Permafrost melting and active layer depth

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1 Permafrost and climate

Permafrost, defined as ground that remains frozen for at least two consecutive years, is found across the world in high latitude and high alpine regions. Importantly for climate, permafrost soils store large amounts of carbon in the form of dead, frozen organic matter. As long as permafrost soils remain perennially frozen, this carbon stays put and cannot be consumed and respired by microbes and released as CO_2 or CH_4, greenhouse gasses. Worryingly, however, human-induced climate change is disproportionately warming the arctic and long-frozen permafrost soils are melting during the warm summer months. As permafrost melts, the organic carbon it stores is consumed by microbes, respired, and released as CO_2 into the atmosphere. Permafrost melting thus provides a positive feedback loop on global warming: as the planet warms, more permafrost melts, releasing more carbon, which via the greenhouse effect acts to further warm the planet. Here, we will explore how changes in air temperature propagate at depth in permafrost settings.

2 The heat equation

The time (t) evolution of heat through a material with thickness z and thermal diffusivity α is governed by the heat equation

$$\frac{\partial T(x,y,z,t)}{\partial t} = \alpha \nabla^2 T(x,y,z,t) \tag{1}$$

$$= \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right). \tag{2}$$

In our study of permafrost melting, we are concerned with temperature variations with depth (z); we will assume the ground is infinite and homogeneous in composition in the x and y directions (see Figure 1), which simplifies the heat equation to one spatial dimension:

$$\frac{\partial T(z,t)}{\partial t} = \alpha \frac{\partial^2 T}{\partial z^2}. \tag{3}$$

where α, the thermal diffusivity, is a constant property of the material (in our case, frozen ground).

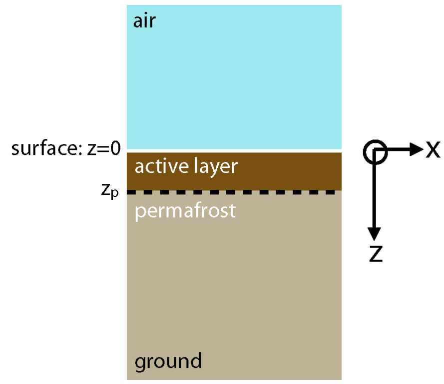

Figure 1: Depth profile through arctic soil. The active layer freezes and thaws seasonally, while permafrost stays frozen year-round.

3 Problems

3.1 Confirm a solution to the heat equation

While solving equation 3 is outside the scope of this class, we can confirm that solutions exist. Please show that

$$T(z,t) = Ae^{(\frac{-z}{z_w})} \sin\left(\omega t - \frac{z}{z_w} + C_1\right) + C_2 \tag{4}$$

is a solution to equation 3 by calculating the derivatives $\frac{\partial T}{\partial t}$ and $\frac{\partial^2 T}{\partial z^2}$. Then, write an expression for the thermal diffusivity (α) in terms of other variables; this term describes how quickly temperature diffuses through a material. (Do the units of α make sense?) The "skin depth" of the material is given by z_w and describes the depth at which temperature has attenuated by a factor of $1/e$. In our coordinate system, z_w is always positive (see Figure 1).

Solution: Calculate derivatives with respect to t and z:

$$\frac{\partial T}{\partial z} = -\frac{A}{z_w} e^{(\frac{-z}{z_w})} \left(sin(\omega t - \frac{z}{z_w} + cos(\omega t - \frac{z}{z_w}) \right) \tag{5}$$

$$\frac{\partial^2 T}{\partial z^2} = 2\frac{A}{z_w^2} e^{(\frac{-z}{z_w})} \left(cos(\omega t - \frac{z}{z_w}) \right) \tag{6}$$

$$\frac{\partial T}{\partial t} = Ae^{(\frac{-z}{z_w})} \omega cos(\omega t - \frac{z}{z_w}) \tag{7}$$

Plug into eq. 3 and solve for α:

$$Ae^{(\frac{-z}{z_w})}\omega cos(\omega t - \frac{z}{z_w}) = \alpha 2\frac{A}{z_w^2}e^{(\frac{-z}{z_w})}\left(cos(\omega t - \frac{z}{z_w})\right) \tag{8}$$

$$\omega = \alpha\frac{2}{z_w^2} \tag{9}$$

$$\alpha = \frac{\omega z_w^2}{2} \tag{10}$$

Units of α: $m^2 s^{-1}$.

