Structural Estimation of Dynamic Discrete Choice Models

The Nested Fixed Point Algorithm (NFXP)

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Dynamic Programming and Structural Econometrics #6

The Nested Fixed Point Algorithm (NFXP)

Rust (ECTA, 1987):

OPTIMAL REPLACEMENT OF GMC BUS ENGINES:
AN EMPIRICAL MODEL OF HAROLD ZURCHER

Harold Alois Zuercher June 16, 1926 - June 21, 2020 (age 94)

Overview of Rust (1987)

This is a path-breaking paper that introduces a methodology to estimate a single-agent dynamic discrete choice (DDC) models.

Main contributions

1. An illustrative application in a simple model of engine replacement.
2. Development and implementation of Nested Fixed Point Algorithm
3. Formulation of assumptions that makes DDC models tractable
4. The first researcher to obtain ML estimates of DDC models
5. Bottom-up approach: Micro-aggregated demand for durable assets

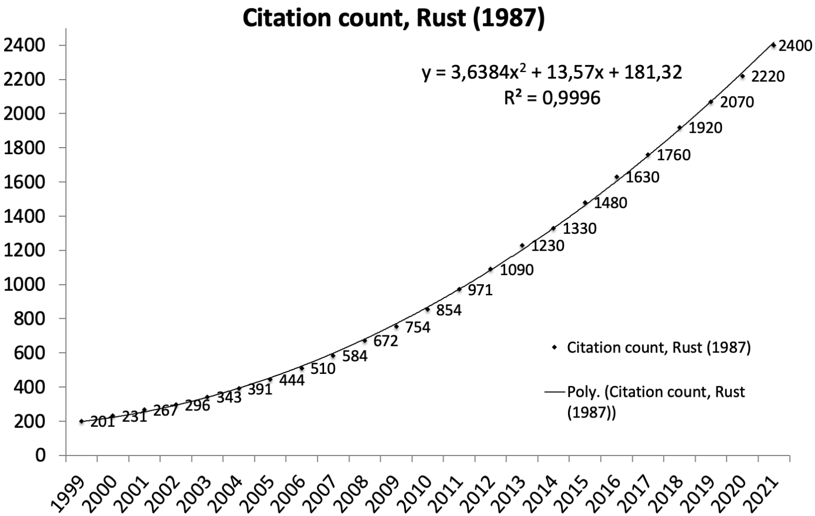
Policy experiments:

- How does changes in replacement cost affect the demand for engines and the equilibrium distribution of mileage?

Who cares about Harold Zurcher?

- Occupational Choice (Keane and Wolpin, JPE 1997)
- Retirement (Rust and Phelan, ECMA 1997)
- Brand choice and advertising (Erdem and Keane, MaScience 1996)
- Choice of college major (Arcidiacono, JoE 2004)
- Individual migration decisions (Kennan and Walker, ECMA 2011)
- High school attendance and work decisions (Eckstein and Wolpin, ECMA 1999)
- Sales and dynamics of consumer inventory behavior (Hendel and Nevo, ECMA 2006)
- Advertising, learning, and consumer choice in experience good markets (Ackerberg, IER 2003)
- Route choice models (Fosgerau et al, Transp. Res. B)
- Fertility and labor supply decisions (Francesconi, JoLE 2002)
- Residential and Work-location choice (Buchinsky et al, ECMA 2015)
- Equilibrium Allocations Under Alternative Waitlist Designs: Evidence From Deceased Donor Kidneys (Argarwal et al, ECMA 2021)
- Equilibrium Trade in Automobiles (Gillingham et al, JPE 2022)
- ...and many more

Big Mac Index of Dynamic Structural Econometrics



Methods for estimating Dynamic Discrete Choice Models

- Rust (1987): MLE using Nested-Fixed Point Algorithm (NFXP)
- Hotz and Miller (1993): CCP estimator - (two step estimator)
- Keane and Wolpin (1994): Simulation and interpolation
- Rust (1997): Randomization algorithm (breaks curse of dimensionality)
- Aguirregabiria and Mira (2002): Nested Pseudo Likelihood (NPL).
- Bajari, Benkard and Levin (2007): Two step-minimum distance (equilibrium inequalities).
- Arcidiacono Miller (2002): CCP with unobserved heterogeneity (EM Algorithm).
- Norets (2009): Bayesian Estimation (allows for serial correlation in ϵ)
- Su and Judd (2012): MLE using constrained optimization (MPEC)
- and MUCH more
- Any estimator method or solution algorithm of DDC models must confront NFXP and Harold Zurcher

