

Agent-based and network models

Lecture 06

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September 2023

Agent-based models (ABM)

Network models

Measures specific to vertices

Measures at the graph level

Agent-based models (ABM)

What are agent-based models

When to use ABM

When not to use ABM

Some examples

Network models

Measures specific to vertices

Measures at the graph level

General context of network models

Agent-based models (ABM)

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ABM \neq IBM

Early in the life of these models, they were called IBM
(individual-based models)

Over the years, a “philosophical” distinction has emerged:

- ▶ IBM are mathematical models that consider individuals as the units; e.g., DTMC, CTMC, branching processes, etc.
- ▶ ABM are computational models whose study is, for the most part, only possible numerically

ABM vs Network models

Network models endow vertices with simple systems and couple them through graphs

Can be ABM, but some networks can also be studied analytically

Agent-based models (ABM)

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When to use ABM

When not to use ABM

Some examples

General context of network models

ABM are very useful to decipher contact processes

Classic mathematical models capture contact by using approximations of what contact could be like

Classic models allow some flexibility (see section about incidence functions in Lecture X but they remain limited

ABM can model actual trajectories of individuals, so given a definition of what a contact is (how close do you need to be for a contact to take place), can count them efficaciously

ABM are very useful to understand behavioural responses

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As with *all* tools, beware!

There is a law of large numbers effects happening often: if you have many units, unless some emergent behaviour arises, you get the same results using ODEs...

With this specific tool, beware!

There is a certain tendency in CS people to create *yet another* system and seek *adoption* by users

Agent-based models (ABM)

What are agent-based models

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When not to use ABM

Some examples

General context of network models

Antibiotic resistance in hospitals

D'Agata, Magal, Olivier, Ruan & Webb. Modeling antibiotic resistance in hospitals: The impact of minimizing treatment duration, Journal of Theoretical Biology (2007)

An IBM that's almost an ABM

This work is a good illustration of the “cultural proximity” between IBM and ABM

Model is stochastic and individual-based, in good enough form that approximating ODE can be derived

Allows for very specific tracking of the status of individuals through the process (almost an ABM in this sense)

The setup

Three processes:

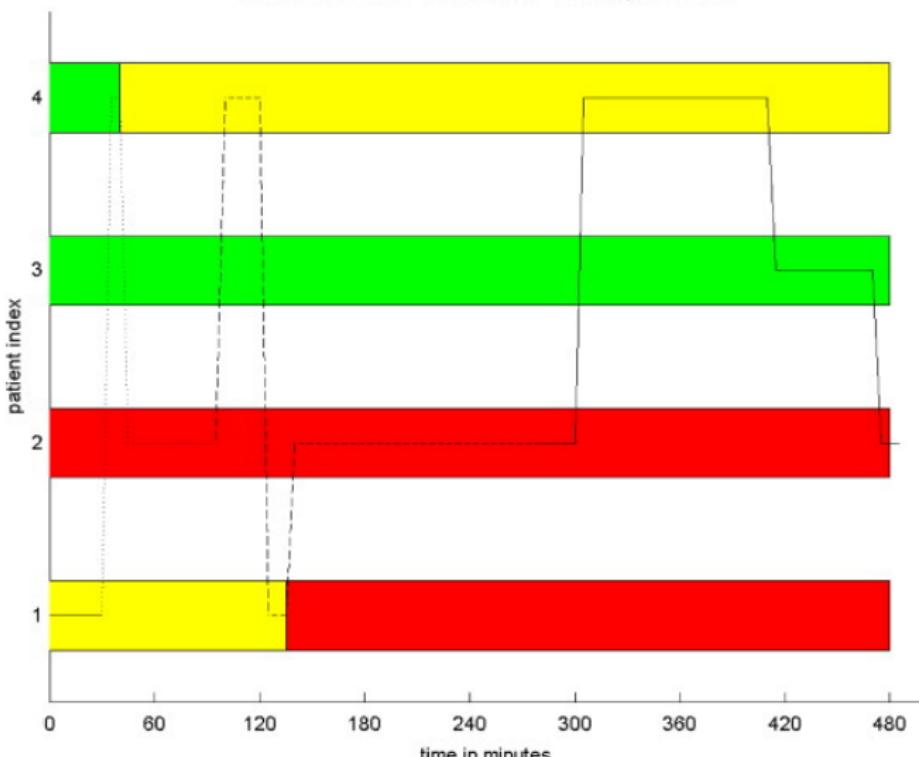
1. admission and exit of patients
2. infection of patients by HCW (health care workers)
3. contamination of HCW by patients

Contamination of HCW is “transient”: they are carriers, if they wash their hands properly, they become OK

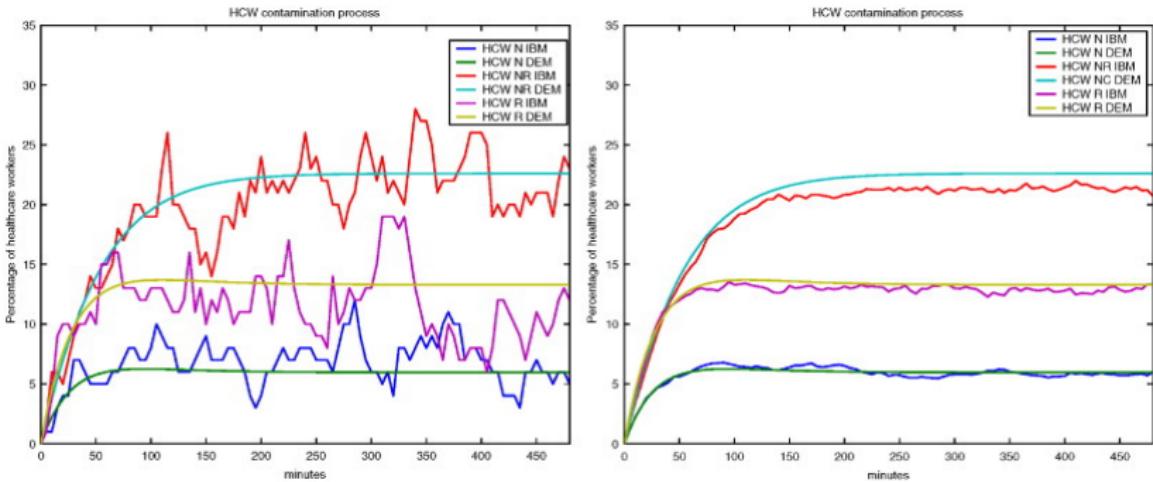
Each day has 3 shifts of 8h for HCW

Patients are put in contact by visits of HCW

Rules for contaminations per unit time



Patient-HCW contact diagram for four patients and one HCW during one shift. Patient status: uninfected (green), infected with the non-resistant strain (yellow), infected with the resistant strain (red). HCW status: uncontaminated (plain), contaminated with the non-resistant strain (dotted), contaminated with the resistant strain (dashed)



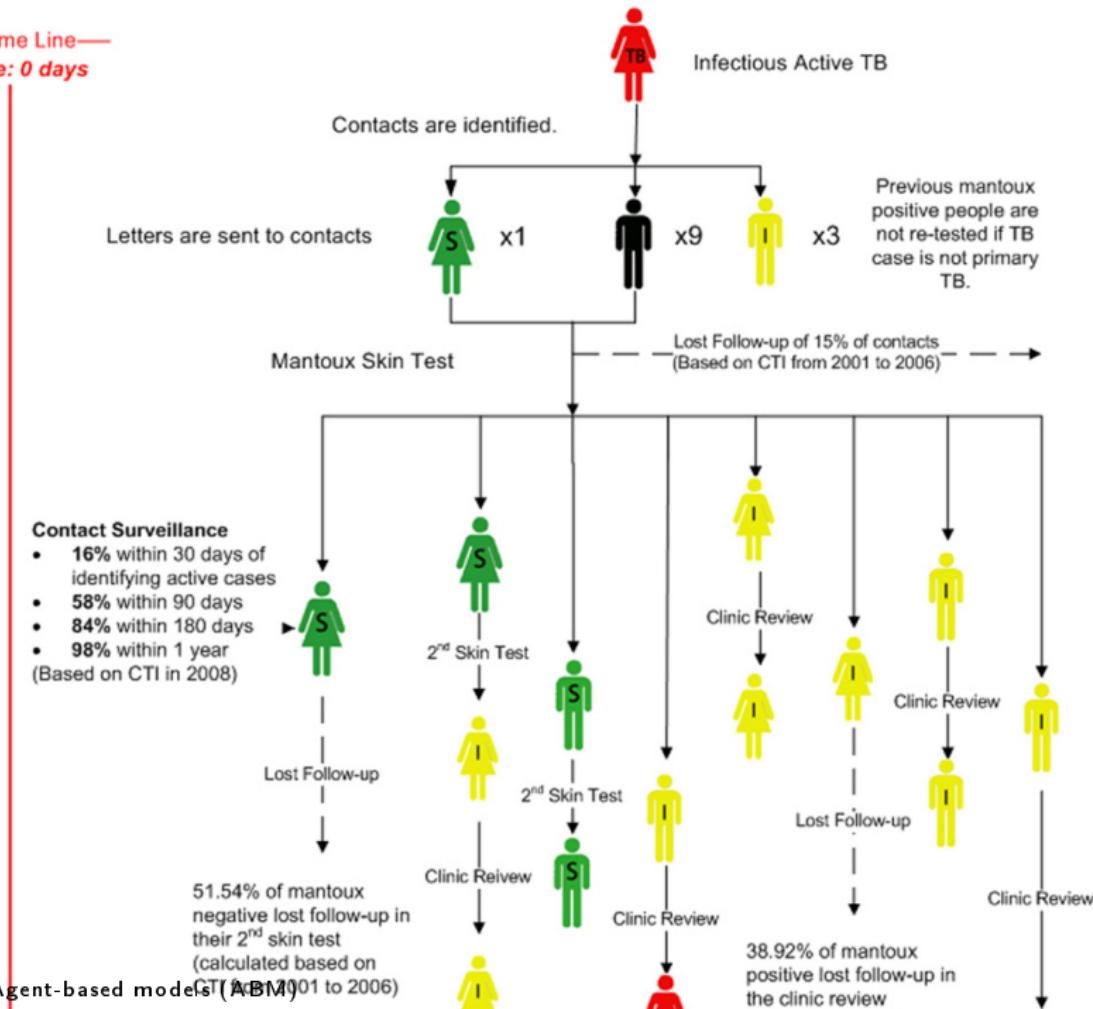
The left (respectively the right) figure corresponds to 1 trajectory (respectively the average over 80 trajectories) of the IBM during one shift, with no exit and admission of patients, and no changes in the infection status of patients