3.1.1 Temperature evolution at the surface

In locations on Earth that experience four seasons, annual surface air temperature can be approximated by a sinusoidal function with a period of one year (3.15576×10^7 s) where temperatures rise from midwinter through midsummer and fall from midsummer through midwinter. We define our coordinate system so that the surface (the interface between air and ground) is at depth $z = 0$ and set $C_1 = \frac{3\pi}{2}$ so that temperatures are at a minimum at $t = 0$. Write an expression for the temperature at the air-ground interface, $z = 0$.

Solution:

$$T(z = 0, t) = A\sin(\omega t + C_1) + C_2 \tag{11}$$

3.2 Permafrost and the active layer

We will now define the top of the permafrost to be the depth z_p below which $T(z, t)$ for all t is less than 0°C. This is ground that will remain frozen all year. Soils between z_p and $z = 0$ are known as the "active layer" and freeze/thaw seasonally: as temperatures fall from summer through winter, the active layer freezes, but will thaw again down to depth z_p as temperatures rise the following year.

Abrupt, seasonal, freezing and thawing in the active layer drives many physical processes that affect the landscape and communities that live there, including ground collapse, rapid erosion, and the formation of short-lived lakes[1]. Determining z_p – and predicting how it will change with our warming climate – is crucial for understanding permafrost melting and its impact on carbon emissions, as well as for local communities' building and planning.

Determining the thickness of the active layer (i.e. the depth to permafrost) is difficult to do analytically, so scientists typically rely on field temperature measurements taken in a borehole (a core through the ground) to identify the active layer. However, we can still use equation 4 to explore interesting behavior of temperature at depth.

3.2.1 Depth to permafrost, z_p

Write an expression for the thickness of the active layer (i.e. depth to permafrost), z_p, at a given time, t_0. (Find $z = z_p(t)$ such that $T(z, t_0) < 0$).

[1]See, for example Schurr et al. 2015

1. Find all z_p such that $T(z = z_p, t = t_0) < 0$.
2. Find the shallowest (smallest) depth, $z_p(t_0)$, at which $T(z = z_p, t = t_0) < 0$.

A will always be positive for our problem set up (check out the Alaskan examples below for real-world values of A). ω (frequency) is equal to one cycle of yearly temperatures ($\omega = 2\pi/t$) and t is one year of time.

Solution: For $T(z,t) = Ae^{(\frac{-z}{z_w})} \sin\left(\omega t - \frac{z}{z_w} + \frac{3*\pi}{2}\right) < 0$, since $Ae^{(\frac{-z}{z_w})}$ is always positive, $\sin\left(\omega t - \frac{z}{z_w} + \frac{3*\pi}{2}\right)$ must be < 0. $sin(x) < 0$ on the interval $(\pi, 2\pi)$. Here, $x = \omega t - \frac{z}{z_w} + \frac{3*\pi}{2}$. Solve $x = \pi$ and $x = 2\pi$ to find the range of depths at which T is always < 0.

$$z_p = \left(z_w(\omega t_0 - \frac{\pi}{2}), z_w(\omega t_0 + \frac{\pi}{2})\right) \tag{12}$$

Since $z_w > 0$ and $\omega t \geq 0$, the smallest z_p is $z_p = z_w(\omega t_0 - \frac{\pi}{2})$.

3.3 Temperature change with depth in Alaska

Based on monthly average temperature data[2], we can approximate the annual (air) temperature curve for two locations in Alaska, Utqiagvik ($\sim 71°$N) and Bettles ($\sim 66°$N), shown in Figure 2:

$$T_{71^\circ}(t) = 16 \times \sin\left(\omega t + \frac{3\pi}{2}\right) - 10 \tag{13}$$

$$T_{66^\circ}(t) = 19 \times \sin\left(\omega t + \frac{3\pi}{2}\right) - 3 \tag{14}$$

$$\omega = \frac{2\pi}{3.15576 \times 10^7}\mathrm{s}^{-1}. \tag{15}$$

We'll now use these real-world examples to explore how temperature propagates with depth.

