The Triviality of Kontsevich's Flow on Quasi-Homogeneous Poisson Structures

Julie Alhosh
Supervisor: Prof. Brent Pym

August 2021

Department of Mathematics and Statistics
McGill University

1 Introduction

M. Kontsevich gives a "deformation quantization" formula for Poisson structures in [4]. It is, however, not the unique quantization formula and its non-uniqueness can be addressed using an object called the graph complex GC that was also introduced by Kontsevich in [3]. The zeroth cohomology of the graph complex, $H^0(\mathsf{GC})$, which is isomorphic to the Lie algebra of the Grothendieck-Teichmüller group, $\mathfrak{grt}$ [5], acts on Poisson structures. Conjecturally, $\mathfrak{grt}$ is freely generated by classes $\sigma_3, \sigma_5, \ldots$ associated to wheels with 3, 5, ... spokes, respectively. The quantization of a Poisson structure, π, is independent of choices if and only if $H^0(\mathsf{GC})$ acts trivially on π.

The triviality of the flow can be determined using Poisson cohomology. Let M be a smooth manifold and let $\pi \in \bigwedge^2 T_M$ be a Poisson structure. Consider the operator δ_π associated with the Lichnerowicz complex of π.

$$\delta_\pi := [\pi, -] : \wedge^k T_M \longrightarrow \wedge^{k+1} T_M \tag{1}$$

where [-,-] is the Schouten bracket. Then $\delta_\pi^2 = 0$ since $[\pi, \pi] = 0$ as π is a Poisson structure. The Poisson cohomology is the cohomology of $H^k(\wedge^k T_M, \delta_\pi)$. The action of the graph complex on π produces a Poisson cocycle. The flow is trivial if and only if the resulting Poisson cocycle is a coboundary of the Lichnerowicz complex of π.

Last summer, I added the functionality to compute the action of the graph cocycles of degree 0, the flow, on Poisson structures. The flow was trivial in all the examples I computed. Currently, it is unknown whether there exists a Poisson structure π such that $H^0(\mathsf{GC})$ acts non-trivially on π. Therefore, it is of interest to know whether the action of $H^0(\mathsf{GC})$ on Poisson structures is trivial.

In this report, we prove that the action of $\sigma_3, \sigma_5, \ldots$ is trivial for quasi-homogeneous Poisson structures on $\mathbb{C}^2$.

2 Graph Cocycles and Kontsevich Quantization Graphs

Let $\mathbf{G}$ be a graph cocycle as described in [3, 2] and let π be a Poisson structure on $\mathbb{C}^n$. $\mathbf{G}$ acts on π by:

$$\mathbf{G} \cdot \pi = \sum_{\mathcal{G} \in \mathbf{G}} \Lambda_{\mathcal{G}} \sum_{\substack{\Gamma:\ \Gamma \text{ is a Kontsevich} \\ \text{Quantization graph of } \mathcal{G}}} \mathrm{w}_\Gamma B_{\Gamma,\pi} \tag{2}$$

where $\Lambda_{\mathcal{G}}$ is the coefficient of a graph $\mathcal{G} \in \mathbf{G}$, Γ is a Kontsevich Quantization graph obtained from $\mathcal{G}$ by adding the external vertices and the sum over Γ takes all possible orientations of such Kontsevich Quantization graphs into consideration, w_Γ is the symmetry factor of the oriented graph Γ , $B_{\Gamma,\pi}$ is the skew-symmetrization of the bidifferential operator introduced in [4].

For simplicity, let $q, r = I(e)$ such that e is the incoming edge to L and R, respectively.

$$B_{\Gamma,\pi}(h_1, h_2) = \sum_{I:E_\Gamma \longrightarrow \{1,\ldots,d\}} \left[\prod_{k=1}^{n} \left(\prod_{\substack{e \in E_\Gamma,\\ e=(*,k)}} \partial_{I(e)} \right) \pi^{I(e_k^1)I(e_k^2)} \right] \partial_q h_1 \partial_r h_2 \quad (3)$$

Remark 1. *We have that for nonzero terms, $q \neq r$.*

Kontsevich Quantization graphs have the following properties:

