that each f_n is constructed means that in fact

$$f(y) = f_n(y) \quad \text{for all } n \geq m \text{ and } y \in [0,1]\backslash C_m.$$

It follows that f is constant on $[0,1]\backslash C_m$.

Exercise 6.2.14. (a) The sequence of real numbers $f_n(x_1)$ is bounded by M. The Bolzano–Weierstrass Theorem implies that there is a convergent subsequence.

(b) Focusing on the sequence $f_{1,k}(x_2)$, we again use the Bolzano–Weierstrass Theorem to conclude that there is a convergent subsequence which we write as $f_{2,k}(x_2)$.

(c) Keep in mind that if $m' > m$ then $(f_{m',k})$ is a subsequence of $(f_{m,k})$. The key idea is to let

$$f_{n_k} = f_{k,k} = (f_{1,1}, f_{2,2}, f_{3,3}, \ldots).$$

The nested quality shows that $(f_{k,k})$ is a subsequence of $f_{1,k}$ and thus $f_{k,k}(x_1)$ converges. But what about $f_{k,k}(x_m)$ for an arbitrary $x_m \in A$? Well, after the first m terms, we see that $f_{k,k}$ becomes a proper subsequence of $f_{m,k}$ (i.e., $f_{k,k}$ is eventually in $f_{m,k}$), and it follows that $f_{k,k}(x_m)$ converges. This shows $f_{k,k}$ converges pointwise on A.

Exercise 6.2.15. (a) If each f_n is uniformly continuous, the choice of δ can be made independently of x but δ will certainly depend on the function f_n. It is possible for different functions f_n to require smaller δ responses, and it may be that there is no single δ that will work simultaneously for all functions in the collection f_n as the definition of equicontinuity requires.

(b) For each n, the function g_n is continuous on the compact set $[0,1]$ and thus it is uniformly continuous. However, the sequence (g_n) is not equicontinuous over the set $[0,1]$. The trouble occurs near 1. To make the discussion concrete, let's take $\epsilon = 1/2$ and set $y = 1$. The definition of equicontinuity requires us to produce a $\delta > 0$ with the property that

$$|x^n - 1| < \frac{1}{2} \quad \text{for all } n \in \mathbf{N} \text{ and } |x-1| < \delta.$$

But notice that δ *cannot* be chosen independently of n because no matter how close to 1 we take our value of x, it will always be possible to find a large value of n that makes $|x^n - 1| \geq 1/2$.

Exercise 6.2.16. (a) Because the set of rational numbers in $[0,1]$ is countable, Exercise 6.2.14 gives us exactly what we need to produce the sequence (g_s).

(b) Consider a fixed r_i from our finite set $\{r_1, r_2, \ldots, r_m\}$. Because (g_s) converges pointwise at every rational, the sequence $(g_s(r_i))$ is a Cauchy sequence. Thus we can choose N_i such that

$$|g_s(r_i) - g_t(r_i)| < \frac{\epsilon}{3} \quad \text{for all } s, t \geq N_i.$$

Letting $N = \max\{N_1, N_2, \ldots, N_m\}$ produces the desired N.

Note that if the set $\{r_1, r_2, \ldots, r_m\}$ were infinite then N would be the maximum of an infinite set which is problematic to say the least.

(c) Given $x \in [0,1]$, we know there exists a rational r_i from our designated set satisfying $|r_i - x| < \delta$. It follows that

$$|g_s(x) - g_s(r_i)| < \frac{\epsilon}{3} \quad \text{for all } s \in \mathbf{N}.$$

Using this fact (twice) and the result in (b) we see that $s, t \geq N$ implies

$$\begin{aligned} |g_s(x) - g_t(x)| &= |g_s(x) - g_s(r_i) + g_s(r_i) - g_t(r_i) + g_t(r_i) - g_t(x)| \\ &\leq |g_s(x) - g_s(r_i)| + |g_s(r_i) - g_t(r_i)| + |g_t(r_i) - g_t(x)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

It follows that (g_s) converges uniformly using the Cauchy criterion in Theorem 6.2.5.

6.3 Uniform Convergence and Differentiation

Exercise 6.3.1. (a) Write

$$|h_n(x) - 0| = \left|\frac{\sin(nx)}{n}\right| \leq \frac{1}{n}.$$

Given $\epsilon > 0$, choose $N > 1/\epsilon$ which is independent of x. Then $n \geq N$ implies $|h_n - 0| < \epsilon$ and we conclude $h_n \to 0$ uniformly.

By contrast, the sequence of derivatives

$$h_n'(x) = \cos(nx)$$

diverges for all values of x except $x = \pi/2 + k\pi$.

(b) The sequence $f_n(x) = \sin(nx)/\sqrt{n}$ has this property. The lesson here is that uniform convergence of a sequence of functions does not, by itself, imply anything particularly useful about the behavior of the sequence of derivatives.

Exercise 6.3.2. (a) First we deduce that $g = \lim g_n = 0$, and the convergence is uniform on $[0,1]$. To prove this, we must find an N such that $n \geq N$ implies $|x^n/n - 0| < \epsilon$. But notice that

$$|\frac{x^n}{n} - 0| \leq \frac{1}{n} \quad \text{for all } x \in [0,1].$$

Given $\epsilon > 0$, pick $N > 1/\epsilon$. Then $n \geq N$ implies $|x^n| < \epsilon$ for all $x \in [0,1]$, as desired.

Because $g(x) = 0$ for all $x \in [0,1]$ it is differentiable and, furthermore, $g'(0) = 0$.

(b) Writing

$$g_n'(x) = \frac{nx^{n-1}}{n} = x^{n-1},$$

we see that the sequence (g_n') converges pointwise on $[0,1]$, to

$$h(x) = \lim_{n\to\infty} g_n'(x) = \begin{cases} 0 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1. \end{cases}$$

The convergence is not uniform over $[0,1]$, and in fact it is not uniform over any set that contains 1 as a limit point. Comparing $h = \lim g_n'$ to g' is illuminating. Note in particular that $h(1) \neq g(1)$, so that it is possible for the sequence of derivatives to converge to the "wrong" value when the convergence of g_n' is not uniform. On the other hand, the convergence of g_n' is uniform on sets of the form $[0,c]$ where $c < 1$, and this is reflected by the fact that $h(x) = g(x)$ on $[0,1)$.

Exercise 6.3.3. We have seen that $f_n \to 0$ uniformly meaning that the limit $f = \lim f_n$ satisfies $f'(x) = 0$ for all values of x.

Taking the derivative we get

$$f_n'(x) = \frac{1 - nx^2}{1 + 2nx^2 + n^2x^4}.$$

If $x \neq 0$ then we can show $\lim f_n'(x) = 0 = f'(x)$. However, for $x = 0$ we get $f_n'(0) = 1$ for all n and thus $f'(0) \neq \lim f_n'(1)$.

Exercise 6.3.4. (a) We have

$$g(x) = \lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \frac{x}{2} + \frac{x^2}{2n} = \frac{x}{2},$$

so $g'(x) = 1/2$.

(b) This time we compute $g_n'(x)$ first to get

$$g_n'(x) = \frac{1}{2} + \frac{x}{n},$$

and note that the pointwise limit of this sequence is $1/2$. For $x \in [-M, M]$ we can write

$$|g_n'(x) - 1/2| = \left|\frac{x}{n}\right| \leq \frac{M}{n}.$$

Given $\epsilon > 0$, choose $N > M/\epsilon$, independent of x. Then $n \geq N$ implies $|g_n'(x) - 1/2| < \epsilon$, and we conclude that $g_n' \to 1/2$ uniformly on $[-M, M]$. It follows from Theorem 6.3.3 that $g'(x) = 1/2$.

(c) Taking the pointwise limit of $f_n(x)$ gives

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^2 + 1/n}{2 + x/n} = \frac{x^2}{2}.$$

Thus, $f'(x) = x$.

Computing the derivative sequence first we get

$$f_n'(x) = \frac{4n^2x + 3nx^2 + 1}{4n^2 + 4nx + x^2},$$

so that

$$\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} \frac{4x + 3x^2/n + 1/n^2}{4 + 4x/n + x^2/n^2} = x.$$

Arguing for uniform convergence on intervals of the form $[-M, M]$ is less elegant for this example but no harder really. For values of x satisfying $|x| < M$ we have

$$|f_n'(x) - x| = \left|\frac{-nx^2 - x^3 + 1}{4n^2 + 4nx + x^2}\right| \leq \frac{nM^2 + M^3 + 1}{4n^2 - 4nM},$$

as long as $n > M$. Because this estimate does not depend on x and tends to zero as $n \to \infty$, it follows that $f_n'(x) \to x$ uniformly on $[-M, M]$.

Exercise 6.3.5. Let $x \in [a, b]$ and assume, without loss of generality, that $x > x_0$. Applying the Mean Value Theorem to the function $f_n - f_m$ on the interval $[x_0, x]$, we get that there exists a point α such that

$$(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0)) = (f_n'(\alpha) - f_m'(\alpha))(b - a).$$

Let $\epsilon > 0$. Because (f_n') converges uniformly, the Cauchy Criterion asserts that there exists an N_1 such that

$$|f_n'(c) - f_m'(c)| < \frac{\epsilon}{2(b-a)} \qquad \text{for all } n, m \geq N \text{ and } c \in [a, b].$$

Our hypothesis states that $(f_n(x_0))$ converges so there exists an N_2 such that

$$|f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2} \qquad \text{for all } n, m \geq N_2.$$

Finally, let $N = \max\{N_1, N_2\}$. Then if $n, m \geq N$ it follows that

$$\begin{aligned}
|f_n(x) - f_m(x)| &\leq |(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| + |f_n(x_0) - f_m(x_0)| \\
&= |(f_n'(\alpha) - f_m'(\alpha))(b - a)| + |f_n(x_0) - f_m(x_0)| \\
&< \frac{\epsilon}{2(b-a)}(b - a) + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Because our choice of N is independent of the point x, the Cauchy Criterion implies that the sequence (f_n) converges uniformly on $[a, b]$.

6.4 Series of Functions

Exercise 6.4.1. Let $\epsilon > 0$. By Theorem 6.4.4, there exists an N such that

$$|g_{m+1}(x) + \cdots + g_n(x)| < \epsilon \qquad \text{whenever } n > m \geq N.$$

Because this holds for all $m \geq N$, we can set $m = n - 1$ to get that

$$|f_n(x)| < \epsilon \quad \text{whenever } n > N.$$

This proves $g_n \to 0$ uniformly.

Exercise 6.4.2. The key idea is to use the Cauchy criterion for convergence of a series of real numbers given in Theorem 2.7.2. Let $\epsilon > 0$ be arbitrary. Because $\sum_{n=1}^{\infty} M_n$ converges, there exists an N such that $n > m \geq N$ implies

$$|M_{m+1} + M_{m+2} + \cdots + M_n| < \epsilon.$$

Because

$$|f_{m+1}(x) + f_{m+2}(x) + \cdots + f_n(x)| \leq M_{m+1} + M_{n+2} + \cdots + M_n,$$

we can appeal to the Cauchy criterion for uniform convergence of series (Theorem 6.4.4) to conclude that $\sum_{n=1}^{\infty} f_n$ converges uniformly.

Exercise 6.4.3. (a) Because

$$\left|\frac{\cos(2^n x)}{2^n}\right| \leq \frac{1}{2^n},$$

and we know $\sum_{n=1}^{\infty} 1/2^n$ converges, it follows by the Weierstrass M–Test that $\sum_{n=1}^{\infty} \cos(2^n x)/2^n$ converges uniformly. Because each of the summands is continuous, the limit is as well according to Theorem 6.4.2.

(b) Again, we have an infinite sum of continuous functions and we would like to conclude that the limit is continuous. This will follow if we can argue the convergence is uniform. On $[-1, 1]$ we can make the estimate $|x^n/n^2| \leq 1/n^2$. Because $\sum_{n=1}^{\infty} 1/n^2$ converges, we may invoke the Weierstrass M–Test to assert that $\sum_{n=1}^{\infty} x^n/n^2$ converges uniformly on $[-1, 1]$. This completes the proof.

Exercise 6.4.4. The "sawtooth" function $h(x)$ satisfies $|h(x)| \leq 1$ for all $x \in \mathbf{R}$. Thus

$$\left|\frac{1}{2^n} h(2^n)\right| \leq \frac{1}{2^n}.$$

Because $\sum_{n=1}^{\infty} 1/2^n$, the Weierstrass M–Test implies that our series for $g(x)$ converges uniformly. The fact that $h(x)$ is continuous allow us to invoke Theorem 6.4.2 to conclude g is continuous.

Exercise 6.4.5. (a) The series for f certainly converges uniformly, but Theorem 6.4.3 requires us to look at the differentiated series

$$\sum_{n=1}^{\infty} \frac{\cos(kx)}{k^2}. \tag{1}$$

For this series we can make the estimate

$$\left|\frac{\cos(kx)}{k^2}\right| \leq \frac{1}{k^2}.$$

Because $\sum_{n=1}^{\infty} 1/k^2$ converges, the M–Test asserts that the series above in (1) converges uniformly. Now Theorem 6.4.3 asserts that f is differentiable and

$$f'(x) = \sum_{n=1}^{\infty} \frac{\cos(kx)}{k^2}.$$

Finally, we note that the uniform convergence also implies (via Theorem 6.4.2) that $f'(x)$ is continuous because each of the summands is.

(b) To use Theorem 6.4.3 to determine whether $f'(x)$ is differentiable requires that we differentiate the series for f' term-by-term and consider

$$\sum_{n=1}^{\infty} \frac{-\sin(kx)}{k}.$$

Unfortunately, the Weierstrass M–Test cannot be used here because $\sum_{n=1}^{\infty} 1/k$ diverges.

The differentiability of f' turns out to be a very deep question that has been studied in depth by Riemann and Hardy, among others.

Exercise 6.4.6. First fix $x_0 \in [0,1)$. Now choose c to satisfy $x_0 < c < 1$ and apply the M–Test on $[0,c]$. Over this interval we get the estimate $|x^n/n| \leq c^n/n$. Because $\sum_{n=1}^{\infty} c^n/n$ converges, the M–Test implies the convergence is uniform and thus f is continuous at $x_0 \in [0,c]$.

Exercise 6.4.7. (a) First observe that the summands are continuous functions and satisfy

$$\left|\frac{1}{x^2+n^2}\right| \leq \frac{1}{n^2} \quad \text{for all } x \in \mathbf{R}.$$

Because $\sum_{n=1}^{\infty} 1/n^2$ converges, the M–Test implies the convergence is uniform and hence $h(x)$ is continuous on $\mathbf{R}$.

(b) To determine if h is differentiable we consider the differentiated series

$$\sum_{n=1}^{\infty} \frac{-2x}{(x^2+n^2)^2}.$$

Restricting our attention to an interval $[-M,M]$, we get the estimate

$$\left|\frac{-2x}{x^2+n^2}\right| \leq \frac{2M}{n^2},$$

and, as before, we note $\sum_{n=1}^{\infty} 2M/n^2$ converges. This proves that the differentiated series converges uniformly to $h'(x)$ and that h' is continuous on $[-M,M]$. Because the interval $[-M,M]$ is arbitrary in this argument, we conclude that h' exists and is continuous on all of $\mathbf{R}$.

Exercise 6.4.8. Using the M–Test and the fact that $|u_n(x)| \leq 1/2^n$, we can show that $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly to $h(x)$. An argument like the one in Exercise 6.2.8 (e) shows that h is increasing. Finally, notice that Theorem 6.2.6 is stated and proved in terms of an individual point c in the domain. In our case, if we take c to be irrational, then we see that every u_n is continuous at c and thus h is as well.

6.5 Power Series

Exercise 6.5.1. (a) The series for g converges for all $x \in (-1, 1]$ and is continuous over this interval. The series does not converge when $x = -1$ and Theorem 6.5.1 implies that it then does not converge for any values of x satisfying $|x| > 1$.

(b) Theorem 6.5.6 implies g is differentiable on $(-1, 1)$ with

$$g'(x) = 1 - x + x^2 - x^3 + \cdots .$$

Notice that our formula for $g'(x)$ no longer converges when $x = 1$ (although the function g is technically differentiable at this point.)

Exercise 6.5.2. (a) $\sum_{n=1}^{\infty} x^n/n^2$

(b) $\sum_{n=1}^{\infty} x^n/n$

(c) $\sum_{n=1}^{\infty} (-1)^n x^{2n}/n$

(d) No. If the series converges absolutely at $x = 1$, then $\sum |a_n|$ converges, it follows that the series converges absolutely at $x = -1$ as well.

Exercise 6.5.3. The set of convergent points for a power series must be **R**, $\{0\}$ or an interval. In the case of an interval, we have seen that the convergence is absolute in the interior of this interval. Thus, the two endpoints are the only candidates for conditional convergence.

Exercise 6.5.4. (a) Let $x \in (-R, R)$ be arbitrary. We want to prove that the power series is continuous at x. To do this, choose $c > 0$ satisfying $0 < |x| < c < R$, and consider the compact set $[-c, c]$ contained in $(-R, R)$. Absolute convergence of the series at $x = c$ implies that we get uniform convergence over the interval $[-c, c]$. Because the summands in a power series are continuous, we may conclude that the series represents a continuous function on $[-c, c]$, and hence is continuous at x.

(b) The content of Abel's Theorem is that convergence at an endpoint $x = R$ implies uniform convergence over the interval $[0, R]$. Once we have established uniform convergence, continuity follows (once again) from Theorem 6.4.2 and the observation that the summands are all continuous.

Exercise 6.5.5. Set $M_n = |a_n x_0^n|$ and note that absolute convergence at x_0 implies

$$\sum_{n=0}^{\infty} |a_n x_0^n| = \sum_{n=0}^{\infty} M_n$$

converges. If $x \in [-c, c]$ then we get the estimate

$$|a_n x^n| \le |a_n x_0^n| = M_n,$$

and the Weierstrass M–Test implies that $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-c, c]$.

Exercise 6.5.6. Assume $\sum a_n x^n$ converges pointwise on the compact set K. Because K is compact, there exist points $x_0, x_1 \in K$ satisfying

$$x_0 \leq x \leq x_1 \quad \text{for all } x \in K.$$

At this point we need to consider a few different cases. If $x_0 > 0$ then $K \subseteq [0, x_1]$ and Theorem 6.5.1 implies we get pointwise convergence on $[0, x_1]$. Then Abel's Theorem implies that the convergence is uniform on $[0, x_1]$ hence over the set $K \subseteq [0, x_1]$.

If $x_1 < 0$ then $K \subseteq [x_0, 0]$ we can make a similar argument along the same lines. If $x_0 \leq 0 \leq x_1$ then Theorem 6.5.1 and Abel's Theorem imply that we the series converges uniformly over each of the intervals $[x_0, 0]$ and $[0, x_1]$. It is a straightforward exercise to show that this implies uniform convergence over $[x_0, x_1]$ and hence over the set $K \subseteq [x_0, x_1]$.