3.3.1 Summer temperature at depth

We have defined C_1 such that "January 1" occurs at $t = 0$ and halfway through the year ($t = \mathrm{year}/2$, "midsummer") surface temperatures are at a maximum. What is the temperature at 1m depth at midsummer at Bettles? How does it compare to the surface temperature at the same time of year? Use $\alpha = 0.2 \times 10^{-6}$ m^{2}s^{-1}.

Surface temperature at midsummer (plug into eq. 15):

$$T_{66^\circ}(t = \mathrm{year}/2) = 19 \times \sin\left(\omega \mathrm{year}/2) + \frac{3\pi}{2}\right) - 3 \tag{16}$$

$$T_{66^\circ}(t = \mathrm{year}/2) = 16°\mathrm{C} \tag{17}$$

[2]Source: http://climate.gi.alaska.edu/

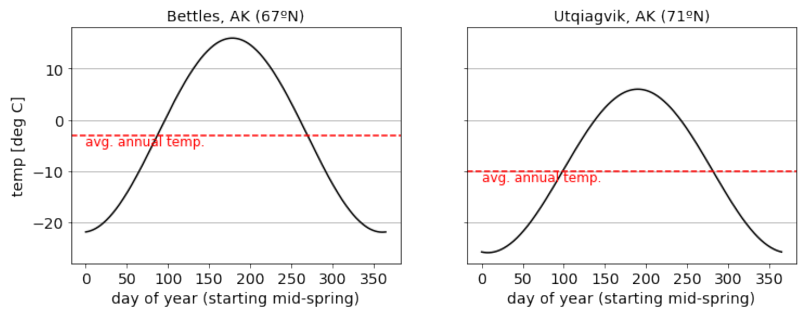

Figure 2: Approximation of annual (mean monthly) temperature variation for two Alaskan localities. See equations 13 and 14.

Temperatures at depth:
Must write $T(z,t)$ given eq. 14.

$$T_{66^\circ}(z,t) = 19e^{-\frac{z}{z_w}} \sin\left(\omega t + \frac{3\pi}{2} - \frac{z}{z_w}\right) - 3 \tag{18}$$

$$T_{66^\circ}(z = 1m, t = \text{year}/2) = 10.4°\text{C} \tag{19}$$

Temperature at depth is less than temperature at surface.

3.3.2 Maximum temperature at depth

Now let's explore the general case of what we just saw with Bettles's midsummer temperatures. Imagine you are standing on the surface in Bettles at midsummer. How many days will elapse (δt) before the ground 1m below your feet ($z = 1m$) reaches its maximum annual temperature? For each meter of depth, how much later in the year does the maximum temperature occur?

$T_{66^\circ}(z,t) = 19e^{-\frac{z}{z_w}} \sin\left(\omega t + \frac{3\pi}{2} - \frac{z}{z_w}\right) - 3$ is greatest when $\sin\left(\omega t + \frac{3\pi}{2} - \frac{z}{z_w}\right)$ is greatest, and the maximum of $\sin\left(\omega t + \frac{3\pi}{2} - \frac{z}{z_w}\right)$ is 1. $\sin(x) = 1$ when $x = \frac{\pi}{2}$, so set $\omega t + \frac{3\pi}{2} - \frac{z}{z_w} = \frac{\pi}{2}$. We want to find how much t changes for each increment in z, and we are concerned with time t^* specifically, the time when T is maximum. So, solve $\omega t + \frac{3\pi}{2} - \frac{z}{z_w} = \frac{\pi}{2}$ for $t^*(z)$.

