

Differential Equations and the Simple and Damped Harmonic Oscillator

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**What is a differential
equation?**

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$$y = e^{\left(\frac{1}{3} x^3 + 6x + C \right)} + 7$$

Why are they useful?

**How are they used in
Physics?**

Solving the Simple Harmonic Oscillator

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We assume $x(t) = e^{rt}$ because we are dealing with linear differential equations. We can replace x with e^{rt}

$$\frac{d^2}{dt^2} e^{rt} + \omega^2 e^{rt} = 0$$

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$$r^2 e^{rt} + \omega^2 e^{rt} = 0 \quad \text{Here, we take the second derivative of } e^{rt}$$

(Most recent step *i* previous slide)

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Now, we use the characteristic equation of :
for $r = \alpha \pm i\beta$, $x(t) = e^{\alpha t} (C_1 \cos(\beta t) + C_2 \sin(\beta t))$
for us, $\alpha = 0 \wedge \beta = \omega$

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$$x(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t)$$

Solving the Damped Harmonic Oscillator

$$\sum \vec{F} = m \vec{a}$$

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$$\frac{d^2 x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega^2 x = 0 \quad \omega^2 = \frac{k}{m}, \gamma = \frac{\beta}{2m}$$

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$$\lambda^2 + 2\gamma \lambda + \omega^2 = 0 \quad \text{factor out } e^{\lambda t}, \text{ exponentials can never be 0}$$

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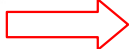
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 linear combination of roots provides general solution

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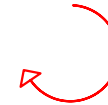
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plug into $e^{\lambda t}$

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$$x(t) = A e^{-\gamma t} \cos\{\omega' t + \phi\} \quad \omega' = \sqrt{\omega^2 - \gamma^2}$$

Thank you! Any questions?