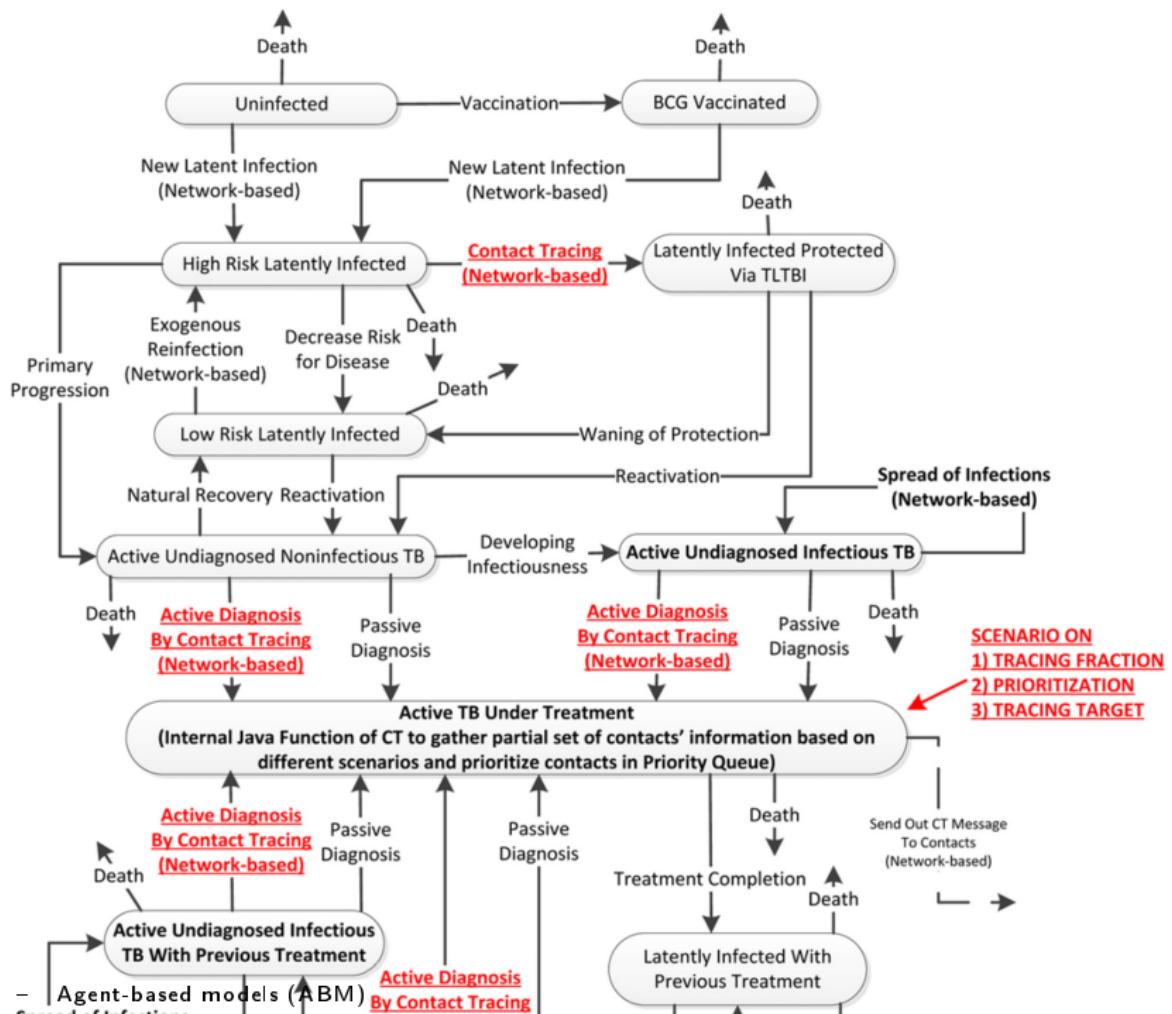
Effectiveness of contact tracing in TB

Tian, Osgood, Al-Azem & Hoeppner. Evaluating the Effectiveness of Contact Tracing on Tuberculosis Outcomes in Saskatchewan Using Individual-Based Modeling, Health Education & Behavior (2013)

—Time Line—

Time: 0 days







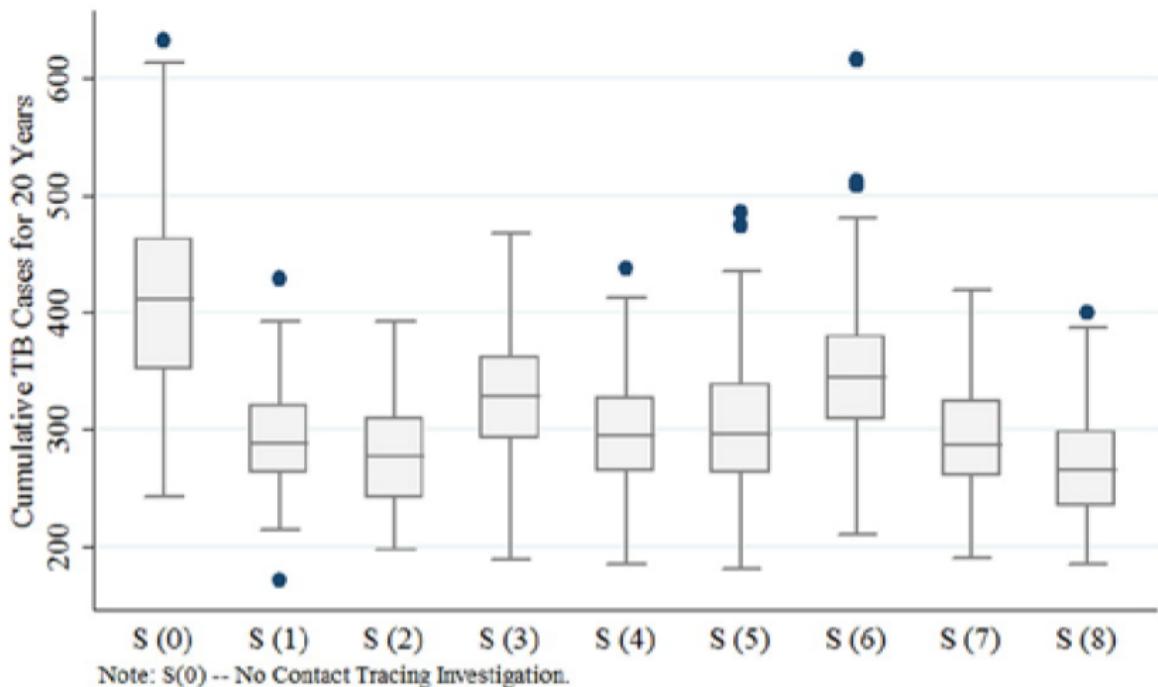
They can then formulate scenarios

Table 1. Scenario Definitions.

ID	Contact Tracing	Tracing Target	Loss to Follow-Up (%)	Priority	Tracing Fraction (%)
S0	Disabled	NA	NA	NA	NA
S1	Enabled	Infectious and primary tuberculosis	30-40	None	90
S2	Enabled	Infectious and primary tuberculosis	10	None	90
S3	Enabled	Infectious and primary tuberculosis	10	None	45
S4	Enabled	Infectious and primary tuberculosis	10	Age	45
S5	Enabled	Infectious and primary tuberculosis	10	Ethnicity	45
S6	Enabled	Infectious and primary tuberculosis	10	Reported times	45
S7	Enabled	Infectious and primary tuberculosis	10	Age and ethnicity	45
S8	Enabled	Infectious and primary tuberculosis	10	None	90 (Fast contact tracing)

They then run these scenarios and compare results

Effect of Contact Tracing Investigation on Cumulative TB Cases (N=100 Per Scenario)



Contacts during Hajj

Tofighi, Asgary, Tofighi, Najafabadi, Arino, Amiche, Rahman, McCarthy, Bragazzi, Thommes, Coudeville, Grunnill, Bourouiba and Wu. Estimating Social Contacts in Mass Gatherings for Disease Outbreak Prevention and Management (Case of Hajj Pilgrimage), Tropical Diseases, Travel Medicine and Vaccines

Contacts during Hajj

In a mass gathering event like Hajj, lots of people come together originating from many countries

So if propagation occurs during the event, this has the capacity to spread infection far and wide when individuals (pilgrims here) return home