Formulating, solving and estimating a dynamic model

Components of the dynamic model

Finite time

- Decision variables: vector describing the choices, $d_t \in C(s_t)$
- State variables: vector of variables, s_t, that describe all relevant information about the modeled decision process
- Instantaneous payoff: utility function, $u(s_t, d_t)$, with time separable discounted utility

  state dependent
- Motion rules: agent's beliefs of how state variable evolve through time, conditional on states and choices. Here formalized by a Markov transition density $p(s_{t+1} \mid s_t, d_t)$

Solution is given by:

- **Value function**: maximum attainable utility $V(s_t)$
- **Policy function**: mapping from state space to action space that returns the optimal choice, $d^\star(s_t)$

Structural Estimation

- Parametrize model: utility function $u(s_t, d_t; \theta_u)$, motion rules for states $p(s_{t+1} \mid s_t, d_t; \theta_p)$, choice sets $C(s_t; \theta_c)$, etc.
- Search for (policy invariant) parameters θ so that model fits targeted aspects of data on (a subset of) decisions, states, payoff's, etc.

Zurcher's Bus Engine Replacement Problem

- Choice set: Binary choice set, $C(x_t) = \{0, 1\}$.
  - Engine replacement ($d_t = 1$) or ordinary maintenance ($d_t = 0$)
- State variables: Harold Zurcher observes $s_t = (x_t, \varepsilon_t)$:
  - x_t: mileage at time t since last engine overhaul/replacement
  - $\varepsilon_t = [\varepsilon_t(d_t = 0), \varepsilon_t(d_t = 1)]$: decision specific state variable
- Utility function: $U(x_t, \varepsilon_t, d_t; \theta_1) =$

$$u(x_t, d_t, \theta_1) + \varepsilon_t(d_t) = \begin{cases} -RC - c(0, \theta_1) + \varepsilon_t(1) & \text{if } d_t = 1 \\ -c(x_t, \theta_1) + \varepsilon_t(0) & \text{if } d_t = 0 \end{cases} \quad (1)$$

theta(RC)

RC: replacement cost

- State variables process
  - ε_t is iid with conditional density $q(\varepsilon_t | x_t, \theta_2)$
  - x_t (mileage since last replacement)

$$p(x_{t+1} | x_t, d_t, \theta_2) = \begin{cases} g(x_{t+1} - 0, \theta_3) & \text{if } d_t = 1 \\ g(x_{t+1} - x_t, \theta_3) & \text{if } d_t = 0 \end{cases} \quad (2)$$

If engine is replaced, state of bus regenerates to $x_t = 0$.

- Parameters to be estimated $\theta = (RC, \theta_1, \theta_3)$
  (Fixed parameters: (β, θ_2))

General Behavioral Framework

The decision problem

- The decision maker chooses a sequence of actions to maximize expected discounted utility over a (in)finite horizon

$$V_\theta\left(s_t\right) = \sup_{\Pi} E\left[\sum_{j=0}^{T} \beta^j U\left(s_{t+j}, d_{t+j}; \theta_1\right) | s_t, d_t\right]$$

where

- $\Pi = \left(f_t, f_{t+1}, .., \right), d_t = f_t\left(s_t, \theta\right) \in C\left(x_t\right) = \{1, 2, .., J\}$
- $\beta \in (0, 1)$ is the discount factor
- $U\left(s_t, d_t; \theta_1\right)$ is a choice and state specific utility function
- We may consider an infinite horizon , i.e. $T = \infty$
- E summarizes expectations of future states given s_t and d_t

Recursive form of the maximization problem

- By Bellman Principle of Optimality, the value function $V(s)$ constitutes the solution of the following functional (Bellman) equation

$$V(x,\varepsilon) \equiv T(V)(x,\varepsilon) = \max_{d \in C(x)} \left\{ u(x,\varepsilon,d) + \beta E\left[V(x',\varepsilon') \middle| x,\varepsilon,d \right] \right\}$$