1. the vertex set is $[n] \sqcup \{L, R\}$ where $[n] = \{1, 2, \ldots, n\}$ are the internal vertices and $\{L, R\}$ is the set of external vertices.
2. each internal vertex is at least trivalent with an out-degree of 2.
3. each external vertex has an in-degree of 1 and an out-degree of 0.
4. there are $2n$ edges, and both endpoints of $2(n-1)$ edges are internal vertices

Lemma 1. *In each Kontsevich Quantization graph Γ involved in the computation in (2), there exists an internal vertex that is trivalent.*

Proof. Assume not, i.e $in - deg(v) \geq 2$ for all $v \in [n]$. Then, the number of edges in between the internal vertices is $\geq 2n$, a contradiction to property 4. On the other hand, for each $\mathcal{G} \in \mathbf{G}$, v is at least trivalent for $v \in V_{\mathcal{G}}$. Then, the internal vertices of Γ are also at least trivalent. □

3 Quasi-Homogeneous Case

Let $\pi = f\ \partial_x \wedge \partial_y$ be a Poisson structure on $\mathbb{C}^2$ with coordinates (x, y). Assume that f is a quasi-homogeneous polynomial of degree D with respect to some weight vector $w = (w_1, w_2)$ corresponding to the weights of (x, y), where $w_1, w_2 \in \mathbb{N}$. Let $Z^k(\pi)$ denote the space of k-cocycles and $B^k(\pi)$ denote the space of k-coboundary of the Lichnerowicz complex of π.

Note that $\mathbf{G} \cdot \pi$ has the form $g\ \partial_x \wedge \partial_y$, where g is a polynomial. We will show that $[\mathbf{G} \cdot \pi] = 0$.

Lemma 2. *The resulting polynomial g is quasi-homogeneous of degree*

$$deg(g) = nD - (n-1)(w_1 + w_2) \quad (4)$$

Proof. By (3), the degree of a nonzero term in $B_{\Gamma,\pi}$ is

$$nD - \sum_{k=1}^{n} \sum_{\substack{e \in E_\Gamma,\\ e=(*,k)}} w_{I(e)} \quad (5)$$

The number of terms in the double sum is equal to the number of edges that are in between the internal vertices, i.e. $2(n-1)$. We notice that there are n constraints for a possible labelling which are of the form $I(e_k^1) \neq I(e_k^2)$ for all $k \in [n]$. Otherwise the term with the labelling I such that $I(e_k^1) = I(e_k^2)$ for any $k \in [n]$ would be zero since $\pi^{I(e_k^1)I(e_k^2)} = \pi^{ii} = 0$ for $i \in \{1,2\}$.

There are two cases regarding the constraints that involve q and r:

1. When $q = I(e_k^i)$ and $r = I(e_k^j)$ for some $k \in [n]$, we have one constraint $q \neq r$.
2. When we have two constraints: $q \neq I(e_k^i)$ and $r \neq I(e_l^j)$ where $k \neq l$. And by remark 1, we then have that $I(e_k^i) \neq I(e_l^j)$.

Now, there are $n-1$ constraints $\{c_1, \ldots, c_{n-1}\}$ that do not involve q and r of the form $I(e_k^i) \neq I(e_l^j)$ which implies that $w_{I(e_k^i)} + w_{I(e_l^j)} = w_1 + w_2$. Therefore,

$$\begin{aligned} deg(g) &= nD - \sum_{k=1}^{n} \sum_{\substack{e \in E_\Gamma, \\ e=(*,k)}} w_{I(e)} \\ &= nD - \sum_{c_1,\ldots,c_{n-1}} w_1 + w_2 \qquad (6) \\ &= nD - (n-1)(w_1 + w_2) \qquad \square \end{aligned}$$

Let I_f be the ideal generated by $\partial_x f$ and $\partial_y f$.