Contacts during part of the event are really specific in their configuration

The setup

Word of warning: I am quite fuzzy on the specifics :)

Pilgrims enter Masjid al-Haram mosque through several gates

Proceed to Mataaf (area around Kaaba), circle the Kaaba 7 times
counterclockwise (process is the *Tawaf*)

Then do seven trips between Safa and Marwah (process is the
Sa'ee)



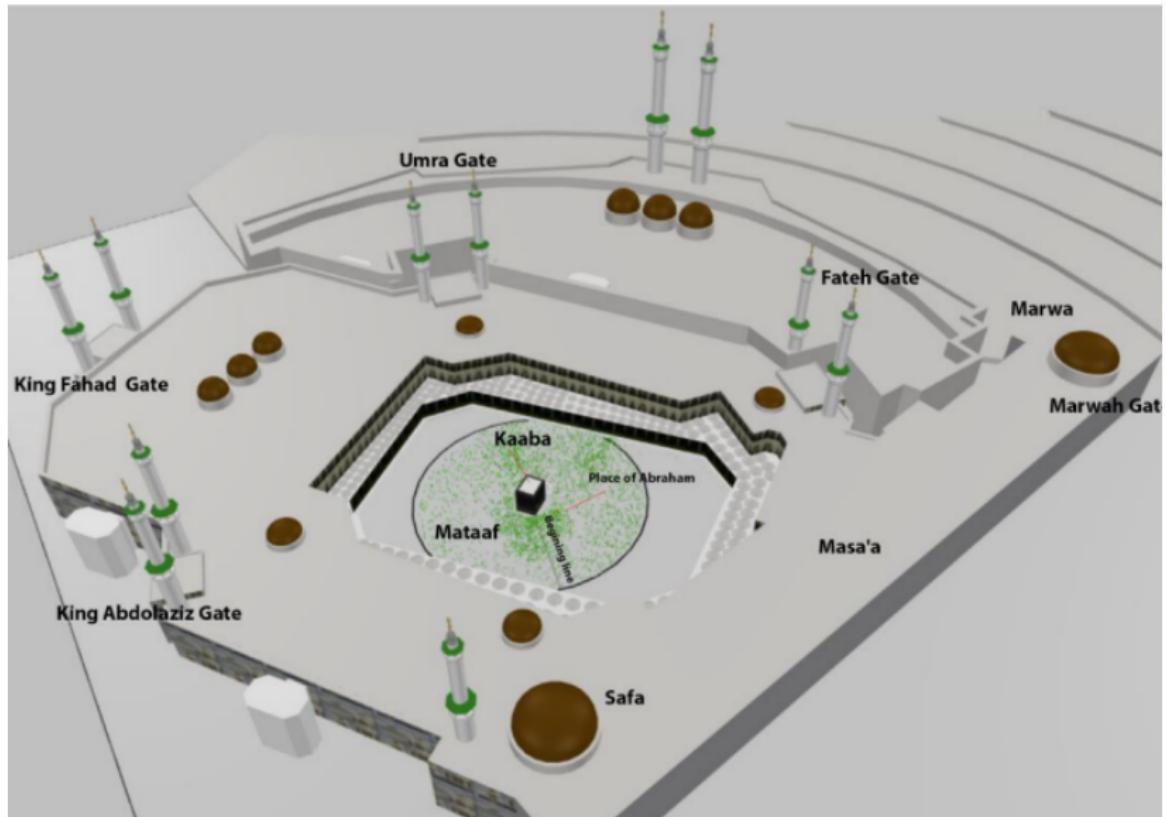
As you can gather from this:

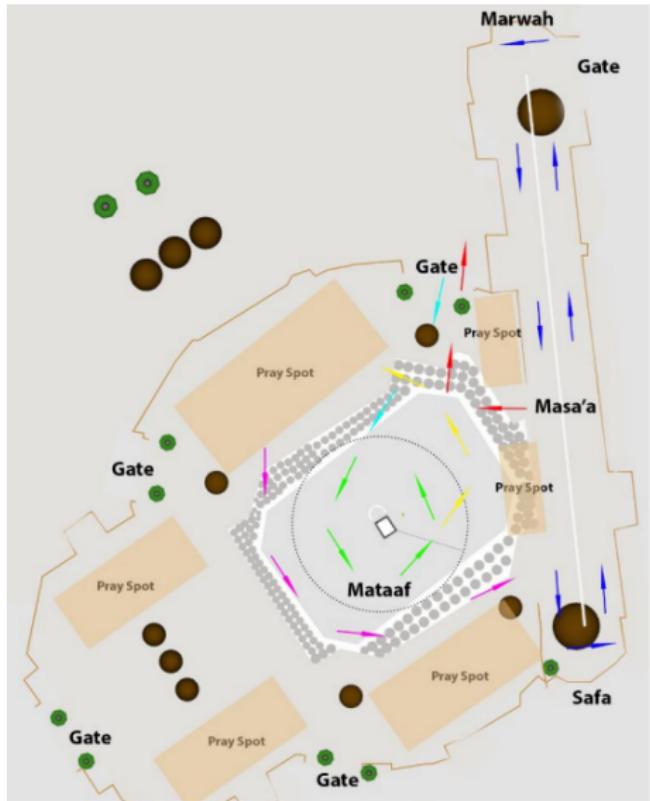
- ▶ Typically high density crowds
- ▶ Very specific mixing patterns

Opportunities for transmission are very high

However, control mechanisms are also available

⇒ understanding contact patterns and frequency would help





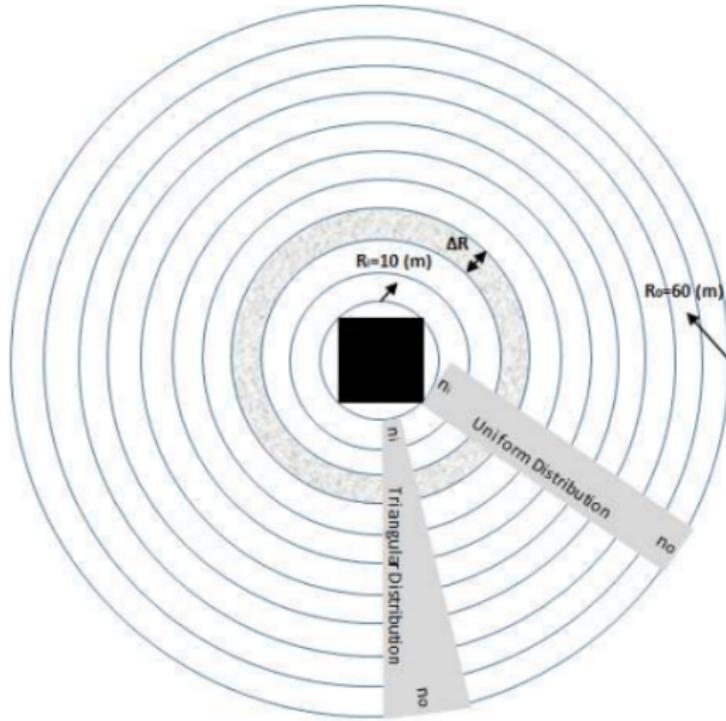


Figure 5. The general setting of Tawaf and pilgrim distribution in social distancing

Agent-based models (ABM)

Network models

Why use network models

Social networks

Measures specific to vertices

Measures at the graph level

General context of network models

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Understand contact processes

Classic models allow a certain degree of flexibility, for instance by using specific incidence functions or group models, but this remains limited and an approximation

Like ABM, network models are used to make more realistic descriptions of the transmission of pathogens

Human life is organised in networks

Family

Friends

Workplace

...

Social network theory has been used for years, e.g., in a professional context (e.g., how to fluidify interactions within a company)

Network models

Why use network models

Social networks

General context of network models

Before considering epidemics in networks, it is useful to learn a few notions of social network theory, as this is very useful to understand networks

Social network methods introduce measures that allow to evaluate some properties of graphs

A network is a (mathematical) graph, oriented or not, in which edges/arcs represent connections (whatever they are) between individuals, who make up the vertices of the graph

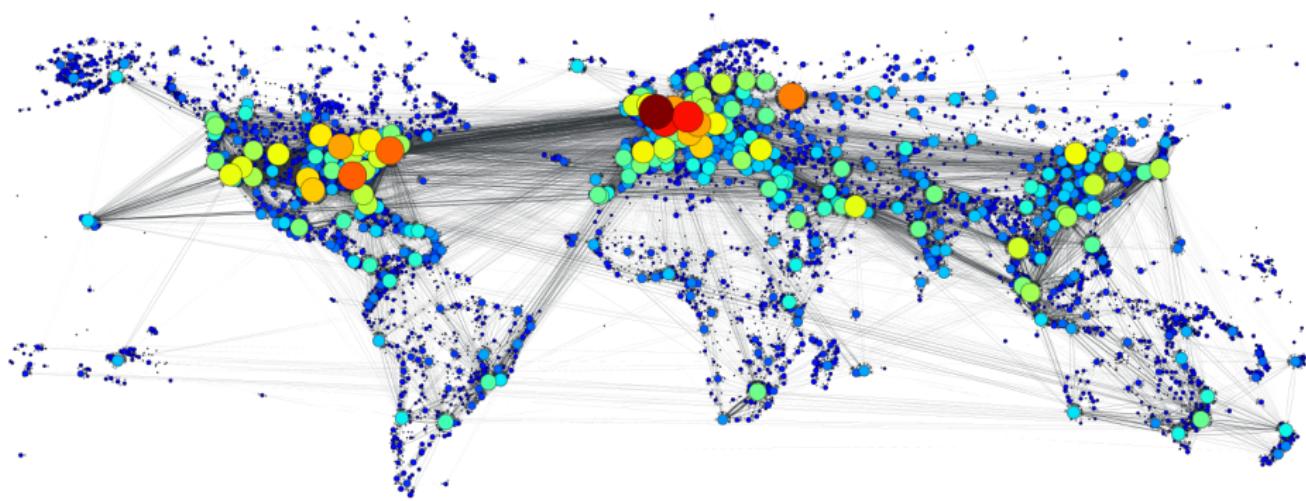
Context

- $\mathcal{G}(\mathcal{V}, \mathcal{E})$ an undirected graph
- $\mathcal{D}(\mathcal{V}, \mathcal{A})$ a digraph (directed graph)
- \mathcal{V} the set of vertices (or nodes)
- \mathcal{E} the set of edges (undirected case)
- \mathcal{A} the set of arcs (directed graph)

