Stationary Equilibrium:

Equilibrium Trade in Automobile Markets

Dynamic Programming and Structural Econometrics #11

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Equilibrium models

Readings:

- Gillingham, Iskhakov, Munk-Nielsen, Rust and Schjerning: "Equilibrium Trade in Automobile Markets", Journal of Political Economy (forthcoming)

Road Map:

1. Introduction and literature overview
2. Stationary equilibrium with transaction cost and consumer heterogeneity
3. Modeling Danish secondary market of automobiles

A brief history of auto models

1. Manski (1982) and Berkovec (1985): were the first to develop models of equilibrium in the used car market by microaggregation, but used static discrete choice models
2. Rust (1985): provided a dynamic framework for equilibrium prices and quantities, and showed that when *transactions costs are zero*, consumers trade every period for an optimal car, so the dynamic problem reduces to a static one
3. Konishi and Sandfort (2002): generalized Rust's analysis to allow for positive transaction costs and proved the existence of equilibrium, allowing also for multiple makes/models of cars
4. Gavazza *et. al.* (2014): numerically calculated equilibrium with one car type and discrete ages/qualities of cars and analyzed the impact of the secondary market with varying levels of transaction costs

James A. Berkovec 1957-2009

James A. Berkovec: "New Car Sales and Used Car Stocks: A Model of the Automobile Market", RAND (1985)

1. Estimated a discrete choice model of demand for cars using the *National Transportation Survey* – micro data on 1095 households in 1978
2. Using estimated demands for cars, he solved for equilibrium in the new and used car market, over 131 type/age classes, with 13 types and 10 ages (vintages) from 1969 to 1978 plus a residual (131st) category of all cars produced prior to 1969 (clunkers)

Berkovec's approach to calculating equilibrium

1. Let P be the 131×1 vector of prices of the 131 car type/age classes.
2. Let $E(P)$ be the 131×1 vector of excess demands for these 131 vehicles implied by the price vector P
3. An equilibrium in the auto market is a vector P^* satisfying $E(P^*) = 0$.
4. Berkovec defined $E(P) = D(P) - S(P) - Q - S(P)$ where Q is the vector of stocks of cars, and $S(P)$ is the number cars scrapped
5. Berkovec used a quasi-Newton method to find a P^* that satisfies $E(P^*) = 0$
$$P' = P - \lambda[\nabla E(P)]^{-1} E(P) \tag{1}$$
where $\lambda \geq 0$ is a scalar *step size.*

A Dynamic Model of the Auto Market: Rust (1985)

John Rust: "Stationary Equilibrium in a Market for Durable Assets'", Econometrica (1985)

- Introduced a *durable asset pricing model* that includes particular durable goods such as cars that trade both on a *primary market* (new cars) and a *secondary market* (used cars)

The essential benefit of a secondary market is to create dynamic trading opportunities similar to a securities market: the consumer can hold the current durable for any desired length of time and has the opportunity of trading it for a new asset or choosing from an array of used assets of different ages and physical conditions." (Rust, 1985, p. 783)

Equilibrium with a continuum of goods/consumers

1. Infinitely elastic supply of new cars at price $\overline{P}$
2. Infinite demand for cars by *scrappers* at price $\underline{P}$.
3. State of car is summarized by its odometer x (no *lemon's problems*).
4. Let $\phi(y|x)$ be the probability density of next period odometer given this period's value x.
   Example: $\phi(y|x) = \lambda \exp\{-\lambda(y-x)\}$ if $y \geq x$, 0 otherwise.
5. Consumers are *heterogeneous, infinitely lived,* must always own 1 car, and have *quasi-linear utility functions*

$$U(x, I, \tau) = a(\tau)I + q(x, \tau)$$

   where τ is the *type* of consumer, I is income, and x is the age/odometer of the car.
6. Let $m(x, \tau)$ be the maintenance costs incurred by consumer τ.

Equivalent cost-minimization formulation

1. Let $P(x)$ be an *equilibrium price function.* It should satisfy $P(0) = \overline{P}$ and $P(x) = \underline{P}$ where $x \geq \gamma$ where γ is the *scrapping threshold*.
2. Let $T(x)$ be the *transactions cost* that a customer incurs when they sell a car of odometer x.
3. Consumer's dynamic utility maximization problem
   $\rightarrow$ cost minimization problem using the "cost function" $G_\tau(x, d)$

$$G_\tau(x, d) = \begin{cases} M(x, \tau) & \text{if} \quad d = x \\ M(z, \tau) + P(z) - P(x) + T(x) & \text{if} \quad d = z \end{cases}$$

   where $M(x, \tau) = m(x, \tau) - q(x, \tau)/a(\tau)$ is the disutility of owning a car of odometer x expressed as a "generalized maintenance cost"

Bellman equation for the consumer

1. Let $V_\tau(x)$ be consumer τ's optimal discounted cost of owning a car of odometer x. The Bellman equation is given by

$$\begin{aligned} V_\tau(x) \quad = \quad \min \Bigg[& M(x,\tau) + \beta E V_\tau(x), \\ & \inf_{0 \le z \le \gamma} M(z,\tau) + P(z) - P(x) + T(x) + \beta E V_\tau(z) \Bigg] \end{aligned}$$

where

$$E V_\tau(x) = \int_x^\infty V_\tau(y) \phi(y|x) dy.$$

2. **Theorem** *Assume that transactions cost are zero, $T(x) = 0 \; \forall x \ge 0$. Then it is optimal for each consumer τ to trade each period for their optimal car $z^*(\tau)$ given by*

$$z^*(\tau) = \underset{0 \le z \le \gamma}{argmin} \left[M(z,\tau) + P(z) + \beta E V_\tau(z) \right].$$

Implication of Zero Transactions Cost

1. **Lemma** If transactions costs are zero, $T(x) = 0$, then we have

   $$V_\tau(x) = \left[M(x,\tau) + P(z^*(\tau)) - \beta EP(z^*(\tau))\right]/(1-\beta) - P(x)$$

   where $z^*(\tau)$ is given by

   $$z^*(\tau) = \underset{0 \le z \le \gamma}{argmin}\left[M(z,\tau) + P(z) - \beta EP(z)\right]. \tag{2}$$

2. What if no secondary market existed? Consumer's problem is now a *regenerative optimal stopping problem*

   $$V_\tau(x) = \min\left[M(x,\tau) + \beta EV_\tau(x), M(0,\tau) + \overline{P} - \underline{P} + \beta EV_\tau(0)\right].$$

   Optimal strategy is to keep the car x unless $x > \gamma(\tau)$ where $\gamma(\tau)$ is the solution to

   $$\overline{P} - \underline{P} + V_\tau(0) = V_\tau(\gamma(\tau)).$$

Definition of Stationary Equilibrium

1. **Definition** The triple $\{P, F, \gamma\}$ is a *stationary equilibrium* if
2. $P(0) = \overline{P}$ and $P(x) = \underline{P}$ if $x \geq \gamma$
3. $F(x|\gamma)$ is a *stationary holdings distribution* the unique solution to

$$F(x|\gamma) = \int_0^\infty \left[1 - \Phi(\gamma|x') + \Phi(x|x')\right] F(dx'|\gamma)$$

4. Each consumer τ chooses an optimal car $z^*(\tau)$ given by equation (2) and $\forall x \in [0, \gamma]$ we have

$$F(x|\gamma) = \int_{\underline{\tau}}^{\overline{\tau}} I\left\{\tau | z^*(\tau) \leq x\right\} H(d\tau) \tag{3}$$

Stationary Equilibrium: Homogeneous Consumers

1. If all consumers have same τ, they must be indifferent about which car to buy

$$M(x,\tau) + P(x) - \beta EP(x) = \text{constant}$$

2. This implies that P is the solution to the following *functional equation*

$$P(x) = \max\left[\underline{P}, \overline{P} - \beta EP(0) - M(0,\tau) - M(x,\tau) + \beta EP(x)\right]$$

3. **Theorem** $P(x) = \overline{P} - [V_\tau(x) - V_\tau(0)]$.