- Expectations are taken over the next period values of state $s' = (x',\varepsilon')$ given it's controlled motion rule, $p(s' \mid s,d)$

$$E\left[V(x',\varepsilon') \middle| x,\varepsilon,d \right] = \int_X \int_\Omega V(x',\varepsilon') p(x',\varepsilon'|x,\varepsilon,d) dx' d\varepsilon'$$

where $\varepsilon = (\varepsilon(1),\ldots,\varepsilon(J)) \in \mathbb{R}^J$

Hard to compute fixed point V such that $T(V) = V$

- x is continuous and ε is continuous and J-dimensional
- $V(x,\varepsilon)$ is high dimensional
- Evaluating E may require high dimensional integration
- Evaluating $V(x',\varepsilon')$ may require high dimensional interpolation/approximation
- $V(x,\varepsilon)$ is non-differentiable

Rust's Assumptions

1. Additive separability in preferences (**AS**):

$$U(s_t, d) = u(x_t, d; \theta_1) + \varepsilon_t(d)$$

2. Conditional independence (**CI**):
   State variables, $s_t = (x_t, \varepsilon_t)$ obeys a (conditional independent) controlled Markov process with probability density

   Episons are not serial correlated

$$p(x_{t+1}, \varepsilon_{t+1} | x_t, \varepsilon_t, d, \theta_2, \theta_3) = q(\varepsilon_{t+1} | x_{t+1}, \theta_2) p(x_{t+1} | x_t, d, \theta_3)$$

3. Extreme value Type I (EV1) distribution of ε (**EV**)
   Each of the choice specific state variables, $\varepsilon_t(d)$ are assumed to be iid. extreme value distributed with CDF Cumulative destribution function

$$F(\varepsilon_t(d); \mu, \lambda) = \exp(-\exp(-(\varepsilon_t(d) - \mu)/\lambda)) \text{ for } \varepsilon_t(d) \in \mathbb{R}$$

   with $\mu = 0$ and $\lambda = 1$

Rust's Assumptions simplifies DP problem

$$V(x,\varepsilon) = \max_{d \in C(x)} \left\{ u(x,d) + \varepsilon(d) + \beta \int_X \int_\Omega V(x',\varepsilon') p(x'|x,d) q(\varepsilon'|x') dx' d\varepsilon' \right\}$$

1. Separate out the deterministic part of choice specific value $v(x,d)$ (assumptions SA and CI)
2. Reformulate Bellman equation on reduced state space (assumption CI)
3. Compute the expectation of maximum using properties of EV1 (assumption EV)

1. DP problem under AS and CI

Separate out the deterministic part of choice specific value $v(x, d)$

$$V(x, \varepsilon) = \max_{d \in C(x)} \Big\{ u(x, d) + \beta \int_X \Big(\int_\Omega V(x', \varepsilon') q(\varepsilon' | x') d\varepsilon' \Big) p(x' | x, d) dx' + \varepsilon(d) \Big\}$$

Not dependt on episilon

So that

$$V(x', \varepsilon') = \max_{d \in C} \big\{ v(x', d) + \varepsilon'(d) \big\}$$

where Choise specific value

$$v(x, d) = u(x, d) + \beta E\big[V(x', \varepsilon') \big| x, d \big]$$

2a. Bellman equation in expected value function space

Let $EV(x,d) = E\left[V(x',\varepsilon')\middle|x,d\right]$ denote the expected value function.