Exercise 6.5.7. (a) Applying the Ratio Test to the sequence $a_n = ns^{n-1}$ we find

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{ns^n + s^n}{ns^{n-1}} \right| = \lim_{n\to\infty} \left| s + \frac{1}{n} s \right| = s.$$

Because $0 < s < 1$, the series $\sum a_n$ converges by the Ratio Test. Therefore, the sequence (ns^{n-1}) converges to zero and thus is bounded.

(b) Let $x \in (-R, R)$ be arbitrary and pick t to satisfy $|x| < t < R$. We will show that $\sum |na_n x^{n-1}|$ converges, implying $\sum na_n x^{n-1}$ converges. First write

$$\sum_{n=1}^{\infty} |na_n x^{n-1}| = \sum_{n=1}^{\infty} \frac{1}{t} \left(n \left| \frac{x}{t} \right|^{n-1} \right) |a_n t^n|.$$

Because $|x/t| < 1$, by part (a) we can pick a bound L satisfying

$$n \left| \frac{x}{t} \right|^{n-1} \leq L \quad \text{for all } n \in \mathbf{N}.$$

Now we have

$$\sum_{n=1}^{\infty} |na_n x^{n-1}| = \sum_{n=1}^{\infty} \frac{1}{t} \left(n \left| \frac{x}{t} \right|^{n-1} \right) |a_n t^n| \leq \frac{L}{t} \sum_{n=1}^{\infty} |a_n t^n|$$

where the last sum converges because $t \in (-R, R)$. Therefore, $\sum_{n=1}^{\infty} na_n x^{n-1}$ converges absolutely and thus converges.

Exercise 6.5.8. (a) For a fixed x, apply the Ratio Test to the series $\sum a_n x^n$ to get

$$\lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| |x| = L|x|.$$

If $|x| < 1/L$ then $L|x| < 1$ and the series converges.

(b) If $L = 0$ then $L|x| = 0 < 1$ for every value of x and the Ratio Test implies that the series converges on all of $\mathbf{R}$.

(c) This will follow using the same proofs if we can prove the following modified version of the Ratio Test:

Given a sequence (b_n), let

$$L' = \lim_{n\to\infty} s_n \quad \text{where} \quad s_n = \sup\left\{\left|\frac{b_{k+1}}{b_k}\right| : k \geq n\right\}.$$

If $L' < 1$ then $\sum b_n$ converges.

The proof is very similar to the proof of the Ratio Test in Exercise 2.7.9. First choose R to satisfy $L' < R < 1$. The sequence (s_n) is decreasing, and because it converges to L' we know there exists an N such that

$$\left|\frac{b_{k+1}}{b_k}\right| \leq R \quad \text{for all } k \geq N.$$

An induction proof like the one before shows

$$|b_k| \leq |b_N| R^{k-N} \quad \text{for all } k \geq N,$$

and then we may compare the series $\sum b_k$ to the convergent geometric series $|a_N| \sum R^k$ to conclude that $\sum b_k$ converges.

(d) The statement in this exercise is false. A condition such as $|a_{n+1}/a_n| \geq 1$ for *all* values of n after some point in the sequence would be sufficient to prove the series diverges.

Exercise 6.5.9. Set $g(x) = \sum_{n=0}^{\infty} a_n x^n$ and $h(x) = \sum_{n=0}^{\infty} b_n x^n$. Because $g(x) = h(x)$ on $(-R, R)$ we see $a_0 = g(0) = h(0) = b_0$. But we also know that g and h are infinitely differentiable. Taking the derivative and setting $x = 0$ yields the formulas

$$a_1 = g'(0) \quad \text{and} \quad b_1 = h'(0).$$

Again, because $g = h$ we see that $a_1 = b_1$. Taking successive derivatives and setting $x = 0$ leads to the conclusion that $a_n = b_n$ for all $n \in \mathbf{N}$. (The upcoming work on Taylor's formula in the next section is very relevant to this discussion.)

Exercise 6.5.10. We are assuming $\sum a_n$, $\sum b_n$ and $\sum d_n$ each converge which, according to Abel's Theorem, tells us that the respective series for f, g, and h converge uniformly on $[0, 1]$. Among other things, this implies that f, g and h are all continuous functions over the closed interval $[0, 1]$.

Fix $x \in [0, 1)$. Because we know we have convergence at 1, Theorem 6.5.1 implies that $\sum a_n x^n$, $\sum b_n x^n$ and $\sum d_n x^n$ each converge absolutely. This fact means that we can invoke the result in Exercise 2.8.8 to assert that

$$h(x) = \sum d_n x^n = f(x)g(x).$$

Because this is true for all $x \in [0, 1)$, and because f, g and h are continuous on the closed interval $[0, 1]$, it follows that $h(1) = f(1)g(1)$ or

$$\sum d_n = \left(\sum a_n\right)\left(\sum b_n\right),$$

as desired.

Exercise 6.5.11. (a) Assume $\sum a_n$ converges to L. If we set $f(x) = \sum a_n x^n$, then Abel's Theorem implies that the series for f converges uniformly on the interval $[0,1]$. Because the summands are continuous polynomials, this proves that f is continuous on $[0,1]$. In particular, this implies $\lim_{x\to 1^-} f(x) = f(1)$. But notice that $f(1) = L$ and thus we have shown that $\sum a_n$ is Abel-summable to L.

(b) Using some familiar facts about geometric series, observe that

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n x^n &= 1 - x + x^2 - x^3 + x^4 - \\ &= \frac{1}{1-(-x)} \\ &= \frac{1}{1+x},\end{aligned}$$

provided $|x| < 1$. Then $\lim_{x\to 1^-} 1/(1+x) = 1/2$ shows that our series is Abel-summable to $1/2$.

Exercise 6.5.12. Begin with the observation that $0 < d_0$. Because G is strictly increasing we see $G(0) < G(d_0)$. But notice $G(0) = d_1$ and $G(d_0) = d_0$ and so we have $d_1 < d_0$. This argument can be repeated. Given $d_r < d_0$ we have

$$d_{r+1} = G(d_r) < G(d_0) = d_0,$$

and it follows that $d_r < d_0$ for all values of r. We conclude that (d_r) converges to d_0—the smaller of the two fixed points.

6.6 Taylor Series

Exercise 6.6.1. Because the series converges when $x = 1$, Abel's Theorem implies that we get uniform convergence over the interval $[0,1]$ and thus the series represents a continuous function over the interval $[0,1]$. Assuming that $\arctan(x)$ is continuous over $[0,1]$, it follows that if these two continuous functions agree for all values of $x \in [0,1)$, then they must also agree when $x = 1$.

Setting $x = 1$ into this formula gives "Leibniz's formula,"

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

Exercise 6.6.2. From equation (1) we get

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots .$$

Then integrating gives

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots .$$

This series converges for all x in the interval $(-1,1]$.

Exercise 6.6.3. The key idea is to take the derivative of each side of equation (2) using a term-by-term approach for the series on the right (this is justified by Theorem 6.5.7). Setting $x = 0$ after n derivatives gives the formula for a_n.

Exercise 6.6.4. We set $a_0 = \sin(0) = 0$, $a_1 = \cos(0)/1! = 1$, $a_2 = -\sin(0)/2! = 0$, $a_3 = -\cos(0)/3! = -1/3!$, and so on. Then substitute these values for a_n and f into the expression in equation (2). It remains to show that the series expression actually equals $\sin(x)$ for any values of x other than $x = 0$.

Exercise 6.6.5. To do this we will show that $E_N(x) \to 0$ uniformly on $[-2, 2]$. By Lagrange's Remainder Theorem we have

$$|E_N(x)| = \left|\frac{f^{(N+1)}(c)}{(N+1)!}x^{N+1}\right| \leq \frac{1}{(N+1)!}\left|x^{N+1}\right| \leq \frac{1}{(N+1)!}2^{N+1}$$

for $x \in [-2, 2]$. From past experience we know that factorials grow faster than exponentials or, put another way, that $\lim_{N\to\infty} 2^{N+1}/(N+1)! = 0$. Thus, given $\epsilon > 0$ we can choose an M so that $N \geq M$ implies $2^{M+1}/(M+1)! < \epsilon$. It follow that

$$|E_M(x)| \leq \frac{2^{M+1}}{(M+1)!} < \epsilon \quad \text{for all } M \geq N,$$

and hence $E_N(x) \to 0$ uniformly on $[-2, 2]$.

Replacing the constant 2 with an arbitrary constant R has no effect on the validity of the argument.

Exercise 6.6.6. (a) Because $f^{(n)}(x) = e^x$ for every n, we get $a_n = e^0/n! = 1/n!$ which yields

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

To show that the series converges uniformly to e^x on any interval of the form $[-R, R]$ use Lagrange's Remainder formula to write

$$|E_N(x)| = \left|\frac{e^c}{(N+1)!}x^{N+1}\right| \leq \frac{e^R}{(N+1)!}R^{N+1}$$

for all $x \in [-R, R]$. Now, just as in the previous exercise, this error bound tends to zero as $N \to \infty$. Because this bound is independent of x, it follows that $E_N(x) \to 0$ uniformly on $[-R, R]$ and we get that $S_N(x) \to e^x$ uniformly on $[-R, R]$ as well.

(b) To verify the formula $f'(x) = e^x$ we differentiate the series representation

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

term-by-term to get

$$\begin{aligned} (e^x)' &= 0 + 1 + 2\frac{x}{2!} + 3\frac{x^2}{3!} + 4\frac{x^3}{4!} + \cdots \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\ &= e^x. \end{aligned}$$

(c) Starting from the formula $e^x = \sum_{n=0}^{\infty} x^n/n!$ we get the formula

$$e^{-x} = \sum_{n=0}^{\infty}(-x)^n/n! = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots .$$

Reviewing the material on Cauchy products from the end of Section 2.8 we now write

$$\begin{aligned}
(e^x)(e^{-x}) &= (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots)(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots) \\
&= 1 + (-1+1)x + (\frac{1}{2!} - 1 + \frac{1}{2!})x^2 + (\frac{-1}{3!} + \frac{1}{2!} - \frac{1}{2!} + \frac{1}{3!})x^3 + \cdots \\
&= 1
\end{aligned}$$

The key to the above calculation is to use the binomial formula to show that the coefficient for x^n is

$$\sum_{k=0}^{n} \frac{1}{k!} \frac{(-1)^{n-k}}{(n-k)!} = \frac{1}{n!}(-1+1)^n = 0 \quad \text{for all } n \geq 1.$$

The point of this exercise is to illustrate that if we take the power series representation for e^x to be the *definition* of the exponential function, then familiar statements such as $(e^x)' = e^x$ and $e^{-x} = 1/e^x$ follow naturally from the definition.

Exercise 6.6.7. Applying the definition of the error function at zero we find

$$\begin{aligned}
E_N^{(n)}(0) &= f^{(n)}(0) - S_N^{(n)}(0) \\
&= f^{(n)}(0) - n!a_n \\
&= f^{(n)}(0) - n!\frac{f^{(n)}(0)}{n!} = 0
\end{aligned}$$

for all $n = 0, 1, 2, \cdots, N$.

Exercise 6.6.8. By applying the Generalized Mean Value Theorem to the functions $E_N(x)$ and x^{N+1} on the interval $[0, x]$ we know that there exists a point $x_1 \in (0, x)$ such that

$$E_N(x) = \frac{x^{N+1}}{(N+1)} \frac{E_N'(x_1)}{x_1^N}.$$

Now apply the Generalized Mean Value Theorem to the functions $E_N'(x)$ and x^N on the interval $[0, x_1]$, to get that there exists a point $x_2 \in (0, x_1)$ where

$$\frac{E_N'(x_1)}{x_1^N} = \frac{E_N''(x_2)}{Nx_2^{N-1}}.$$

Substituting this observation into our earlier result gives

$$E_N(x) = \frac{x^{N+1}}{(N+1)} \frac{E_N'(x_1)}{x_1^N} = \frac{x^{N+1}}{(N+1)(N)} \frac{E_N''(x_2)}{x_2^{N-1}}.$$

Continuing in this manner we find

$$\begin{aligned} E_N(x) &= \frac{x^{N+1}}{(N+1)}\frac{E_N'(x_1)}{x_1^N} = \frac{x^{N+1}}{(N+1)(N)}\frac{E_N''(x_2)}{x_2^{N-1}} = \cdots \\ &= \frac{x^{N+1}}{(N+1)!}\frac{E_N^{(N+1)}(x_{N+1})}{x_{N+1}^{N-N}} \end{aligned}$$

where $x_{N+1} \in (0, x_N) \subseteq \cdots \subseteq (0, x)$. Now set $c = x_{N+1}$ and, noting that $c^{N-N} = 1$, write

$$E_N(x) = \frac{E_N^{(N+1)}(c)}{(N+1)!}x^{N+1}.$$

Finally, because $S_N^{(N+1)} = 0$ we have that $E_N^{(N+1)}(x) = f^{(N+1)}(x)$ and it follows that

$$E_N(x) = \frac{E_N^{(N+1)}(c)}{(N+1)!}x^{N+1} = \frac{f^{(N+1)}(c)}{(N+1)!}x^{N+1}.$$

This proves Lagranges's Remainder Theorem.

Exercise 6.6.9. By the definition of the derivative, we have

$$g'(0) = \lim_{x\to 0}\frac{g(x)-g(0)}{x-0} = \lim_{x\to 0}\frac{e^{-1/x^2}}{x} = \lim_{x\to 0}\frac{1/x}{e^{1/x^2}}$$

where both numerator and denominator tend to ∞ as x approaches zero. Applying the ∞/∞ version of L'Hospital's rule we can write

$$g'(0) = \lim_{x\to 0}\frac{-1/x^2}{e^{1/x^2}(-2/x^3)} = \lim_{x\to 0}\frac{x}{2e^{1/x^2}} = 0.$$

Exercise 6.6.10. Computing the derivatives for $x \neq 0$ we find

$$g'(x) = \frac{2e^{-1/x^2}}{x^3}, \qquad g''(x) = -\frac{6e^{-1/x^2}}{x^4} + \frac{4e^{-1/x^2}}{x^6},$$

$$g'''(x) = \frac{24e^{-1/x^2}}{x^5} - \frac{36e^{-1/x^2}}{x^7} + \frac{8e^{-1/x^2}}{x^9},$$

and in general we can write

$$g^{(n)}(x) = \sum_{k=1}^{n}\frac{f(n,k)e^{-1/x^2}}{x^{2k+n}}$$

where $f(n,k)$ describes the coefficients.

Exercise 6.6.11. To compute $g''(0)$ from the definition we substitute the formula for $g'(x)$ away from zero to get

$$g''(0) = \lim_{x\to 0}\frac{g'(x)}{x} = \lim_{x\to 0}\frac{2e^{-1/x^2}}{x^4} = \lim_{x\to 0}\frac{2/x^4}{e^{1/x^2}}.$$

Applying L'Hospital's rule we can write

$$g''(0) = \lim_{x\to 0} \frac{-8/x^5}{-2e^{1/x^2}/x^3} = \lim_{x\to 0} \frac{4/x^2}{e^{1/x^2}}.$$

One more application of L'Hospital's rule lets us conclude

$$g''(0) = \lim_{x\to 0} \frac{-8/x^3}{-2e^{1/x^2}/x^3} = \lim_{x\to 0} \frac{-4}{e^{1/x^2}} = 0.$$

In general, whenever we have a quotient of the form $x^{-m}/e^{1/x^2}$, what we discover is that by repeated applications of L'Hospital's rule we can show

$$\lim_{x\to 0} \frac{1/x^m}{e^{1/x^2}} = 0.$$

An induction argument now proves that $g^{(n)}(0) = 0$ for all n. To see this explicitly, observe that if $g^{(n)}(0) = 0$ then our formula from the previous exercise yields

$$\begin{aligned} g^{(n+1)}(0) &= \lim_{x\to 0} \frac{g^{(n)}(x) - g^{(n)}(0)}{x - 0} \\ &= \lim_{x\to 0} \frac{g^{(n)}(x)}{x} \\ &= \lim_{x\to 0} \sum_{k=1}^{n} \frac{f(n,k)/x^{2k+n+1}}{e^{1/x^2}}. \end{aligned}$$

Because this limit is zero for each term in the sum, we see that $g^{(n+1)}(0) = 0$, and it follows that $g^{(n)}(0) = 0$ for all values of n.

Exercise 6.6.12. We have discussed the fact that g is an infinitely differentiable function as long as e^x has this property. This means that g has a Taylor series. Because $g^{(n)}(0) = 0$ for all n, every coefficient in the series expansion is 0. Thus the Taylor series exists and converges at every value of x to zero. But notice that $g(x) \neq 0$ whenever $x \neq 0$. *The Taylor series for $g(x)$ exists and converges, but it does not converge to $g(x)$ apart from the center point $x = 0$.* Thus, every infinitely differentiable function cannot be represented by its Taylor series.

Chapter 7

The Riemann Integral

7.1 Discussion: How Should Integration be Defined?

7.2 The Definition of the Riemann Integral

Exercise 7.2.1. Momentarily fix the partition P. Then Lemma 7.2.4 implies

$$L(f, P) \leq U(f, P') \quad \text{for all partitions } P'.$$

Because $L(f, P)$ is a lower bound for the set of upper sums, it must be less than the greatest lower bound for this set; i.e., $L(f, P) \leq U(f)$. But P is arbitrary in this discussion meaning that $U(f)$ is an upper bound on the set of lower sums. From the definition of the supremum we get $L(f) \leq U(f)$ as desired.

Exercise 7.2.2. (a) $L(f, P) = 17/2$, $U(f, P) = 23/2$, and $U(f, P) - L(f, P) = 3$.

(b) In this case $U(f, P) - L(f, P) = 2$.

(c) Adding any new point to $\{1, 3/2, 2, 5/2, 3\}$ will do it.

Exercise 7.2.3. For any partition P of $[a, b]$ we have

$$L(f, P) = \sum_{k=1}^{n} k(x_k - x_{k-1}) = k(b - a),$$

as well as

$$U(f, P) = \sum_{k=1}^{n} k(x_k - x_{k-1}) = k(b - a).$$

Thus $L(f) = k(b - a)$ and $U(f) = k(b - a)$. Because the upper and lower integrals are equal, the function $f(x) = k$ is integrable with $\int_a^b f = k(b - a)$.

Exercise 7.2.4. (a) ($\Rightarrow$) Assume there exists a sequence of partitions (P_n) satisfying

$$\lim_{n\to\infty}[U(f,P_n)-L(f,P_n)]=0.$$

Given $\epsilon>0$, choose P_N from this sequence so that $U(f,P_N)-L(f,P_N)<\epsilon$. Then Theorem 7.2.8 implies f is integrable.

($\Leftarrow$) Conversely, if f is integrable then given $\epsilon_n=1/n$, Theorem 7.2.8 implies that there exists a partition P_n satisfying $U(f,P_n)-L(f,P_n)<1/n$. It follows that

$$\lim_{n\to\infty}[U(f,P_n)-L(f,P_n)]=0,$$

as desired.