Stationary Equilibrium: Homogeneous Consumers

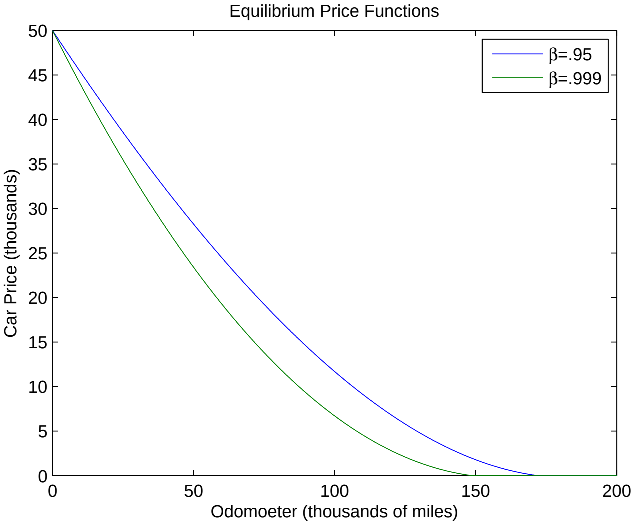
1. **Theorem** If $\Phi(y|x) = \max[0, 1 - \exp\{-\lambda(y - x)\}]$, then

$$\begin{aligned} P(x) &= \max\Big[\underline{P}, \underline{P} + \\ &\qquad \frac{1}{1-\beta}\int_x^\gamma M'_\tau(y)\,[1 - \beta\exp\{-\lambda(1-\beta)(y-x)\}]\,dy\Big] \\ F(x|\gamma) &= \frac{1+\lambda x}{1+\lambda\gamma} \quad x \in [0,\gamma] \end{aligned}$$

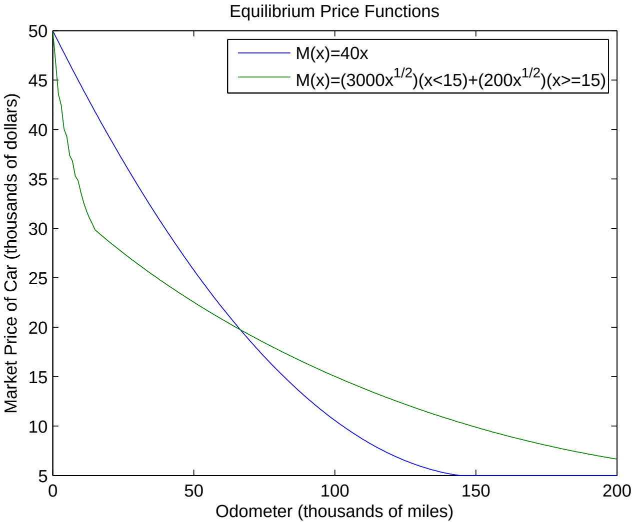
where γ is the unique solution to

$$\overline{P} - \underline{P} = \frac{1}{1-\beta}\int_0^\gamma M'(y,\tau)\,[1 - \beta\exp\{-\lambda(1-\beta)(y-x)\}]\,dy$$

Equilibrium Price Functions, effect of discount factor β



Equilibrium Price Functions, effect of maintenance costs M



Part II: Stationary equilibrium with transaction costs and consumer heterogeneity: theory

Introduction

- **Policy question:** *what are the effects of switching car taxes from purchase to usage (fuel)?*
- **Methodologically:** Complex primary/secondary market interactions
  - Plus: prices for used cars are often not observed.
- **Theory contribution:** characterize and prove existence of equilibrium
- **Applied contribution:** tractable model with
  - Transactions, scrappage, consumer/car heterogeneity,
  - Flexible utility: estimating 131 parameters with 39m obs. in under 15 min
- **Conclusion (preview):** High Danish taxes above the Laffer curve's top point...
  - ... but a "naive" model overestimates the strength of this effect,
  - possibly leading to detrimental policies for tax revenues and the environment

Modeling equilibrium trade in automobile markets

We extend Rust (1985) stationary flow equilibrium model

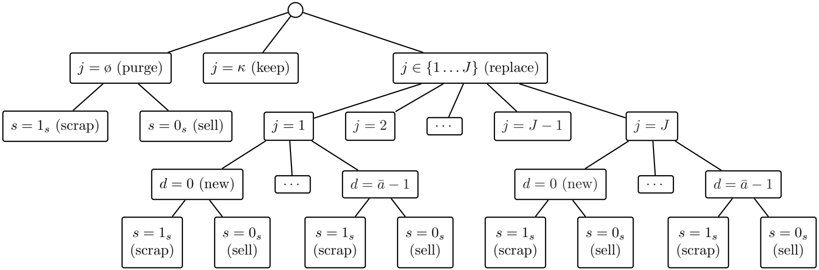
- Dynamic discrete choice equilibrium model
- Unit mass of (homogeneous) consumers
- Stationary flow equilibrium
- No transactions costs
- Continuous quality of durable good (odometer reading)

1. Add outside option
2. Add transactions cost
3. Add endogenous scrappage decisions at any car age
4. Allow for several types of consumer heterogeneity
5. Allow for multiple makes/models of cars of different vintages
6. Flexible utility specification with the possibility for driving and demand for gasoline in the applications

Baseline setup: idiosyncratic consumer heterogeneity

- Unit mass of consumers who live in the infinite horizon
- $i \in \{1, \dots, J\}$ make/models of cars, age $a \in \{0, 1, \dots, \bar{a}\}$
- Cars must be scrapped at age $\bar{a}$
- Car owners can:
    1. trade their current car (i, a) for a car (j, d)
    2. keep their current car, $(d = \kappa)$ (if $a < \bar{a}$)
    3. purge their current car (i, a) and remain without a car $i = \emptyset$
- When existing car (i, a) is replaced or purged, there is additional *endogenous scrappage* choice:
    1. sell $s = 0_s$ the existing car (i, a) for the market price $P(i, a)$
    2. scrap $s = 1_s$ the existing car (i, a) at fixed buy-in price $\underline{P}$

Consumer choice tree - car owner



Notes: The figure presents an example of the choice tree for a consumer who owns a car under the nested logit specification of the GEV distribution of the idiosyncratic heterogeneity term ϵ. Note three choice nests at the top level (to purge, keep or trade the existing car), two intermediate levels of nesting in the case of trading, and a additional scrappage choice of the existing car in the cases when it is traded.

Timing of events

1. Consumer enter the period with a car of make/model i and age a, or without a car $i = \o$
2. Discrete trading/keeping $j \in \{\kappa, \o, 1, \dots, J\}$, $d \in \{0, 1, \dots, \bar{a} - 1\}$, and scrappage $s \in \{0_s, 1_s\}$ choice immediately at the start of the period
3. The chosen car (j, d) is then utilized during the period, but can be involved in the accident with probability α
4. By the start of the next period:
   - with probability $1 - \alpha$ car has aged $a = d + 1$
   - with probability α car reach scrappage state $a = \bar{a}$

- Hence, it is impossible to start the period with a brand new car $\rightarrow$ the state variable a takes values from $a \in \{1, \dots, \bar{a}\}$

Utility of car ownership and consumer heterogeneity

$$\text{Utility} = u(i, a) - \mu\big[\text{operating costs} + \text{trade and transaction costs}\big] + \epsilon$$

- Car utility $u(i, a)$ is a decreasing function of car age a that reflects
  - decreasing utility of car services
  - increasing cost of maintenance
- Marginal utility of money μ

Unit mass of idiosyncratically heterogeneous consumers

- **Extreme value** consumer types (taste shifters)
- GEV specification for $\epsilon \to$ nested choices to allow correlation between alternatives
- Logit choice probabilities and analytic expectations

Consumer's trading problem: no car or terminal age car

$$V(\emptyset, \epsilon) = \max \left[v(\emptyset, \emptyset) + \epsilon(\emptyset); \max_{\substack{j \in \{1,\dots,J\}, \\ d \in \{0,1,\dots,\bar{a}-1\}}} \left[v(\emptyset, j, d) + \epsilon(j, d) \right] \right]$$

$$V(i, \bar{a}, \epsilon) = \max \left[v(i, \bar{a}, \emptyset) + \epsilon(\emptyset); \max_{\substack{j \in \{1,\dots,J\}, \\ d \in \{0,1,\dots,\bar{a}-1\}}} \left[v(i, \bar{a}, j, d) + \epsilon(j, d) \right] \right]$$

where $v(\emptyset, j, d)$ and $v(i, \bar{a}, j, d)$ are values of trading to car of make/model j and age $d \in \{1, \dots, \bar{a} - 1\}$

Consumer's trading problem: car owners

$$V(i,a,\epsilon) = \max \left\{ \begin{array}{l} v(i,a,\kappa) + \epsilon(\kappa); \\ \\ \max_{s\in\{1_s,0_s\}} \left[v(i,a,\emptyset,s) + \epsilon(\emptyset,s) \right]; \\ \\ \max_{\substack{j\in\{1,\dots,J\}, \\ d\in\{0,1,\dots,\bar{a}-1\}, \\ s\in\{1_s,0_s\}}} \left[v(i,a,j,d,s) + \epsilon(j,d,s) \right] \end{array} \right\}$$

When existing car (i,a) is replaced, there is additional scrappage choice $s \in \{0_s, 1_s\}$: to sell or to scrap the replaced car.