Because of CI we can now express the Bellman equation in expected value function space

$$EV(x,d) = \Gamma(EV)(x,d) \equiv \int_X \int_\Omega \left[V(x',\varepsilon')q(\varepsilon'|x')d\varepsilon'\right] p(x'|x,d)dx'$$

where

Choise specific value

$$V(x',\varepsilon') = \max_{d' \in C(x')} \left[u(x',d') + \beta EV(x',d') + \varepsilon'(d')\right]$$

- Γ is a contraction mapping with unique fixed point EV, i.e. $\|\Gamma(EV) - \Gamma(W)\| \leq \beta \|EV - W\|$
- Global convergence of VFI
- $EV(x,d)$ is lower dimensional: does not depend on ε

x and d: descrete choices

2b. Bellman equation in integrated value function space

Let $\bar{V}(x) = E\left[V(x,\varepsilon)\middle|x\right]$ denote the integrated value function

Because of CI we can express Bellman equation in integrated value function space

$$\bar{V}(x) = \bar{\Gamma}(\bar{V})(x) \equiv \int_{\Omega} V(x,\varepsilon)q(\varepsilon|x)d\varepsilon$$

where

$$V(x,\varepsilon) = \max_{d \in C(x)} [u(x,d) + \varepsilon(d) + \beta \int_{X} \bar{V}(x')p(x'|x,d)dx']$$

- $\bar{\Gamma}$ is a contraction mapping with unique fixed point $\bar{V}$, i.e. $\left\|\bar{\Gamma}\left(\bar{V}\right) - \bar{\Gamma}\left(W\right)\right\| \leq \beta \left\|\bar{V} - W\right\|$
- Global convergence of VFI
- $\bar{V}(x)$ is lower dimensional: does not depend on ε and d

3. Compute the expectation of maximum under EV

Closed form expectations

We can express expectation of maximum using properties of EV1 distribution (assumption EV)

Expectation of maximum, $\bar{V}(x)$, can be expressed as "the log-sum"

$$\bar{V}(x) = E\left[\max_{d\in\{1,\dots,J\}}\{v(x,d)+\lambda\varepsilon(d)\} \mid x\right] = \lambda \log \sum_{j=1}^{J} \exp(v(x,d)/\lambda)$$

Conditional choice probability, $P(x,d)$ has closed form logit expression

$$P(d \mid x) = E\left[\mathbb{1}\left\{d = \arg\max_{j\in\{1,\dots,J\}}\{v(x,j)+\lambda\varepsilon(j)\}\right\} \mid x\right]$$
$$= \frac{\exp(v(x,d)/\lambda)}{\sum_{j=1}^{J}\exp(v(x,j)/\lambda)}$$

HUGE benefits

- Avoids J dimensional numerical integration over ε
- $P(d \mid x)$, $\bar{V}(x)$ and $EV(x,d)$ are smooth functions.

Givet de er eksponentielle, så ingen hak og sådan —> nemt at differentiere

The DP problem under AS, CI and EV

Putting all this together

- Conditional Choice Probabilities (CCPs) are given by

$$P\left(d|x,\theta\right)=\frac{\exp\left\{u\left(x,d,\theta_1\right)+\beta EV_\theta\left(x,d\right)\right\}}{\sum_{j\in C(x)}\exp\left\{u\left(x,j,\theta_1\right)+\beta EV_\theta\left(x,j\right)\right\}}$$

- The expected value function can be found as the unique fixed point to the contraction mapping Γ_θ, defined by

$$\begin{aligned}EV_\theta\left(x,d\right) &= \Gamma_\theta(EV_\theta)\left(x,d\right)\\ &= \int_y \ln\left[\sum_{d'\in C(y)}\exp\left[u(y,d';\theta_1)+\beta EV_\theta\left(y,d'\right)\right]\right]\\ &\quad p\left(dy|x,d,\theta_2\right)\end{aligned}$$

- We have used the subscript θ to emphasize that the Bellman operator,Γ_θ depends on the parameters.
- In turn, the fixed point, EV_θ, and the resulting CCPs, $P\left(d|x,\theta\right)$ are implicit functions of the parameters we wish to estimate.

How to deal with continuous mileage state?

Rust discretize the mileage state space x into n grid points

$$X = \{x_1, ..., x_n\} \text{ with } x_1 = 0$$

Mileage transition probability: for $l = 0, \ldots, L$

$$p(x'|\hat{x}_k, d, \theta_2) = \begin{cases} Pr\{x' = x_{k+l}|\theta_2\} = \pi_l \text{ if } d = 0 \\ Pr\{x' = x_{1+l}|\theta_2\} = \pi_l \text{ if } d = 1 \end{cases}$$

- where $\theta_2 = [\pi_1, \ldots, \pi_L]$, $\pi_0 = 1 - \sum_{l=1}^{L} \pi_l$, and $\pi_l \geq 0$
- Mileage in the next period x' can move up at most L grid points.
- L is determined by the empirical distribution of mileage.