Remark 2. *Lemma 1 implies that $g \in I_f$.*

Lemma 3. *If f is quasi-homogeneous with degree $D \neq w_1+w_2$, then $[\mathbf{G}\cdot\pi] = \mathbf{0}$.*

Proof. Assume that $deg(g) = nD - (n-1)(w_1 + w_2) = 2D - w_1 - w_2$. Then, $D = w_1 + w_2$, a contradiction. Therefore, g does not contain any component of degree $2D - w_1 - w_2$ and by remark 2 and in [1, Lemma 2.5.11.], $\mathbf{G} \cdot \pi = g\, \partial_x \wedge \partial_y \in B^2(\pi)$. $\square$

Now, let $D = w_1 + w_2$. We know that

$$f = \sum_{i,j} a_{ij} x^i y^j$$

$D = w_1 + w_2 = iw_1 + jw_2$. Then, $(i-1)w_1 + (j-1)w_2 = 0$. Since $w_1, w_2 \in \mathbb{N}$, we have that either $i = 0$ and $j = 0$, $i = 0$ and $j = \frac{w_1}{w_2} + 1$, or $j = 0$ and $i = \frac{w_2}{w_1} + 1$ and therefore f is of the form $f = axy + bx^{\frac{w_2}{w_1}+1} + cy^{\frac{w_1}{w_2}+1}$. We can notice three cases:

1. $w_1 = w_2$ with $f = ax^2 + bxy + cy^2$
2. $w_2 = kw_1$ and $f = axy + bx^{k+1}$.
3. $w_1 = kw_2$ and $f = axy + cy^{k+1}$.

If $a \neq 0$, then cases 2 and 3 are isomorphic to 1 by a change of coordinates. Otherwise, the bidifferential operator would be 0 due to the possible labellings of the graphs and the fact that f is a single variable polynomial. It, therefore, suffices to treat 1.

Remark 3. *For $\sigma \in Z^0(\mathsf{GC}_2)$, $A \in GL_n$, and $\pi \in \bigwedge^2 T_{\mathbb{C}^n}$, we have that:*

$$\sigma(A \cdot \pi) = A(\sigma \cdot \pi) \tag{7}$$

Remark 4. *Let $\sigma_3, \sigma_5, \ldots \in Z^0(\mathsf{GC}_2)$ associated with the wheels with 3, 5, ... spokes, respectively. For $n \in \{3, 5, \ldots\}$, σ_n is a linear combination of graphs with an even number of vertices.*

Now, let $\pi = xy\, \partial_x \wedge \partial_y$, $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and $B = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$.

$$\begin{aligned} A(\sigma \cdot \pi) &\underset{Remark\ 3}{=} \sigma(A \cdot \pi) \\ &= \sigma(-\pi) \\ &\underset{Remark\ 4}{=} (-1)^m \sigma(\pi) \qquad \text{where } m \text{ is even} \\ &= \sigma(\pi) \end{aligned} \tag{8}$$

By applying (7) to B, we get that

$$B \cdot \sigma(\pi) = \sigma \cdot (B\pi) = \sigma\pi \tag{9}$$

Therefore, $\sigma\pi$ is B-invariant. We know that $\sigma\pi = (ax^2 + bxy + cy^2)\ \partial_x \wedge \partial_y$ since $w_1 = w_2 = 1$. Then, $B \cdot \sigma(\pi) = (at^2x^2 + bxy + ct^{-2}y^2)\ \partial_x \wedge \partial_y$. So $\sigma\pi = bxy\ \partial_x \wedge \partial_y$ since (9) is satisfied if and only if $a = c = 0$. But we know that:

$$A \cdot (bxy\partial_x \wedge \partial_y) = -bxy\partial_x \wedge \partial_y \tag{10}$$

Finally, we have:

$$-\sigma\pi \underset{(10)}{=} A\sigma\pi \underset{(8)}{=} \sigma\pi \tag{11}$$

Hence $\sigma\pi = 0$.

References

[1] Jean-Paul Dufour and Nguyen Tien Zung. *Poisson structures and their normal forms*, volume 242 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2005.

[2] Maxim Kontsevich. Formal (non)commutative symplectic geometry. In *The Gelfand Mathematical Seminars, 1990–1992*, pages 173–187. Birkhäuser Boston, Boston, MA, 1993.

[3] Maxim Kontsevich. Formality conjecture. In *Deformation theory and symplectic geometry (Ascona, 1996)*, volume 20 of *Math. Phys. Stud.*, pages 139–156. Kluwer Acad. Publ., Dordrecht, 1997.

[4] Maxim Kontsevich. Deformation quantization of Poisson manifolds. *Lett. Math. Phys.*, 66(3):157–216, 2003.

[5] Thomas Willwacher. M. Kontsevich's graph complex and the Grothendieck-Teichmüller Lie algebra. *Invent. Math.*, 200(3):671–760, 2015.