(b) For the partition P_n we have $x_k=k/n$, $m_k=(k-1)/n$ and $M_k=k/n$. Then

$$U(f,P_n)=\sum_{k=1}^{n}\frac{k}{n}(1/n)=\frac{1}{n^2}\sum_{k=1}^{n}k=\frac{1}{n^2}\left(\frac{n(n+1)}{2}\right),$$

and

$$L(f,P_n)=\sum_{k=1}^{n}\frac{(k-1)}{n}(1/n)=\frac{1}{n^2}\sum_{k=1}^{n}(k-1)=\frac{1}{n^2}\left(\frac{(n-1)n}{2}\right).$$

(c) Now we may compute

$$\lim_{n\to\infty}[U(f,P_n)-L(f,P_n)]=\lim_{n\to\infty}\left[\frac{n(n+1)}{2n^2}-\frac{(n-1)n}{2n^2}\right]=\lim_{n\to\infty}\left[\frac{1}{n}\right]=0.$$

The result in (a) now implies that $f(x)=x$ is integrable.

Exercise 7.2.5. We shall use the criterion in Theorem 7.2.8. The shape of the proof is determined by the triangle inequality estimate

$$\begin{aligned}U(f,P)-L(f,P) &= U(f,P)-U(f_N,P)+U(f_N,P)-L(f_N,P)\\ &\qquad +L(f_N,P)-L(f,P)\\ &\leq |U(f,P)-U(f_N,P)|+(U(f_N,P)-L(f_N,P))\\ &\qquad +|L(f_N,P)-L(f,P)|.\end{aligned}$$

Let $\epsilon>0$ be arbitrary. Because $f_n\to f$ uniformly, we can choose N so that

$$|f_N(x)-f(x)|\leq\frac{\epsilon}{3(b-a)}\quad\text{for all } x\in[a,b].$$

Now the function f_N is integrable and so there exists a partition P for which

$$U(f_N,P)-L(f_N,P)<\frac{\epsilon}{3}.$$

Let's consider a particular subinterval $[x_{k-1},x_k]$ from this partition. If

$$M_k=\sup\{f(x):x\in[x_{k-1},x_k]\}\quad\text{and}\quad N_k=\sup\{f_N(x):x\in[x_{k-1},x_k]\},$$

then our choice of f_N guarantees that

$$|M_k - N_k| \leq \frac{\epsilon}{3(b-a)}.$$

From this estimate we can argue that

$$\begin{aligned} |U(f,P) - U(f_N,P)| &= \left| \sum_{k=1}^{n} (M_k - N_k)\Delta x_k \right| \\ &\leq \sum_{k=1}^{n} \frac{\epsilon}{3(b-a)} \Delta x_k = \frac{\epsilon}{3}. \end{aligned}$$

Similarly we can show

$$|L(f_N,P) - L(f,P)| \leq \frac{\epsilon}{3}.$$

Putting this altogether, we see that using our choices of f_N and P in the preliminary estimate gives

$$U(f,P) - L(f,P) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

By the criterion in Theorem 7.2.8 we conclude that the uniform limit of integrable functions is integrable.

Exercise 7.2.6. As in the previous exercise, we shall use the criterion in Theorem 7.2.8. Let P be a partition where all the subintervals have equal length $\Delta x = x_k - x_{k-1}$. Because the function is increasing, on each subinterval $[x_{k-1}, x_k]$ we have

$$M_k = f(x_k) \quad \text{and} \quad m_k = f(x_{k-1}).$$

Thus,

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{k=1}^{n} (M_k - m_k)\Delta x \\ &= \Delta x \sum_{k=1}^{n} (f(x_k) - f(x_{k-1})) \\ &= \Delta x(b-a). \end{aligned}$$

Given $\epsilon > 0$, choose a partition P_ϵ to have equal subintervals with common length satisfying $\Delta x < \epsilon/(b-a)$. The previous calculation then shows

$$U(f,P_\epsilon) - L(f,P_\epsilon) = \Delta x(b-a) < \frac{\epsilon}{b-a}(b-a) = \epsilon.$$

7.3 Integrating Functions with Discontinuities

Exercise 7.3.1. (a) Let P be an arbitrary partition of $[0,1]$. On any subinterval $[x_{k-1}, x_k]$, it must be that $m_k = \inf\{h(x) : x \in [x_{k-1}, x_k]\} = 1$, and it follows that

$$L(h, P) = \sum_{k=1}^{n} m_k \Delta x_k = \sum_{k=1}^{n} \Delta x_k = 1.$$

(b) Consider the partition $P = \{0, .95, 1\}$. Then

$$U(h, P) = (1)(.95) + (2)(.05) = 1.05.$$

(c) Consider the partition $P_\epsilon = \{0, 1 - \epsilon/2, 1\}$. Then

$$U(h, P_\epsilon) = (1)\left(1 - \frac{\epsilon}{2}\right) + (2)\left(\frac{\epsilon}{2}\right) = 1 + \frac{\epsilon}{2}.$$

The implication is that for this partition we have $U(h, P_\epsilon) - L(h, P_\epsilon) < \epsilon$, proving that h is integrable.

Exercise 7.3.2. First write

$$\mathbf{Q} \cap [0,1] = \{r_1, r_2, r_3, \ldots\},$$

which is allowed because $\mathbf{Q} \cap [0,1]$ is a countable set. For each $n \in \mathbf{N}$ define

$$g_n(x) = \begin{cases} 1 & \text{if } x \in \{r_1, r_2, \ldots, r_n\} \\ 0 & \text{otherwise.} \end{cases}$$

Because g_n has only a finite number of discontinuities we know it is integrable, and $g_n \to g$ pointwise is easy to verify. This example is discussed explicitly in the Epilogue to Chapter 7.

Exercise 7.3.3. Assume f is not continuous on the finite set $\{z_1, z_2, \ldots, z_N\}$. We shall build a partition P in two steps: first handling the "bad" or discontinuous points, and then handling the remainder of the interval $[a, b]$.

Assume f is bounded by M and let $\epsilon > 0$. Around each z_i construct disjoint subintervals small enough so that the sum of the lengths of all N of these comes to less than $\epsilon/(4M)$. Focusing on just these subintervals we see that

$$\sum_{\text{bad pts}} (M_k - m_k)\Delta x_k \leq \sum_{\text{bad pts}} 2M \Delta x_k = 2M\left(\frac{\epsilon}{4M}\right) = \frac{\epsilon}{2}.$$

If O is the union of the open subintervals that surround each z_i, then $[a, b] \backslash O$ is a compact set. Because f is continuous on this set, it is uniformly continuous and so there exists a $\delta > 0$ with the property that

$$|f(x) - f(y)| < \frac{\epsilon}{2(b-a)} \qquad \text{whenever } |x - y| < \delta.$$

Focusing on the intervals that make up $[a,b]\backslash O$ (the "good points"), we partition these so that all the resulting subintervals have length less than δ. This puts us into a situation like the one in Theorem 7.2. In particular we get that

$$\sum_{\text{good pts}} (M_k - m_k)\Delta x_k < \frac{\epsilon}{2(b-a)} \sum_{\text{good pts}} \Delta x_k < \frac{\epsilon}{2(b-a)}(b-a) = \frac{\epsilon}{2}.$$

Putting these two parts together we see

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{\text{bad pts}} (M_k - m_k)\Delta x_k + \sum_{\text{good pts}} (M_k - m_k)\Delta x_k \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

and by the criterion in Theorem 7.2.8, f is integrable.

Exercise 7.3.4. (a) Assume f is integrable so that $U(f) = L(f) = \int_a^b f$. Now let f_0 be the modified function where we have changed the value of f at x_0. Set $D = |f(x_0) - f_0(x_0)|$. We want to prove that $U(f_0) = U(f)$ and $L(f_0) = L(f)$.

Let $\epsilon > 0$ be arbitrary. To argue that $U(f_0) = U(f)$, it is sufficient to find a partition for which $U(f_0, P) < U(f) + \epsilon$. Because $U(f) = \inf\{U(f,P) : P \in \mathcal{P}\}$, we know there exists a partition P where

$$U(f,P) < U(f) + \epsilon/2.$$

The first step is let P' be a refinement of P with the property that the interval(s) containing x_0 have width less than $\epsilon/(4D)$. Because $P \subseteq P'$ we know $U(f,P') \leq U(f,P)$. Now observe that because f and f_0 agree everywhere except at x_0 it follows that

$$|U(f,P') - U(f_0,P')| < D(2\Delta x) < \frac{\epsilon}{2}.$$

(The extra 2 is needed in case the point x_0 is an endpoint of an interval in P' and is thus contained in two subintervals.) Finally, we see that

$$U(f_0,P') < U(f,P') + \frac{\epsilon}{2} \leq \left(U(f) + \frac{\epsilon}{2}\right) + \frac{\epsilon}{2} = U(f) + \epsilon,$$

and we conclude that $U(f_0) = U(f)$.. The proof that $L(f_0) = L(f)$ is similar.

(b) This follows using an induction argument.

(c) Dirichlet's function differs from the zero function in only a countable number of points but is not integrable.

Exercise 7.3.5. Every interval contains points where $f(x) = 0$, and thus it follows that $L(f,P) = 0$ for every partition P. This implies that $L(f) = 0$. It remains to show that $U(f) = 0$.

Let $\epsilon > 0$ be arbitrary and consider the *finite* set $\{1, 1/2, 1/3, \ldots, 1/N\}$ consisting of points of the form $1/n$ that satisfy $1/n \geq \epsilon/2$. Because this set is finite, we may construct a set of disjoint intervals around each of these points with the property that the sum of the lengths of these intervals comes to less than $\epsilon/2$. Letting P be the partition that results from taking the union of these intervals together with the interval $[0, \epsilon/2]$, it follows that

$$U(f, P) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

and f integrates to zero.

Exercise 7.3.6. (a) This proof is nearly indentical to the argument in Exercise 7.3.3. In particular, we shall build a partition P in two steps: first handling the "bad" or discontinuous points, and then handling the "good" or continuous parts of the interval $[a, b]$.

Assume f is bounded by M and let $\epsilon > 0$. Because our set of discontinuities has content zero, we may let $\{O_1, \ldots, O_N\}$ be a collection of open intervals that covers the set of discontinuous points and satisfies

$$\sum_{n=1}^{N} |O_n| \leq \frac{\epsilon}{4M}.$$

Focusing on just these subintervals we see that $|O_n| = \Delta x_n$ and

$$\sum_{\text{bad pts}} (M_n - m_n)\Delta x_n \leq \sum_{\text{bad pts}} 2M\Delta x_n = 2M\left(\frac{\epsilon}{4M}\right) = \frac{\epsilon}{2}.$$

If $O = \bigcup_{n=1}^{N} O_n$ then $[a, b]\backslash O$ is a compact set. Because f is continuous on this set, it is uniformly continuous and so there exists a $\delta > 0$ with the property that

$$|f(x) - f(y)| < \frac{\epsilon}{2(b-a)} \qquad \text{whenever } |x - y| < \delta.$$

Focusing on the intervals that make up $[a, b]\backslash O$ (the "good points"), we partition these so that all the resulting subintervals have length less than δ. This puts us into a situation like the one in Theorem 7.2. In particular we get that

$$\sum_{\text{good pts}} (M_k - m_k)\Delta x_k < \frac{\epsilon}{2(b-a)} \sum_{\text{good pts}} \Delta x_k < \frac{\epsilon}{2(b-a)}(b - a) = \frac{\epsilon}{2}.$$

Putting these two parts together we see

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{\text{bad pts}} (M_k - m_k)\Delta x_k + \sum_{\text{good pts}} (M_k - m_k)\Delta x_k \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

and by the criterion in Theorem 7.2.8, f is integrable.

(b) Given a finite set $\{z_1, z_2, \ldots, z_N\}$ and $\epsilon > 0$, let $O_n = V_{\epsilon'}(z_n)$ where $\epsilon' = \epsilon/(2N)$. Then $|O_n| = \epsilon/N$ and the sum of these lengths is equal to ϵ, as desired.

(c) Recall that we defined the Cantor set C as the intersection

$$C = \bigcap_{n=0}^{\infty} C_n,$$

where C_n consists of 2^n closed intervals of length $1/3^n$. Given $\epsilon > 0$, choose m so that $2^m(1/3^m) < \epsilon/2$. Now it would be nice if we could just use the intervals that make up C_m as our covering set. However, the definition of content zero requires that we use *open* intervals. To fix this, we can imbed each of the 2^m closed intervals that make up C_m in a slightly larger open interval whose length is equal to $1/3^m + (\epsilon/2)2^{-m}$. This collection of open intervals will then contain C (because C_m does) and the lengths will sum to

$$2^m[1/3^m + \frac{\epsilon}{2}2^{-m}] < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

as desired.

(d) The fact that h is integrable follows immediately from (a), (c), and the result in Exercise 4.3.12. Because C contains no intervals, we see that $L(h, P) = 0$ for every partition P and so it must be that $\int_0^1 h = 0$.

7.4 Properties of the Integral

Exercise 7.4.1. (a) Let $\epsilon > 0$ be arbitrary and choose x_1 and x_2 so that $M' - \epsilon/2 < |f(x_1)|$ and $m' + \epsilon/2 > |f(x_2)|$. Then using Exercise 1.2.5 (b) we can write

$$\begin{aligned}(M' - m') - \epsilon &\leq |f(x_1)| - |f(x_2)| \\ &\leq |f(x_1) - f(x_2)| \leq M - m.\end{aligned}$$

(b) Let $\epsilon > 0$. Because f is integrable, there exists a partition P satisfying $U(f, P) - L(f, P) < \epsilon$. But now from part (a) it follows that

$$U(|f|, P) - L(|f|, P) \leq U(f, P) - L(f, P) < \epsilon,$$

and the result follows.

(c) Because $-|f| \leq f \leq |f|$ and all of these functions are integrable, we know from Theorem 7.4.2 (iv) and (ii) that

$$-\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f|.$$

Exercise 7.4.2. From Theorem 7.4.1 we get

$$\int_c^b f = \int_c^a f + \int_a^b f.$$

Then Definition 7.4.3 allows us to write $\int_c^a f = -\int_a^c f$ which, when substituted into the first statement, gives us

$$\int_a^c f + \int_c^b f = \int_a^b f.$$

Exercise 7.4.3. The properties of the integral in Theorem 7.4.2 allow us to write

$$\left|\int_a^b f_n - \int_a^b f\right| = \left|\int_a^b (f_n - f)\right| \leq \int_a^b |f_n - f|.$$

Let $\epsilon > 0$ be arbitrary. Because $f_n \to f$ uniformly, there exists an N such that

$$|f_n(x) - f(x)| < \epsilon/(b-a) \quad \text{for all } n \geq N \text{ and } x \in [a,b].$$

Thus, for $n \geq N$ we see that

$$\begin{aligned} \left|\int_a^b f_n - \int_a^b f\right| &\leq \int_a^b |f_n - f| \\ &\leq \int_a^b \frac{\epsilon}{b-a} = \epsilon, \end{aligned}$$

and the result follows.

Exercise 7.4.4. (a) False. Dirichlet's function is a counterexample.

(b) False. The functions in Exercise 7.3.5 and Exercise 7.3.6 are counterexamples.

(c) True. Because g is continuous at x_0 with $g(x_0) > 0$, there exists a δ-neighborhood $V_\delta(x_0)$ with the property that $g(x) \geq g(x_0)/2$ for all $x \in V_\delta(x_0)$. Now let P be a partition that contains the interval $V_\delta(x_0)$. When we compute the lower sum $L(f,P)$ with respect to this partition, the contribution from the subinterval $V_\delta(x_0)$ is at least $[g(x_0)/2]2\delta > 0$. The assumption that $g(x) \geq 0$ on the rest of $[a,b]$ guarantees that there are no negative terms in the sum $L(f,P)$, and it follows that

$$\int_a^b f = L(f) \geq L(f,P) > 0.$$

(d) True. We again argue using lower sums. Because the value of the integral is strictly positive, there must exist a partition P such that $L(f,P) > 0$. But this implies that there is at least one subinterval $[c,d]$ in the partition P where the product $m(d-c)$ is strictly positive. Because $m = \inf\{f(x) : x \in [c,d]\}$, setting $\delta = m$ gives the result.

Exercise 7.4.5. (a) Consider a particular subinterval $[x_{k-1}, x_k]$ of P and let

$$M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}, \quad M_k' = \sup\{g(x) : x \in [x_{k-1}, x_k]\}, \text{ and}$$

$$M_k'' = \sup\{f(x) + g(x) : x \in [x_{k-1}, x_k]\}.$$

Because $M_k + M'_k$ is an upper bound for the set $\{f(x) + g(x) : x \in [x_{k-1}, x_k]\}$ it follows that $M''_k \leq M_k + M'_k$. This inequality leads directly to the conclusion that $U(f + g, P) \leq U(f, P) + U(g, P)$.

The two sides are usually not equal because the functions f and g could easily take on their larger values in different places of each subinterval. For example, consider $f(x) = x$ and $g(x) = 1 - x$ on the interval $[0, 1]$. Then

$$M_k = 1, \quad M'_k = 1 \quad \text{and} \quad M''_k = 1,$$

so we have $M''_k < M_k + M'_k$.

The inequality for lower sums takes the form $L(f+g, P) \geq L(f, P)+L(g, P)$.

(b) Because f and g are integrable, there exist sequences of partitions (P_n) and (Q_n) such that

$$(1) \qquad \lim_{n\to\infty}[U(f, P_n) - L(f, P_n)] = 0 \quad \text{and} \quad \lim_{n\to\infty}[U(g, Q_n) - L(g, Q_n)] = 0.$$

For each n, let R_n be the common refinement $R_n = P_n \cup Q_n$. Then part (a) of this exercise and Lemma 7.2.3 imply

$$\begin{aligned} U(f + g, R_n) - L(f + g, R_n) &\leq [U(f, R_n) + U(g, R_n)] - [L(f, R_n) + L(g, R_n)] \\ &\leq [U(f, P_n) + U(g, Q_n)] - [L(f, P_n) + L(g, Q_n)] \\ &\leq [U(f, P_n) - L(f, P_n)] + [U(g, Q_n) - L(g, Q_n)]. \end{aligned}$$

From (1) it now follows that

$$\lim_{n\to\infty}[U(f + g, R_n) - L(f + g, R_n)] = 0,$$

and the result follows.

Exercise 7.4.6. (a) Set

$$f_n(x) = \begin{cases} (-1)^n n & \text{if } 0 < x < 1/n \\ 0 & \text{if } x = 0 \text{ or } x \geq 1/n. \end{cases}$$

Then $\int_0^1 f_n = (-1)^n$, and the limit of these integrals does not exist.

(b) Set

$$f_n(x) = \begin{cases} n^2 & \text{if } 0 < x < 1/n \\ 0 & \text{if } x = 0 \text{ or } x \geq 1/n. \end{cases}$$

Then $\int_0^1 f_n = n$ which is unbounded as $n \to \infty$.