Transition matrix for mileage is sparse

Transition matrix conditional on keeping engine

$$\Pi(d = \text{keep})_{nxn} = \begin{pmatrix} \pi_0 & \pi_1 & \pi_2 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \pi_0 & \pi_1 & \pi_2 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & \pi_0 & \pi_1 & \pi_2 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ 0 & & & & \pi_0 & \pi_1 & \pi_2 & 0 \\ 0 & & & & & \pi_0 & \pi_1 & \pi_2 \\ 0 & & & & & & \pi_0 & 1 - \pi_0 \\ 0 & 0 & & & & & & 1 \end{pmatrix}$$

Givet at den er øvre triangulær: bussen kører ikke baglæns

Transition matrix for mileage is sparse

Transition matrix conditional on replacing engine

$$\Pi(d = \text{replace})_{nxn} = \begin{pmatrix} \pi_0 & \pi_1 & \pi_2 & 0 & \cdot & \cdot & \cdot & 0 \\ \pi_0 & \pi_1 & \pi_2 & 0 & \cdot & \cdot & \cdot & 0 \\ \pi_0 & \pi_1 & \pi_2 & 0 & \cdot & \cdot & \cdot & 0 \\ \pi_0 & \pi_1 & \pi_2 & 0 & \cdot & \cdot & \cdot & 0 \\ \pi_0 & \pi_1 & \pi_2 & 0 & \cdot & \cdot & \cdot & 0 \\ \pi_0 & \pi_1 & \pi_2 & 0 & \cdot & \cdot & \cdot & 0 \\ \pi_0 & \pi_1 & \pi_2 & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix}$$

Bellman equation in matrix form

Bellman equation in expected value function space

$$EV(d) = \Gamma(EV) = \Pi(d) \ln \left[\sum_{d'} \exp[u(d') + \beta EV(d')] \right]$$

Bellman equation in integrated value function space

$$\bar{V} = \bar{\Gamma}(\bar{V}) = \ln \left[\sum_{d'} \exp[u(d') + \beta \Pi(d') \bar{V}] \right]$$

where

- $u(d) = [u(x_1, d), .., u(x_n, d)]$
- $EV(d) = [EV(x_1, d), .., EV(x_n, d)]$
- $\bar{V} = [\bar{V}(x_1), .., \bar{V}(x_n)]$
- $\Pi(d)$ is a $n \times n$ state transition matrix conditional on decision d

Structural Estimation

Se zurcher.m (matlab kode).
Se README for detaler af hver function

Data: $(d_{i,t}, x_{i,t})$, $t = 1, ..., T_i$ and $i = 1, ..., N$

Log likelihood function

$$L(\theta, EV_\theta)) = \sum_{i=1}^{N} \ell_i^f(\theta, EV_\theta)$$

$$\ell_i^f(\theta, EV_\theta) = \sum_{t=2}^{T_i} log(P(d_{i,t}|x_{i,t}, \theta)) + \sum_{t=2}^{T_i} log\left(p(x_{i,t}|x_{i,t-1}, d_{i,t-1}, \theta_3)\right)$$

where

$$P(d|x, \theta) = \frac{\exp\{u(x, d, \theta_1) + \beta EV_\theta(x, d)\}}{\sum_{d' \in \{0,1\}} \{u(x, d', \theta_1) + \beta EV_\theta(x, d')\}}$$

and

$$\begin{aligned} EV_\theta(x, d) &= \Gamma_\theta(EV_\theta)(x, d) \\ &= \int_y \ln\left[\sum_{d' \in \{0,1\}} \exp[u(y, d'; \theta_1) + \beta EV_\theta(y, d')]\right] p(dy|x, d, \theta_3) \end{aligned}$$

The Nested Fixed Point Algorithm

Since the contraction mapping Γ_θ always has a unique fixed point, the constraint $EV_\theta = \Gamma(EV_\theta)$ implies that the fixed point EV_θ is an implicit function of θ.