Example of the global air transport network

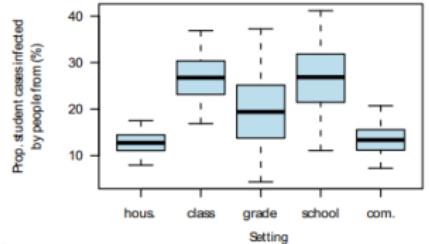
- Je vais illustrer avec des données du réseau de transport aérien mondial - Données assez bonnes (très bonnes parfois), et un avantage flagrant: - Quand un avion part de quelque part et arrive ailleurs, c'est quelque chose d'assez .. déterministe

The global air transportation network

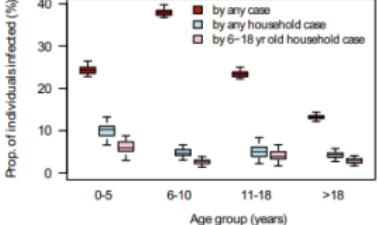


Example of spread of p-H1N1

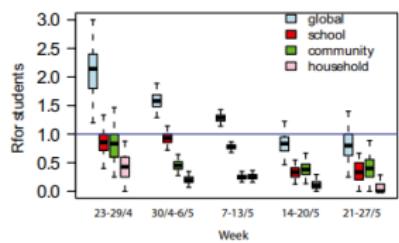
A



B



C



D

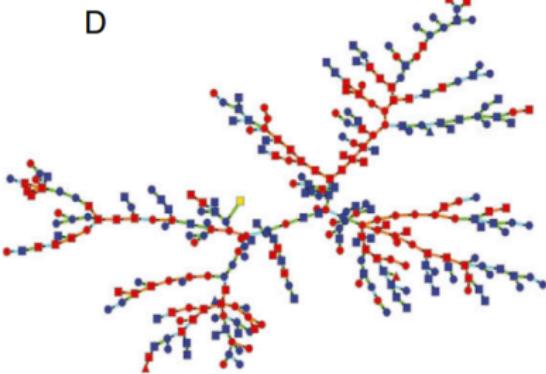
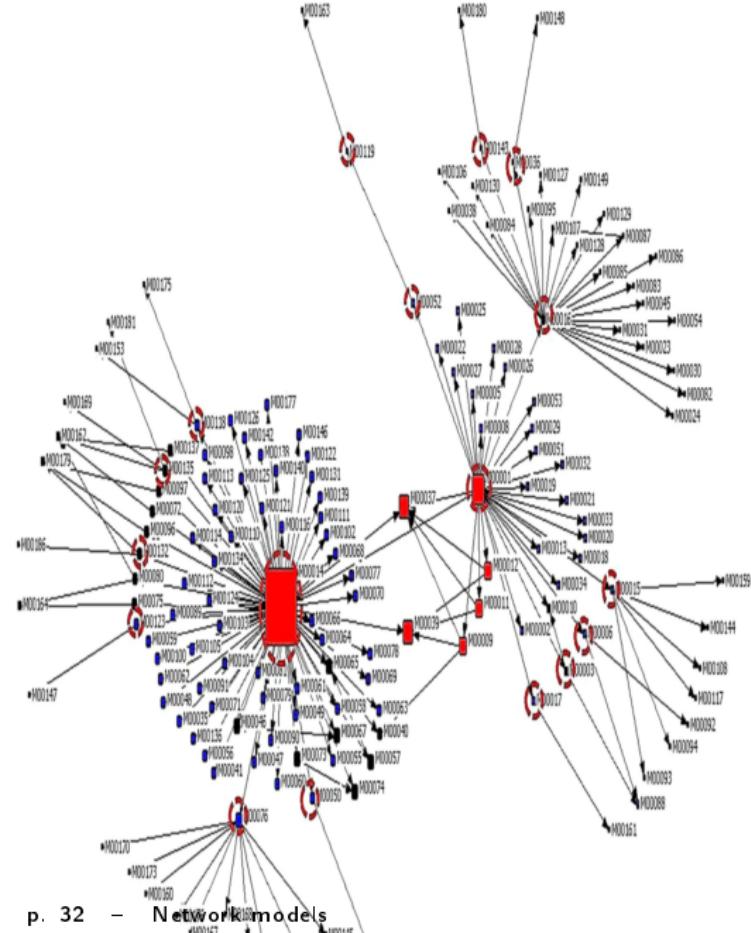


Fig. 4. Reconstruction of the transmission tree. (A) Proportion of student cases infected by people from their household, class, grade, school, or from the community. (B) Proportion of individuals infected by any other case (red), by any household case (blue), or by a household case aged 6–18 y (pink), as a function of the age of the individual. (C) Weekly estimates of the effective reproduction number in the outbreak (“global” R) and in places (school, household, and community). (D) Reconstructed transmission tree drawn from its predictive distribution (color of the nodes, yellow, first case; red, student of the school; blue, household member of a student; color of the lines for the type of transmission, orange, among students of the school; light blue, among household members; green, in the community; shape of the nodes, circle, female; square, male; triangle, sex is missing). Boxplots give percentiles 2.5%, 25%, 50%, 75%, and 97.5% of the predictive distribution.

Example of spread of MERS



Topological dynamics of the 2015 South Korea MERS-CoV spread-on-contact networks, Yang & Jung, Scientific Reports 10:4327 (2020)

Some “measures” concern the vertices, others the graph as a whole
In all that follows, unless otherwise indicated, $G = (X, A)$ is a
digraph. If undirected, we write $G = (X, E)$.

Agent-based models (ABM)

Network models

Measures specific to vertices

Centre of a graph

Centrality – Betweenness and closeness

Periphery of a graph

Degree distribution

Measures at the graph level

General context of network models

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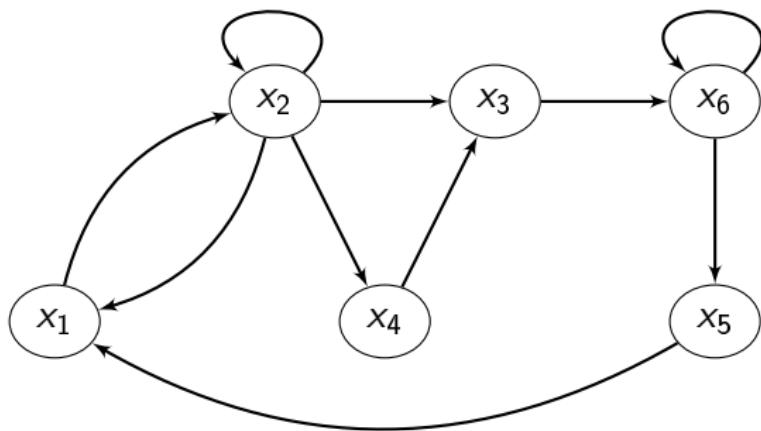
Geodesic distance

Definition 1 (Geodesic distance)

For $x, y \in X$, the **geodesic distance** $d(x, y)$ is the length of the shortest path from x to y , with $d(x, y) = \infty$ if no such path exists

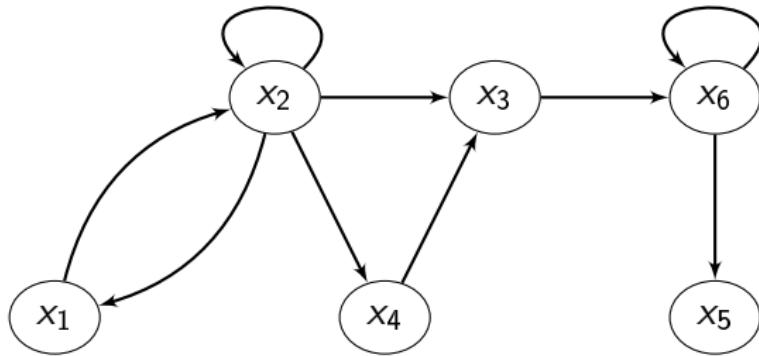
- ▶ $d(x_1, x_2) = 1$
- ▶ $d(x_1, x_3) = 2$
- ▶ ...

$$\begin{pmatrix} 0 & 1 & 2 & 2 & 4 & 3 \\ 1 & 0 & 1 & 1 & 3 & 2 \\ 3 & 4 & 0 & 5 & 2 & 1 \\ 4 & 5 & 1 & 0 & 3 & 2 \\ 1 & 2 & 3 & 3 & 0 & 4 \\ 2 & 3 & 4 & 4 & 1 & 0 \end{pmatrix}$$