Choice specific value functions for every state and choice

$$
\begin{aligned}
v(\emptyset, \emptyset) &= u(\emptyset) + \beta EV(\emptyset) \\
v(\emptyset, j, d) &= u(j, d) - \mu\big[P(j, d) + T_b(P, j, d)\big] + \\
&\quad + \beta\big(1-\alpha\big) EV(j, d+1) + \beta\alpha EV(j, \bar{a}) \\
v(i, \bar{a}, \emptyset) &= u(\emptyset) + \mu\, \underline{P}(i) + \beta EV(\emptyset) \\
v(i, \bar{a}, j, d) &= u(j, d) - \mu\big[P(j, d) - \underline{P}(i) + T_b(P, j, d)\big] + \\
&\quad + \beta\big(1-\alpha\big) EV(j, d+1) + \beta\alpha EV(j, \bar{a}) \\
v(i, a, \emptyset, 1_s) &= u(\emptyset) + \mu\, \underline{P}(i) + \beta EV(\emptyset) \\
v(i, a, \emptyset, 0_s) &= u(\emptyset) + \mu[P(i, a) - T_s(P, i, a)] + \beta EV(\emptyset) \\
v(i, a, \kappa) &= u(i, a) + \beta\big(1-\alpha\big) EV(i, a+1) + \beta\alpha EV(i, \bar{a}) \\
v(i, a, j, d, 1_s) &= u(j, d) - \mu[P(j, d) - \underline{P}(i) + T_b(P, j, d)] + \\
&\quad + \beta\big(1-\alpha\big) EV(j, d+1) + \beta\alpha EV(j, \bar{a}) \\
v(i, a, j, d, 0_s) &= u(j, d) - \mu[P(j, d) - P(i, a) + T_s(P, i, a) + T_b(P, j, d)] + \\
&\quad + \beta\big(1-\alpha\big) EV(j, d+1) + \beta\alpha EV(j, \bar{a})
\end{aligned}
$$

States Choices $\rightarrow$ Current period utility Future value

Expected value function

- Expected value function

$$EV(i,a) = \int_{\epsilon} V(i,a,\epsilon) f(\epsilon|i,a) d\epsilon$$

- Has analytic form under GEV: composition of logsum functions
- In the simplest case (GEV $\rightarrow$ EV1)

$$EV(i,a) = \sigma \log \left(\sum_{j,d,s} \exp \frac{v(i,a,j,d,s)}{\sigma} \right)$$

Bellman operator

After plugging in expressions for choice specific value function $v(i, a, j, d, s)$

$$EV(P) = \Gamma\big(EV(P), P\big)$$

- Rust engine replacement model on steroids!
- The EV vector is a fixed point to a contraction mapping
- Can be calculated by globally convergent method of successive approximations (VFI)
- Much better to use **Newton's method** in functional space $\rightarrow$ Newton-Kantorovich iterations

Implied choice probabilities

1. Solution to DP problem $(EV(i,1),\ldots,EV(i,\bar{a})) \rightarrow$
2. Choice specific values $v(i,a,j,d,s)$ for all choices $j,d,s \rightarrow$
3. Analytic nested logit choice probabilities

- Simplest case:

$$\Pi(j,d,s|i,a) = \frac{\exp\left\{v(i,a,j,d,s)/\sigma\right\}}{\sum_{j',d',s'}\exp\left\{v(i,a,j',d',s')/\sigma\right\}}$$

- Choice probability for sell/scrap decision is separable under the additively separable transaction costs
- Implicitly depend on the market prices P

Ownership distribution

- Let $0 \leq q_{i,a} \leq 1$ denote the fraction of owners of make/model i cars aged a, $i \in \{1, \ldots, J\}$, $a \in \{1, \ldots, \bar{a}\}$
- Let $0 \leq q_{\emptyset} \leq 1$ denote the fraction of consumers without a car

- **Ownership distribution**

$$q = \Big((q_{11}, \ldots, q_{1\bar{a}}), \ldots, (q_{J1}, \ldots, q_{J\bar{a}}), q_{\emptyset}\Big) \in \mathbb{R}^{J\bar{a}+1}.$$

- q is measured in the beginning of the period before trading
- q changes from period to period due to trading, deterministic aging of cars, and stochastic accidents

Equilibrium

- Infinitely elastic supply of new cars at price $\overline{P}(j)$
- Infinitely elastic demand for scrap cars at price $\underline{P}(j)$
- $J(\bar{a}-1)$ tradable cars at prices $P = \big(P(1,1), \ldots, P(J, \bar{a}-1)\big)$

Definition (Stationary equilibrium)

Equilibrium is the pair (q, P) such that

1. Consumers maximize their expected discounted utility
2. Secondary market clears for all tradable cars
3. Ownership distribution is time invariant

Demand

Demand for cars of make/model j of age d:

$$D_{jd}(P, q) = \Pi(j, d | \o, P) q_{\o} + \sum_{i=1}^{J} \sum_{a=1}^{\bar{a}} \Pi(j, d | i, a, P) q_{ia}$$

- In equilibrium all intended choices take place $\rightarrow$
- Sum up choice probabilities from all owners
- Economy with continuum of consumers but finite number of tradable goods $\rightarrow$
- Choice probabilities = market shares

Supply

Supply of cars of make/model j of age d:

$$S_{jd}(P, q) = \big(1 - \Pi(\kappa | j, d, P)\big)\big(1 - \Pi(1_s | j, d, P)\big) q_{jd}$$

- Cars are supplied by owners who don't keep $1 - \Pi(\kappa | j, d, P) \rightarrow$
- And don't voluntarily scrap $1 - \Pi(1_s | j, d, P)$

Market clearing

$$D(P,q) = \Big((D_{11},\ldots,D_{1,\bar{a}-1}),\ldots,(D_{J1},\ldots,D_{J,\bar{a}-1})\Big) \in \mathbb{R}^{J(\bar{a}-1)}$$
$$S(P,q) = \Big((S_{11},\ldots,S_{1,\bar{a}-1}),\ldots,(S_{J1},\ldots,S_{J,\bar{a}-1})\Big) \in \mathbb{R}^{J(\bar{a}-1)}$$

Market clearing $\Leftrightarrow$ Zero excess demand condition:

$$ED(P,q) = D(P,q) - S(P,q) = 0$$

- System of $J(\bar{a}-1)$ equations: linear in q, non-linear in prices
- $J\bar{a}+1$ unknowns in the ownership distribution vector
- $J(\bar{a}-1)$ unknown prices

Trade transition probability matrix

$$\Omega(P) = \qquad J\bar{a}+1 \times J\bar{a}+1 \text{ matrix}$$

$$\begin{bmatrix} \Delta_{11}(P)+\Lambda_1(P) & \Delta_{12}(P) & \dots & \Delta_{1J}(P) & \Delta_{1\emptyset}(P) \\ \Delta_{21}(P) & \Delta_{22}(P)+\Lambda_2(P) & \dots & \Delta_{2J}(P) & \Delta_{2\emptyset}(P) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Delta_{J1}(P) & \Delta_{J2}(P) & \dots & \Delta_{JJ}(P)+\Lambda_J(P) & \Delta_{J\emptyset}(P) \\ \Delta_{\emptyset 1}(P) & \Delta_{\emptyset 2}(P) & \dots & \Delta_{\emptyset J}(P) & \Pi(\emptyset|\emptyset, P) \end{bmatrix}$$

Then $q \cdot \Omega(P)$ is distribution of cars after the trading phase

- $\Delta_{ij}(P)$ = trading components
- $\Lambda_{ij}(P)$ = keeping components

Matrix blocks for j-type car

$$\Delta_{ij}(P) = \begin{bmatrix} \Pi(j,1|i,1) & \dots & \Pi(j,\bar{a}-1|i,1) & \Pi(j,0|i,1) \\ \vdots & \ddots & \vdots & \vdots \\ \Pi(j,1|i,\bar{a}) & \dots & \Pi(j,\bar{a}-1|i,\bar{a}) & \Pi(j,0|i,\bar{a}) \end{bmatrix},$$

$$\Lambda_i(P) = \begin{bmatrix} \Pi(\kappa|i,1) & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \Pi(\kappa|i,\bar{a}-1) & 0 \\ 0 & \dots & 0 & 0 \end{bmatrix}.$$

- $\bar{a} \times \bar{a}$ matrices
- Last column pertains to the car of age $\bar{a}$, but collects choice probabilities for the new cars