Hence, NFXP solves the unconstrained optimization problem

$$\max_{\theta} L(\theta, EV_\theta)$$

Outer loop (Hill-climbing algorithm):

- Likelihood function $L(\theta, EV_\theta)$ is maximized w.r.t. θ
- Quasi-Newton algorithm: Usually BHHH, BFGS or a combination.
- Each evaluation of $L(\theta, EV_\theta)$ requires solution of EV_θ

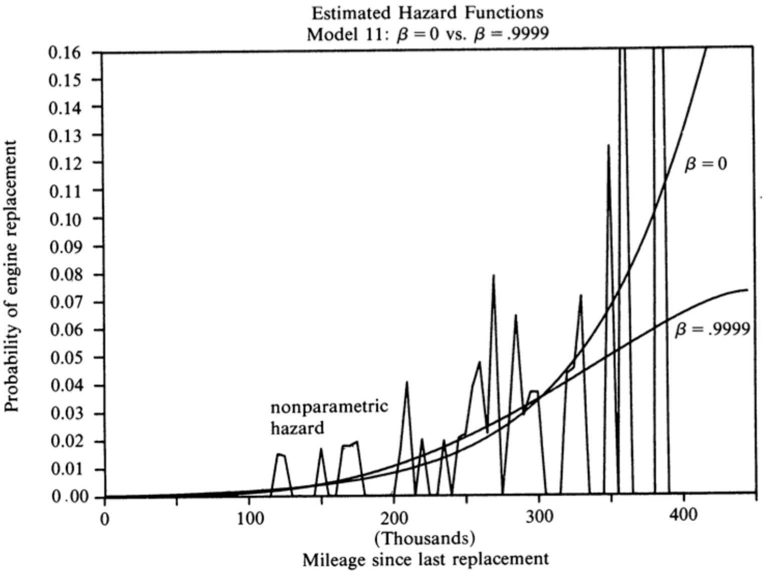
Inner loop (fixed point algorithm):
The implicit function EV_θ defined by $EV_\theta = \Gamma(EV_\theta)$ is solved by:

- Successive Approximations (SA) aka VFI
- Newton-Kantorovich (NK) Iterations

Data

- Harold Zurcher's Maintenance records of 162 busses
- Monthly observations of mileage on each bus (odometer reading)
- Data on maintenance replacement decisions

Estimated Hazard Functions



Structural Estimates, n=90

TABLE IX

STRUCTURAL ESTIMATES FOR COST FUNCTION $c(x, \theta_1) = .001\theta_{11}x$

FIXED POINT DIMENSION = 90

(Standard errors in parentheses)

Parameter		Data Sample			Heterogeneity Test	
Discount Factor	Estimates/ Log-Likelihood	Bus groups Groups 1, 2, 3 3864 Observations	Group 4 4292 Observations	Groups 1, 2, 3, 4 8156 Observations	LR Statistic ($df = 4$)	Marginal Significance Level
$\beta = .9999$	RC	11.7270 (2.602)	10.0750 (1.582)	9.7558 (1.227)	85.46	1.2E−17
	θ_{11}	4.8259 (1.792)	2.2930 (0.639)	2.6275 (0.618)		
	θ_{30}	.3010 (.0074)	.3919 (.0075)	.3489 (.0052)		
	θ_{31}	.6884 (.0075)	.5953 (.0075)	.6394 (.0053)		
	LL	−2708.366	−3304.155	−6055.250		
$\beta = 0$	RC	8.2985 (1.0417)	7.6358 (0.7197)	7.3055 (0.5067)	89.73	1.5E−18
	θ_{11}	109.9031 (26.163)	71.5133 (13.778)	70.2769 (10.750)		
	θ_{30}	.3010 (.0074)	.3919 (.0075)	.3488 (.0052)		
	θ_{31}	.6884 (.0075)	.5953 (.0075)	.6394 (.0053)		
	LL	−2710.746	−3306.028	−6061.641		
Myopia test:	LR Statistic ($df = 1$)	4.760	3.746	12.782		
$\beta = 0$ vs. $\beta = .9999$	Marginal Significance Level	0.0292	0.0529	0.0035		