- ▶ $d(x_5, x_1) = \infty$
- ▶ $d(x_3, x_1) = \infty$
- ▶ ...

$$\begin{pmatrix} 0 & 1 & 2 & 2 & 4 & 3 \\ 1 & 0 & 1 & 1 & 3 & 2 \\ \infty & \infty & 0 & \infty & 2 & 1 \\ \infty & \infty & 1 & 0 & 3 & 2 \\ \infty & \infty & \infty & \infty & 0 & \infty \\ \infty & \infty & \infty & \infty & 1 & 0 \end{pmatrix}$$



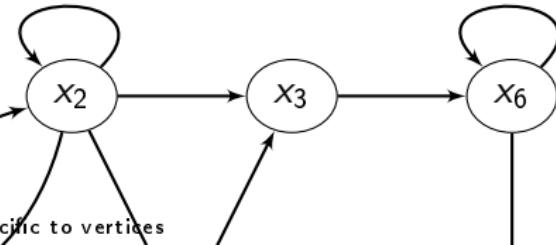
Eccentricity

Definition 2 (Vertex eccentricity)

The **eccentricity** $e(x)$ of vertex $x \in X$ is

$$e(x) = \max_{\substack{y \in X \\ y \neq x}} d(x, y)$$

$$\begin{pmatrix} 0 & 1 & 2 & 2 & \textcolor{red}{4} & 3 \\ 1 & 0 & 1 & 1 & \textcolor{red}{3} & 2 \\ 3 & 4 & 0 & \textcolor{red}{5} & 2 & 1 \\ 4 & \textcolor{red}{5} & 1 & 0 & 3 & 2 \\ 1 & 2 & 3 & 3 & 0 & \textcolor{red}{4} \\ 2 & 3 & \textcolor{red}{4} & \textcolor{red}{4} & 1 & 0 \end{pmatrix}$$



Central points, radius and centre

Definition 3 (Central point)

A **central point** of G is a vertex x_0 with smallest eccentricity

Definition 4 (Radius)

The **radius** of G is $\rho(G) = e(x_0)$, where x_0 is a centre of G . In other words,

$$\rho(G) = \min_{x \in X} e(x)$$

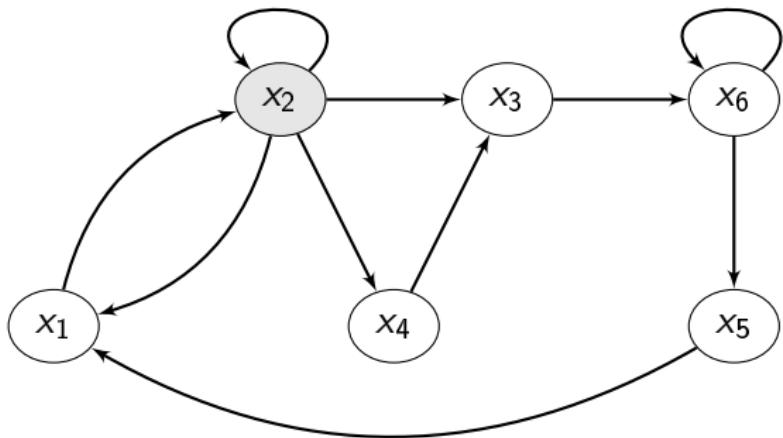
Definition 5 (Centre)

The **centre** of G is the set of vertices that are central points of G , i.e.,

$$\{x \in X : e(x) = \rho(G)\}$$

0	1	2	2	4	3
1	0	1	1	3	2
3	4	0	5	2	1
4	5	1	0	3	2
1	2	3	3	0	4
2	3	4	4	1	0

Radius is 3, x_2 is a central point
 (the only one) and the centre is
 $\{x_2\}$



Measures specific to vertices

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How *central* is a vertex?

Centrality tries to answer the question: what are the most influent vertices?

We have seen central vertices and vertices on the periphery, let us consider two other measures of centrality

- ▶ Betweenness centrality
- ▶ Closeness centrality

Many other forms (we will come back to this, e.g., degree centrality)

Betweenness

Definition 6 (Betweenness)

$G = (X, U)$ a graph, $x \in X$. The **betweenness** of v is

$$b_D(v) = \sum_{s \neq t \neq v \in X} \frac{\sigma_{st}(v)}{\sigma_{st}}$$

where

- ▶ σ_{st} number of shortest geodesic paths from s to t
- ▶ $\sigma_{st}(v)$ number of shortest geodesic paths from s to t through v

In other words

- ▶ For each pair of vertices (s, t) , compute the shortest paths between them
- ▶ For each pair of vertices (s, t) , determine the fraction of shortest paths that pass through vertex v
- ▶ Sum this fraction over all pairs of vertices (s, t)

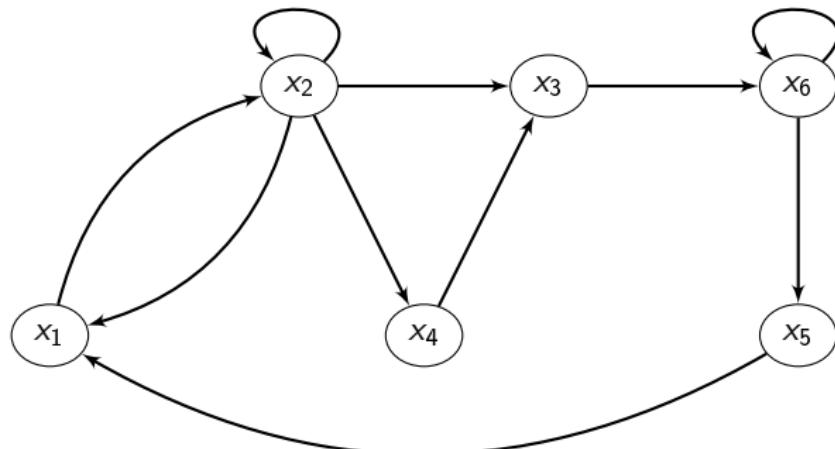
Normalising betweenness

Betweenness may be normalized by dividing through the number of pairs of vertices not including v :

- ▶ for directed graphs, $(n - 1)(n - 2)$
- ▶ for undirected graphs, $(n - 1)(n - 2)/2$

Example of betweenness
distances(G,
mode="out")

$$\begin{pmatrix} 0 & 1 & 2 & 2 & 4 & 3 \\ 1 & 0 & 1 & 1 & 3 & 2 \\ 3 & 4 & 0 & 5 & 2 & 1 \\ 4 & 5 & 1 & 0 & 3 & 2 \\ 1 & 2 & 3 & 3 & 0 & 4 \\ 2 & 3 & 4 & 4 & 1 & 0 \end{pmatrix}$$



Number of shortest paths

Recall we found `distances(G, mode="out")`

$$\mathcal{D} = \begin{pmatrix} 0 & 1 & 2 & 2 & 4 & 3 \\ 1 & 0 & 1 & 1 & 3 & 2 \\ 3 & 4 & 0 & 5 & 2 & 1 \\ 4 & 5 & 1 & 0 & 3 & 2 \\ 1 & 2 & 3 & 3 & 0 & 4 \\ 2 & 3 & 4 & 4 & 1 & 0 \end{pmatrix}$$

To find the number of shortest paths between pairs of vertices, we can use powers of the adjacency matrix

Write $\mathcal{D} = [d_{ij}]$, for a given (i, j) ($i \neq j$), if $d_{ij} = k$, then pick the (i, j) in A^k

We find

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Recall that betweenness of v is

$$b_{\mathcal{D}}(v) = \sum_{s \neq t \neq v \in X} \frac{\sigma_{st}(v)}{\sigma_{st}}$$

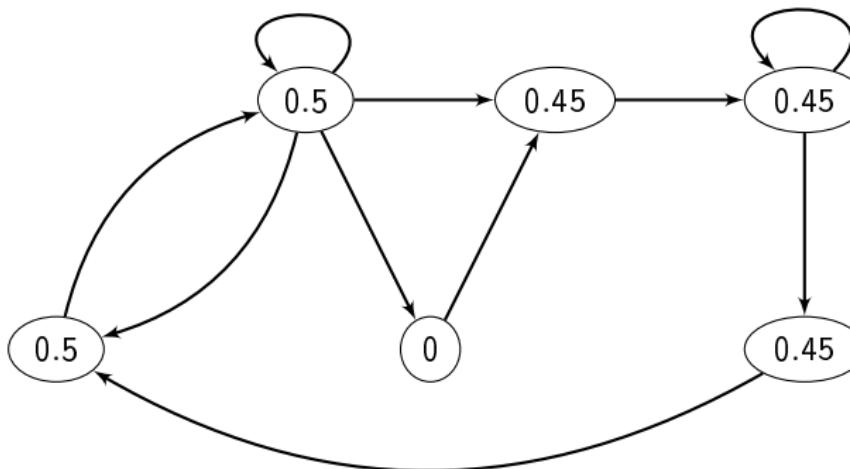
σ_{st} (# shortest paths from s to t) is found in the matrix above

What about $\sigma_{st}(v)$, # of those shortest paths that go through v ?