Last column and bottom row of $\Omega(P)$

- Probabilities of buying (j, d)-cars by people without cars

$$\Delta_{\o j}(P) = \Big[\Pi(j, 0|\o), \ldots, \Pi(j, \bar{a}-1|\o), \Pi(j, 0|\o)\Big] \in \mathbb{R}^{\bar{a}}$$

- Probabilities of transition to no car state by owners of (i, a)-cars

$$\Delta_{j\o}(P) = \begin{bmatrix} \Pi(\o|i, 1) \\ \vdots \\ \Pi(\o|i, \bar{a}) \end{bmatrix} \in \mathbb{R}^{\bar{a}}$$

Physical transition probability matrix

$Q =$ $\qquad J\bar{a}+1 \times J\bar{a}+1$ matrix

$$\begin{bmatrix} Q_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & Q_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & Q_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & Q_J & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix} \quad Q_j = \begin{bmatrix} 0 & 1-\alpha & \dots & 0 & \alpha \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1-\alpha & \alpha \\ 0 & 0 & \dots & 0 & 1 \\ 1-\alpha & 0 & \dots & 0 & \alpha \end{bmatrix}$$

- Ageing of cars with probability $1-\alpha$
- Total loss accidents with probability α
- Last column in each block is applies for the new cars, restores the order

$q \cdot \Omega(P)Q$ is ownership distribution in the next period

The stationary holdings distribution

$$\underbrace{q}_{t} \quad \to \quad \underbrace{q\Omega(P)}_{\text{after trading}} \quad \to \quad \underbrace{q\Omega(P)Q}_{t+1}$$

Condition for time invariance of the ownership distribution:

$$q = q\Omega(P)Q$$

Theorem (Uniqueness of stationary ownership distribution)

Let $\sigma > 0$. Then stationary ownership distribution is unique.

Proof.

With positive GEV scale parameter σ, the choice probabilities have full support, therefore transition matrix $\Omega(P)Q$ is irreducible and aperiodic. Uniqueness follows from the Fundamental theorem of Markov chains. □

Existence of stationary equilibrium

Theorem (Equilibrium existence)

The stationary equilibrium for the automobile economy with the idiosyncratically heterogeneous consumers $(q^\star, P^\star)$ *exists, and in equilibrium it holds:*

$$\begin{aligned} q^\star &= q^\star \Omega(P^\star) Q, \\ 0 &= ED(P^\star, q^\star). \end{aligned}$$

- Only existence: $q^\star$ is unique, but unclear about $P^\star$
- However, have not seen any signs of multiplicity in computations

Equilibrium flow property

Theorem (Stationary flow equilibrium)

The stationary equilibrium $(q^\star, P^\star)$ in previous theorem has the flow property:

$$\underbrace{\sum_{a=1}^{\bar{a}-1} \Pi(1_s|j,a,P)\big(1-\Pi(\kappa|j,a,P)\big)q^\star_{ja} + q^\star_{j\bar{a}}}_{\text{all scrapped cars of make/model } j} = \underbrace{\sum_{i=1}^{J}\sum_{a=1}^{\bar{a}} \Pi(j,0|i,a,P)q^\star_{ia}}_{\text{all new cars of make/model } j}$$

- Follows from stationarity of ownership distribution $q^\star$, including its subvectors
- Also holds for the outside good (not owning the car), follows algebraically from the structure of $\Omega(P^\star)$ and Q
- So, we have stationary flow equilibrium

How to compute stationary flow equilibrium?

Straightforward algorithm following the existence Theorem:

1. Equilibrium ownership distribution q is an implicit function of P from the stationarity condition $q = q\Omega(P)Q$
2. Express excess demand as a function of prices only $ED\big(P, q(P)\big)$
3. Numerically solve $ED\big(P, q(P)\big) = 0$ in prices
4. Finally, compute the corresponding stationary ownership distribution

The key to success:

- Use of the efficient gradient based solver (Newton method)
- Analytic derivatives $\rightarrow$ quick and precise
- Precise theoretically founded starting values $\rightarrow$ robust

Smoothness

To be able to apply Newton method we prove in a series of Lemmas that the following model object are continuously differentiable in prices P (*and* structural parameters + other fundamentals as α, $\overline{P}$, $\underline{P}$):

- Fixed point of Bellman equation EV
- Choice specific functions $v(i, a, j, d, s)$
- Choice probabilities $\Pi(j, d, s|i, a)$
- Invariant distribution of transition probability matrix $\Omega(P)Q$
- Excess demand $ED(P, q)$

Newton method is therefore applied:

1. When solving the DP problem (Newton-Kantorovich)
2. When solving for equilibrium prices
3. Later again when maximizing likelihood

Computational algorithm for $ED\big(P, q(P)\big)$

Use Newton method to solve the nonlinear system of $ED\big(P, q(P)\big) = 0$ with $J(\bar{a} - 1)$ equations and $J(\bar{a} - 1)$ unknown prices P.

Single computation of $ED\big(P, q(P)\big)$ for given prices P:

1. Solve the dynamic problem to find the fixed point of Bellman equation EV
2. Compute choice probabilities and form trade transition probability matrix $\Omega(P)$
3. Compute the invariant distribution q of $\Omega(P)Q$
4. Compute excess demand $ED(P, q)$

- Derivatives are computed by repeated application of chain rule
- Originate with derivatives of the utility function and trade costs
- Fit like "LEGO blocks" from the inner steps

Numerical performance

Good numerical performance already in Matlab

	$J = 1$	$J = 2$	$J = 5$	$J = 10$	$J = 25$
$\bar{a}$=20	0.081s	0.141s	0.674s	4.103s	1m 6.865s
$\bar{a}$=50	0.162s	0.720s	11.566s	1m 12.317s	
$\bar{a}$=100	0.680s	4.407s	1m 29.818s		

Possible to nest into iterative algorithms:

- Structural estimation
- Primary market modelling
- etc.

Persistent consumer heterogeneity

We extend the model to allow for:

1. Time-invariant consumer heterogeneity
2. Time-variant consumer heterogeneity
3. The combination of the two

- Existence theorems
- Computational algorithm is linear in the number of types
- Allows for sorting of consumers into the ages and types of cars
  - Rich hold newer better cars, poor hold older worse cars
  - Gains from trade and longer surviving cars
  - High transaction costs suppress trade and lower scrappage age

Updates in the heterogeneous consumer case

- Consumer types $\tau = 1, \ldots, N$
- Type fractions (time invariant types) or stationary distribution (time variant types) $f = (f_1, \ldots, f_N)$
- Consumer choice model has to be solved separately for each type τ
- Type-specific trading matrix $\Omega_\tau(P)$
- Type-specific ownership distribution q_τ
- Type-specific excesses demand $ED(P, q_\tau)$
- The equilibrium conditions become

$$\begin{aligned} \forall \tau \; q^\star_\tau &= q^\star_\tau \Omega(P^\star) Q, \\ 0 &= \sum_{\tau=1}^{N} ED(P^\star, q^\star_\tau). \end{aligned}$$

- Market clearing condition integrated over types

Gains from trade between rich and poor consumers

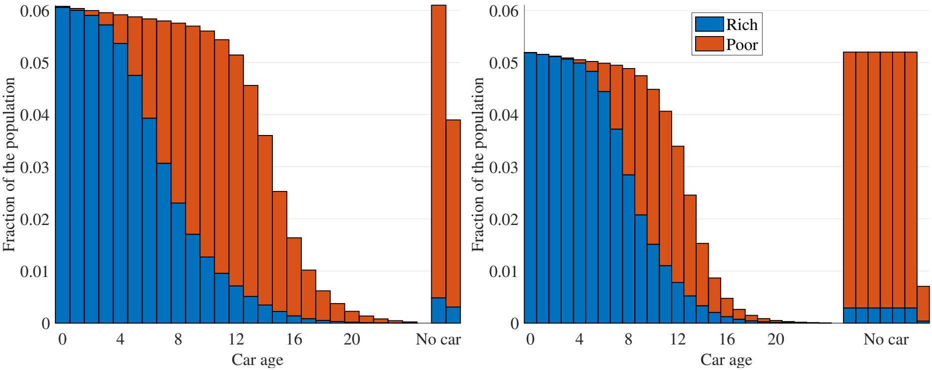
Rich mans Volvo

Poor mans Volvo

Illustrative example: ownership by two consumer types

Normal transactions costs

High transactions costs



- Sorting of consumers in each regime
- Heterogeneous effects of transaction costs

Double fixed point estimator

Full solution estimator for the equilibrium model

- Let θ denote the vector of structural parameters
- Data on car ownership and observed household characteristics x
- Likelihood function

$$L(\theta) = \sum_{\tau} \sum_{x'} \sum_{x} \log \left(\Pi(x'|x, \tau, \theta) \right) \underbrace{\Pi(x'|x, \tau) q_\tau(x) f(\tau)}_{\sim \text{observed } x \to x'}$$