$$\omega t^* + \frac{3\pi}{2} - \frac{z}{z_w} = \frac{\pi}{2} \tag{20}$$

$$t^*(z) = \frac{z}{\omega z_w} - \frac{\pi}{\omega} \tag{21}$$

and differentiate:

$$\frac{dt^*}{dz} = \frac{1}{\omega z_w} \tag{22}$$

$$\frac{\delta t}{\delta z} = \frac{1}{\omega z_w} \tag{23}$$

$$\delta t = \delta z \frac{1}{\omega z_w} \tag{24}$$

For each meter of depth ($\delta z = 1$), maximum temperature (which occurs at t^*) occurs $\frac{1}{\omega z_w}$ days later. Plug in α and ω (as above) into solution of question 3.1 to find z_w, and then find

$$\delta t = 3542508s. \tag{25}$$

Convert to days (divide by $60 \times 60 \times 24$) and find $\delta t = 41$ days.

In colder climates, where more of the ground is ice, α approaches 1×10^{-6} m^{2}s^{-1}. Find δt if $\alpha = 1 \times 10^{-6}$ m^{2}s^{-1}. How would Figure 3 change for this value of α?

Same as above, but re-calculate z_w and δt and find $\delta t = 18.3$ days. “Smears” of cold/hot temperatures propagating at depth would be more vertical and less skewed right. Coldest day at depth would be closer in time to coldest day at surface. (See Figure 4)

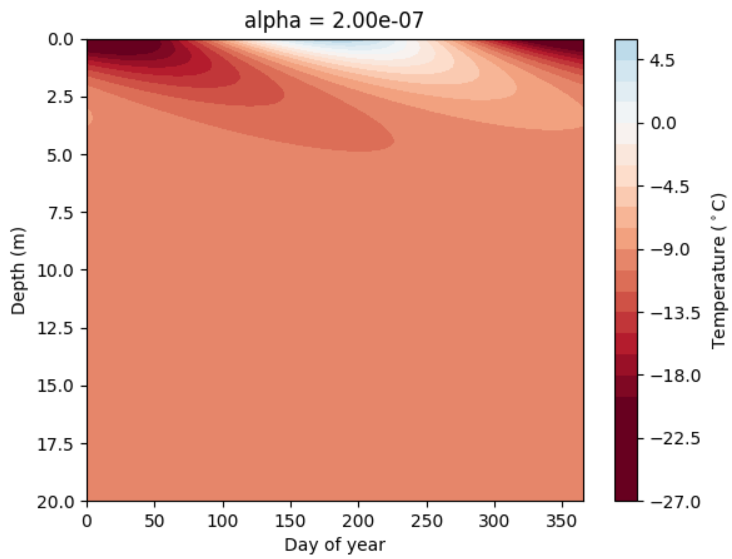

Figure 3: Temperature with time and depth for $\alpha = 0.2 \times 10^{-6}$ m^{2}s^{-1}.

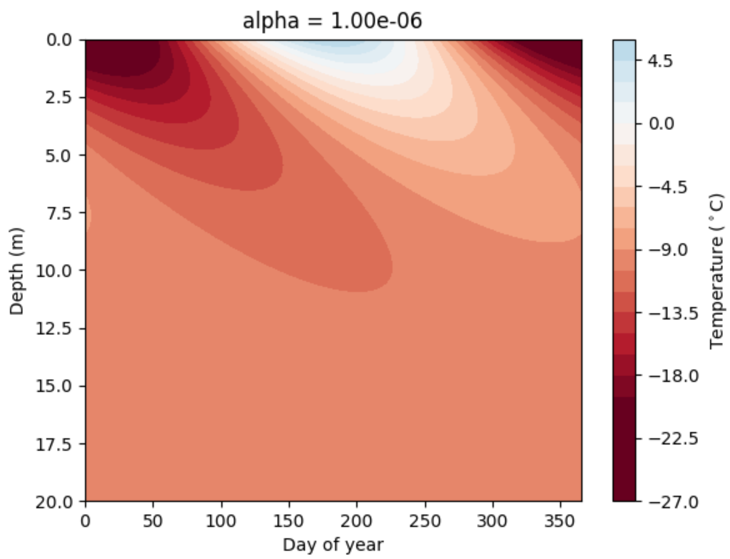

Figure 4: Temperature with time and depth for $\alpha = 1.0 \times 10^{-6}$ m^{2}s^{-1}.