We can use `all_shortest_paths(G, from = s, to = t, mode = "out")`

Example of betweenness

```
betweenness(G,  
directed = FALSE,  
normalized = TRUE)  
Values shown in the ver-  
tices.
```



Closeness

Definition 7

$G = (X, U)$, $v \in X$. The **closeness** of v is

$$c_D(v) = \frac{1}{n-1} \sum_{t \in X \setminus \{v\}} d_D(v, t)$$

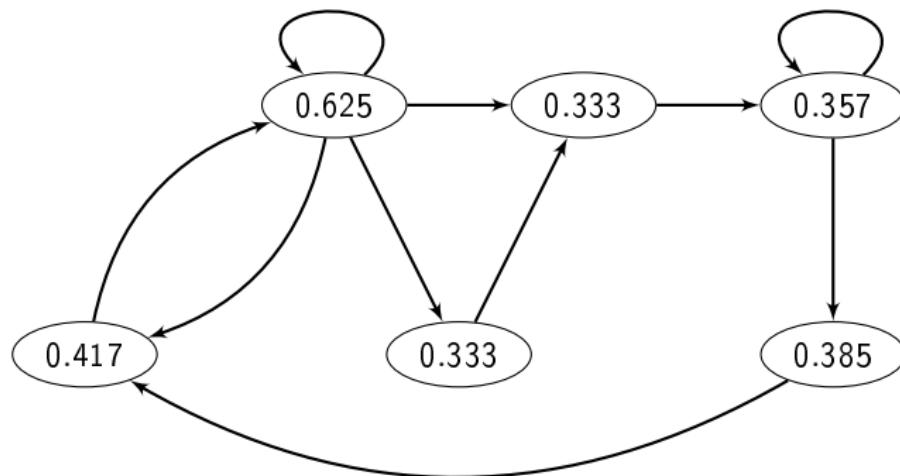
i.e., mean geodesic distance between a vertex v and all other vertices it has access to

Another definition is

$$c_D(v) = \frac{1}{\sum_{t \in X \setminus \{v\}} d_D(v, t)}$$

Example of (out) closeness

```
closeness(G, normalized = TRUE, mode="out")
```



Measures specific to vertices

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Periphery of a graph

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General context of network models

Diametre and periphery of a graph

Definition 8 (Diametre of a graph)

The **diametre** of G is

$$\delta(G) = \max_{\substack{x,y \in X \\ x \neq y}} d(x,y)$$

or, in other words,

$$\delta(G) = \max_{x \in X} e(x)$$

$\delta(G) < \infty \iff G$ strongly connected

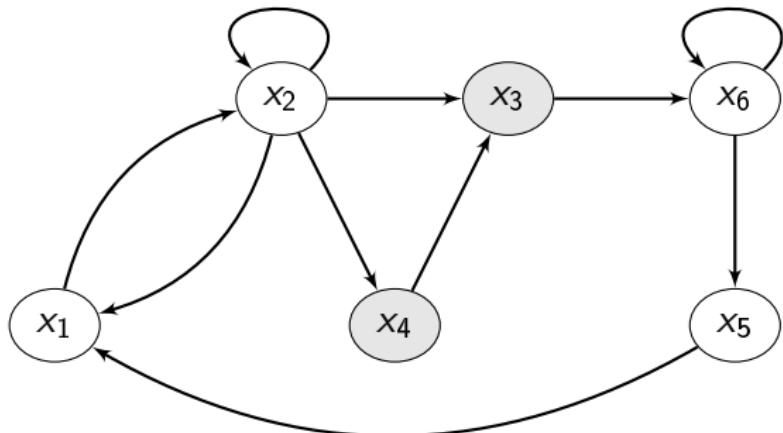
Definition 9 (Periphery)

The **periphery** of a graph is the set of vertices whose eccentricity achieves the diameter, i.e.,

$$\{x \in X : e(x) = \delta(G)\}$$

0	1	2	2	4	3
1	0	1	1	3	2
3	4	0	5	2	1
4	5	1	0	3	2
1	2	3	3	0	4
2	3	4	4	1	0

Diametre is 5 and the periphery is
 $\{x_3, x_4\}$



Definition 10 (Antipodal vertices)

Measures specific to vertices

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Periphery of a graph

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Degree distribution

Definition 11 (Arc incident to a vertex)

If a vertex x is the initial endpoint of an arc u , which is not a loop, the arc u is **incident out of vertex x**

The number of arcs incident out of x plus the number of loops attached to x is denoted $d_G^+(x)$ and is the **outer demi-degree** of x

An arc **incident into vertex x** and the **inner demi-degree** $d_G^-(x)$ are defined similarly

Definition 12 (Degree)

The **degree** of vertex x is the number of arcs with x as an endpoint, each loop being counted twice. The degree of x is denoted $d_G(x) = d_G^+(x) + d_G^-(x)$

If each vertex has the same degree, the graph is **regular**

Definition 13 (Isolated vertex)

A vertex of degree 0 is **isolated**.

Definition 14 (Average degree of G)

$$d(G) = \frac{1}{|V|} \sum_{v \in V} \deg_G(v).$$

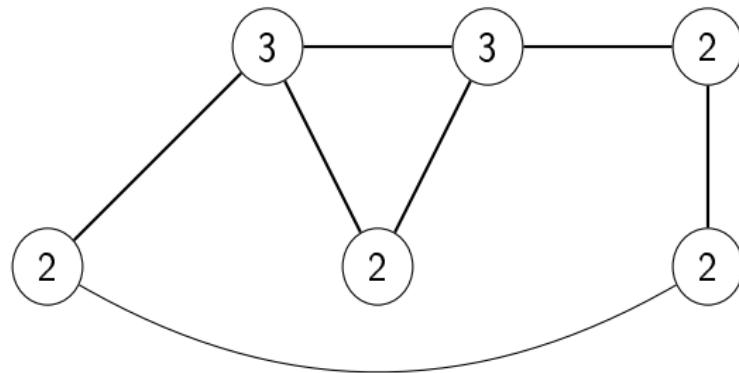
Definition 15 (Minimum degree of G)

$$\delta(G) = \min\{\deg_G(v) | v \in V\}.$$

Definition 16 (Maximum degree of G)

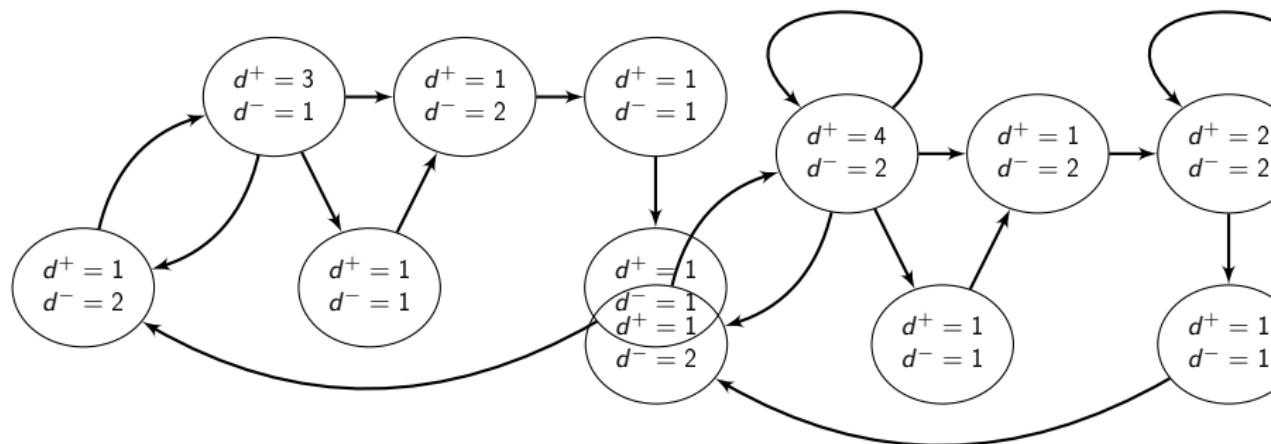
$$\Delta(G) = \max\{\deg_G(v) | v \in V\}.$$

Degrees in an undirected graph



Here, vertices are labelled using the degree

Degrees in a directed graph



What to consider about degrees?

Degrees are often considered as a measure of popularity

Often write $k(i)$ (or k_i) for “degree of vertex i ”, $k^-(i)$ and $k^+(i)$ for in- and out-degree

- ▶ Minimum and maximum degree
- ▶ Minimum and maximum in/out-degree. E.g., if you consider the global air transportation network and the in/out-degree of airports, in-degree is a measure of a location's “popularity” as a travel destination
- ▶ Range of degrees in a graph: are there large discrepancies in connectivity between vertices in the graph?
- ▶ Average degree (often denoted $\langle k \rangle$ because of physicists)
- ▶ Average in/out-degree
- ▶ Variance of the degrees or in/out-degrees

- ▶ Average (nearest) neighbour degree, to encode for *preferential attachment* (one prefers to hang out with popular people)

$$k_i^{nn} = \frac{1}{k(i)} \sum_{j \in \mathcal{N}(i)} k(j)$$

or, in terms of the adjacency matrix $A = [a_{ij}]$,

$$k_i^{nn} = \frac{1}{k(i)} \sum_j a_{ij} k(j)$$

- ▶ *Excess degree*: take nearest neighbour degree but do not consider the edge/arc followed to get to the neighbour
- ▶ Degree, nearest neighbour and excess degree distributions

Degrees in igraph

- ▶ `degree` gives the degrees of the vertices
- ▶ `degree_distribution` gives numeric vector of the same length as the maximum degree plus one. The first element is the relative frequency zero degree vertices, the second vertices with degree one, etc.
- ▶ `knn` calculate the average nearest neighbor degree of the given vertices and the same quantity in the function of vertex degree
- ▶ `strength` sums up the edge weights of the adjacent edges for each vertex