(c) Sure. Rather than putting in step functions over the intervals $[0, 1/n]$ we could put taller and taller triangular "tents" that would be continuous and still create the same effect.

(d) This is a delicate question that requires a deeper study of the integral to work out in any satisfactory way. In all of the examples in this exercise, the sequence of badly behaving functions has been unbounded. This turns out to be a requirement. With a stronger integral it is possible to prove that if f_n are integrable functions that are uniformly bounded, and if $f_n \to f$ pointwise on $[a, b]$, then $\int_a^b f_n \to \int_a^b f$.

Exercise 7.4.7. This exercise requires the stronger hypothesis (not included in earlier editions) that g_n and g are uniformly bounded; i.e., that there exists $M > 0$ satisfying

$$|g(x)| \leq M \quad \text{and} \quad |g_n(x)| \leq M \quad \text{for all } n \text{ and } x \in [0,1].$$

As a first step we use the properties of the integral proved in this section to write

$$\left| \int_0^1 g_n - \int_0^1 g \right| \leq \int_0^1 |g_n - g| = \int_0^\delta |g_n - g| + \int_\delta^1 |g_n - g|.$$

Let $\epsilon > 0$. Let's first pick $\delta < \epsilon/(4M)$. Having chosen δ, we know $g_n \to g$ uniformly on $[\delta, 1]$, so there exists an N such that $|g_n - g| < \epsilon/2$ for all $n \geq N$. It follows that if $n \geq N$ then

$$\begin{aligned} \left| \int_0^1 g_n - \int_0^1 g \right| &\leq \int_0^\delta |g_n - g| + \int_\delta^1 |g_n - g| \\ &\leq \int_0^\delta 2M + \int_\delta^1 \epsilon/2 \\ &\leq (2M)\delta + \epsilon/2 < \epsilon, \end{aligned}$$

and the result follows.

7.5 The Fundamental Theorem of Calculus

Exercise 7.5.1. Assume g is continuous on $[a,b]$ and set $G(x) = \int_a^x g(t)\,dt$. By part (ii) of the Fundamental Theorem, g is the derivative of G.

Exercise 7.5.2. (a) For $f(x) = |x|$ we get

$$F(x) = \begin{cases} -x^2/2 + 1/2 & \text{if } x < 0 \\ x^2/2 + 1/2 & \text{if } x \geq 0 \end{cases}$$

In this case, F is continuous and differentiable with $F'(x) = f(x)$ for all $x \in \mathbf{R}$. This follows from FTC but it is interesting to check this directly from the formula for F, especially at $x = 0$ where we get $F'(0) = 0$ from both sides.

(b) This time we get

$$F(x) = \begin{cases} x + 1 & \text{if } x < 0 \\ 2x + 1 & \text{if } x \geq 0 \end{cases}$$

A sketch of F is valuable and illustrates in particular that F is continuous on all of $\mathbf{R}$ but fails to be differentiable at $x = 0$ due to the "corner" on the graph. If $x \neq 0$, then we certainly get $F'(x) = f(x)$ as predicted by FTC.

Exercise 7.5.3. The Mean Value Theorem does not require $F(x)$ to be differentiable at the endpoints so we could get by with assuming that F is continuous on $[a,b]$ and differentiable on (a,b) with $F'(x) = f(x)$ for all $x \in (a,b)$. By appealing to Theorem 7.4.1 we could in fact weaken the hypothesis even more to allow $F'(x) = f(x)$ to fail at an arbitrary finite number of points.

Exercise 7.5.4. (a) $H(1) = 0$. Using FTC we see that $H'(x) = 1/x$ for all $x > 0$.

(b) Given $x < y$, apply the Mean Value Theorem to H on the interval $[x, y]$ to get $H(y) - H(x) = H'(t)(y - x)$ for some $t \in (x, y)$. Because $H'(t) = 1/t > 0$, it follows that $H(y) > H(x)$.

(c) Using the Chain Rule, we see that

$$g'(x) = H'(cx) \cdot c = \frac{1}{cx} \cdot c = \frac{1}{x}.$$

Thus g and H have the same derivative and so by Corollary 5.3.4 to the Mean Value Theorem we know that $g(x) = H(x) + k$, or

$$H(cx) = H(x) + k,$$

for some constant k. To determine k, set $x = 1$ to get $H(c) = H(1) + k = k$, and the result follows.

Exercise 7.5.5. Because $f_n' \to g$ uniformly on any interval of the form $[a, x]$, it follows from Theorem 7.4.4 that

$$\lim_{n\to\infty} \int_a^x f' = \int_a^x g.$$

Taking the limit as $n \to \infty$ on each side of the equation $\int_a^x f_n' = f_n(x) - f_n(a)$ leads to the equation

$$f(x) = f(a) + \int_a^x g.$$

But g is the uniform limit of continuous functions and so g must also be continuous. Part (ii) of the Fundamental Theorem of Calculus then implies that $f'(x) = g(x)$, as desired.

Exercise 7.5.6. This exercise requires that f be continuous. If we set $G(x) = \int_a^x f$, then given this extra assumption about f, it follows from part (ii) of FTC that $G'(x) = f(x)$. Because $F'(x) = f(x)$ as well, F and G have the same derivative and a corollary to the Mean Value Theorem implies

$$G(x) = F(x) + k, \tag{1}$$

for some constant k. To compute k, set $x = a$ in equation (1) to get $0 = F(a)+k$ or $k = -F(a)$. Substituting this back into (1) and setting $x = b$ we find

$$\int_a^b f = G(b) = F(b) - F(a).$$

Exercise 7.5.7. The idea is to apply the Mean Value Theorem to the function $G(x) = \int_a^x g$ on the interval $[a, b]$. Note that g is continuous and so G is differentiable and thus MVT can be employed. In this case we get that there exists a point $c \in (a, b)$ where

$$G'(c) = \frac{G(b) - G(a)}{b - a} = \frac{1}{b - a} \int_a^b g.$$

Because $G'(c) = g(c)$, this gives the desired result.

Exercise 7.5.8. (a) Let P be a partition of $[a, b]$ and consider a particular subinterval $[x_{k-1}, x_k]$ of P. Because f' is continuous, we may use FTC to write

$$f(x_k) - f(x_{k-1}) = \int_{x_{k-1}}^{x_k} f'.$$

Computing the variation with respect to this particular partition, we get

$$\begin{aligned} \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| &= \sum_{k=1}^{n} \left| \int_{x_{k-1}}^{x_k} f' \right| \\ &\leq \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} |f'| = \int_a^b |f'|. \end{aligned}$$

What we discover is that $\int_a^b |f'|$ is an upper bound on the set of variations, and it follows that $Vf \leq \int_a^b |f'|$ because Vf is the least upper bound of this set.

(b) Given a partition P, this time we apply MVT to an arbitrary subinterval $[x_{k-1}, x_k]$ to get

$$f(x_k) - f(x_{k-1}) = f'(c_k)\Delta x_k \quad \text{for some } c_k \in (x_{k-1}, x_k).$$

Because lower sums are computed by taking the infimum over each subinterval, this allows us to write

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| = \sum_{k=1}^{n} |f'(c_k)|\Delta x_k \geq L(|f'|, P).$$

It follows that Vf is an upper bound for the set of lower sums for $|f'|$ and we immediately get $Vf \geq \int_a^b |f'|$.

Parts (a) and (b) then imply $Vf = \int_a^b |f'|$.

Exercise 7.5.9. In this case, $H(x) = x$. Any doubts about whether this formula holds when $x = 1$ should be alleviated by the fact that we know H is continuous on all of $\mathbf{R}$. It is evident, then, that H is differentiable everywhere. The point to make is that the statement in FTC part (ii) (if g is continuous then G is differentiable) does not have a converse unless we are more specific about the type of discontinuity in g.

Exercise 7.5.10. Let $L_1 = \lim_{x\to c^-} f(x)$. If we insist that $f(c) = L_1$, then the argument in the text for FTC part (ii) can be used to show

$$\lim_{x\to c^-} \frac{F(x) - F(c)}{x - c} = f(c) = L_1.$$

On the other hand, if we let $L_2 = \lim_{x\to c+} f(x)$, and set $f(c) = L_2$, then the same argument also shows that

$$\lim_{x\to c^+} \frac{F(x) - F(c)}{x - c} = f(c) = L_2.$$

Because $L_1 \neq L_2$ the result is that the graph of F has a "corner" at $x = c$ and is not differentiable.

Exercise 7.5.11. Let $h(x) = \sum_{n=1}^{\infty} u_n(x)$ be the function defined in Exercise 6.4.8. Note that $0 < h(x) < 1$ and h is increasing. By Exercise 7.2.6, h is integrable over any interval and thus we can set

$$H(x) = \int_0^x h(t)\, dt.$$

Part (ii) of FTC implies that H is continuous (and differentiable at every irrational point.) Also, if $x < y$ then

$$H(y) - H(x) = \int_x^y h(t) \geq 0,$$

and it follows that H is increasing. Now fix a rational number r_N from the enumeration in Exercise 6.4.8. The fact that h is increasing implies that both $\lim_{x\to r_N^-} h(x)$ and $\lim_{x\to r_N^+} h(x)$ exist, and we can show that they must differ by r_N. Then Exercise 7.5.10 implies that H is not differentiable at r_N, and hence at any rational point in $\mathbf{R}$.

7.6 Lebesgue's Criterion for Riemann Integrability

Exercise 7.6.1. (a) Because $t(x) = 0$ for every irrational and the irrationals are dense in $\mathbf{R}$, it follows that $L(t, P) = 0$ for every partition P.

(b) If $x \in D_{\epsilon/2}$ then x must be a rational number of the form $x = m/n$ with $n \leq 2/\epsilon$. The number of such points in the interval $[0, 1]$ is finite.

(c) Let $\{x_1, x_2, \dots, x_N\}$ be the finite set of points in $D_{\epsilon/2} \cap [0, 1]$. Now build a partition P by constructing small, disjoint intervals around each x_k with length less than $\epsilon/(2N)$. Because $|t(x)| \leq 1$, the contribution of all of the intervals containing "bad points" to the upper sum will be at most $N \cdot (\epsilon/(2N)) \cdot 1 = \epsilon/2$. On all of the other intervals we have $|t(x)| \leq \epsilon/2$ and so, taken altogether, these contribute at most $\epsilon/2$ to the value of the upper sum. It follows that $U(t, P) \leq \epsilon/2 + \epsilon/2 = \epsilon$. Thus, t is integrable and $\int_0^1 t = 0$.

Exercise 7.6.2. Because C contains no intervals, $g(x)$ will equal zero at least once in every subinterval of every partition P. It follows that $L(g, P) = 0$. Therefore, given $\epsilon > 0$, our task is to find a partition where $U(g, P) < \epsilon$. From this we will be able to conclude that g is integrable and $\int_0^1 g = 0$.

The set C_m consists of 2^m intervals of length 3^m, so choose m large enough so that $2^m/3^m < \epsilon/2$. It would be nice to simply use the intervals that make up C_m to construct our partition, but we need to worry a bit that the endpoints of these intervals are in C. To fix this, we can imbed each of the 2^m closed intervals that make up C_m in a slightly larger interval whose length is equal to $1/3^m + (\epsilon/2)2^{-m}$. This collection of intervals then contains C in its interior and the lengths sum to

$$2^m[1/3^m + \frac{\epsilon}{2}2^{-m}] < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Now two things follow if we let P be the partition obtained from these slightly enlarged intervals from C_m. First, because $|g(x)| \leq 1$, the contribution of all of the intervals containing points of C to $U(g, P)$ is bounded by ϵ. Second, on all of the other intervals, our function is zero and there are no contributions to $U(g, P)$. It follows that $U(g, P) \leq \epsilon$, as desired.

Exercise 7.6.3. Let $A = \{a_1, a_2, a_3, \ldots\}$ be a countable set. Given $\epsilon > 0$, let $O_n = V_{\epsilon_n}(a_n)$ where $\epsilon_n = \epsilon/2^{n+1}$. Clearly the collection $\{O_n : n \in \mathbf{N}\}$ covers A and we have

$$\sum_{n=1}^{\infty} |O_n| = \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon.$$

Exercise 7.6.4. In Exercise 7.3.6 we proved that C has content zero which immediately implies C has measure zero.

Exercise 7.6.5. Given $\epsilon > 0$, let $\{O_n : n \in \mathbf{N}\}$ be a collection of open intervals that cover A with the property that $\sum_{n=1}^{\infty} |O_n| \leq \epsilon/2$. Likewise, let $\{P_n : n \in \mathbf{N}\}$ be a collection of open intervals that cover B satisfying $\sum_{n=1}^{\infty} |P_n| \leq \epsilon/2$. Then the collection $\{O_n, P_n : n \in \mathbf{N}\}$ is still countable (the union of countable sets is countable), it forms a cover for the union $A \cup B$, and

$$\sum_{n=1}^{\infty} |O_n| + |P_n| = \sum_{n=1}^{\infty} |O_n| + \sum_{n=1}^{\infty} |P_n| \leq \epsilon$$

as desired.

Now assume we are given a countable collection $\{A_1, A_2, A_3, \ldots\}$ of sets of measure zero. Let $\epsilon > 0$. For each A_k, let $\{O_{k,n} : n \in \mathbf{N}\}$ be a countable collection of open intervals that cover A_k and satisfies $\sum_{n=1}^{\infty} |O_{k,n}| \leq \epsilon/2^k$. It follows that $\{O_{k,n} : n, k \in \mathbf{N}\}$ is a countable collection of open intervals (Theorem 1.4.13 (ii)) whose union certainly covers $\bigcup_{k=1}^{\infty} A_k$. Finally, taking the sum of the lengths of all of the intervals in $\{O_{k,n} : k, n \in \mathbf{N}\}$ involves reordering this set, but the content of Theorem 2.8.1 is that we are justified in simply computing the iterated sum

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} |O_{k,n}| = \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon.$$

This shows $\bigcup_{k=1}^{\infty} A_k$ has measure zero.

Exercise 7.6.6. See Exercise 4.6.8.

Exercise 7.6.7. See Exercises 4.6.9 and 4.6.10

Exercise 7.6.8. See Exercise 4.6.7

Exercise 7.6.9. Assume, for contradiction, that f is not uniformly α-continuous on K. This means that given $\delta_n = 1/n$, there must exist points $x_n, y_n \in K$ such that

$$|x_n - y_n| < \frac{1}{n} \quad \text{and} \quad |f(x_n) - f(y_n)| \geq \alpha.$$

Because $(x_n) \subseteq K$ and K is compact, there exists a convergent subsequence (x_{n_k}). Set $x = \lim x_{n_k}$ and note that $x \in K$. If we consider the corresponding subsequence (y_{n_k}) we see that

$$\begin{aligned} \lim y_{n_k} &= \lim[x_{n_k} + (y_{n_k} - x_{n_k})] \\ &= x + \lim(y_{n_k} - x_{n_k}) = x. \end{aligned}$$

Because (x_{n_k}) and (y_{n_k}) both converge to x, it follows that given any $\delta > 0$ we can find points $x_{n_K}, y_{n_K} \in (x - \delta, x + \delta)$ satisfying $|f(x_{n_K}) - f(y_{n_K})| \geq \alpha$. But this contradicts the assumption that f is α-continuous at x, and we conclude that f must be uniformly α-continuous on K.

Exercise 7.6.10. See Exercise 3.3.8 (c) and (d).

Exercise 7.6.11. Because D has measure zero, we know there exists a *countable* collection of open intervals $\{G_1, G_2, \ldots\}$ whose union contains D and that satisfies

$$\sum_{n=1}^{\infty} |G_n| < \frac{\epsilon}{4M}. \tag{1}$$

But $D_a \subseteq D$ is closed and hence compact. This means we can find a finite collection $\{G_1, \ldots, G_N\}$ that covers D_α and the inequality above in (1) is certainly true for this smaller set.

Exercise 7.6.12. If $x \in K$ then $x \notin D_\alpha$ and it follows that f is α-continuous at x. Because we are removing open intervals from $[a, b]$, we see that K is a closed set (it is a finite union of closed intervals). By Exercise 7.6.9, f is uniformly α-continuous on K.

Exercise 7.6.13. As a first step in constructing P_ϵ we include the intervals from the open cover $\{G_1, G_2, \ldots, G_N\}$. Because $\sum_{n=1}^{N} |G_n| < \epsilon/(4M)$ the contribution of these subintervals to $U(f, P_\epsilon) - L(f, P_\epsilon)$ can be estimated by

$$\sum (M_k - m_k)\Delta x_k < (2M) \sum \Delta x_k \leq (2M)\left(\frac{\epsilon}{4M}\right) = \frac{\epsilon}{2}.$$

Now consider the set $K = [a,b]\backslash\bigcup_{n=1}^N G_n$. The function f is uniformly α-continuous on K and so there exists $\delta > 0$ such that $|f(x) - f(y)| < \alpha$ whenever $|x - y| < \delta$. To finish constructing the partition P_ϵ we take each interval in K and subdivide it until all of the subintervals have length less than δ. The implication here is that on each of these subintervals we get $M_k - m_k \leq \alpha$. Thus, the contribution of all of the subintervals that make up K is less than

$$\sum (M_k - m_k)\Delta x_k \leq \alpha \sum \Delta x_k < \left(\frac{\epsilon}{2(b-a)}\right)(b-a) = \frac{\epsilon}{2}.$$

Altogether then we get

$$U(f, P_\epsilon) - L(f, P_\epsilon) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

and it follows that f is Riemann-integrable.

Exercise 7.6.14. (a) To produce a cover for D_a, let $\{G_1, G_2, \ldots, G_N\}$ be the collection of closed intervals from the partition P_ϵ that contain points of D_α. On each subinterval G_k, it follows that $M_k - m_k \geq \alpha$. This enables us to write

$$\begin{aligned} \alpha\epsilon &> \sum (M_{k'} - m_{k'})\Delta x_{k'} \\ &\geq \sum_{k=1}^N (M_k - m_k)|G_k| \\ &\geq \alpha \sum_{k=1}^N |G_k|. \end{aligned}$$

What we immediately see is that $\sum_{k=1}^N |G_k| < \epsilon$. Now our definition of measure zero requires that our cover for D_α consist of open intervals. To remedy this, we can take each G_k to be open and cover the finite number of endpoints we have lost with intervals chosen small enough to keep the sum less than ϵ.

(b) From (a) we see that D_α has measure zero. Using Exercise 7.6.7, we can argue that the set D is a countable union of D_α sets. With a nod to Exercise 7.6.5, we conclude that D has measure zero.

An issue discussed in Exercise 7.6.5 is that the proof in the countable case requires a result about absolute convergence of double summations. To show that $D = \bigcup_{n=1}^\infty D_{1/n}$ has measure zero we can avoid this complication because the cover for each $D_{1/n}$ consists of a finite collection of open intervals. Thus, we have a double summation but one of the sums is finite and we can use the Algebraic Limit Theorem to justify the manipulations we need.