Structural Estimates, n=175

TABLE X

STRUCTURAL ESTIMATES FOR COST FUNCTION $c(x, \theta_1) = .001\theta_{11}x$

FIXED POINT DIMENSION = 175

(Standard errors in parentheses)

Parameter		Data Sample			Heterogeneity Test	
Discount Factor	Estimates Log-Likelihood	Groups 1, 2, 3 3864 Observations	Group 4 4292 Observations	Groups 1, 2, 3, 4 8156 Observations	LR Statistic (df = 6)	Marginal Significance Level
$\beta = .9999$	RC	11.7257 (2.597)	10.896 (1.581)	9.7687 (1.226)	237.53	1.89E − 48
	θ_{11}	2.4569 (.9122)	1.1732 (0.327)	1.3428 (0.315)		
	θ_{30}	.0937 (.0047)	.1191 (.0050)	.1071 (.0034)		
	θ_{31}	.4475 (.0080)	.5762 (.0075)	.5152 (.0055)		
	θ_{32}	.4459 (.0080)	.2868 (.0069)	.3621 (.0053)		
	θ_{33}	.0127 (.0018)	.0158 (.0019)	.0143 (.0013)		
	LL	−3993.991	−4495.135	−8607.889		
$\beta = 0$	RC	8.2969 (1.0477)	7.6423 (.7204)	7.3113 (0.5073)	241.78	2.34E − 49
	θ_{11}	56.1656 (13.4205)	36.6692 (7.0675)	36.0175 (5.5145)		
	θ_{30}	.0937 (.0047)	.1191 (.0050)	.1070 (.0034)		
	θ_{31}	.4475 (.0080)	.5762 (.0075)	.5152 (.0055)		
	θ_{32}	.4459 (.0080)	.2868 (.0069)	.3622 (.0053)		
	θ_{33}	.0127 (.0018)	.0158 (.0019)	.0143 (.0143)		
	LL	−3996.353	−4496.997	−8614.238		
Myopia tests:	LR Statistic ($df = 1$)	4.724	3.724	12.698		
$\beta = 0$ vs. $\beta = .9999$	Marginal Significance Level	0.0297	0.0536	.00037		

MATLAB implementation, n=175, (replication of Table X)

Output from `run_busdata.m`:

```
fzp386 — cefhelper (Renderer) ◂ MATLAB_maci64 -nodesktop — 89×35

>> run_busdata
Structural Estimation using busdata from Rust(1987)
Bustypes        = [ 1  2  3  4 ]
Beta            =    0.99990
n               =  175.00000
Sample size     = 8156.00000


Method nfxp (pmle)
    Param.                         Estimates          s.e.        t-stat
-----------------------------------------------------------------------------------------
    RC                                9.7910        1.2684        7.7190
    c                                 1.3486        0.3458        3.8996
-----------------------------------------------------------------------------------------
log-likelihood     = -300.57017
runtime (seconds) =     0.07795
g'*inv(h)*g       = 2.65552e-09


Method nfxp (mle)
    Param.                         Estimates          s.e.        t-stat
-----------------------------------------------------------------------------------------
    RC                                9.7915        1.2689        7.7168
    c                                 1.3488        0.3460        3.8982
    p                  (1)            0.1070        0.0034       31.2111
    p                  (2)            0.5152        0.0055       93.0533
    p                  (3)            0.3622        0.0053       68.0413
    p                  (4)            0.0143        0.0013       10.8947
    p                  (5)            0.0009        0.0003        2.6469
-----------------------------------------------------------------------------------------
log-likelihood     = -8607.88844
runtime (seconds) =     0.07484
g'*inv(h)*g       = 7.26854e-09
>> 
```

Equilibrium bus mileage and demand for enigines

- Let π be the long run stationary (or equilibrium) distribution of the controlled process $\{i_t, x_t\}$
- π is then given by the unique solution to the functional equation