Degree from adjacency matrix

Suppose adjacency matrix take the form $A = [a_{ij}]$ with $a_{ij} = 1$ if there is an arc from the vertex indexed i to the vertex indexed j and 0 otherwise. (Could be the other way round, using A^T , just make sure)

Let $e = (1, \dots, 1)^T$ be the vector of all ones

$$Ae = (d_G^+(1), \dots, d_G^+(1))^T \text{ (out-degree)}$$

$$e^T A = (d_G^-(1), \dots, d_G^-(1)) \text{ (in-degree)}$$

Agent-based models (ABM)

Network models

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Measures at the graph level

Circumference & Girth

Graph density

Graph connectivity

Cliques

k -cores

General context of network models

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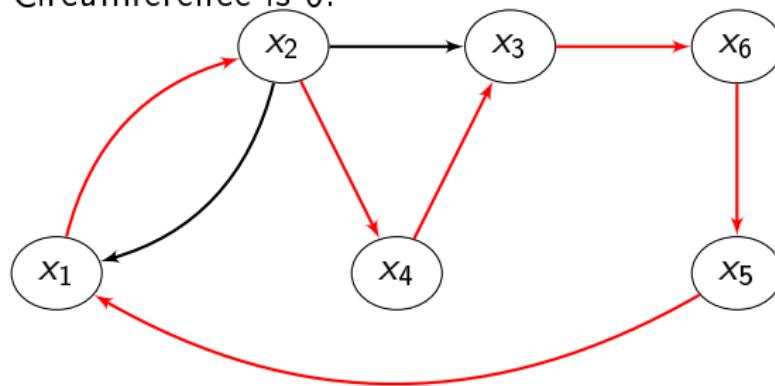
General context of network models

Circumference

Definition 17 (Circumference)

In an undirected (resp. directed) graph, the total number of edges (resp. arcs) in the longest cycle of graph G is the **circumference** of G

Circumference is 6.

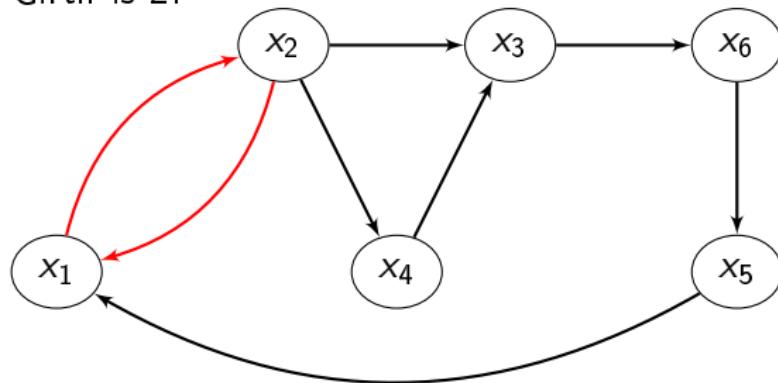


Girth

Definition 18 (Girth)

The total number of edges in the shortest cycle of graph G is the **girth** $g(G)$

Girth is 2.



Measures at the graph level

Circumference & Girth

Graph density

Graph connectivity

Cliques

k -cores

General context of network models

General context of network models

Completeness

Definition 19 (Complete undirected graph)

An undirected graph is complete if every two of its vertices are adjacent.

Definition 20 (Complete digraph)

A digraph $D(V, A)$ is complete if $\forall u, v \in V, uv \in A$.

In case of simple graphs, completeness effectively means that “information” can be transmitted from every vertex to every other vertex quickly (1 step)

It can be useful to know how far away we are from being complete

Number of edges/arcs in a complete graph

$G = (X, E)$ undirected and simple of order n has at most

$$\frac{n(n - 1)}{2}$$

edges, while $G = (X, A)$ directed and simple of order n has at most

$$n(n - 1)$$

arcs

Density of a graph

Definition 21 (Density)

The fraction of maximum number of edges or arcs present in the graph is the **density** of the graph.

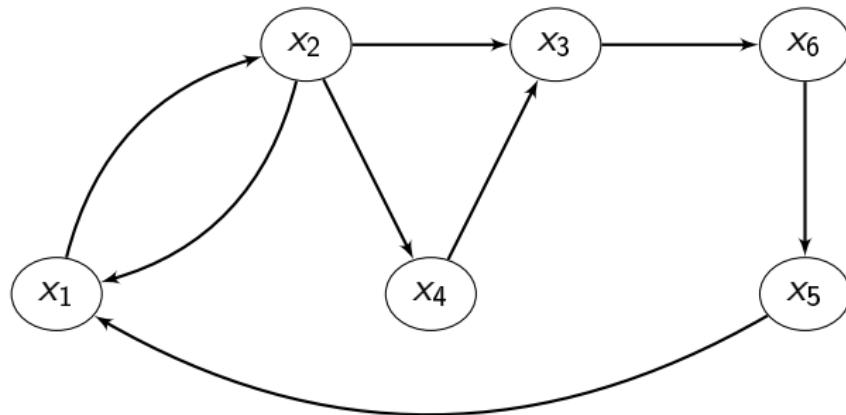
If the graph has p edges or arcs, then its density is, respectively,

$$\frac{2p}{n(n-1)}$$

or

$$\frac{p}{n(n-1)}$$

Example of density



Graph has order 6
and thus a max of
30 arcs. Here, 8 arcs
 \implies density 0.267
(26.7% of arcs are
present)

Measures at the graph level

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General context of network models

General context of network models

Connectedness

We have already seen connectedness (quasi- or strong in the oriented case)

Connectedness is important in terms of characterising graph properties, as it shows the capacity of the graph to convey information to all the members of the graph (the vertices)

Definition 22 (Connected graph)

A **connected graph** is a graph that contains a chain $\mu[x, y]$ for each pair x, y of distinct vertices

Denote $x \equiv y$ the relation “ $x = y$, or $x \neq y$ and there exists a chain in G connecting x and y ”. \equiv is an equivalence relation since

1. $x \equiv y$ [reflexivity]
2. $x \equiv y \implies y \equiv x$ [symmetry]
3. $x \equiv y, y \equiv z \implies x \equiv z$ [transitivity]

Definition 23 (Connected component of a graph)

The classes of the equivalence relation \equiv partition X into connected sub-graphs of G called **connected components**

Articulation set

Definition 24 (Articulation set)

For a connected graph, a set A of vertices is called an **articulation set** (or a **cutset**) if the subgraph of G generated by $X - A$ is not connected

`articulation_points(G)` in igraph (assumes the graph is undirected, makes it so if not)

Strongly connected graphs

$G = (X, U)$ connected. A **path of length 0** is any sequence $\{x\}$ consisting of a single vertex $x \in X$

For $x, y \in X$, let $x \equiv y$ be the relation “there is a path $\mu_1[x, y]$ from x to y as well as a path $\mu_2[y, x]$ from y to x ”. This is an equivalence relation (it is reflexive, symmetric and transitive)

Definition 25 (Strong components)

Sets of the form

$$A(x_0) = \{x : x \in X, x \equiv x_0\}$$

are equivalence classes; they partition X and are the **strongly connected components** of G

Definition 26 (Strongly connected graph)

G strongly connected if it has a single strong component

Definition 27 (Minimally connected graph)

G is **minimally connected** if it is strongly connected and removal of any arc destroys strong-connectedness

Definition 28 (Contraction)

$G = (X, U)$. The **contraction** of the set $A \subset X$ of vertices consists in replacing A by a single vertex a and replacing each arc into (resp. out of) A by an arc with same index into (resp. out of) a

Quasi-strong connectedness

Definition 29 (Quasi-strong connectedness)

G **quasi-strongly connected** if $\forall x, y \in X$, exists $z \in X$ (denoted $z(x, y)$ to emphasize dependence on x, y) from which there is a path to x and a path to y

Strongly connected \implies quasi-strongly connected (take $z(x, y) = x$); converse not true

Quasi-strongly connected \implies connected

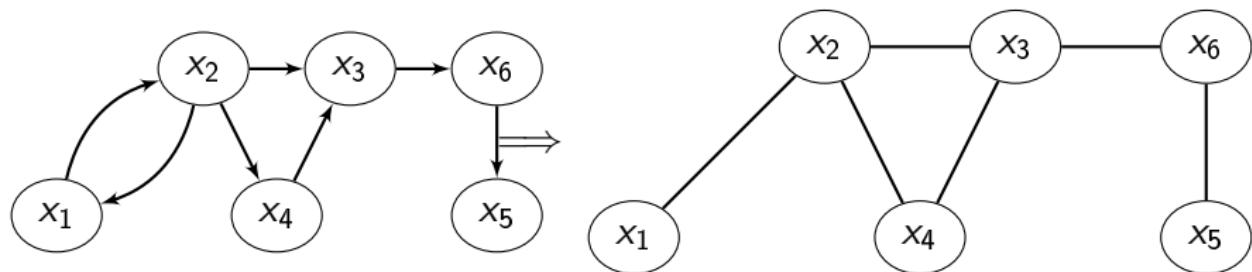
Lemma 30

$G = (X, U)$ has a root $\iff G$ quasi-strongly connected

Weak-connectedness

Definition 31 (Weakly connected graph)

$G = (X, U)$ **weakly connected** if $G = (X, E)$ connected, where E is obtained from U by ignoring the direction of arcs



Weak components

Define for $x, y \in X$ the relation $x \equiv y$ as “ $x = y$ or $x \neq y$ and there is a chain in G connecting x and y ” [like for components in an undirected graph, except the graph is directed here]