Exercise 7.6.15. (a) From the definition of the derivative we get $g'(0) = \lim_{x\to 0} g(x)/x$. For $x < 0$ we get $g(x)/x = 0$ and for $x > 0$ we get $g(x)/x = x\sin(1/x)$. In both cases we see $\lim_{x\to 0} g(x)/x = 0$, so $g'(0) = 0$.

(b) The chain rule and product rule yield can be applied when $x > 0$ and this gives us the formula

$$g'(x) = \begin{cases} -\cos(1/x) + 2x\sin(1/x) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

(c) The $\cos(1/x)$ term in the formula for $g'(x)$ oscillates between $+1$ and -1 as $x \to 0$. Because the other term in this formula converges to zero, the net effect is that $g'(x)$ attains every value between $+1$ and -1 as $x \to 0$ from the right.

Exercise 7.6.16. (a) If $c \in C$ then $f_n(c) = 0$ for all $n \in \mathbf{N}$. It follows that $\lim_{n\to\infty} f_n(c) = 0$.

(b) If $x \notin C$ then choose N to be the smallest natural number for which $x \notin C_N$. Then, by its construction, $f_n(x) = f_N(x)$ for all $n \geq N$ and $\lim_{n\to\infty} f_n(x) = f_N(x)$.

Exercise 7.6.17. If $x \notin C$ then, as in the previous exercise, there is a smallest natural number N such that $x \notin C_N$. This means that x is part of an open interval $O \subseteq C_N^c$ where

$$f(y) = f_N(y) \quad \text{ for all } y \in O.$$

Because f_N is differentiable everywhere and O is open, we can be sure that f is differentiable at x.

(b) Fix $c \in C$ and let $x \in [0, 1]$ be arbitrary. If $x \in C$ then $f(x) = 0$ so $|f(x)| \leq (x - c)^2$ is trivially true. If $x \notin C$, then either

$$f(x) = (x - c')^2 \sin(1/(x - c'))$$

for some $c' \in C$ or—because of the "splicing together" process—we at least have

$$|f(x)| \leq (x - c')^2$$

where c' is an endpoint of an interval that makes up some C_n. The point to emphasize is that there are no elements of C between x and c' which means $|x - c'| \leq |x - c|$ and consequently

$$|f(x)| \leq (x - c')^2 \leq (x - c)^2,$$

as desired.

Turning our attention toward computing $f'(c)$, we now have

$$\left|\frac{f(x) - f(c)}{x - c}\right| = \left|\frac{f(x)}{x - c}\right| \leq \frac{|x - c|^2}{|x - c|} = |x - c|,$$

from which it follows that

$$f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c} = 0.$$

(c) Let C_E consist of the countable set of points that appear as endpoints of the intervals that make up $C_1, C_2, C_3, \ldots$. The content of Exercise 7.6.15 is not only that $f'(x)$ fails to be continuous at each $c_E \in C_E$ but that $f'(x)$ attains every value between 1 and -1 in every neighborhood of c_E. Given an arbitrary $c \in C$ an argument like the one in Exercise 3.4.3 shows that there exists a sequence $(c_n) \subseteq C_E$ with $c_n \to c$. Let $\delta > 0$ be arbitrary. Choose N so that $|c_N - c| < \delta/2$ so that $V_{\delta/2}(c_N) \subseteq V_\delta(c)$. Because $c_N \in C_E$, we know that f' attains every value between 1 and -1 in the neighborhood $V_{\delta/2}(c_N)$ and therefore the same is true inside the neighborhood $V_\delta(c)$. Because δ was arbitrary, we conclude that f' is not continuous at c.

Exercise 7.6.18. The set of discontinuities of f' is precisely the Cantor set C. Because C has measure zero (see Exercise 7.6.4), Lebesgue's Theorem (Theorem 7.6.5) implies f' is Riemann-integrable.

Exercise 7.6.19. We start with the interval $[0, 1]$. To form C_1 we remove 1 interval of length $1/9$. To form C_2 we then remove two intervals of length $1/27$. In general, to form C_n we remove 2^{n-1} intervals of length $1/3^{n+1}$. If we take the sum of the lengths of all of the intervals to be removed we get

$$\frac{1}{9} + 2\left(\frac{1}{27}\right) + 4\left(\frac{1}{81}\right) + \cdots = \frac{1/9}{1 - 2/3} = \frac{1}{3}.$$

This implies that the lengths $|C_1|, |C_2|, |C_3|, \ldots$ satisfy

$$\lim_{n\to\infty} |C_n| = 1 - \frac{1}{3} = \frac{2}{3}.$$

Chapter 8

Additional Topics

8.1 The Generalized Riemann Integral

Exercise 8.1.1. (a) For any tagged partition $(P, \{c_k\})$, it is certainly true that $m_k \leq f(c_k) \leq M_k$, and this is enough to conclude

$$L(f, P) \leq R(f, P) \leq U(f, P).$$

The fact that $L(f, P) \leq \int_a^b f \leq U(f, P)$ follows from Definition 7.2.7.

(b) Because P' is a refinement of P_ϵ, we can use Lemma 7.2.3 to argue

$$U(f, P') - L(f, P') \leq U(f, P_\epsilon) - L(f, P_\epsilon) < \frac{\epsilon}{3}.$$

Exercise 8.1.2. This again follows from Lemma 7.2.3 and the fact that P' is a refinement of P.

Exercise 8.1.3. (a) To form P' from P we added the points of P_ϵ. This means adding $(n-1)$ potentially new points to the interior of $[a, b]$. Now each new point adds two terms to $U(f, P')$ that do not appear in $U(f, P)$ and also creates one term in $U(f, P)$ that is no longer in $U(f, P')$. Thus, there can be at most $3(n-1)$ terms of the form $M_k \Delta x_k$ that appear in one of $U(f, P')$ or $U(f, P)$ but not the other.

(b) Because P is assumed to be δ-fine and $P \subseteq P'$, any term from either $U(f, P')$ or $U(f, P)$ can be estimated by

$$|M_k(x_x - x_{k-1})| \leq M\delta = \frac{\epsilon}{9n}.$$

Using our conclusion from (a), we then get

$$U(f, P) - U(f, P') \leq 3(n-1)\frac{\epsilon}{9n} < \frac{\epsilon}{3}.$$

Exercise 8.1.4. (a) For each subinterval $[x_{k-1}, x_k]$ from a partition P, we use $M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$ to compute the upper sum. By the Extreme Value Theorem, there exist points $c_k \in [x_{k-1}, x_k]$ where $f(c_k) = M_k$. Using the set $\{c_k\}$ as our tags, it follows that $U(f, P) = R(f, P)$.

(b) Assume our partition P has n subintervals. Using Lemma 1.3.7, we can pick points $c_k \in [x_{k-1}, x_k]$ so that

$$M_k - f(c_k) < \frac{\epsilon}{n\Delta x_k} \quad \text{for each } k \in \{1, \dots, n\}.$$

Then

$$U(f,P) - R(f,P) = \sum_{k=1}^{n}(M_k - f(c_k))\Delta x_k < \sum_{k=1}^{n} \frac{\epsilon}{n\Delta x_k}\Delta x_k = \epsilon.$$

Exercise 8.1.5. We shall prove f is integrable using the criterion in Theorem 7.2.8. Let $\epsilon > 0$. From our hypothesis we know that there exists a $\delta > 0$ such that

$$|R(f,P) - A| < \frac{\epsilon}{4} \tag{1}$$

for all δ-fine partitions P regardless of the choice of tags. So let P_ϵ be δ-fine and use the previous exercise to pick tags $\{c_k\}$ so that

$$U(f, P_\epsilon) - R(f, P_\epsilon, \{c_k\}) < \frac{\epsilon}{4}.$$

Now we can also pick tags $\{d_k\}$ so that

$$R(f, P_\epsilon, \{d_k\}) - L(f, P_\epsilon) < \frac{\epsilon}{4},$$

and from (1) above it must be that

$$|R(f, P_\epsilon, \{c_k\}) - R(f, P_\epsilon, \{d_k\})| < \frac{\epsilon}{2}.$$

A triangle inequality argument then implies

$$U(f, P_\epsilon) - L(f, P_\epsilon) < \frac{\epsilon}{4} + \frac{\epsilon}{2} + \frac{\epsilon}{4} = \epsilon,$$

and we conclude that f is integrable. A second implication from this string of inequalities is that

$$L(f, P_\epsilon) \leq A \leq U(f, P_\epsilon),$$

from which may conclude that $\int_a^b f = A$.

Exercise 8.1.6. (a) Take $P = \{0, .1, .2, .3, .4, .5, .6, .7, .8, .9, 1\}$. The choice of tags does not matter because $\Delta x_k = 1/10 < 1/9 = \delta(c_k)$ for every choice of c_k.

(b) One such partition could be

$$P = \{0, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{3}{4}, 1\}.$$

For the tags we let $c_1 = 0$ on the first subinterval $[0, 1/5]$. For every other subinterval we take c_k to be the right-hand endpoint: $c_2 = 1/4$, $c_3 = 1/3$ and so on.

Exercise 8.1.7. Assume, for contradiction, that this process does not terminate after a finite number of steps. Then we obtain a sequence of nested intervals (I_n) satisfying $|I_n| \to 0$ and

$$\delta(x) \leq |I_n| \quad \text{for all } x \in I_n.$$

From the Nested Interval Property we know that there exists a point $x_0 \in \bigcap_{n=1}^{\infty} I_n$. But then $\delta(x_0) \leq |I_n|$ for all $n \in \mathbf{N}$, and it follows that $\delta(x_0) = 0$. Because this is not allowed in the definition of a gauge, we conclude that the algorithm does terminate after a finite number of steps and we obtain a $\delta(x)$-fine tagged partition.

Exercise 8.1.8. Let $\delta(x) = \min\{d_1(x), \delta_2(x)\}$. It is clear that $\delta(x) > 0$ so this function qualifies a gauge. From Theorem 8.1.5, there exists a tagged partition $(P, \{c_k\})$ that is $\delta(x)$-fine and, consequently, it is also $\delta_1(x)$-fine and $\delta_2(x)$-fine. It follows that

$$\begin{aligned} |A_1 - A_2| &\leq |A_1 - R(f, P)| + |R(f, P) - A_2| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

and we conclude that $A_1 = A_2$.

Exercise 8.1.9. Looking at Theorem 8.1.2, we just observe that the constant δ can also serve as the gauge function $\delta = \delta(x)$ required in Definition 8.1.6.

Exercise 8.1.10. Let $(P, \{c_k\})$ be $\delta(x)$-fine. If $c \notin \mathbf{Q}$ then $g(c_k)\Delta x_k = 0$. If $c_k = r_{k'}$ for some k', then

$$g(c_k)\Delta x_k = \Delta x_k < \delta(r_{k'}) = \frac{\epsilon}{2^{k'+1}}.$$

Because it is possible for $r_{k'}$ to be a tag in at most two partitions, it follows that

$$\sum_{k=1}^{n} g(c_k)\Delta x_k < 2 \sum_{k'=1}^{\infty} \delta(r_{k'}) = 2\left(\frac{\epsilon}{2}\right) = \epsilon.$$

Thus $R(g, P) < \epsilon$ and it follows that $\int_0^1 g = 0$.

Exercise 8.1.11. This is due to the fact that we have a “telescoping” sum. Writing out the terms in the finite sum $\sum_{k=1}^{n} F(x_k) - F(x_{k-1})$, we can check that all of the summands cancel out except $F(x_n) = F(b)$ and $-F(x_0) = -F(a)$.

Exercise 8.1.12. We are assuming F is differentiable with $F'(c) = f(c)$. This means that

$$f(c) = \lim_{x \to c} \frac{F(x) - F(c)}{x - c}.$$

The ϵ–δ criterion for functional limits then asserts the existence of the $\delta(c) > 0$ described in the exercise.

Exercise 8.1.13. Let's first apply the result in Exercise 8.1.12 with $x = x_k$ and $c = c_k$. Because our tagged partition is assumed to be $\delta(c)$-fine we know that $(x_k - c_k) \leq (x_k - x_{k-1}) < \delta(c_k)$ and so

$$\left| \frac{F(x_k) - F(c_k)}{x_k - c_k} - f(c_k) \right| < \epsilon.$$

Multiplying by the positive number $(x_k - c_k)$ gives the first requested inequality. To obtain the second one we again apply Exercise 8.1.12, this time with $x = x_{k-1}$ and $c = c_k$.

An equivalent way to write these two inequalities is

$$-\epsilon(x_k - c_k) < F(x_k) - F(c_k) - f(c_k)(x_k - c_k) < \epsilon(x_k - c_k)$$

$$-\epsilon(c_k - x_{k-1}) < F(c_k) - F(x_{k-1}) - f(c_k)(c_k - x_{k-1}) < \epsilon(c_k - x_{k-1}),$$

and adding along the respective columns yields

$$-\epsilon\Delta x_k < F(x_k) - F(x_{k-1}) - f(c_k)\Delta x_k < \epsilon\Delta x_k.$$

Now this is equivalent to $|F(x_k) - F(x_{k-1}) - f(c_k)\Delta x_k| < \epsilon\Delta x_k$ and taking a sum over k gives us

$$\sum_{k=1}^{n} |F(x_k) - F(x_{k-1}) - f(c_k)\Delta x_k| < \epsilon(b - a).$$

Looking back at the beginning of the proof in the text, we see that we have now derived the inequality requested in (2) albeit with $\epsilon(b - a)$ in place of ϵ. This completes the proof.

Exercise 8.1.14. (a) One implication of Theorem 8.1.9 is that every derivative has a generalized Riemann integral.

(b) A second implication of Theorem 8.1.9 is

$$\int_a^b (F \circ g)' = F(g(b)) - F(g(a)).$$

By the Chain Rule,

$$\begin{aligned} (F \circ g)'(x) &= F'(g(x)) \cdot g'(x) \\ &= f(g(x)) \cdot g'(x) = (f \circ g) \cdot g'(x) \end{aligned}$$

which implies

$$\int_a^b (f \circ g) \cdot g' = F(g(b)) - F(g(a)).$$

(c) Because $f = F'$ on the interval $g([a, b])$, Theorem 8.1.9 implies that f has generalized Riemann integral

$$\int_{g(a)}^{g(b)} f = F(g(b)) - F(g(a)).$$

Combining this with the last equation in (b) gives

$$\int_a^b (f \circ g) \cdot g' = \int_{g(a)}^{g(b)} f,$$

as desired.

8.2 Metric Spaces and the Baire Category Theorem

Exercise 8.2.1. (a) This is a metric. In fact this is the standard Euclidean distance function on $\mathbf{R}^2$. Conditions (i) and (ii) are straightforward. The most common way to prove (iii) is to introduce the scalar product from vector calculus. Squaring both sides of (iii) gives in equivalent inequality that can be derived using the so-called Schwartz inequality. An alternative proof can be derived by first considering the special case where the point z falls on the line

$$l(t) = (x_1, x_2) + t(y_1 - x_1, y_2 - x_2), \qquad t \in \mathbf{R}$$

through the points x and y. In this case $z = l(t_0)$ for some $t_0 \in \mathbf{R}$ and it follows that $d(x,z) = |t_0| d(x,y)$ and $d(z,y) = |1 - t_0| d(x,y)$. Then the triangle inequality in $\mathbf{R}$ implies

$$\begin{aligned} d(x,y) &= (t_0 + 1 - t_0)\, d(x,y) \\ &\leq (|t_0| + |1 - t_0|)\, d(x,y) \\ &= d(x,z) + d(z,y). \end{aligned}$$

To prove the general case, we let $z \in \mathbf{R}^2$ be arbitrary, and pick z_t to be the point on the line $l(t)$ such that the line through z and z_t is perpendicular to $l(t)$. Because x and y are both on the line $l(t)$, we can use the Pythagorean Theorem to show that $d(x, z_t) \leq d(x,z)$ and $d(y, z_t) \leq d(y,z)$. Applying the previous result about collinear points we get

$$d(x,y) \leq d(x,z_t) + d(z_t,y) \leq d(x,z) + d(z,y).$$

(b) This is a metric. Again, conditions (i) and (ii) can be easily verified. For (iii), we must consider 5 distinct cases. First, suppose that x, y and z are all distinct. Then

$$d(x,y) = 1 < 2 = d(x,z) + d(z,y).$$

If $x = y \neq z$, then

$$d(x,y) = 0 < 2 = d(x,z) + d(z,y).$$

If $x \neq y = z$, then

$$d(x,y) = 1 \leq 1 = d(x,z) + d(z,y),$$

which is identical to the case where $y \neq x = z$. Finally, if $x = y = z$, then

$$d(x, y) = 0 \leq 0 = d(x, z) + d(z, y).$$

Thus the triangle inequality holds for all possible scenarios.

(c) This is a metric. It is clear that $d(x, y) \geq 0$. Also, if $\max\{|x_1 - y_1|, |x_2 - y_2|\} = 0$, then $|x_1 - y_1| = 0$ and $|x_2 - y_2| = 0$. But this is true if and only if $x_1 = y_1$ and $x_2 = y_2$. This proves (i). Because $|x_i - y_i| = |y_i - x_i|$, condition (ii) holds. For (iii), consider the case where $\max\{|x_1 - y_1|, |x_2 - y_2|\} = |x_1 - y_1|$. The triangle inequality from $\mathbf{R}^1$ implies

$$|x_1 - y_1| \leq |x_1 - z_1| + |z_1 - y_1|.$$

Because $|x_1 - z_1| \leq d(x, z)$ and $|z_1 - y_1| \leq d(z, y)$, it follows that

$$|x_1 - y_1| \leq d(x, z) + d(z, y),$$

and similar argument works in the other case.

(d) This is not a metric, for it fails conditions (i) and (iii). This example fails (i) because we can have have $d(x, y) = 0$ where $x \neq y$. For instance, let $x = (1, -1)$ and let $y = (1, 1)$. Then $d(x, y) = |1(-1)+1(1)| = 0$, but $x_2 \neq y_2$, so $x \neq y$. Part (iii) also does not hold in general. Consider $x = (1, -1), y = (4, -1)$, and $z = (1, 1)$. Then

$$d(x, y) = 6 > 5 = d(x, z) + d(z, y),$$

which violates the triangle inequality.

Exercise 8.2.2. (a) This is a metric. Clearly $d(f, g) \geq 0$ and $\sup\{|f(x) - g(x)|\} = 0$ if and only if $f(x) = g(x)$ for all $x \in [0, 1]$. We also have that $|f(x) - g(x)| = |g(x) - f(x)|$, so condition (ii) holds. For (iii), we want to show that

$$\sup\{|f(x) - g(x)|\} \leq \sup\{|f(x) - h(x)|\} + \sup\{|h(x) - g(x)|\}.$$

Because $| - g|$ is a continuous function on the compact set $[0, 1]$, the Extreme Value Theorem asserts that there exists an $x_0 \in [0, 1]$ where $|f(x_0) - g(x_0)|$ is maximum. It follows that

$$\begin{aligned} |f(x_0) - g(x_0)| &= |f(x_0) - h(x_0) + h(x_0) - g(x_0)| \\ &\leq |f(x_0) - h(x_0)| + |h(x_0) - g(x_0)| \\ &\leq \sup\{|f(x) - h(x)|\} + \sup\{|h(x) - g(x)|\}. \end{aligned}$$

Hence (iii) is true and we have a metric.

