4 Integration





4.1 Antiderivatives and Indefinite Integration

Objectives

- Write the general solution of a differential equation and use indefinite integral notation for antiderivatives.
- Use basic integration rules to find antiderivatives.
- Find a particular solution of a differential equation.

Antiderivatives

Antiderivatives

To find a function F whose derivative is $f(x) = 3x^2$, you might use your knowledge of derivatives to conclude that

$$F(x) = x^3 \text{ because } \frac{d}{dx}[x^3] = 3x^2.$$

The function F is an *antiderivative* of f.

Definition of Antiderivative

A function F is an **antiderivative** of f on an interval I when $F'(x) = f(x)$ for all x in I.

Antiderivatives

Note that F is called *an* antiderivative of f rather than *the* antiderivative of f.

To see why, observe that

$$F_1(x) = x^3, \quad F_2(x) = x^3 - 5, \quad \text{and} \quad F_3(x) = x^3 + 97$$

are all antiderivatives of $f(x) = 3x^2$.

In fact, for any constant C, the function $F(x) = x^3 + C$ is an antiderivative of f.

Antiderivatives

THEOREM 4.1 Representation of Antiderivatives

If F is an antiderivative of f on an interval I, then G is an antiderivative of f on the interval I if and only if G is of the form $G(x) = F(x) + C$ for all x in I where C is a constant.

Using Theorem 4.1, you can represent the entire family of antiderivatives of a function by adding a constant to a *known* antiderivative.

Antiderivatives

For example, knowing that

$$D_x[x^2] = 2x$$

you can represent the family of *all* antiderivatives of $f(x) = 2x$ by

$$G(x) = x^2 + C$$ Family of all antiderivatives of $f(x) = 2x$

where C is a constant. The constant C is called the **constant of integration**.

Antiderivatives

The family of functions represented by G is the **general antiderivative** of f, and $G(x) = x^2 + C$ is the **general solution** of the *differential equation*

$$G\boxed{?}(x) = 2x.$$

Differential equation

A **differential equation** in x and y is an equation that involves x, y, and derivatives of y. For instance,

$$y\boxed{?} = 3x \quad \text{and} \quad y\boxed{?} = x^2 + 1$$

are examples of differential equations.

Example 1 – *Solving a Differential Equation*

Find the general solution of the differential equation $y' = 2$.

Solution:

To begin, you need to find a function whose derivative is 2. One such function is

2*x* is an antiderivative of 2.

$$y = 2x.$$

Now, you can use Theorem 4.1 to conclude that the general solution of the differential equation is

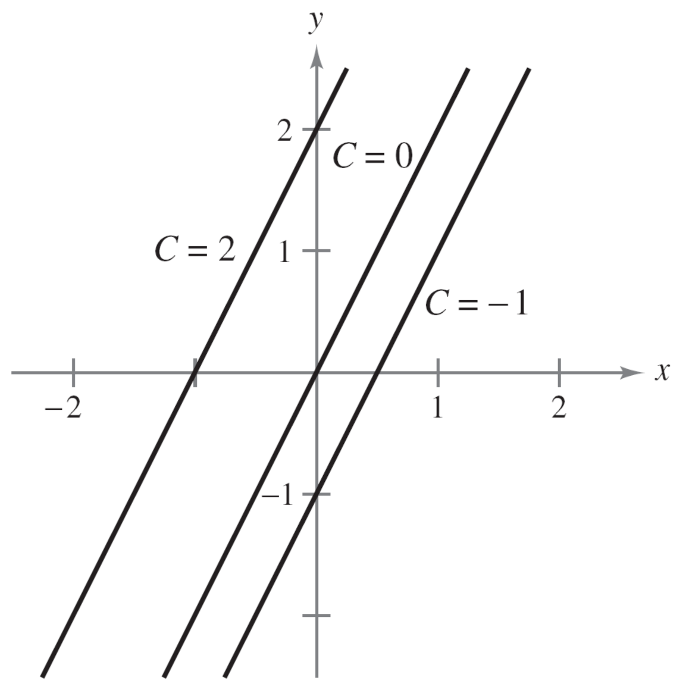
General solution

$$y = 2x + C.$$

Example 1 – *Solution*

cont'd

The graphs of several functions of the form $y = 2x + C$ are shown in Figure 4.1.