Three loops:

Outer Maximization of likelihood with respect to θ

Middle Equilibrium solver for price vector $P(\theta)$

Inner DP solver for fixed point of Bellman operator $EV(P, \theta)$

Summary of the theoretical part

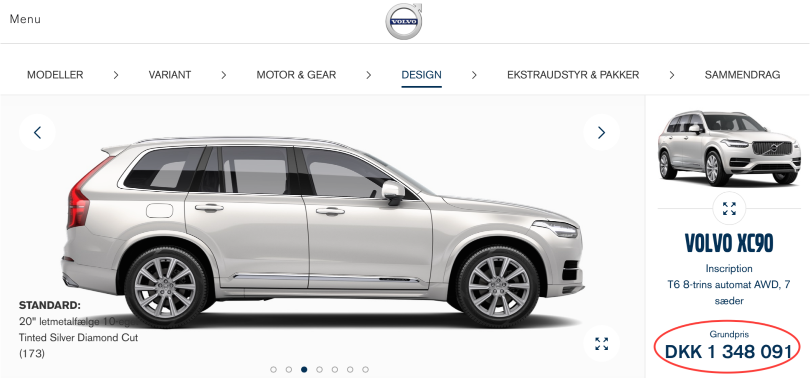
Empirical engine for analyzing equilibrium models of secondary markets of durable goods

- Computationally tractable in real empirical applications!
- Flexible specification of preferences and transaction costs $\rightarrow$
- Recover consumer heterogeneous preferences towards each type of car of every age
- Possible to use with no secondary market price data!
- Identity of car owners is not required: panel microdata on car ownership and trading can be represented by counts of transitions conditional on consumer types

Part III: Modeling the Danish secondary market for automobiles

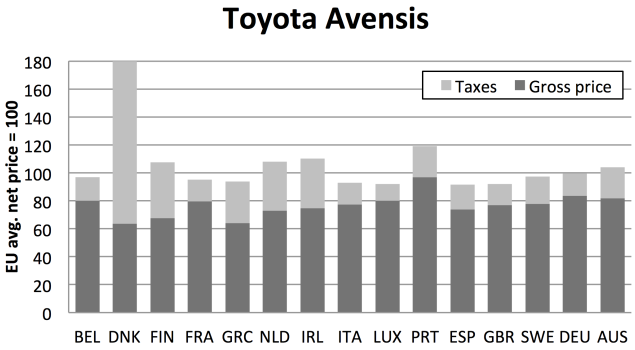
How much is a Volvo in Denmark?

204,816 US Dollars!



MSRP in US: $62,350

Danish car registration tax: 180%!
..... plus 25% VAT!!!



Car taxes in Denmark

Annual Revenue

- 30–50 billion DKK
- $\cong$ 2–3 pct. of GDP
- $\cong$ 4-7 pct. of total tax revenue
- Most revenue originate from taxation of ownership and registration of new cars.

It is also widely understood that transport externalities are rarely appropriately priced (e.g., Parry and Small (2005)).

- Underpricing congestion.
- Incorrect taxation of gasoline.

Simulating the effects of a hypothetical tax reform

Proposed Danish IRUC reform:

- lowers registration taxes, and
- raises usage taxes (road charging or gas tax).

Outcomes of interest:

- Equilibrium dynamics of car ownership and type choice:
  - new car sales and trade in secondary markets
  - fleet age and scrappage,
  - value of the car stock.
- Driving, fuel demand, and emissions.
- Redistribution and welfare.
- Need to capture these effects simultaneously

To implement the counterfactual simulation:

1. Estimate the model using Danish register data
2. Cut the registration tax rates for new vehicles by half
3. Simultaneously increases the fuel tax rate such that revenue is unchanged
4. Compute economic/welfare/environmental implications

Utility specification

Consider a utility function of the form

$$u(a, x) = u_{\text{car}}(a) + u_{\text{drive}}(a, x) + \mu[\text{trade + transaction cost}]$$

Utility of ownership when $x = 0$

$$u_{\text{car}}(a) = \alpha_0 + \alpha_1 a + \alpha_2 a^2$$

Net utility from driving

$$u_{\text{drive}}(a, x) = (\gamma_0 + \gamma_1 a)x - \mu p x + \frac{\phi}{2} x^2$$

- x is vehicle kilometers driven each period
- p is the per kilometer cost of fuel plus any taxes,
- parameter vector $\theta = (\alpha_0, \alpha_1, \alpha_2, \gamma_0, \gamma_1, \mu, \phi)$
- θ may be specific each consumer type/car type (for notational simplicity we have suppressed this dependence)

Demand for driving and gasoline

Assumption: *The probability of an accident and other physical deterioration in an automobile is independent of driving, x.*

This implies driving is a static subproblem of the overall DP problem that can be solved independently.

Structural driving equation implied by the direct utility function $u(x, a)$

$$x^*(p, a) = -\frac{1}{\phi}\left[\gamma_0 + \gamma_1 a - \mu p\right]. \tag{4}$$

Identification

We want to consider the identification of parameters $\theta = (\alpha_0, \alpha_1, \alpha_2, \gamma_0, \gamma_1, \mu, \phi)$ from observations on:

1. consumer trading of automobiles
2. observed driving.

Reduced form driving equation

Structural driving equation implied by the direct utility function $u(x, a)$

$$x^*(p, a) = -\frac{1}{\phi}\left[\gamma_0 + \gamma_1 a - \mu p\right]. \tag{5}$$

Reduced form driving equation

$$x = d_0 + d_1 a + d_2 p. \tag{6}$$

where

$$\begin{aligned} d_0 &= -\frac{\gamma_0}{\phi} \\ d_1 &= -\frac{\gamma_1}{\phi} \\ d_2 &= \frac{\mu}{\phi} \end{aligned} \tag{7}$$

- reduced form parameters d_0, d_1 and d_2 are separately identified from data on driving, car age a and price per kilometer p of cars.
- Identifying variation in p across car types mainly due to different fuel efficiency of different types and different ages

Reduced form indirect utility of car ownership

Now plug the optimal driving into utility to obtain the indirect utility

$$\begin{aligned} u(a, x^*(p,a)) = v(a,p,\tau) &= \alpha_0 + \alpha_1 a + \alpha_2 a^2 - \frac{1}{2\phi}\left[\gamma_0 + \gamma_1 a - \mu p\right]^2 \\ &= u_0 + u_1 a + u_2 a^2 \end{aligned}$$

where

$$\begin{aligned} u_0 &= \alpha_0 - \frac{1}{2\phi}\left[\gamma_0 - \mu p\right]^2 \\ u_1 &= \alpha_1 - \frac{\gamma_1}{\phi}\left[\gamma_0 - \mu p\right] \\ u_2 &= \alpha_2 - \frac{1}{2\phi}\left[\gamma_1^2\right]. \end{aligned}$$

- (u_0, u_1, u_2) and μ are identified from an unrestricted or "reduced form" dynamic discrete choice model of car trading.
- Given μ, (u_0, u_1, u_2) and (d_0, d_1, d_2) are identified, then $(\alpha_0, \alpha_1, \alpha_2, \gamma_0, \gamma_1, \phi)$ can be separately identified

Estimation of parameters

Step 1: Least Squares Estimation of Reduced form driving parameters, (d_0, d_1, d_2)

Step 2: Maximum Likelihood Estimates of Equilibrium Model

- Reduced from utility parameters, (u_0, u_1, u_2)
- Marginal utility of money μ
- Purchase transaction cost
- Coefficients of binary logit model of accident rates: $\alpha(a) = \Lambda(\alpha_j + \alpha_j^a a)$
- Sales transaction cost