(b) This is not a metric. It fails condition (i), because it is possible for $f(1)-g(1) = 0$ when $f \neq g$. For instance, let $f(x) = 2x-1$ and let $g(x) = 3x-2$. Then $f(1) - g(1) = 0$ but $f(x) \neq g(x)$.

(c) This is a metric. We can immediately verify that $\int_0^1 |f-g| \geq 0$. That $\int_0^1 |f-g| = 0$ implies $f=g$ is a consequence of the fact that $|f-g|$ is nonnegative and continuous. The details of this argument are contained in Exercise 7.4.4 (c). This shows condition (i) holds. Clearly $|f-g| = |g-f|$, so (ii) is true as well. For (iii), we know that

$$\begin{aligned} |f-g| &= |f-h+h-g| \\ &\leq |f-h| + |h-g|. \end{aligned}$$

It then follows from Theorem 7.4.2 (i) and (iv) that

$$\int_0^1 |f-g| \leq \int_0^1 |f-h| + \int_0^1 |h-g|.$$

Thus (iii) holds, and we have a metric.

Exercise 8.2.3. See Exercise 8.2.1 (b). In this exercise, we did not use the fact that $X = \mathbf{R}^2$. Hence our argument holds for any set X.

Exercise 8.2.4. Let (X,d) be a metric space and let $(x_n) \subseteq X$ converge to $x \in X$. Given $\epsilon > 0$, there exists an N such that $d(x_n, x) < \epsilon/2$ whenever $n \geq N$. Now if $n, m \geq N$ we can use the triangle inequality to write

$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus (x_n) is a Cauchy sequence.

Exercise 8.2.5. (a) By considering values of ϵ less than one, we can show that Cauchy sequences in this metric space are eventually constant sequences. Because such a sequence converges (to this constant value), $\mathbf{R}^2$ is complete with respect to this metric.

(b) Assume that (f_n) is a Cauchy sequence in the metric of Exercise 8.2.2 (a). Then given $\epsilon > 0$, there exists an N such that $d(x_m, x_n) < \epsilon$ for all $m, n \geq N$. This implies that

$$|f_m(x) - f_n(x)| < \epsilon \quad \text{for all } m, n \geq N \text{ and } x \in [0,1].$$

Thus (f_n) converges uniformly according to the Cauchy Criterion for Uniform Convergence (Theorem 6.2.5). What is really happening here is that convergence with respect to this metric is equivalent to uniform convergence on $[0,1]$. If $f = \lim_{n\to\infty} f_n$ uniformly, then f is continuous by Theorem 6.2.6. Hence f is an element of $C[0,1]$ and the metric is complete.

(c) Let's start with a Cauchy sequence (f_n) in $C[0,1]$. In (b) we saw that there exists $f \in C[0,1]$ such that $f_n \to f$ uniformly, but it does not have to be the case that $f \in C^1[0,1]$. A counterexample appears in Example 6.2.2 (iii).

(d) Convergent sequences in the discrete metric are eventually constant sequences.

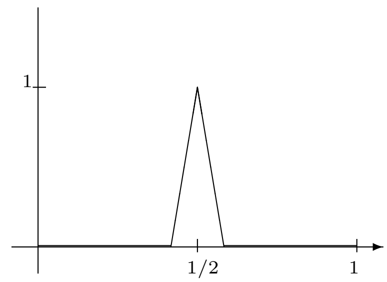


Figure 8.1: SKETCH OF $h_\delta(x)$.

Exercise 8.2.6. (a) Let $\epsilon > 0$. We need to find a $\delta > 0$ such that

$$|g(f) - g(h)| < \epsilon \quad \text{whenever } d(f,h) < \delta.$$

Because $k \in C[0,1]$, there exists a constant $K > 0$ satisfying $|k(x)| \leq K$ for all $x \in [0,1]$. Now the properties of the Riemann integral allow us to write

$$\begin{aligned} |g(f) - g(h)| &= \left| \int_0^1 fk - \int_0^1 hk \right| \\ &= \left| \int_0^1 (f-h)k \right| \\ &\leq \int_0^1 |f-h||k| \leq K \int_0^1 |f-h|. \end{aligned}$$

Now pick $\delta = \epsilon/K$. Then $d(f,h) < \delta$ implies

$$\begin{aligned} |g(f) - g(h)| &\leq K \int_0^1 |f-h| \\ &< K \int_0^1 \frac{\epsilon}{K} = \epsilon, \end{aligned}$$

and g is continuous on $C[0,1]$.

(b) Let $\epsilon > 0$. We need to find a $\delta > 0$ such that $d(f,h) < \delta$ implies $|g(f) - g(h)| < \epsilon$. In this case it works to take $\delta = \epsilon$. To see why, note that if $d(f,h) < \epsilon$ then

$$\begin{aligned} |g(f) - g(h)| = |f(1/2) - h(1/2)| &\leq \sup\{|f(x) - h(x)| : x \in [0,1]\} \\ &= d(f,h) < \epsilon, \end{aligned}$$

as desired.

(c) Let f be the zero function and for small $\delta > 0$ let h_δ be the function pictured in Figure 8.1. Note that $h(x) = 0$ on $[0, 1/2 - \delta]$ and $[1/2 + \delta, 1]$. On

$(1/2-\delta, 1/2+\delta)$ define h to be the piecewise linear "tent" satisfying $h(1/2) = 1$. Then $d(f, h_\delta) = \int_0^1 |h_\delta| = \delta$. Now observe that for all $\delta > 0$ we have

$$|f(1/2) - h_\delta(1/2)| = 1.$$

Given $\epsilon_0 = 1/2$, for instance, the functions h_δ can be chosen arbitrarily close to f and still satisfy $|g(f) - g(h_\delta)| \geq \epsilon_0$. Thus g is not continuous at f, and a similar argument shows it is not continuous at any other point.

Exercise 8.2.7. (a) The ϵ-neighborhoods of the metric in (a) are discs with center x and radius ϵ. For the metric in (b), $V_\epsilon(x) = \mathbf{R}^2$ if $\epsilon \geq 1$. If $\epsilon < 1$, then $V_\epsilon(x)$ is a singleton point. The metric in part (c) has ϵ-neighborhoods that form a square with sides of length 2ϵ and x in the center. In the discrete metric, $V_\epsilon(x) = X$ if $\epsilon \geq 1$. If $\epsilon < 1$, then $V_\epsilon(x)$ is a singleton point.

(b) Using the discrete metric in $\mathbf{R}$, $V_\epsilon(x)$ is the entire real line if $\epsilon \geq 1$. If $\epsilon < 1$ then $V_\epsilon(x)$ is a singleton point.

Exercise 8.2.8. (a) Let $a \in V_\epsilon(x)$. We want to show that there exists an $\epsilon' > 0$ such that $V_{\epsilon'}(a) \subseteq V_\epsilon(x)$. According to Definition 8.2.6, $d(x, a) < \epsilon$. Let $\epsilon' = \epsilon - d(x, a)$. If $b \in V_{\epsilon'}(a)$ then $d(a, b) < \epsilon'$ and the triangle inequality imlies

$$\begin{aligned} d(x,b) &\leq d(x,a) + d(a,b) \\ &< d(x,a) + \epsilon' = \epsilon. \end{aligned}$$

This implies that $b \in V_\epsilon(x)$, so $V_{\epsilon'}(a) \subseteq V_\epsilon(x)$. Hence $V_\epsilon(x)$ is open.

The set $C_\epsilon(x)$ is closed. Assume that y is a limit point of $C_\epsilon(x)$. If $\delta > 0$, then $V_\delta(y)$ intersects $C_\epsilon(x)$ at a point $z \neq y$. So $d(x, z) \leq \epsilon$ and $d(z, y) < \delta$. By the triangle inequality,

$$\begin{aligned} d(x,y) &\leq d(x,z) + d(z,y) \\ &\leq \epsilon + \delta. \end{aligned}$$

Because $\delta > 0$ is arbitrary, it must be that $d(x, y) \leq \epsilon$. Therefore $y \in C_\epsilon(x)$ and thus $C_\epsilon(x)$ is closed.

(b) Let h be a limit point of Y, and let $\epsilon > 0$ be arbitrary. Then $V_\epsilon(h)$ intersects Y at a point $g \neq h$. So $d(g, h) < \epsilon$, and $g \in Y$ meaning $|g(x)| \leq 1$ for all $x \in [0, 1]$. It follows that

$$\begin{aligned} |h(x)| &= |h(x) - g(x) + g(x)| \\ &\leq |h(x) - g(x)| + |g(x)| \\ &< \epsilon + 1. \end{aligned}$$

Because $\epsilon > 0$ is arbitrary, $|h(x)| \leq 1$ for all $x \in [0, 1]$ and thus $h \in Y$. This proves Y is closed.

(c) This set is closed. Suppose that g is a limit point of T. Then, given $\epsilon > 0$, we know $V_\epsilon(g)$ intersects T at a point $h \neq g$. So

$$|g(0) - h(0)| \leq d(g, h)) < \epsilon.$$

But $h \in T$ so $h(0) = 0$ and this implies $|g(0)| < \epsilon$. Because ϵ is arbitrary, we conclude that $g(0) = 0$, and hence $g \in T$. Thus T is closed.

Exercise 8.2.9. (a) A subset K of a metric space (X, d) is *bounded* if there exists $M > 0$ and $x \in X$ such that $d(x, k) \leq M$ for all $k \in K$.

(b) First let's prove that K is bounded. Assume, for contradiction, that K is not bounded. Our goal is to produce a sequence in K that does not have a convergent subsequence. Because K is not bounded, there must exist elements $x_1, x_2 \in K$ satisfying $d(x_1, x_2) \geq 1$. Having picked x_2, there exists an element $x_3 \in K$ such that $d(x_2, x_3) \geq 2$. In general, given $x_n \in K$, we can pick $x_{n+1} \in K$ satisfying $d(x_n, x_{n+1}) \geq 2^n$. An extended triangle inequality argument shows that it must be that $d(x_n, x_m) \geq 1$ for all $m \neq n$.

Now, because K is assumed to be compact, (x_n) has a convergent subsequence (x_{n_k}). But the elements of the subsequence (x_{n_k}) must necessarily satisfy $d(x_{n_k}, x_{n_{k'}}) \geq 1$, and consequently (x_{n_k}) is not Cauchy and cannot converge. This contradiction proves K is bounded.

To show that K is closed let x be an arbitrary limit point of K. For each $\delta_n = 1/n$, the neighborhood $V_{\delta_n}(x)$ intersects K so we can choose $x_n \in K \cap V_{\delta_n}(x)$. It follows that $(x_n) \to x$ and $(x_n) \subseteq K$. By compactness, there is a subsequence $(x_{n_k}) \to y$ with $y \in K$. But every subsequence of a convergent sequence converges to the same limit, so $y = x$ which implies $x \in K$. Thus, K is closed.

(c) We have already shown that Y is closed, and the fact that $d(f, 0) \leq 1$ for all $f \in Y$ shows that Y is bounded.

To see that Y is not compact, consider the sequence $f_n(x) = x^n$ and check that $f_n \in Y$. Now the pointwise limit $f(x) = \lim f_n(x)$ is *not* continuous and every subsequence of (f_n) will necessarily converge pointwise to $f \notin C[0, 1]$. Because convergence in the metric space $C[0, 1]$ means uniform convergence and uniform limits of continuous functions are continuous, we see there is no way to find a convergent subsequence of (f_n).

Exercise 8.2.10. (a) ($\Rightarrow$) Assume that E is closed. Then E contains its limit points, so $E \cup L = E$, where L is the set of limit points of E. Therefore $E = \overline{E}$. ($\Leftarrow$) Now assume that $\overline{E} = E$. Then $E = E \cup L$, so E contains its limit points and hence it is closed.

($\Rightarrow$) Assume that E is open. Then for each $x \in E$, there exists an $\epsilon > 0$ such that $V_\epsilon(x) \subseteq E$. Hence $E = E^\circ$. ($\Leftarrow$) For the other direction, if $E^\circ = E$, then for each $x \in E$, $V_\epsilon(x) \subseteq E$. Hence E is open.

(b) Let $x \in \overline{E}^c$. Then $x \notin \overline{E}$. Hence there exists $\epsilon > 0$ such that $V_\epsilon(x)$ does not intersect E, so $V_\epsilon(x) \subseteq E^c$. By Definition 8.2.8, $x \in (E^c)^\circ$. This shows $\overline{E}^c \subseteq (E^c)^\circ$. To prove the other inclusion let $x \in (E^c)^\circ$. Then there exists $V_\epsilon(x) \subseteq E^c$. So $x \notin \overline{E}$, and hence $x \in \overline{E}^c$. Thus $(E^c)^\circ \subseteq \overline{E}^c$ and $\overline{E}^c = (E^c)^\circ$.

To prove the second statement let $x \in (E^\circ)^c$. Then $x \notin E^\circ$, so every $V_\epsilon(x)$ fails to be contained in E. Thus every $V_\epsilon(x)$ intersects E^c, and therefore $x \in \overline{E^c}$. This shows $(E^\circ)^c \subseteq \overline{E^c}$. Now assume $x \in \overline{E^c}$. Then every $V_\epsilon(x)$ intersects E^c, and so $V_\epsilon(x)$ is not contained in E. Thus $x \notin E^\circ$ implying $x \in (E^\circ)^c$. This proves $\overline{E^c} \subseteq (E^\circ)^c$ and hence we have $(E^\circ)^c = \overline{E^c}$.

Exercise 8.2.11. Set $\epsilon = 1$ and consider the discrete metric on an arbitrary space X consisting of at least two points. If we fix $x \in X$, then $V_\epsilon(x)$ is just the singleton set $\{x\}$. Because this set has no limit points we also get $\overline{V_\epsilon(x)} = \{x\}$. On the other hand, the set $\{y \in X : d(x,y) \leq 1\}$ is the entire space X.

Note that an important consequence of Exercise 8.2.8 (a) is that we always have the inclusion

$$\overline{V_\epsilon(x)} \subseteq \{y \in X : d(x,y) \leq \epsilon\}.$$

This fact is used implicitly in the proof of Theorem 8.2.10.

Exercise 8.2.12. ($\Rightarrow$) Let E be nowhere-dense in X. Then $\overline{E}^\circ$ is empty. This means that given $x \in \overline{E}$, every $V_\epsilon(x)$ intersects $\overline{E}^c$. So x is a limit point of $\overline{E}^c$. It follows that $\overline{\overline{E}^c} = X$, and hence $\overline{E}^c$ is dense.

($\Leftarrow$) Now assume that $\overline{E}^c$ is dense. Then $\overline{\overline{E}^c} = X$. So every point $x \in X$ is either an element of $\overline{E}^c$ or a limit point of $\overline{E}^c$. This implies that for all $\epsilon > 0$, $V_\epsilon(x)$ is not contained in $\overline{E}$, which means that $\overline{E}^\circ$ is empty. Hence E is nowhere dense.

Exercise 8.2.13. (a) Pick $x_1 \in O_1$. Because O_1 is open, there exists an $\epsilon_1 > 0$ such that $V_{\epsilon_1}(x_1) \subseteq O_1$. Since O_2 is dense, there exists an $x_2 \in V_{\epsilon_1}(x_1) \cap O_2$. We also have that $V_{\epsilon_1}(x_1) \cap O_2$ is open, so there exists an $\epsilon_2 > 0$ such that $V_{\epsilon_2}(x_2) \subseteq V_{\epsilon_1}(x_1) \cap O_2$, and let's also insist that ϵ_2 satisfy $\epsilon_2 < \epsilon_1/2$. Now certainly $V_{\epsilon_2}(x_2) \subseteq O_2$, but we want $\overline{V_{\epsilon_2}(x_2)} \subseteq V_{\epsilon_1}(x_1)$ to be true. By shrinking ϵ_2 we can ensure that the closure of $V_{\epsilon_2}(x_2)$ is contained in $V_{\epsilon_1}(x_1)$. (The result in Exercise 8.2.8 (a) and the discussion in the solution of Exercise 8.2.11 contain the justification for this last claim.)

(b) In general, following (a) we can produce a sequence (x_n) with

$$\overline{V_{\epsilon_{n+1}}(x_{n+1})} \subseteq V_{\epsilon_n}(x_n) \subseteq O_n \quad \text{ where } \epsilon_{n+1} < \epsilon_1/2^n.$$

This last condition on (ϵ_n) ensures that (x_n) is a Cauchy sequence and so $x = \lim_{n\to\infty} x_n$ exists because our space is complete. For each $m \in \mathbf{N}$, our sequence (x_n) is eventually contained in the set $\overline{V_{\epsilon_{m+1}}(x_{m+1})} \subseteq O_m$. It follows that $x \in O_m$ and the intersection $\bigcap_{m=1}^\infty O_m$ is not empty.

Exercise 8.2.14. If E is nowhere-dense in X, then $(\overline{E})^c$ is dense. Although we have not explicitly proved it to this point, we can also show that the complement of a closed set (such as $\overline{E}$) is open.

Now suppose that E_n is a collection of nowhere dense sets and assume, for contradiction, that $X = \bigcup_{n=1}^\infty E_n$. Then certainly it is true that $X = \bigcup_{n=1}^\infty \overline{E_n}$. By De Morgan's Law, this implies that $\bigcap_{n=1}^\infty (\overline{E_n})^c$ is empty. But since $(\overline{E_n})^c$ is dense and open, this intersection is not empty by Theorem 8.2.10, so we have reached a contradiction.

Exercise 8.2.15. Assume f is differentiable at x so that

$$f'(x) = \lim_{t \to x} \frac{f(x) - f(t)}{x - t}.$$

Choose $n > |f'(x)|$. Applying the definition of functional limits with $\epsilon_0 = n - |f'(x)|$, it follows that there exists a $\delta > 0$ such that

$$\left| \frac{f(x) - f(t)}{x - t} - f'(x) \right| < \epsilon_0 \quad \text{whenever } 0 < |x - t| < \delta.$$

Now choose m large enough so that $1/m < \delta$. Then we can show $0 < |x - t| < 1/m$ implies

$$\left| \frac{f(x) - f(t)}{x - t} \right| \leq n,$$

and we conclude that $f \in A_{m,n}$.

Exercise 8.2.16. (a) The sequence (x_k) is contained in $[0, 1]$ and so the Bolzano–Weierstrass Theorem can be applied to argue that there is a convergent subsequence.