Functions of the form $y = 2x + C$

Figure 4.1

Antiderivatives

When solving a differential equation of the form

$$\frac{dy}{dx} = f(x)$$

it is convenient to write it in the equivalent differential form

$$dy = f(x)\, dx.$$

The operation of finding all solutions of this equation is called **antidifferentiation** (or **indefinite integration**) and is denoted by an integral sign $\int$.

Antiderivatives

The general solution is denoted by

$$y = \int \underbrace{f(x)}_{\text{Integrand}} \, \overbrace{dx}^{\text{Variable of integration}} = \underbrace{F(x)}_{\text{An antiderivative of } f(x)} + \overbrace{C}^{\text{Constant of integration}}.$$

The expression $\int f(x)\,dx$ is read as the *antiderivative of f with respect to x*. So, the differential *dx* serves to identify *x* as the variable of integration. The term **indefinite integral** is a synonym for antiderivative.

Basic Integration Rules

Basic Integration Rules

The inverse nature of integration and differentiation can be verified by substituting $F'(x)$ for $f(x)$ in the indefinite integration definition to obtain

$$\int F'(x)\,dx = F(x) + C.$$

Integration is the "inverse" of differentiation.

Moreover, if $\int f(x)\,dx = F(x) + C$, then

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x).$$

Differentiation is the "inverse" of integration.

Basic Integration Rules

These two equations allow you to obtain integration formulas directly from differentiation formulas, as shown in the following summary.

Basic Integration Rules

Differentiation Formula	*Integration Formula*	
$\frac{d}{dx}[C] = 0$	$\int 0\,dx = C$	
$\frac{d}{dx}[kx] = k$	$\int k\,dx = kx + C$	
$\frac{d}{dx}[kf(x)] = kf'(x)$	$\int kf(x)\,dx = k\int f(x)\,dx$	
$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$	$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$	
$\frac{d}{dx}[x^n] = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$	Power Rule
$\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x\,dx = \sin x + C$	
$\frac{d}{dx}[\cos x] = -\sin x$	$\int \sin x\,dx = -\cos x + C$	

Basic Integration Rules

cont'd

Basic Integration Rules

Differentiation Formula	*Integration Formula*
$\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
$\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$
$\frac{d}{dx}[\cot x] = -\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$
$\frac{d}{dx}[\csc x] = -\csc x \cot x$	$\int \csc x \cot x\,dx = -\csc x + C$
$\frac{d}{dx}[e^x] = e^x$	$\int e^x\,dx = e^x + C$
$\frac{d}{dx}[a^x] = (\ln a)a^x$	$\int a^x\,dx = \left(\frac{1}{\ln a}\right)a^x + C$
$\frac{d}{dx}[\ln x] = \frac{1}{x},\ x > 0$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$

Example 2 – *Describing Antiderivatives*

$$\int 3x\,dx = 3\int x\,dx$$ Constant Multiple Rule

$$= 3\int x^1\,dx$$ Rewrite x as x^1.

$$= 3\left(\frac{x^2}{2}\right) + C$$ Power Rule ($n = 1$)

$$= \frac{3}{2}x^2 + C$$ Simplify.

The antiderivatives of $3x$ are of the form $\frac{3}{2}x^2 + C$, where C *is* any constant.

Basic Integration Rules

When indefinite integrals are evaluated, a strict application of the basic integration rules tends to produce complicated constants of integration.

For instance, in Example 2, the solution could have been written as

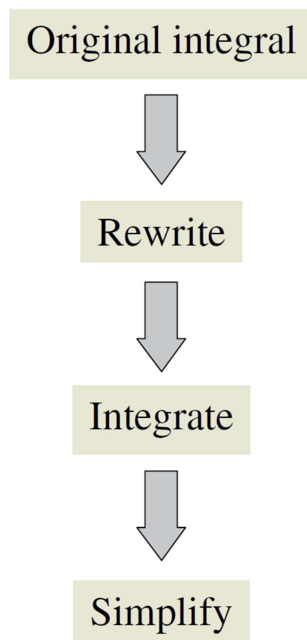
$$\int 3x\,dx = 3\int x\,dx = 3\left(\frac{x^2}{2} + C\right) = \frac{3}{2}x^2 + 3C.$$

Because C represents *any* constant, it is both cumbersome and unnecessary to write $3C$ as the constant of integration.

Basic Integration Rules

So, $\frac{3}{2}x^2 + 3C$ is written in the simpler form $\frac{3}{2}x^2 + C$.

In Example 2, note that the general pattern of integration is similar to that of differentiation.