Step 3: Back out structural parameters

Table: Driving Regression Parameter Estimates

	Dependent variable: driving (1,000 km per year)		
γ_0	Intercept	19.07	(0.58)
$\hat{\gamma}_1^a/\phi_\tau$	Car age	-0.1325	(0.03)
$\hat{\gamma}_2^a/\phi_\tau$	Car age squared	-0.001975	(0.00)
$\hat{\gamma}_\tau/\phi_\tau$	Intercept, Low WD, Couple, Rich	4.43	(0.70)
$\hat{\gamma}_\tau/\phi_\tau$	Intercept, Low WD, Single, Poor	-3.752	(1.24)
$\hat{\gamma}_\tau/\phi_\tau$	Intercept, Low WD, Single, Rich	-0.0325	(0.79)
$\hat{\gamma}_\tau/\phi_\tau$	Intercept, High WD, Couple, Poor	9.825	(0.79)
$\hat{\gamma}_\tau/\phi_\tau$	Intercept, High WD, Couple, Rich	12.33	(0.74)
$\hat{\gamma}_\tau/\phi_\tau$	Intercept, High WD, Single, Poor	6.436	(1.52)
$\hat{\gamma}_\tau/\phi_\tau$	Intercept, High WD, Single, Rich	12.23	(1.27)
$\hat{\gamma}_j/\phi_\tau$	Car dummy: light, green	-1.994	(0.15)
$\hat{\gamma}_j/\phi_\tau$	Car dummy: heavy, brown	4.345	(0.15)
$\hat{\gamma}_j/\phi_\tau$	Car dummy: heavy, green	3.606	(0.14)
μ/ϕ	Price (common)	-7.074	(0.84)
$\hat{\mu}_\tau/\phi_\tau$	Price, Low WD, Couple, Rich	-4.111	(1.02)
$\hat{\mu}_\tau/\phi_\tau$	Price, Low WD, Single, Poor	4.732	(1.84)
$\hat{\mu}_\tau/\phi_\tau$	Price, Low WD, Single, Rich	0.2781	(1.16)
$\hat{\mu}_\tau/\phi_\tau$	Price, High WD, Couple, Poor	-6.41	(1.17)
$\hat{\mu}_\tau/\phi_\tau$	Price, High WD, Couple, Rich	-9.892	(1.09)
$\hat{\mu}_\tau/\phi_\tau$	Price, High WD, Single, Poor	-1.714	(2.29)
$\hat{\mu}_\tau/\phi_\tau$	Price, High WD, Single, Rich	-9.007	(1.91)
N	Driving periods	19,635,940	

Table: Maximum Likelihood Estimates of Equilibrium Model

μ_τ: marginal utility of money		
Low WD, Couple, Poor	0.1131	(0.0007)
Low WD, Couple, Rich	0.1119	(0.0007)
Low WD, Single, Poor	0.0941	(0.0007)
Low WD, Single, Rich	0.1077	(0.0007)
High WD, Couple, Poor	0.1036	(0.0006)
High WD, Couple, Rich	0.1155	(0.0007)
High WD, Single, Poor	0.0920	(0.0008)
High WD, Single, Rich	0.1081	(0.0008)

Table: Maximum Likelihood Estimates of Equilibrium Model

$u_{\tau,j,0}$: intercept in quadratic indirect utility for car ownership				
	light, brown	light, green	heavy, brown	heavy, green
Low WD, Couple, Poor	3.6490	3.1132	5.1535	4.7760
	(0.0178)	(0.0179)	(0.0281)	(0.0257)
Low WD, Couple, Rich	4.0324	3.4657	5.7492	5.3478
	(0.0172)	(0.0173)	(0.0271)	(0.0247)
Low WD, Single, Poor	2.4042	2.1504	3.5823	3.2068
	(0.0191)	(0.0178)	(0.0298)	(0.0274)
Low WD, Single, Rich	3.2454	2.8222	4.6934	4.2829
	(0.0175)	(0.0174)	(0.0275)	(0.0252)
High WD, Couple, Poor	3.8821	3.4351	5.3199	5.0561
	(0.0164)	(0.0164)	(0.0258)	(0.0235)
High WD, Couple, Rich	4.7620	4.3185	6.5617	6.2375
	(0.0182)	(0.0183)	(0.0285)	(0.0260)
High WD, Single, Poor	2.6685	2.4290	3.7554	3.5129
	(0.0206)	(0.0188)	(0.0324)	(0.0295)
High WD, Single, Rich	3.5538	3.1741	4.9946	4.6985
	(0.0204)	(0.0194)	(0.0319)	(0.0291)

Table: Maximum Likelihood Estimates of Equilibrium Model

$u_{\tau,j,1}$: coefficient of age in quadratic indirect utility for car ownership				
	light, brown	light, green	heavy, brown	heavy, green
Low WD, Couple, Poor	-0.1459	-0.0922	-0.2196	-0.1717
	(0.0009)	(0.0011)	(0.0014)	(0.0012)
Low WD, Couple, Rich	-0.1586	-0.0985	-0.2411	-0.1953
	(0.0009)	(0.0011)	(0.0013)	(0.0012)
Low WD, Single, Poor	-0.0984	-0.0615	-0.1600	-0.1128
	(0.0009)	(0.0010)	(0.0014)	(0.0012)
Low WD, Single, Rich	-0.1308	-0.0841	-0.2056	-0.1544
	(0.0009)	(0.0011)	(0.0014)	(0.0011)
High WD, Couple, Poor	-0.1390	-0.0825	-0.2134	-0.1696
	(0.0008)	(0.0010)	(0.0013)	(0.0011)
High WD, Couple, Rich	-0.1642	-0.1069	-0.2551	-0.2074
	(0.0009)	(0.0011)	(0.0014)	(0.0012)
High WD, Single, Poor	-0.1108	-0.0707	-0.1732	-0.1273
	(0.0010)	(0.0010)	(0.0016)	(0.0013)
High WD, Single, Rich	-0.1493	-0.1007	-0.2208	-0.1735
	(0.0010)	(0.0011)	(0.0016)	(0.0013)

Table: Maximum Likelihood Estimates of Equilibrium Model

utility costs of transacting		
	common	no car
Low WD, Couple, Poor	6.5944	1.7899
	(0.0226)	(0.0029)
Low WD, Couple, Rich	6.4425	1.0719
	(0.0229)	(0.0030)
Low WD, Single, Poor	6.5457	3.0816
	(0.0186)	(0.0045)
Low WD, Single, Rich	6.6036	2.5769
	(0.0212)	(0.0031)
High WD, Couple, Poor	6.2644	0.7793
	(0.0207)	(0.0036)
High WD, Couple, Rich	6.4237	0.1742
	(0.0237)	(0.0045)
High WD, Single, Poor	6.0303	2.3383
	(0.0182)	(0.0066)
High WD, Single, Rich	6.2786	1.7884
	(0.0220)	(0.0064)

Table: Maximum Likelihood Estimates of Equilibrium Model

coefficients of binary logit model of accident rates: $\alpha(a) = \Lambda(\alpha_j + \alpha_j^a a)$				
	light, brown	light, green	heavy, brown	heavy, green
intercept	-5.6248	-6.0443	-5.6728	-5.7826
	(0.0119)	(0.0105)	(0.0095)	(0.0088)
age slope	0.1804	0.2216	0.2020	0.2048
	(0.0015)	(0.0011)	(0.0009)	(0.0009)

Preferences and willingness to pay for cars

- Example: A rich couple with low work distance is willing to pay (e.g. rent for one year) a new light brown car for 36,040 DKK compared to 32,256 DKK for a poor household.
- Rich households are willing to pay more for any type of car compared to poor households
- All households preferred the heavy cars to the light ones and brown cars to green ones resulting in the following preference ordering: heavy brown $\succ$ heavy green $\succ$ light brown $\succ$ light green,
- Willingness to pay for cars by high work distance households exceeds that of low work distance ones
- Couples generally have higher willingness to pay for cars than singles

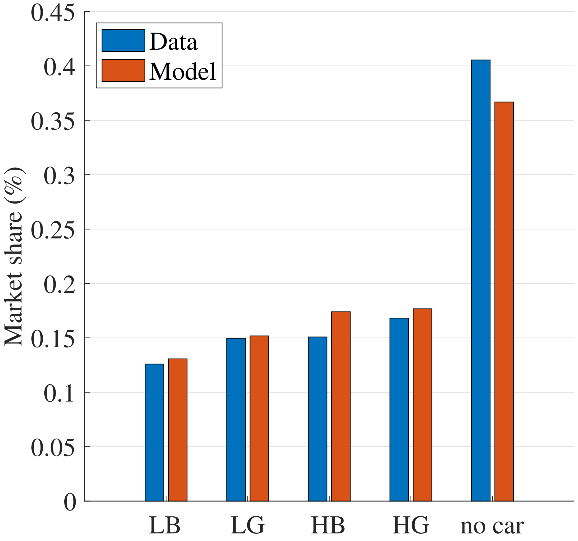
Model successfully captures key features of Danish households

- Poor households are significantly more likely not to own a car than rich ones
- Couples are more likely to own cars than singles
- High work distance households are relatively more likely to own cars than those with low work distance

