

Statistical learning

1. Linear regression and stochastic gradient descent

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Notations

- ▶ n – sample size
- ▶ k – number of features
- ▶ $X \in \mathbf{R}^{n \times k}$ – input data $(x_i)_{i=1, \dots, n}$
- ▶ $\hat{\beta} \in \mathbf{R}^k$ – estimator of the weight parameters to build
- ▶ $y \in \mathbf{R}^n$ – output data
- ▶ $\hat{y} \in \mathbf{R}^n$ – predicted output

Least squares approximation

- ▶ The fit of a model to a data point is measured by its **residuals**

$$y_i - \hat{y}_i, \quad i = 1, \dots, n$$

- ▶ The least squared loss (or cost) function focuses in the **mean squared error** of this measure, that is

$$L(w) = \frac{1}{n} \sum_{i=1}^n \ell(y_i, \langle x_i, w \rangle) \quad \text{where} \quad \ell(y, y') = \frac{1}{2} |y - y'|^2$$

- . We immediately note that

$$\nabla L(w) = \frac{1}{n} \sum_{i=1}^n \ell'(y_i, \langle x_i, w \rangle) x_i \quad \text{with} \quad \ell'(y, y') = \partial_{y'} \ell(y, y')$$

Least squares estimator

- ▶ That leads us to the **optimization** problem :

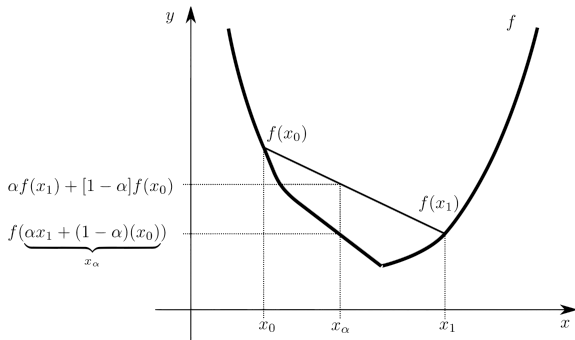
$$\hat{\beta} = \operatorname{argmin}_w L(w) \Leftrightarrow \nabla L(\hat{\beta}) = \vec{0} \quad (1)$$

- ▶ What are the conditions for uniqueness?
 - ▶ Why is (1) true?
- ▶ We are looking for the estimators $\hat{y}_i = \langle x_i, \hat{\beta} \rangle$.

Convex functions

A function $f : \mathbf{R}^d \rightarrow \mathbf{R}$ is **convex** if for all $x, y \in \mathbf{R}^d$ and all $\alpha \in [0, 1]$

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y)$$



Gradient Descent

Convergence of GD

One can show that this algorithm :

$$w_{t+1} = w_t - \gamma_t \cdot \nabla f(w_t) \quad (2)$$

converges to some $w_* \in \{\nabla f = 0\}$ for some **smooth functions** f with certain properties and some step size $\gamma_t := \gamma_t(f)$.

In our case, **convexity** is enough, but know that there is a more general class of functions that makes it still true.

Stochastic gradient descent (SGD)

If n is large, computing ∇L is expensive : it requires to to on the whole data set for just a step of the descent algorithm !

The idea of stochastic gradient is to build an **unbiased** estimator of ∇L : Choosing uniformly at random $N \in \{1, \dots, n\}$, then

$$\partial_w \mathbf{E}[\ell(y_N, \langle x_N, w \rangle)] = \frac{1}{n} \sum_{i=1}^n \ell'(y_i, \langle x_i, w \rangle) x_i = \nabla L(w)$$

- Question : Can we use this estimator in (2) ?

Convergence of SGD

Theorem : Robbins-Monroe

Let ℓ be defined as previously, and $N_t, t \in \mathbf{N}$ a sequence of *i.i.d.* random variables uniformly distributed on $\{1, \dots, n\}$. The algorithm

$$w_{t+1} = w_t - \gamma_t \cdot \partial_w \ell(y_{N_t}, \langle x_{N_t}, w_t \rangle)$$

converges to some $w_* \in \{\nabla L = 0\}$