(b) Let $\epsilon > 0$. Because $f_{k_l} \to f$ uniformly, we can pick L_1 so that $l \geq L_1$ implies $|f_{k_l}(y) - f(y)| < \epsilon/2$ for all $y \in [0, 1]$. Now the limit function f is continuous at x and so there exists a $\delta > 0$ such that

$$|f(x_{k_l}) - f(x)| < \frac{\epsilon}{2} \quad \text{whenever } |x_{k_l} - x| < \delta.$$

Because $x_{k_l} \to x$, we can pick L_2 so that $|x_{k_l} - x| < \delta$ for all $l \geq L_2$. Finally, set $L = \max\{L_1, L_2\}$. Then $l \geq L$ implies

$$\begin{aligned} |f_{k_l}(x_{k_l}) - f(x)| &\leq |f_{k_l}(x_{k_l}) - f(x_{k_l})| + |f(x_{k_l}) - f(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

(c) If t satisfies $|x - t| < 1/m$, then there exists an L such that $|x_{k_l} - x| < 1/m$ for all $l \geq L$. In this case we have

$$\left| \frac{f_{k_l}(x_{k_l}) - f_{k_l}(t)}{x_{k_l} - t} \right| \leq n.$$

Now taking the limit as $l \to \infty$ and using (b) together with the Algebraic Limit Theorem and the Order Limit Theorem gives

$$\left| \frac{f(x) - f(t)}{x - t} \right| \leq n.$$

The conclusion is that $f \in A_{m,n}$ meaning that $A_{m,n}$ contains its limit points and thus is closed.

Exercise 8.2.17. (a) Because f is continuous on [0,1], it is uniformly continuous. Thus, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$(1) \qquad |f(x) - f(y)| < \epsilon/4 \quad \text{whenever } |x - y| < \delta.$$

Now let $\{0 = x_0 < x_1 < \cdots < x_n = 1\}$ be a partition of $[0,1]$ where every subinterval satisfies $x_k - x_{k-1} < \delta$. Our function p is going to satisfy $p(x_k) = f(x_k)$ for all $k = 0, 1, \ldots, n$. On each subinterval $[x_{k-1}, x_k]$ we define $p(x)$ to be the line segment connecting the endpoints $(x_{k-1}, f(x_{k-1}))$ and $(x_k, f(x_k))$. It's straightforward to check that p is piecewise linear and continuous. Also, given a point $x \in [x_{k-1}, x_k]$, statement (1) above implies

$$\begin{aligned} |f(x) - p(x)| &\leq |f(x) - f(x_k)| + |f(x_k) - p(x)| \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}. \end{aligned}$$

It follows that $\|f - p\| + \infty < \epsilon/2$.

(b) Assume $|h(x)| \leq 1$ for all $x \in [0,1]$. Then

$$\begin{aligned} |f(x) - g(x)| &= |f(x) - p(x) + \frac{\epsilon}{2}\, h(x)| \\ &\leq |f(x) - p(x)| + \frac{\epsilon}{2}|h(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

It follows that $d(f, g) < \epsilon$ and thus $g \in V_\epsilon(f)$.

(c) Because p is piecewise linear, we can let M be the maximum of the absolute values of the slopes of each segment that make up p. Now consider the sawtooth function $h(x)$ from Section 5.4 and sketched in Figure 5.6. For any choice of $N \in \mathbf{N}$, the function

$$g_N(x) = p(x) + \frac{\epsilon}{2}\, h(Nx)$$

is continuous, piecewise linear and, by part (b), falls in the ϵ-neighborhood $V_\epsilon(f)$. Now if we choose $N > 2(n+M)/\epsilon$, we can argue that every line segment that makes up g_N has slope greater than n in absolute value. The result of this is that $g_N \notin A_{m,n}$ and consequently $V_\epsilon(f)$ is not contained in $A_{m,n}$. Because ϵ and f were arbitrary, it follows that $A_{m,n}$ has no interior points and thus it is nowhere dense.

We conclude that D is a subset of the countable union of the nowhere dense sets $\{A_{m,n}\}$ and thus D is a set of first category in $C[0,1]$.

8.3 Fourier Series

Exercise 8.3.1. (a) Taking partial derivatives yields

$$\frac{\partial^2 u}{\partial x^2} = -b_n \sin(nx)\cos(nt) \cdot n^2 = \frac{\partial^2 u}{\partial t^2}.$$

Also

$$u(0,t) = b_n \sin(0)\cos(nt) = 0 \quad \text{and}$$
$$u(\pi,t) = b_n \sin(\pi n)\cos(nt) = 0.$$

Note that this second statement requires n be an integer. Finally,

$$\frac{\partial u}{\partial t} = -b_n \sin(nx)\sin(nt)\cdot n,$$

and setting $t = 0$ gives $\frac{\partial u}{\partial t}(x,0) = 0$.

(b) The derivative is a linear transformation meaning that the derivative of the sum of functions is the sum of the derivatives of each one. This property makes (1) and (3) true for a sum of solutions, and (2) is easy to check as well.

Exercise 8.3.2. (a)

$$\int_{-\pi}^{\pi} \cos(nx)\,dx = \frac{1}{n}\sin(nx)\Big|_{-\pi}^{\pi} = 0.$$

(b) Using a trigonometric identity for $\cos^2\theta$ we get

$$\int_{-\pi}^{\pi} \cos^2(nx)\,dx = \frac{x}{2} + \frac{1}{4n}\sin(2nx)\Big|_{-\pi}^{\pi} = \left(\frac{\pi}{2}+0\right) - \left(\frac{-\pi}{2}+0\right) = \pi.$$

(c) Using a trigonometric identity for $\cos\theta\sin\beta$ we get

$$\int_{-\pi}^{\pi} \cos(mx)\sin(nx)\,dx = -\frac{\cos(n-m)x}{2(n-m)} - \frac{\cos(n+m)x}{2(n+m)}\Big|_{-\pi}^{\pi} = 0,$$

where the zero occurs because the cosine function is even and gives the same value at $x = \pi$ and $x = -\pi$.

The other integrals in (a), (b) and (c) can be done in a similar fashion.

Exercise 8.3.3. Start with equation (6) in the text and multiply each side of this equation by $\cos(mx)$ to get

$$f(x)\cos(mx) = a_0\cos(mx) = \sum_{n=1}^{\infty} a_n\cos(nx)\cos(mx) + b_n\sin(nx)\cos(mx).$$

Now take the integral of each side of this equation from $-\pi$ to π and, as before, distribute the integral through the infinite sum. Using Exercise 8.3.2, we see that for a_0 and for every value of $n \in \mathbf{N}$ we get an integral that equals zero *except* the one where $n = m$. When $n = m$ we get

$$\int_{-\pi}^{\pi} a_m\cos^2(mx)\,dx = a_m\pi$$

and it follows that

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = a_m\pi.$$

The formula for a_m is immediate. To get the formula for b_m we multiply across equation (6) by $\sin(mx)$ and follow the same procedure.

Exercise 8.3.4. (a) The approximating functions are trigonometric functions, which are continuous. The limit function $f(x)$ is not continuous. Because the uniform limit of continuous functions is continuous, we know the convergence in this case cannot be uniform.

(b) The function $g(x) = |x|$ is even and so a symmetry argument shows that $b_n = 0$ for all $n \geq 1$. For a_0 we write

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|dx = \frac{1}{\pi}\int_0^{\pi} xdx = \frac{1}{\pi}\left(\frac{\pi^2}{2}\right) = \frac{\pi}{2}.$$

For a_n with $n \geq 1$ we use integration by parts to compute

$$\begin{aligned} a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos(nx)dx &= \frac{2}{\pi}\int_0^{\pi} x\cos(nx)dx \\ &= \frac{2}{\pi}\left(\frac{x}{n}\sin(nx) + \frac{1}{n^2}\cos(nx)\Big|_0^{\pi}\right) \\ &= \frac{2}{n^2\pi}(\cos(n\pi) - 1) \\ &= \begin{cases} -4/(n^2\pi) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

Plugging these results into equation (6) in the text we get

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2}\cos((2m+1)x).$$

Before constructing any graphs, we can observe that the coefficients in this case go to zero like $1/n^2$. More specifically, we have

$$|a_n \cos(nx)| < (4/\pi)(1/n^2)$$

and because $\sum 1/n^2$ converges we can use the Weierstrass M-Test to conclude that our series converges uniformly to some continuous function. In fact, S_n does converge to g is this case as suggested by the sketch of S_3 and g in Figure 8.2.

(c) Taking the term-by-term derivative of the series for $g(x) = |x|$ in (b) gives the Fourier series for the square-wave $f(x)$ derived in Example 8.3.1. This makes intuitive sense because away from zero we have $g'(x) = f(x)$. Using $S_N(x)$ to denote the partial sums of the Fourier series for $g(x)$ in (b), we have $g(x) = \lim S_N(x)$. Then graphical evidence suggests that, for all $x \neq n\pi$,

$$g'(x) = f(x) = \lim S_N'(x). \tag{1}$$

In order to use Theorem 6.4.3 to prove something rigorous, we would need to know that S_N' (the Fourier series for $f(x)$) converges uniformly. The Weierstrass M-Test is of no use because the Fourier coefficients for $f(x)$ go to zero like $1/n$ and $\sum 1/n$ diverges. We remarked in (a) that the convergence to $f(x)$ is not

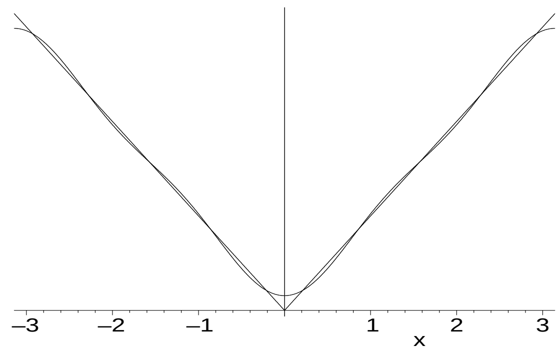


Figure 8.2: $g(x) = |x|$ AND S_3 ON $[-\pi, \pi]$.

uniform on intervals containing $x = 0$. This is reassuring since g' does not exist here. On compact sets that do not contain points of the form $x = n\pi$, it turns out that the convergence of the series for $f(x)$ is uniform meaning statement (1) above could be proved using Theorem 6.4.3.

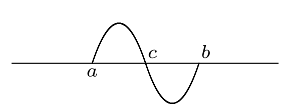
Differentiating the series for $f(x)$ term-by-term gives a series of the form

$$\frac{4}{\pi}\sum_{m=0}^{\infty}\cos((2m+1)x) = \frac{4}{\pi}(\cos(x) + \cos(3x) + \cos(5x) + \cdots).$$

When $x = \pm\pi/2$ the series converges to zero, but otherwise the series diverges because the terms do not tend to zero. Thus, even though f' exists (away from zero), we cannot obtain a valid representation for f' by differentiating the Fourier series for f in a term-by-term fashion. This predicament should be contrasted with the situation for power series where term-by-term differentiation always yields a valid series.

Exercise 8.3.5. Recall that any function continuous on a compact set is uniformly continuous. Thus h is uniformly continuous over, say, $[\pi, 3\pi]$. This means that given $\epsilon > 0$, there exists a $\delta > 0$ that "works" for all pairs x, y in this set. Now the fact that h is periodic implies that this δ suffices on all of $\mathbf{R}$.

Exercise 8.3.6. Let c be the midpoint of $[a, b]$, and let's assume $[a, b]$ is chosen so that $\sin(na) = \sin(nb) = \sin(nc) = 0$ with $\sin(nx) \geq 0$ on $[a, c]$ and $\sin(nx) \leq 0$ on $[c, b]$.



Then

$$\int_a^b h(x)\sin(nx)\,dx = \int_a^c h(x)\sin(nx)\,dx + \int_c^b h(x)\sin(nx)\,dx,$$

and the trick is to argue that because h does not change very much over this interval, the two integrals on the right mostly cancel out. To make this explicit, note that $\sin(n(x+\pi/n)) = -\sin(nx)$ so

$$\int_c^b h(x)\sin(nx)\,dx = \int_a^c h(x+\pi/n)\sin(n(x+\pi/n))\,dx = -\int_a^c h(x+\pi/n)\sin(nx)\,dx.$$

Then we can write

$$\begin{aligned}
\left|\int_a^b h(x)\sin(nx)\,dx\right| &= \left|\int_a^c (h(x)-h(x+\pi/n))\sin(nx)\,dx\right| \\
&\leq \int_a^c |h(x)-h(x+\pi/n)|\sin(nx)\,dx \\
&< \frac{\epsilon}{2}\int_a^c \sin(nx)\,dx \\
&= \frac{\epsilon}{2}\left(\frac{2}{n}\right) = \frac{\epsilon}{n}.
\end{aligned}$$

Over the interval $[-\pi,\pi]$ there are exactly n intervals of length $2\pi/n$ like the interval $[a,b]$. Thus it follows that

$$\left|\int_{-\pi}^{\pi} h(x)\sin(nx)\,dx\right| \leq n\cdot\left|\int_{a_n}^{b_n} h(x)\sin(nx)\,dx\right| < n\left(\frac{\epsilon}{n}\right) = \epsilon,$$

for all $n \geq N$. This completes the proof.

Exercise 8.3.7. (a) Because f is continuous, the function $q_x(u) = f(u+x) - f(x)$ is continuous. It follows from the Riemann–Lebesgue Lemma (Theorem 8.3.2) that

$$\int_{-\pi}^{\pi} q_x(u)\cos(Nx)\,dx \to 0 \quad \text{as } N\to\infty.$$

(b) The idea here is to show that the discontinuity of $p_x(u)$ at zero is removable; that is, that $p_x(u)$ can be defined at $u=0$ in such a way that makes p_x continuous. To see how to do this write

$$\begin{aligned}
p_x(u) &= \frac{(f(u+x)-f(x))\cos(u/2)}{\sin(u/2)} \\
&= 2\,\frac{f(u+x)-f(x)}{u}\cdot\frac{(u/2)}{\sin(u/2)}\cdot\cos(u/2).
\end{aligned}$$

The fact that f is differentiable at x and the well-known limit $\lim_{t\to 0}\sin(t)/t = 1$ imply

$$\lim_{u\to 0} p_x(u) = 2f'(x).$$

Thus, defining $p_x(0) = 2f'(x)$ makes p_x continuous on $(-\pi, \pi]$ and it now follows from the Riemann–Lebesgue Lemma that

$$\int_{-\pi}^{\pi} p_x(u)\sin(Nu)\,du \to 0 \qquad \text{as } N \to \infty.$$

Exercise 8.3.8. This exercise appeared as Exercise 2.3.11.

Exercise 8.3.9. For $k = 1, 2, \ldots, N$ write

$$D_k(\theta) = \frac{1}{2}\left[\cos(k\theta) + \frac{\sin(k\theta)\cos(\theta/2)}{\sin(\theta/2)}\right]$$

as in the proof of Theorem 8.3.3. Then,

$$\begin{aligned}
\frac{1}{N+1}\left[\frac{1}{2} + \sum_{k=1}^{N} D_k(\theta)\right] &= \frac{1}{2(N+1)}\left[1 + \sum_{k=1}^{N}\cos(k\theta) + \frac{\cos(\theta/2)}{\sin(\theta/2)}\sum_{k=1}^{N}\sin(k\theta)\right] \\
&= \frac{1}{2(N+1)}\left[\frac{1}{2} + D_N(\theta) + \frac{\cos(\theta/2)}{\sin(\theta/2)}\frac{\sin(N\theta/2)\sin((N+1)\theta/2)}{\sin(\theta/2)}\right] \\
&= \frac{1}{2(N+1)\sin^2(\theta/2)}\left[\mathbf{B}\right],
\end{aligned}$$

where

$$\mathbf{B} = \frac{\sin^2(\theta/2)}{2} + \frac{\sin(\theta/2)\sin(N\theta + \frac{\theta}{2})}{2} + \cos(\theta/2)\sin(N\theta/2)\sin((N+1)\theta/2).$$

To finish the proof we must show that $\mathbf{B} = \sin^2((N+1)\theta/2)$. Using the identity $\sin(t)\cos(t) = (1/2)\sin(2t)$, we can write

$$\begin{aligned}
\sin^2((N+1)\theta/2) &= [\sin(N\theta/2)\cos(\theta/2) + \cos(N\theta/2)\sin(\theta/2)]^2 \\
&= \sin^2(N\theta/2)\cos^2(\theta/2) + \frac{\sin(N\theta)\sin(\theta)}{2} + \cos^2(N\theta/2)\sin^2(\theta/2).
\end{aligned}$$

Now we use Fact 1(b) from the text together with the identities $\sin(t)\cos(t) = (1/2)\sin(2t)$ and $1 + \cos(t) = 2\cos^2(t/2)$ to write

$$\begin{aligned}
\mathbf{B} &= \frac{\sin^2(\theta/2)}{2} + \frac{\sin(\theta/2)}{2}\left[\cos(N\theta)\sin(\theta/2) + \sin(N\theta)\cos(\theta/2)\right] \\
&\qquad + \cos(\theta/2)\sin(N\theta/2)\left[\cos(N\theta/2)\sin(\theta/2) + \sin(N\theta/2)\cos(\theta/2)\right] \\
&= \frac{\sin^2(\theta/2)}{2}[1 + \cos(N\theta)] + \frac{\sin(N\theta)\sin(\theta)}{4} \\
&\qquad + \frac{\sin(N\theta)\sin(\theta)}{4} + \sin^2(N\theta/2)\cos^2(\theta/2) \\
&= \sin^2(\theta/2)\cos^2(N\theta/2) + \frac{\sin(N\theta)\sin(\theta)}{2} + \sin^2(N\theta/2)\cos^2(\theta/2).
\end{aligned}$$

This completes the derivation.

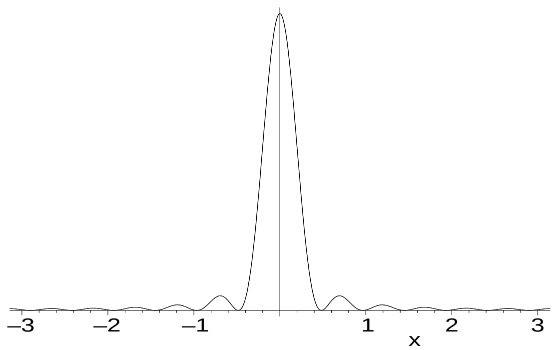


Figure 8.3: F_{16} ON $[-\pi, \pi]$.

