Differential Equations and the Simple and Damped Harmonic Oscillator

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What is a differential equation?

Differential Equation Example:

$$\frac{dy}{dx} = (x^2 + 6)(y - 7)$$

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$$\boxed{y=e^{\left(\frac{1}{3}x^3+6x+C\right)}+7}$$

Why are they useful?

How are they used in Physics?

Solving the Simple Harmonic Oscillator

$$m\frac{d^2 x}{dt^2}=-k\,x$$

$$m\frac{d^2x}{dt^2} = -kx$$

$$m\frac{d^2x}{dt^2} + kx = 0$$

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$$Note:\omega^2=\frac{k}{m}$$

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We assume $x(t) = e^{rt}$ *because we are dealing with linear differential equations. We can replace* x *with* e^{rt}

$$\frac{d^2}{dt^2}e^{rt} + \omega^2 e^{rt} = 0$$

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$$\frac{d^2}{dt^2}e^{rt}+\omega^2 e^{rt}=0$$

$$r^2 e^{rt}+\omega^2 e^{rt}=0 \qquad Here, we\ take\ the\ second\ derivative\ of\ e^{rt}$$

$\left(Most\ recent\ step\ ¿\ previous\ slide\right)$ $$r^2 e^{rt} + \omega^2 e^{rt} = 0$$ $Here, we\ take\ the\ second\ derivative\ of\ e^{rt}$

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$$e^{rt}(r^2 + \omega^2) = 0$$

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$for\ r = \alpha \pm i\beta,\ x(t) = e^{\alpha t}(C_1 \cos(\beta t) + C_2 \sin(\beta t))$

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Solving the Damped Harmonic Oscillator

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$$\lambda^2 + 2\gamma\lambda + \omega^2 = 0 \qquad \textit{factor out } e^{\lambda t}\textit{, exponentials can never be } 0$$

$$\lambda^2+2\gamma\lambda+\omega^2=0$$

$$\lambda^2 + 2\gamma\lambda + \omega^2 = 0$$

$$\lambda_{\pm} = \frac{-2\gamma \pm \sqrt{4\gamma^2 - 4\omega^2}}{2\,(1)}$$

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linear combination of roots provides general solution

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plug into $e^{\lambda t}$

$$x = a\, e^{(-\gamma + i\sqrt{\omega^2 - \gamma^2})t} + b\, e^{\left(-\gamma - i\sqrt{\omega^2 - \gamma^2}\right)t}$$

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$$x(t) = A\,e^{-\gamma t}\cos\left\{\left(\sqrt{\omega^2-\gamma^2}\right)t+\phi\right\}$$

factor out $e^{-\gamma t}$*, simplify with euler's formula*

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$$\boxed{x(t) = A\, e^{-\gamma t} \cos\{\omega' t + \phi\}}$$

$$\omega' = \sqrt{\omega^2 - \gamma^2}$$

Thank you! Any questions?