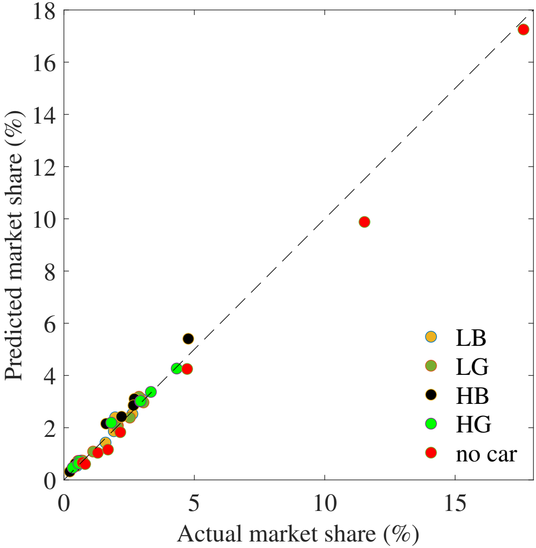
Implications for driving

- Households with high work distances drive much more than those with low, and more so for the rich.
- Model implies fuel price elasticities between -0.10 and -0.60 across households
- Driving response similar to Gillingham and Munk-Nielsen (2015) who find an average elasticity of -0.30

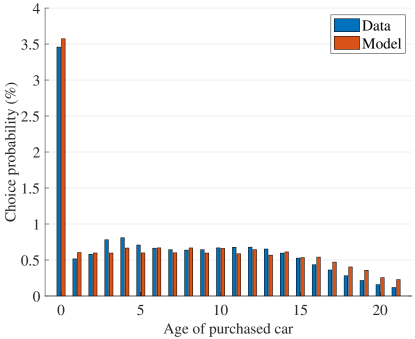
Model fit: aggregate market shares



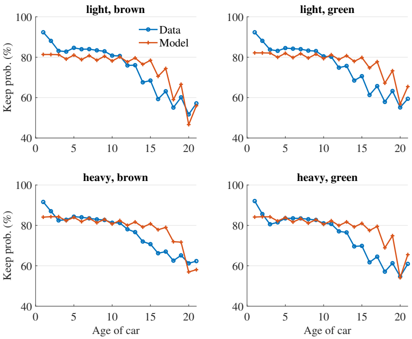
Household-specific Market Shares



Actual and predicted probability of keep and purchase

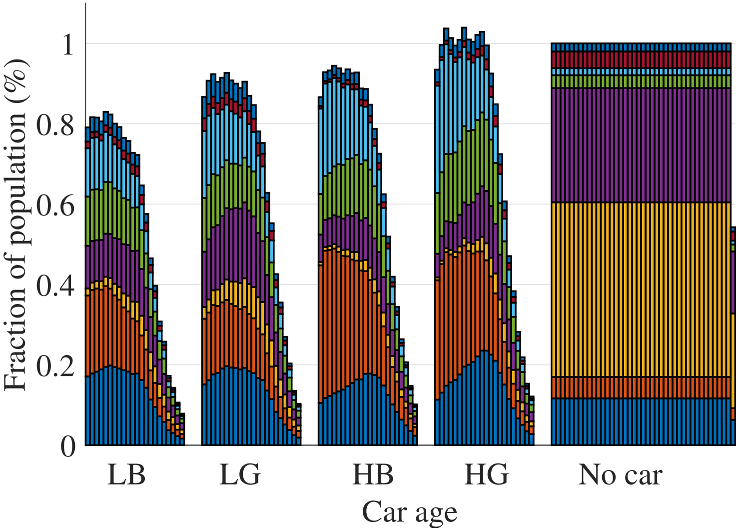


(a) Probability of purchase

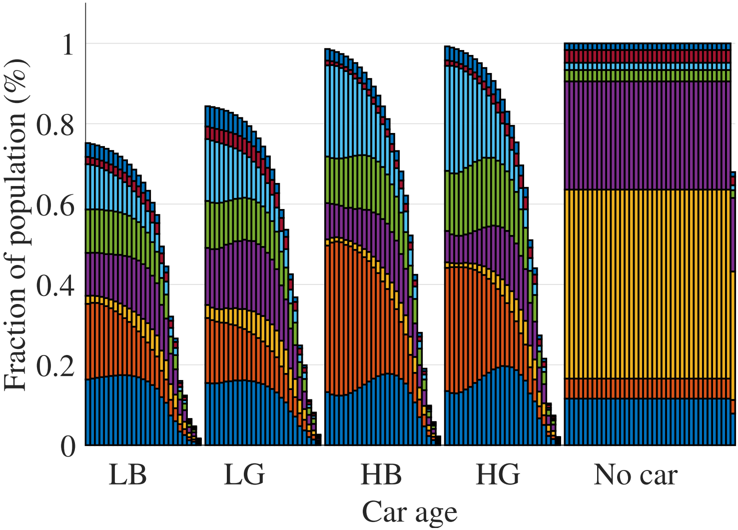


(b) Probability of keeping by car type

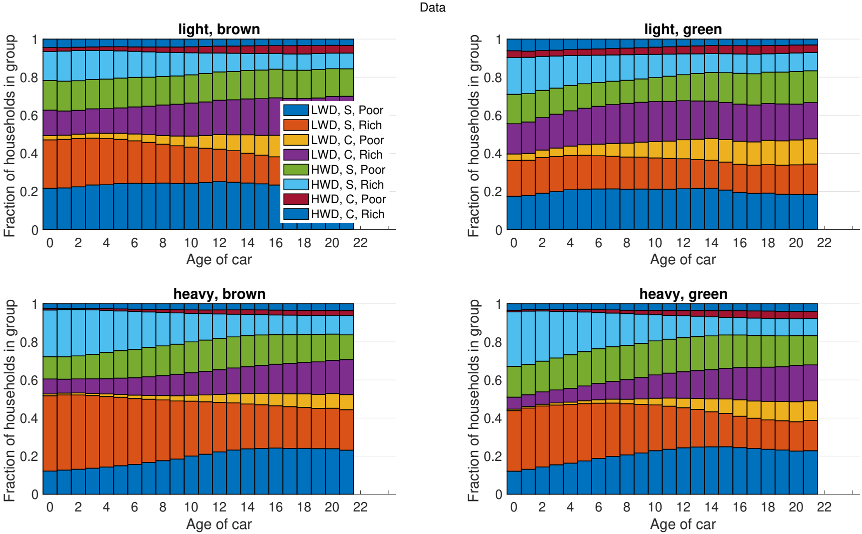
Model fit: observed ownership distribution



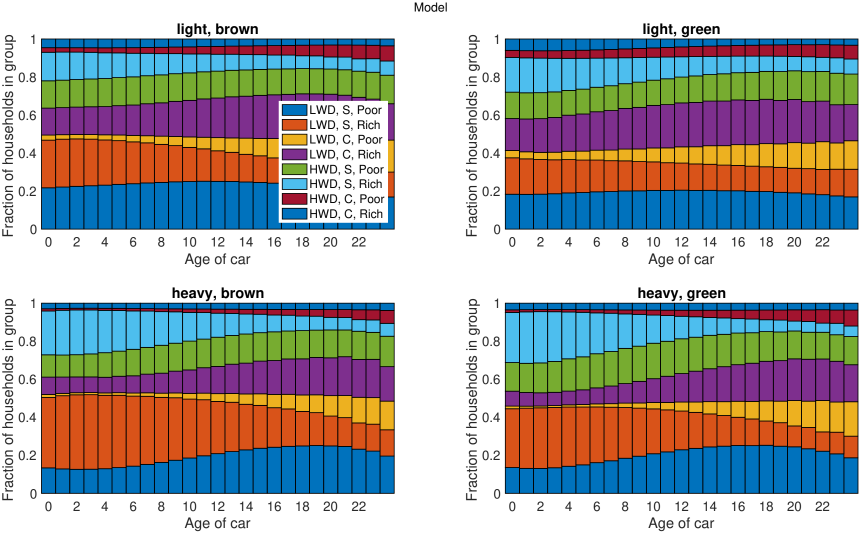
Model fit: predicted ownership distribution



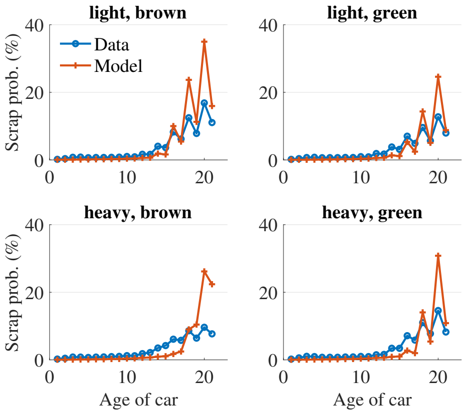
Model fit: observed sorting



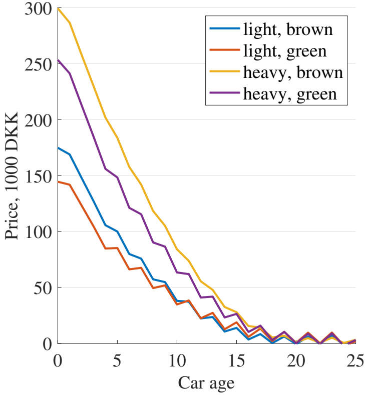
Model fit: predicted sorting



Zig-Zag Patterns in Scrappage and Equilibrium Prices



(c) Scrappage rates: data (blue) vs model (red)