This defines an equivalence relation

Definition 32 (Weak components)

Sets of the form

$$A(x_0) = \{x : x \in X, x \equiv x_0\}$$

are equivalence classes partitioning X into the **weakly connected components** of G

$G = (X, U)$ is weakly connected if there is a single weak component

Components in igraph

- ▶ `is_connected` decides whether the graph is weakly or strongly connected
- ▶ `components` finds the maximal (weakly or strongly) connected components of a graph
- ▶ `count_components` does almost the same as `components` but returns only the number of clusters found instead of returning the actual clusters
- ▶ `component_distribution` creates a histogram for the maximal connected component sizes
- ▶ `decompose` creates a separate graph for each component of a graph
- ▶ `subcomponent` finds all vertices reachable from a given vertex, or the opposite: all vertices from which a given vertex is reachable via a directed path

Measures at the graph level

Circumference & Girth

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k-cores

General context of network models

General context of network models

Cliques

Definition 33 (Clique in undirected graphs)

$G = (X, E)$ a simple undirected graph. A **clique** is a subgraph G' of G such that all vertices in G' are adjacent

Definition 34 (n -clique)

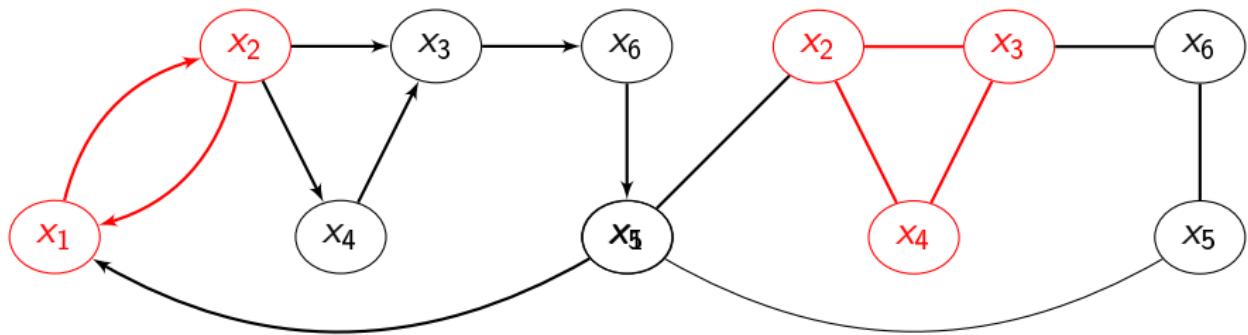
A simple, complete graph on n vertices is called an **n -clique** and is often denoted K_n

Definition 35 (Clique in directed graphs)

$G = (X, U)$ a simple directed graph. A **clique** is a subgraph G' of G such that all vertices in G' are mutually adjacent

Definition 36 (Maximal clique)

A **maximal clique** is a clique that cannot be extended by adding another adjacent vertex



Cliques in igraph

- ▶ `cliques` finds all complete subgraphs in the input graph, obeying the size limitations given in the `min` and `max` arguments
- ▶ `largest_cliques` finds all largest cliques in the input graph
- ▶ `max_cliques` finds all maximal cliques in the input graph
(The largest cliques are always maximal, but a maximal clique is not necessarily the largest)
- ▶ `count_max_cliques` counts the maximal cliques
- ▶ `clique_num` calculates the size of the largest clique(s)

Measures at the graph level

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General context of network models

General context of network models

k -core

Definition 37 (k -core of a graph)

$G = (X, U)$ a graph. The **k -core** of G is a maximal subgraph in which each vertex has degree at least k

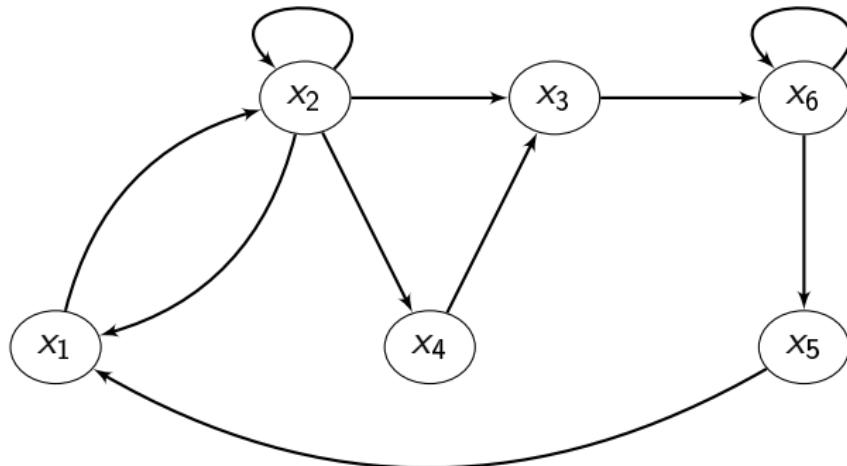
Definition 38 (Coreness of a vertex)

$G = (X, U)$ a graph, $x \in X$. The **coreness** of x is k if x belongs to the k -core of G but not to the $k + 1$ core of G

For directed graphs, in-cores or out-cores depending on whether in-degree or out-degree is used

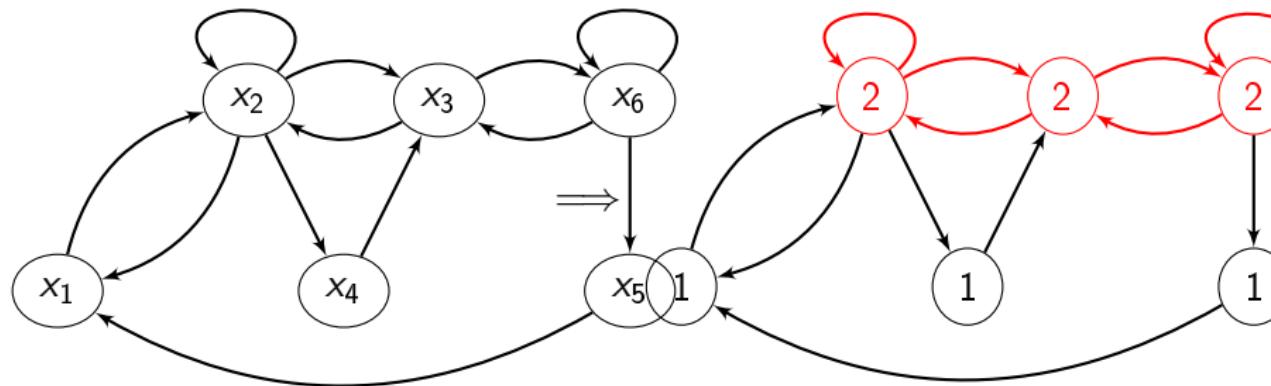
In igraph: coreness

Coreness in the directed case

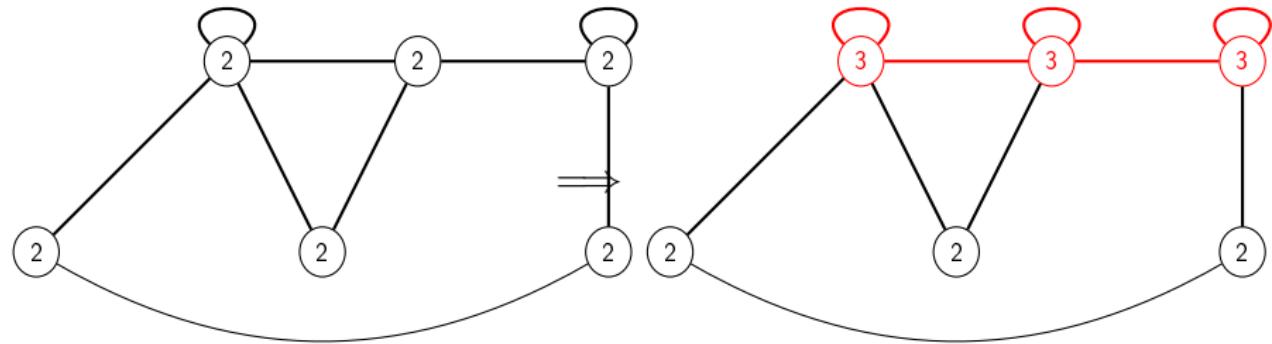


G has only a 1-in-core and 1-out-core: there is no (maximal) subgraph in which the in- or out-degree is larger than 1

In-coreness in the directed case



Coreness in the undirected case



Measures at the graph level

Circumference & Girth

Graph density

Graph connectivity

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k -cores

General context of network models

General context of network models

See for instance...

- ▶ Newman. Spread of epidemic disease on networks, 2002
- ▶ Keeling & Eames. Networks and epidemic models, 2005
- ▶ Meyers, Pourbohloul, Newman, Skowronski & Brunham. Network theory and SARS: predicting outbreak diversity, 2005
- ▶ Meyers, Newman & Pourohloul. Predicting epidemics on directed contact networks, 2006
- ▶ Bansal, Read, Pourbohloul & Meyers. The dynamic nature of contact networks in infectious disease epidemiology, 2010

La librairie EpiModel

Jenness SM, Goodreau SM and Morris M. EpiModel: An R Package for Mathematical Modeling of Infectious Disease over Networks. *Journal of Statistical Software*. 2018; 84(8): 1-47

EpiModel

R library providing tools to simulate and analyse network epidemiological models

Provides two types of approaches

- ▶ Simulation of ODE compartmental models (not so interesting)
- ▶ Simulation of network models

Their website has several useful tutorials

Part of the statnet meta-library