Initial Conditions and Particular Solutions

Initial Conditions and Particular Solutions

We know that the equation $y = \int f(x)\,dx$ has many solutions (each differing from the others by a constant).

This means that the graphs of any two antiderivatives of f are vertical translations of each other.

Initial Conditions and Particular Solutions

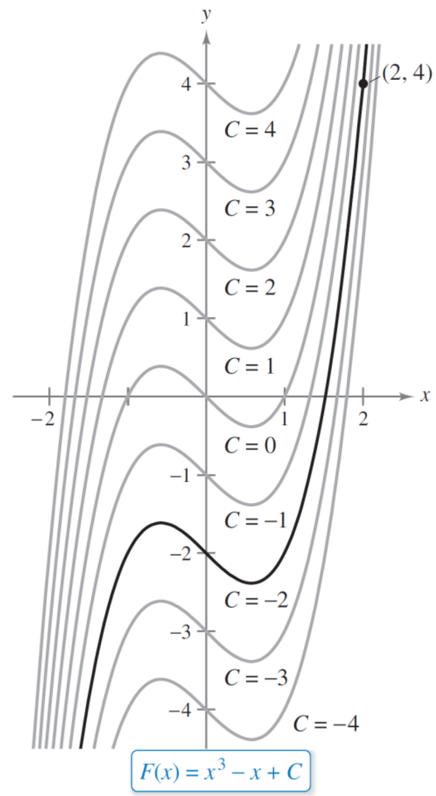
For example, Figure 4.2 shows the graphs of several antiderivatives of the form

$$y = \int (3x^2 - 1)\, dx$$

$$= x^3 - x + C \qquad \text{General solution}$$

for various integer values of C.

Each of these antiderivatives is a solution of the differential equation $\frac{dy}{dx} = 3x^2 - 1.$



The particular solution that satisfies the initial condition $F(2) = 4$ is $F(x) = x^3 - x - 2$.

Figure 4.2

Initial Conditions and Particular Solutions

In many applications of integration, you are given enough information to determine a **particular solution**.

To do this, you need only know the value of $y = F(x)$ for one value of x.

This information is called an **initial condition**. For example, in Figure 4.2, only one curve passes through the point (2, 4).

Initial Conditions and Particular Solutions

To find this curve, you can use the general solution

$F(x) = x^3 - x + C$ General solution

and the initial condition

$F(2) = 4.$ Initial condition

By using the initial condition in the general solution, you can determine that

$F(2) = 8 - 2 + C = 4$

which implies that $C = -2$. So, you obtain

$F(x) = x^3 - x - 2.$ Particular solution

Example 8 – *Finding a Particular Solution*

Find the general solution of

$$F'(x) = e^x$$ Differential equation

and find the particular solution that satisfies the initial condition

$$F(0) = 3.$$ Initial condition

Solution:

To find the general solution, integrate to obtain

$$F(x) = \int e^x \, dx$$

$$= e^x + C.$$ General solution

Example 8 – *Solution*

cont'd

Using the initial condition $F(0) = 3$, you can solve for C as follows.

$$F(0) = e^0 + C$$

$$3 = 1 + C$$

$$2 = C$$

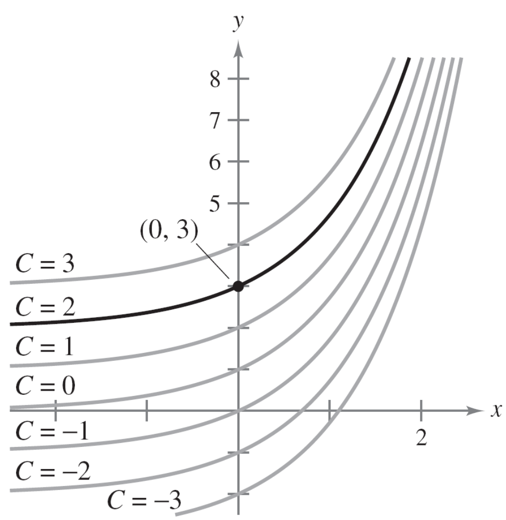
Example 8 – *Solution*

cont'd

So, the particular solution is

$$F(x) = e^x + 2$$ Particular solution

as shown in Figure 4.3.



The particular solution that satisfies the initial condition $F(0) = 3$ is $F(x) = e^x + 2$.

Figure 4.3

Initial Conditions and Particular Solutions

So far in this section, you have been using *x* as the variable of integration. In applications, it is often convenient to use a different variable.