Mathematical Oncology

Travelling Wave Analysis

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Disclaimer: We only discuss the one-dimensional case although higher dimensional extensions are possible.

The idea behind *travelling wave analysis* is to investigate the transient behaviour of solutions to the reaction diffusion equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + F(u) \tag{1}$$

where $t \geq 0$, $x \in \mathbb{R}$, $D > 0$ and F sufficiently smooth. The main argument though is the following:

> depending on the problem at hand, (1) might posses special solutions that don't change in shape, they rather shift horizontally (i.e. parallel to the $x'x$-axis) with time.

Clearly, these *travelling wave solutions* do not always exist; still, they are quite common in problems arising in Biology (and beyond).

To further investigate this class of solutions we need to quantify the notion of a "shifting". To this end, we consider a function $u : [0, \infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ and make the following assumption:

$$\exists\ w : \mathbb{R} \longrightarrow \mathbb{R} \ \text{ and } c > 0 \ : \ u(x, t) = w(x - ct)\ \forall t \geq 0,\ x \in \mathbb{R} \tag{2}$$

The meaning of this assumption becomes clear when we consider $t > 0$ (fixed), and define the graphs $G_u,\ G_w \subset \mathbb{R}^2$ of the functions u and w as:

$$G_u = \left\{(x, y) \in \mathbb{R}^2 \ : \ y = u(t, x)\right\}, \qquad G_w = \left\{(x, y) \in \mathbb{R}^2 \ : \ y = w(x)\right\}.$$

The graph G_u is a horizontal shift of the graph G_w, to the right by an amount ct, when the following holds

$$(x, y) \in G_u \text{ iff } (x - ct, y) \in G_w$$

or, according to the definition of G_u and G_w, if

$$y = u(t, x) \text{ iff } y = w(x - ct)$$

Example: Sketch the (indicative) functions $w(x) = x^3$ and $u(x) = w(x - 2) = (x - 2)^3$ and note that the graph of u is a horizontal shift of the graph of w to the right by 2.

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Back to the PDE (1). Assuming that (1) possesses a *travelling wave solution* u—as we said previously, this is a very strong, albeit common, assumption—then

$$\exists\, w : \mathbb{R} \longrightarrow \mathbb{R} \text{ and } c > 0 \;\; : \;\; u(x,t) = w(x-ct)\ \forall t > 0,\ x \in \mathbb{R}.$$

If both u and w are sufficiently smooth then we can write

$$\begin{aligned}
\frac{\partial}{\partial t}u(t,x) &= \frac{\partial}{\partial t}\left(w(x-ct)\right) = w'(x-ct)\frac{\partial}{\partial t}(x-ct) \overset{z=x-ct}{=} -cw'(z),\\
\frac{\partial}{\partial x}u(t,x) &= \frac{\partial}{\partial x}\left(w(x-ct)\right) = w'(x-ct)\frac{\partial}{\partial x}(x-ct) \overset{z=x-ct}{=} w'(z),\\
\frac{\partial^2}{\partial x^2}u(t,x) &= \frac{\partial}{\partial x}\left(w'(x-ct)\right) = w''(x-ct)\frac{\partial}{\partial x}(x-ct) \overset{z=x-ct}{=} w''(z).
\end{aligned}$$

Based on the above we can perform a change of dependent variables, from u to w, and transform the PDE (1) into a second order ODE (!)

$$-cw'(z) = Dw''(z) + F(w(z))$$

or

$$Dw''(z) + cw'(z) + F(w(z)) = 0 \tag{3}$$

or even, after setting $p(z) = w'(z)$, as a 2×2 system of ODEs

$$\begin{cases} w' = p \\ p' = -\dfrac{c}{D}p - \dfrac{F(w)}{D} \end{cases} \tag{4}$$

The steady states (SS) (w^*, p^*) of the last system satisfy the algebraic system

$$\begin{cases} F(w^*) = 0 \\ p^* = 0 \end{cases}$$

To identify the stability of these SSs we evaluate the Jacobian at the SS $(w^*, 0)$

$$J^* = \begin{pmatrix} 0 & 1 \\ -\dfrac{F'(w^*)}{D} & -\dfrac{c}{D} \end{pmatrix}$$

and find

$$\operatorname{tr} J^* = -\frac{c}{D}, \qquad \det J^* = \frac{F'(w^*)}{D}.$$

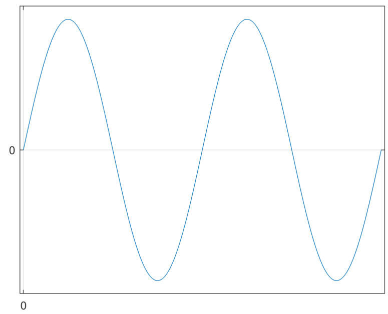
Clearly, since $\operatorname{tr} J^* < 0$, the stability of the SS $(w^*, 0)$ is characterised as

$$\begin{cases} \text{saddle point,} & \text{if } F'(w^*) < 0, \\ \text{asymptotically stable,} & \text{if } F'(w^*) > 0, \\ \text{line of stable fixed points,} & \text{if } F'(w^*) = 0. \end{cases}$$

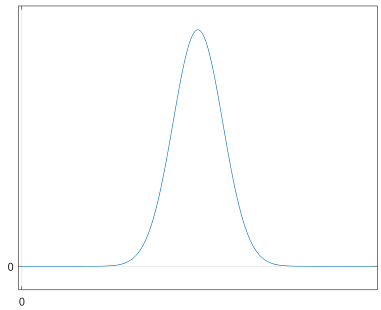
Remark 1. *What is here even more interesting (if that is ever possible) is that the spatially uniform SS, m^*, of the original PDE* (1)*, satisfies $F(m^*) = 0$ and is stable if $F'(m^*) < 0$, and unstable if $F'(m^*) > 0$. What an unexpected "contradiction"..... or is it? Well, on one hand we do understand the SSs m^* of the original PDE* (1) *to correspond to the large (physical) time t asymptotic behaviour of the solution m. On the other hand, the SSs (w^*, p^*) of the ODE system* (4) *correspond to large values of the travelling wave variable $z\,(= x - ct)$.*

Asymptotic values. Still the basic question remains unanswered. What is the shape of the (fixed) travelling wave solution w? It should be made clear that the existence of such solutions is by far not guaranteed within the mathematical realm. On the contrary, they constitute, whenever they exist, a rather positive surprise. So, why do we deal with them? Well, travelling wave solutions might be a "rare" occurrence in Mathematics but they are "quite" common in Biology (and Physics for that matter).

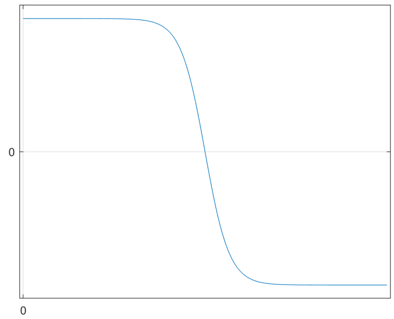
In effect, if it is not possible to investigate the shape of the travelling waves analytically, one gets inspired from Biology. As an example, consider the three cases depicted below, where the represented functions correspond to different phenomena in biology



(a) predator-prey-like



(b) infectious disease-like



(c) invasion-like