(d) Predicted secondary prices

Counterfactual simulation

Halving registration tax corresponds to a drop in new car price by:
25.6% (new normal car) and 27.1% (new luxury car)

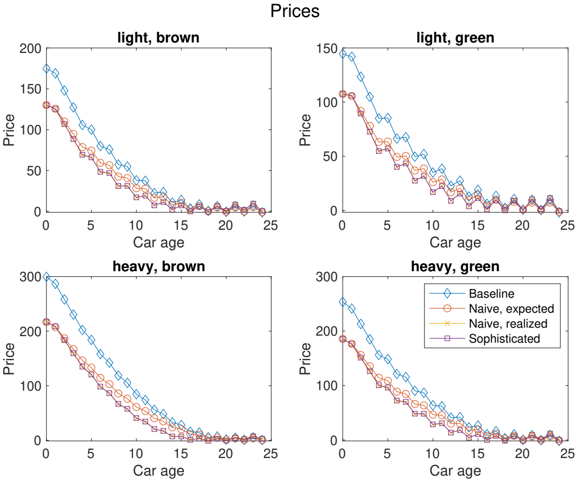
We consider the following four scenarios:

1. **Baseline:** Calibration under Danish tax rates from 2008.
2. **Naive, expected:** Non-equilibrium simulation: Both new and used car prices drops proportionally: between 25.6% (cheapest), 26.8% (most expensive car)
   Fuel taxes increase to keep total tax revenue neutral
   from 50% to 72.3% of the price at the pump
3. **Naive, expected:** Equilibrium simulation: As above + market equilibrium imposed. Not revenue neutral
4. **Sophisticated policy maker:** Equilibrium and revenue-neutral: Fuel taxes increase only to 68.4%

Policy Simulation Results

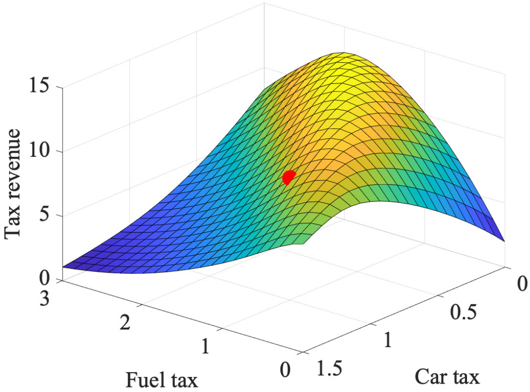
	Baseline	Naive, expected	Naive, realized	Sophisticated
Policy choice variables				
Registration tax (bottom rate)	1.050	0.525	0.525	0.525
Registration tax (top rate)	1.800	0.900	0.900	0.900
Fuel tax (share of pump price)	0.573	0.760	0.760	0.728
Exogeneous prices				
Price, light, brown (1000 DKK)	174.902	130.110	130.110	130.110
Price, light, green (1000 DKK)	144.551	107.532	107.532	107.532
Price, heavy, brown (1000 DKK)	299.452	216.760	216.760	216.760
Price, heavy, green (1000 DKK)	253.397	185.508	185.508	185.508
Fuel price (DKK/l)	8.322	14.824	14.824	13.054
Outcomes				
Social surplus (1000 DKK)	8.831	10.851	7.705	9.632
Total tax revenue (1000 DKK)	9.302	9.302	7.113	9.302
Fuel tax revenue (1000 DKK)	4.281	5.039	4.788	6.191
Car tax revenue (1000 DKK)	5.021	4.263	2.325	3.111
Non-CO2 externalities (1000 DKK)	6.749	3.312	3.182	4.782
Externalities (1000 DKK)	7.372	3.622	3.476	5.234
Consumer surplus (1000 DKK)	6.901	5.172	4.069	5.564
CO2 (ton)	2.148	1.069	1.016	1.559
VKT (1000 km)	10.858	5.329	5.118	7.693
E(car age)	6.505	2.911	4.228	5.473
Pr(no car)	0.367	0.546	0.548	0.415

Simulation of tax reform: Equilibrium prices



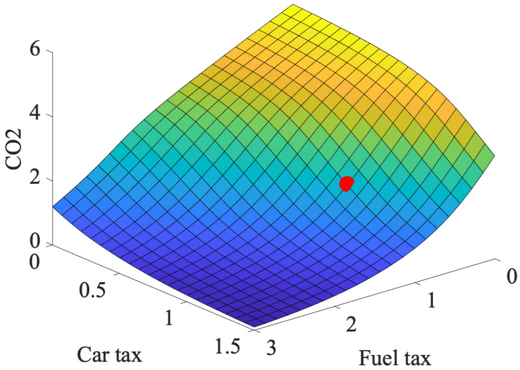
- Naive: based on proportional change in all prices
- But in equilibrium used car prices fall non-proportionally

Laffer curves for new car registration tax and fuel tax



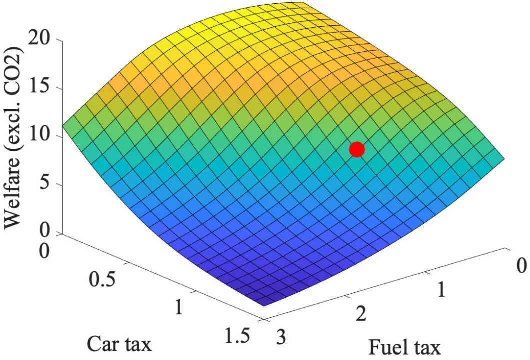
The new car registration and the fuel tax are measured relative to the baseline level of 1. The total tax revenue comes from new car tax (from new car sales only) and fuel tax.

CO2 emissions vs. new car registration tax and fuel tax



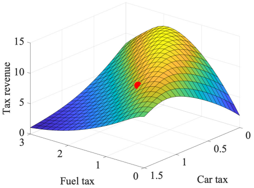
The new car registration and the fuel tax are measured relative to the baseline level of 1. The total tax revenue comes from new car tax (from new car sales only) and fuel tax.

Social welfare (ex CO2) vs. new car registration tax and fuel tax

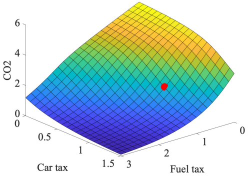


The new car registration and the fuel tax are measured relative to the baseline level of 1. The total tax revenue comes from new car tax (from new car sales only) and fuel tax.

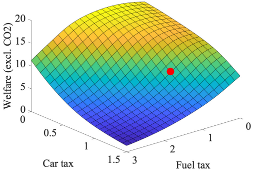
The Effects of Varying the Fuel and Registration Tax Rates



(e) Total tax revenue

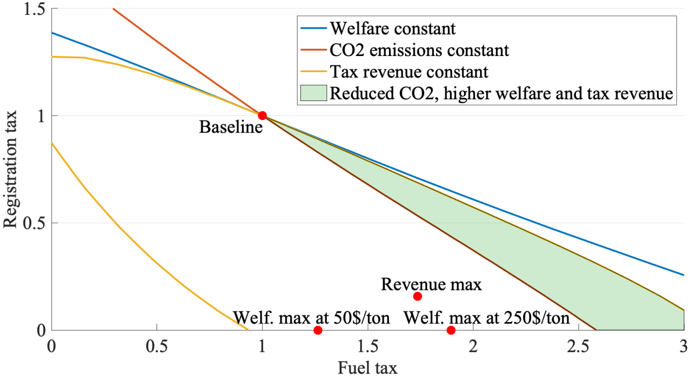


(f) CO_2 Emissions



(g) Social Surplus

Trade-off between CO2 emissions and welfare



Conclusion

- **Theory contribution:** characterize and prove existence of equilibrium in a tractable model of primary and secondary markets.
- **Applied contribution:** tractable model with
  - Transactions, scrappage, consumer/car heterogeneity,
  - Flexible utility: estimating 131 parameters with 39m obs. in under 15 min
- **Conclusion:** High Danish taxes above the Laffer curve's top point...
  - ... but a "naive" model overestimates the strength of this effect,
  - possibly leading to detrimental policies for tax revenues and the environment