Exercise 8.3.10. (a) Setting $D_0 = 1/2$, we get that

$$S_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(u+x) D_n(u)\, du \qquad \text{for all } n \geq 0$$

as in the proof of Theorem 8.3.3. Then,

$$\begin{aligned} \sigma_N(x) = \frac{1}{N+1} \sum_{n=0}^{N} S_n(x) \quad &= \quad \frac{1}{N+1} \sum_{n=0}^{N} \left(\frac{1}{\pi} \int_{-\pi}^{\pi} f(u+x) D_n(u)\, du \right) \\ &= \quad \frac{1}{\pi} \int_{-\pi}^{\pi} f(u+x) \left(\frac{1}{N+1} \sum_{n=0}^{N} D_n(u) \right) du \\ &= \quad \frac{1}{\pi} \int_{-\pi}^{\pi} f(u+x) F_N(u)\, du. \end{aligned}$$

(b) Looking at Figure 8.3, we see that F_N, like D_N, has a spike at the origin. However, unlike D_N, $F_N \geq 0$ and as N gets larger we can observe that away from zero the magnitude of the oscillations actually dies off to zero. To make this observation explicit, we can refer to the formula

$$F_N(u) = \frac{1}{2(N+1)} \left[\frac{\sin((N+1)\frac{\theta}{2})}{\sin(\frac{\theta}{2})} \right]^2.$$

If $\delta \leq |u| \leq \pi$, then $|\sin(u/2)| \geq \sin(\delta/2)$ and we see

$$|F_N(u)| \leq \frac{1}{2(N+1)} \left(\frac{1}{\sin(\delta/2)} \right)^2.$$

Because this estimate tends to zero as $N \to \infty$ and is independent of u, we see that $F_N \to 0$ uniformly on the set $\delta \leq |u| \leq \pi$.

(c) This follows from the fact that $\int_{-\pi}^{\pi} D_k(u)\,du = \pi$ for $k = 0, 1, \ldots, N$.
(d) From (c) we are able to write

$$f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)F_N(u)\,du,$$

so that

$$\begin{aligned}
\sigma_N(x) - f(x) &= \frac{1}{\pi}\int_{-\pi}^{\pi}(f(u+x) - f(x))F_N(u)\,du \\
&= \frac{1}{\pi}\int_{-\delta}^{\delta}(f(u+x) - f(x))F_N(u)\,du + \int_{|u|\geq\delta}(f(u+x) - f(x))F_N(u)\,du.
\end{aligned}$$

Given $\epsilon > 0$, use the uniform continuity of f to choose $\delta > 0$ so that

$$|f(x+u) - f(x)| < \epsilon \quad \text{whenever } |u| < \delta.$$

It follows that

$$\begin{aligned}
\left|\frac{1}{\pi}\int_{-\delta}^{\delta}(f(u+x) - f(x))F_N(u)\,du\right| &\leq \frac{\epsilon}{\pi}\int_{-\delta}^{\delta} F_N(u)\,du \\
&< \frac{\epsilon}{\pi}\int_{-\pi}^{\pi} F_N(u)\,du = \epsilon.
\end{aligned}$$

Having chosen δ, now pick N_0 large enough so that $N \geq N_0$ implies $|F_N(u)| \leq \epsilon$ for all $|u| \geq \delta$. Letting M be an upper bound on the size of $|f|$ we see

$$|f(u+x) - f(x)|F_N(u) \leq 2M\epsilon$$

as long as $|u| \geq \delta$, and it follows that

$$\frac{1}{\pi}\left|\int_{|u|\geq\delta}(f(u+x) - f(x))F_N(u)\,du\right| \leq \frac{1}{\pi}(2M\epsilon)\int_{-\pi}^{\pi} du = 4M\epsilon.$$

Combining the estimates on each of these two integrals, we get that

$$|\sigma_N(x) - f(x)| \leq \epsilon + 4M\epsilon \quad \text{for all } x \in (-\pi, \pi] \text{ and } N \geq N_0.$$

Because ϵ is arbitrary, we conclude that $\sigma_N \to f$ uniformly, and the proof is complete.

Exercise 8.3.11. Fix $e_N = 1/N$. If we can find a polynomial $p_N(x)$ such that

$$|p_N(x) - f(x)| < \epsilon_N \quad \text{for all } x \in [0, \pi],$$

it will follow that $p_N \to F$ uniformly, as desired.

From Fejér's theorem, we know there exists N such that

$$|\sigma_N(x) - f(x)| < \frac{\epsilon_N}{2} \quad \text{for all } x \in [0, \pi].$$

But $\sigma_N(x)$ is a linear combination of the partial sums $S_n(x)$ and each $S_n(x)$ is a linear combination of functions of the form $\cos(kx)$ and $\sin(kx)$. From the previous discussion about Taylor series, we know it is possible to find polynomials that are arbitrarily and uniformly close to the trigonometric functions that constitute each S_n. Because the sums in question are all finite, a repeated application of the triangle inequality implies that we can find a polynomial p_N satisfying

$$|p_N(x) - \sigma_N(x)| < \frac{\epsilon_N}{2} \quad \text{for all } x \in [0, \pi].$$

Finally, one last triangle inequality argument shows

$$\begin{aligned} |p_N(x) - f(x)| &\leq |p_N(x) - \sigma_N(x)| + |\sigma_N(x) - f(x)| \\ &< \frac{\epsilon_N}{2} + \frac{\epsilon_N}{2} = \epsilon_N. \end{aligned}$$

This proves the result on the interval $[0, \pi]$.

(b) To prove the general case we just use the change of variables $t = \pi(x - a)/(b-a)$ and observe that polynomials are preserved under this transformation.

8.4 A Construction of R From Q

Exercise 8.4.1. (a) We have to show C_r possesses the three properties of a cut. Property (c1) can be verified by noticing that C_r contains all rational $t < r$ and hence, it is not the empty set. Also, the set $C_r \neq \mathbf{Q}$ since all rational numbers greater than r are not contained in C_r.

To prove property (c2), fix $t \in C_r$ and assume $q < t$. Because $t \in C_r$ we have $q < t < r$ and thus q is an element of C_r, as desired.

Finally, let's show property(c3) holds for C_r. Note that for any $t \in C_r$ we can produce $q \in C_r$ with $t < q < r$ by letting $q = (t+r)/2$ .This shows C_r does not have a maximum.

(b) The set S is not a cut because it has a maximum.

(c) The set T is a cut.

(d) The set U is also a cut. It may seem as though $\sqrt{2}$ is a maximum, but our definition of a cut deals exclusively with rational numbers. At the moment there is no such thing as $\sqrt{2}$. In fact, this cut (which is equal to the cut in (c)) is to become $\sqrt{2}$ when we are finished.

Exercise 8.4.2. Because A is a cut, all rational $q < r$ are also in A. Hence, a rational number $s \notin A$ must be greater than $r \in A$ because if $s \leq r$ then s would be an element of A.

Exercise 8.4.3. The operations of addition and multiplication are commutative and associative on all of these sets, and the distributive property holds. The set of natural numbers is not a field because there is no additive identity and no additive inverses. Although $\mathbf{N}$ has a multiplicative identity, it also fails to have multiplicative inverses. The set of integers is an improvement in that $\mathbf{Z}$ has an additive identity and additive inverses. However, multiplicative inverses do not exist for elements of **Z** (except for the numbers -1 and 1). The set of rational numbers **Q** possesses all the properties of a field.

Exercise 8.4.4. In order to prove property (o1), we have to show that, for every pair of real numbers A and B, at least one of the statements $A \subseteq B$ or $B \subseteq A$ is true. This means either A is a subset of B or B be is a subset of A. If A is a subset of B then we are done, so let's assume that A is not a subset of B. Our goal is to show that $B \subseteq A$. Because A is not a subset of B there must exist an element $a \in A$ where $a \notin B$. Now let $b \in B$ be arbitrary. Because $a \notin B$, we know from Exercise 8.4.2 that $b < a$. Then property (c2) implies $b \in A$ which shows $B \subseteq A$.

Property (o2) is verified by noting that $A \subseteq B$ and $B \subseteq A$ is true if and only if $A = B$. In fact, showing containment in each direction is the standard way to prove two sets are equal.

Property (o3) is also straightforward because $A \subseteq B$ and $B \subseteq C$ certainly implies $A \subseteq C$.

Exercise 8.4.5. (a) The set $A+B$ is not the empty set because A is not empty and B is not empty. To argue $A + B \neq \mathbf{Q}$ pick $r_1 notin A$ and $l_2 \notin B$. Given an arbitrary elements $a \in A$ and $b \in B$, we again use Exercise 8.4.2 to say that $a < l_1$ and $b < l_2$. This implies $l_1 + l_2$ is an upper bound on $A + B$ meaning $A + B$ cannot be all of **Q**.

To show that $A + B$ does not have a maximum, fix $c \in A + B$ and write $c = a + b$ where $a \in A$ and $b \in B$. By property (c3) we know that there exists $s \in A$ with $a < s$. Also, there exists $r \in B$ with $b < r$. We can now conclude $s + r \in A + B$ with $c < s + r$.

(b) To show that addition is commutative we can write

$$\begin{aligned} A + B &= \{a + b : a \in A, b \in B\} \\ &= \{b + a : a \in A, b \in B\} = B + A. \end{aligned}$$

The proof that addition is associative is similar in that it follows directly from the fact that addition of rational numbers is associative. In particular, we can show that $x \in (A + B) + C$ if and only if $x = a + b + c$ where $a \in A$, $b \in B$ and $c \in C$. Then $(a + b) + c = a + (b + c)$ and the rest is clear sailing.

(c) (Note that some early editions of the text erroneously suggest showing $A + O = O$ instead of $A + O = A$.)

Let's follow the advice to prove inclusions in each direction, starting with $A + O \subseteq A$. Given $a + b \in A + O$ where $a \in A$ and $b \in O$, we know $b < 0$. Thus, $a + b < a$, and by property (c2), $a + b \in A$.

To prove the reverse inclusion, fix $a \in A$. By property (c3) there must exist $s \in A$ satisfying $a < s$, from which it follows that $a - s \in O$. Then

$$a = s + (a - s) \in A + O,$$

which proves $A \subseteq A + O$. These two inclusions together show $A = A + O$.

Exercise 8.4.6. (a) Let's verify property (c1). Because $A \neq \mathbf{Q}$, there exists $t \notin A$. Since $t < t+1$ we can conclude $-(t+1) \in -A$ by the definition of $-A$, and thus $-A \neq \emptyset$. To show $-A \neq \mathbf{Q}$ we start by noting A is not empty and picking $a \in A$. If $r \in -A$ we know there exists $t \notin A$ with $t < -r$. Then $t \notin A$ implies $a < t$ and it follows that $r < -a$. This proves that $-A$ is bounded above by $-a$ and thus $-A \neq \mathbf{Q}$.

To prove property (c2), we let $r \in -A$ and consider $s \in \mathbf{Q}$ satisfying $s < r$. Because $r \in -A$ there exists $t \notin A$ with $t < -r$. Since $s < r$ implies $-r < -s$ we have $t < -s$, which means $s \in -A$.

Finally, let's prove property (c3). If we let $r \in -A$, then there exists $t \notin A$ with $t < -r$. By the density property of the rational numbers we can choose $s \in \mathbf{Q}$ such that $t < s < -r$. This implies $-s \in -A$ and, because $r < -s$, we see that $-A$ does not possess a maximum.

(b) If we set $-A = \{r \in \mathbf{Q} : -r \notin A\}$ then $-A$ will not necessarily be a cut. In particular, property (c3) may fail to hold. For instance, if we let $A = \{r : r < 0\}$ then $-A = \{r : r \leq 0\}$ has a maximum value.

(c) Because $r \in -A$ we know there exists $t \notin A$ with $t < -r$. By Exercise 8.4.2 we have $a < t$, which implies $a < -r$. Thus, $a + r < 0$ and so $a + r \in O$. This shows $A + (-A) \subseteq O$.

Now, let's prove the reverse inclusion by fixing $o \in O$ and finding $a \in A$ and $b \in -A$ so that $a + b = o$. Set $\epsilon = |o|/2 = -o/2$. Now choose a rational number $t \notin A$ with the property that $t - \epsilon \in A$. (Here we are relying on properties (c1) and (c2) of a cut. In particular, we could show that if no such t existed then either $A = \mathbf{Q}$ or $A = \emptyset$.) Now the fact that $t \notin A$ implies $-(t+\epsilon) \in -A$. Then

$$o = -2\epsilon = -(t+\epsilon) + (t-\epsilon) \in -A + A,$$

and we conclude $O \subseteq -A + A$. This proves (f4).

(d) (Early versions of the text ask to prove property (o3) which has already been done. Later versions ask for proofs of (o4) earlier and (o5) later in the section.)

Exercise 8.4.7. (a) We must show AB has the properties of a cut. Let's first verify property (c1). The set $AB \neq \emptyset$ because all rational $q < 0$ are in AB. Furthermore, because A and B are bounded above then so are products of the form ab where both $a, b \geq 0$ with $a \in A$ and $b \in B$. This implies $AB \neq \mathbf{Q}$.

To prove property (c2), we let $t \in AB$ be arbitrary and let $s \in \mathbf{Q}$ satisfy $s < t$. If $s < 0$ then $s \in AB$ by the way we have defined the product. For the case $0 \leq s < t$ it must be that $t = ab$ where $a \in A$ and $b \in B$ satisfy $a > 0$ and $b > 0$. Because $s < ab$ we have $s/b < a$ which implies $s/b \in A$. Then

$$s = \left(\frac{s}{b}\right)(b) \in AB,$$

and (c2) is proved.

To verify property (c3), consider $t \in AB$. If $t < 0$ then $t < t/2$ and $t/2 \in AB$ because $t/2 < 0$ as well. If $t \geq 0$ then $t = ab$ for some $a \in A$ and $b \in B$. Applying property (c3) to A and B we get $s \in A$ and $r \in B$ with $a < s$ and $b < r$. We conclude $sr \in A + B$ with $ab < sr$.

(b) Let A, B and C be cuts and assume $A \leq B$ meaning $A \subseteq B$. To show $A + C \leq B + C$ we let $x \in A + C$ be arbitrary. Then $x = a + c$ where $a \in A$ and $c \in C$. Because $A \subseteq B$ we have $a \in B$ as well and it follows that $x \in B + C$. This proves (o4). Property (o5) follows immediately from our definition of the product of two positive cuts.

(c) The cut $I = \{p \in \mathbf{Q} : p < 1\}$ is the multiplicative identity. Exercise 8.4.1 contains the argument that $I = C_1$ is actually a cut. We now show $AI = A$ for all $A \geq O$ by demonstrating inclusion both ways.

Fix $q \in AI$. Because $I \geq 0$, then either $q < 0$ or $q = ab$ where $a, b \geq 0$ with $a \in A$ and $b < 1$. If $q < 0$ then $q \in A$ because $A \geq 0$. In the other case we have $q = ab < a$ and property (c2) implies $ab \in A$. Thus, $AI \subseteq A$.

In the other direction we consider $a \in A$. If $a < 0$ then $a \in AI$ by our definition of the product of positive cuts. If $a \geq 0$, then property (c3) says that we can pick a rational $p \in A$ with $a < p$. This implies $a/p < 1$ and hence $a/p \in I$. But then,

$$a = \left(\frac{a}{p}\right)(p) \in AI,$$

which shows $AI \subseteq A$, and we conclude that $A = AI$.

(d) To show $AO \subseteq O$ we let $b \in AO$ be arbitrary. Because there are no positive elements of O, we see from our definition of the product AO that we must have $b < 0$. This implies $b \in O$ and we conclude $AO \subseteq O$. The reverse inclusion is true because $a \in O$ means $a < 0$ which implies $a \in AO$.

Exercise 8.4.8. (a) In order to prove $S \in R$ we have to show S possesses the three properties of a cut. Consider property (c1). Since $\mathcal{A} \neq \emptyset$, the set S, which is the union of all $A \in \mathcal{A}$, cannot be the empty set. In addition, because $\mathcal{A}$ is bounded above by some cut B, we have that $S \leq B$. Since, $B \neq \mathbf{Q}$ we conclude $S \neq \mathbf{Q}$ as well.

To prove property (c2), we let $a \in S$ and consider $r \in \mathbf{Q}$ satisfying $r < a$. Because $a \in S$, it follows that $a \in A$ for some $A \in \mathcal{A}$. Since A is a cut, $r \in A$ which implies $r \in S$ as well.

Finally, to verify property (c3), let's fix an arbitrary $a \in S$ and show that there exists an element q in S with $a < q$. As before, if $a \in S$ then $a \in A$ for some $A \in \mathcal{A}$. Since A is a cut we can find $q \in A$, and hence in S, with $a < q$.

(b) By definition, S is the union of all $A \in \mathcal{A}$ which implies $A \subseteq S$ or $A \leq S$. This shows S is an upper bound for $\mathcal{A}$. Now, let B be an arbitrary upper bound for $\mathcal{A}$. To show $S \leq B$, consider an arbitrary $s \in S$. As we have seen several times now, it must be that $s \in A$ for some A in $\mathcal{A}$ and this implies $a \in B$ because $A \subseteq B$. Therefore, $S \subseteq B$ or $S \leq B$, and our proof is complete.

Exercise 8.4.9. (a) We first show $C_r + C_s = C_{r+s}$ by showing inclusion both ways. For the forward inclusion, let $t + p \in C_r + C_s$ where $t \in C_r$ and $p \in C_s$. Then $t < r$ and $p < s$, and we see $t + p < r + s$. This implies $t + p \in C_{r+s}$ and thus $C_r + C_s \subseteq C_{r+s}$.

For the reverse inclusion we start with $p \in C_{r+s}$. Then $p < r+s$ implies $r+s-p > 0$. Letting $\epsilon = r+s-p$, a little algebra yields $p = (r-\epsilon/2)+(s-\epsilon/2)$. Observe that $r-\epsilon/2 \in C_r$ and $s-\epsilon/2 \in C_s$, and this implies $p \in C_r + C_s$, as desired. We conclude $C_{r+s} \subseteq C_r + C_s$ and therefore the sets are equal.

To verify $C_rC_s = C_{rs}$ for positive r and s we fix $q \in C_rC_s$. If $q < 0$ then $q < rs$ which implies $q \in C_{rs}$. If $q \geq 0$ then $q = ap$ for some $a \in C_r$ and $p \in C_s$ where both $a, p \geq 0$. Because everything is positive, we get $ap < rs$ which implies $q = ap \in C_{rs}$. This shows $C_rC_s \subseteq C_{rs}$.

For the other inclusion we consider $p \in C_{rs}$. If $p < 0$ then the way we have defined the product ensures $p \in C_rC_s$. If $p \geq 0$ then observe that $p < rs$ implies $p/s < r$ from which we conclude that $p/s \in C_r$. Then

$$p = \left(\frac{p}{s}\right)(s) \in C_rC_s,$$

and it follows that $C_{rs} \subseteq C_rC_s$. Thus $C_rC_s = C_{rs}$.

(b) ($\Rightarrow$) For each $n \in \mathbf{N}$ the rational number $r - (1/n) \in C_r$. Because $C_r \subseteq C_s$, we see $r - (1/n) \in C_s$. This means

$$r - \frac{1}{n} < s \quad \text{for all } n \in \mathbf{N},$$

and a short contradiction argument shows $r \leq s$.

($\Leftarrow$) Conversely, assume $r \leq s$. If $a \in C_r$ then $a < r \leq s$ which implies $a \in C_s$. Therefore, $C_r \subseteq C_s$ or, equivalently, $C_r \leq C_s$.