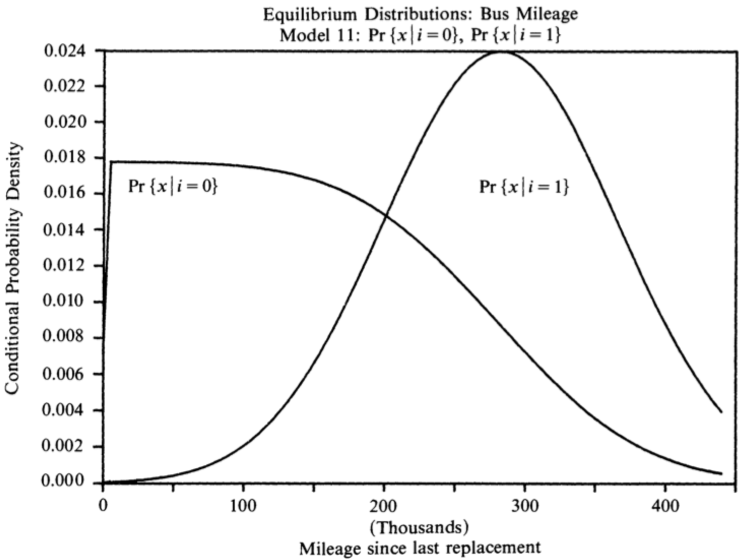
$$\pi(x, i) = \int_y \int_j P(i|x, \theta)p(x|y, j, \theta_3)\pi(dy, dj)$$

- π is the ergodic distribution of the controlled state transition matrix
- Carly the equilibrium distribution of π is an implicit function of the structural parameters θ, which we emphasize by the notation π_θ
- Given π_θ, we can also obtain the following simple formula for annual equilibrium demand for engines as a function of RC

$$d(RC) = 12M \int_0^\infty \pi_\theta(dx, 1)$$

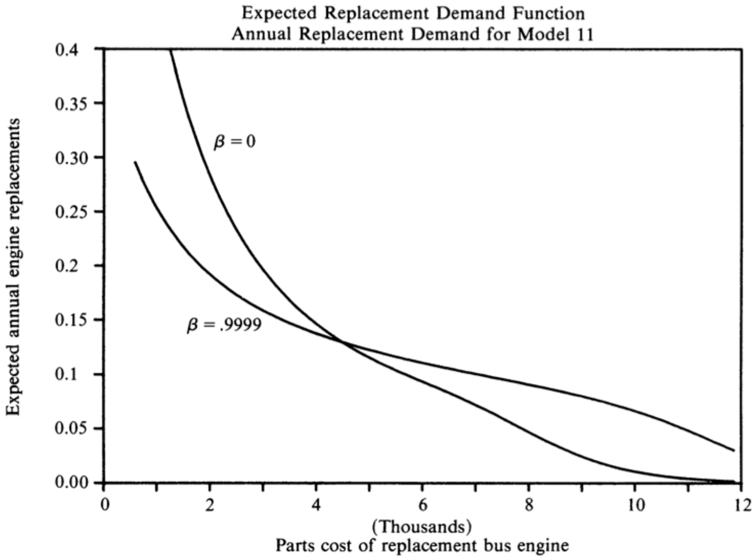
Equilibrium Bus mileage, bus group 4

Se: run_dem.m



Demand Function, bus group 4

Se: run_dem.m



Why not a reduced form for demand?

<u>Reduced form</u>

- Regress engine replacements on replacement costs

Problem: Lack of variation in replacement costs

- Data would be clustered around the intersection of the demand curves for $\beta = 0$ and $\beta = 0.9999$
  (both models predict that RC is around the actual RC of $4343)
- Demand also depends on how operating costs varies with mileage
- Need exogenous variation in RC
  .... that doesn't vary with operating costs
- Even if we had exogenous variation, this does not help us to understand the underlying economic incentives

Structural Approach

Attractive features

- Structural parameters have a transparent interpretation
- Evaluation of (new) policy proposals by counterfactual simulations.
- Economic theories can be tested directly against each other.
- Economic assumptions are more transparent and explicit (compared to statistical assumptions)

Less attractive features

- We impose more structure and make more assumptions
- Truly "structural" (policy invariant) parameters may not exist
- The curse of dimensionality
- The identification problem
- The problem of multiplicity and indeterminacy of equilibria
- Intellectually demanding and a huge amount of work