# Mathematical Oncology

2x2 system of ODEs

Nikolaos Sfakianakis*

We will consider the homogenous 2x2 system of of ODEs with constant coefficients:

$$\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t), \quad \mathbf{A} = \left(a_{ij}\right)_{i,j=1,2} \in \mathbb{R}^{2x2} \tag{1}$$

**Remark 1.** *The treatment of non-linear systems extends beyond the scope of these notes. Still, if needed, a non-linear system could be linearised around the a specific state.*

The solution of the system of ODEs (1) depends on the type of eigenvalues of the matrix $A$—to be more precise, the solution depends of the diagonalization of the matrix $A$; hence on the eigenvalues and eigenvectors.

> *Question:* Why do the eigenvalues and eigenvectors of $A$ provide a solution to the system (1)?
> *Answer:* Consider an eigenvalue $\lambda \in \mathbb{R}$ of $\mathbf{A}$ and $\mathbf{v} \in \mathbb{R}^2$ the corresponding eigenvector. Since $\mathbf{A}$ is a matrix of constants, $\lambda$ and $\mathbf{v}$ are constants as well. By the definition of eigenvalues/-vectors it holds:
>
> $$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}.$$
>
> Consider now the function of $t \geq 0$ given by the specific eigenvalue $\lambda$ and eigenvector $\mathbf{v}$
>
> $$\mathbf{y}(t) = e^{\lambda t}\mathbf{v},$$
>
> and note that the time derivative of $\mathbf{y}$ reads (since $\lambda$ and $\mathbf{v}$ constants)
>
> $$\frac{d}{dt}\mathbf{y}(t) = \frac{d}{dt}\left(e^{\lambda t}\mathbf{v}\right) = \lambda e^{\lambda t}\mathbf{v} = e^{\lambda t}\mathbf{A}\mathbf{v} = \mathbf{A}\mathbf{y}.$$
>
> This implies that $\mathbf{y}(t) = e^{\lambda t}\mathbf{v}$ is indeed a solution of the ODE system (1).

The *fundamental theorem of Algebra* states that $\mathbf{A}$ has exactly two eigenvalues, calculated by e.g. solving the characteristic equation

$$\lambda^2 - \operatorname{tr}(\mathbf{A})\lambda + \det(\mathbf{A}) = 0 \implies \lambda_{1,2} = \frac{\operatorname{tr}(\mathbf{A}) \pm \sqrt{\operatorname{tr}(\mathbf{A})^2 - 4\det(\mathbf{A})}}{2} = \frac{\operatorname{tr}(\mathbf{A}) \pm \sqrt{\Delta}}{2}$$

where $\operatorname{tr}(\mathbf{A})$, $\det(\mathbf{A})$ are the trace and determinant of $\mathbf{A}$ and where $\Delta = \operatorname{tr}(\mathbf{A})^2 - 4\det(\mathbf{A})$ is the discriminant of the quadratic equation above. The eigenvalues $\lambda_{1,2}$ can either be

*School of Mathematics and Statistics, University of St. Andrews, Scotland, UK, n.sfakianakis@st-andrews.ac.uk

real and distinct $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 \neq \lambda_2$ , or real and equal $\lambda_1 = \lambda_2 \in \mathbb{R}$, or complex conjugate $\lambda_1 = \bar{\lambda}_2 \in \mathbb{C}$. As the dimension of the solution space of the ODE (1) is 2, it suffices that we found 2 linearly independent solutions, $\mathbf{y}_1$, $\mathbf{y}_2$. Then, their linear combination constitutes the general solution

$$\mathbf{y}_{\text{ho}} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2$$

of the (homogenous) ODE (1). In more detail:

**Case 1:** $\mathbf{A}$ has two real eigenvalues $\lambda_1$, $\lambda_2$ with the corresponding eigenvectors $\mathbf{v}_1$,$\mathbf{v}_2$ (refresh your method of calculating eigenvectors).
The functions $\mathbf{y}_1 = e^{\lambda_1 t}\mathbf{v}_1$, $\mathbf{y}_2 = e^{\lambda_2 t}\mathbf{v}_2$ are linearly independent solutions of the problem and, hence, the general solution of (1) reads

$$\mathbf{y}_{\text{ho}} = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2, \quad c_1, c_2 \in \mathbb{R}.$$

**Case 2:** $\mathbf{A}$ has one double real eigenvalue $\lambda$ with eigenvector $\mathbf{v}$.
One special solution of the ODE system is

$$\mathbf{y}_1(t) = e^{\lambda t}\mathbf{v}.$$

A second special solution, linearly independent to $\mathbf{y}_1$, is

$$\mathbf{y}_2 = te^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{u}$$

where $\mathbf{u}$ is the solution of $(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{u} = \mathbf{v}$. The general solution of the ODE system (1) is then given as before by:

$$\mathbf{y}_{\text{ho}} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2$$

**Case 3:** $\mathbf{A}$ has two conjugate complex eigenvalues $\lambda$, $\bar{\lambda}$ with eigenvectors that are also conjugate $\mathbf{v}$, $\bar{\mathbf{v}}$. The generic solution is

$$\mathbf{y}_{\text{ho}} = c_1 e^{\lambda t}\mathbf{v} + c_2 e^{\bar{\lambda} t}\bar{\mathbf{v}}, \quad c_1, c_2 \in \mathbb{C}, \quad \text{(YES, complex coefficients!)}$$

**Note:** The imaginary part of $\mathbf{y}_{\text{ho}}$ can be avoided by selecting conjugate constants, i.e. $c_2 = \bar{c}_1 \in \mathbb{C}$. Namely, for $c_1 = c_1^R + ic_1^I$, $\lambda = \lambda^R + i\lambda^I$, and $\mathbf{v} = \mathbf{v}^R + i\mathbf{v}^I$, $\mathbf{y}_{\text{ho}}$ reads

$$\begin{aligned}
\mathbf{y}_{\text{ho}} &= \left(c_1^R + ic_1^I\right) e^{\lambda^R t + i\lambda^I t} \left(\mathbf{v}^R + i\mathbf{v}^I\right) + \left(c_1^R - ic_1^I\right) e^{\lambda^R t - i\lambda^I t} \left(\mathbf{v}^R - i\mathbf{v}^I\right) \\
&\overset{\text{Euler}}{=} \left(c_1^R + ic_1^I\right) e^{\lambda^R t} \left(\cos(\lambda^I t) + i\sin(\lambda^I t)\right) \left(\mathbf{v}^R + i\mathbf{v}^I\right) \\
&\qquad + \left(c_1^R - ic_1^I\right) e^{\lambda^R t} \left(\cos(\lambda^I t) - i\sin(\lambda^I t)\right) \left(\mathbf{v}^R - i\mathbf{v}^I\right) \\
&= \cdots \text{(some algebraic manipulations)} \cdots \\
&= e^{\lambda^R t} \left(2c_1^R \cos(\lambda^I t)\mathbf{v}^R - 2c_1^R \sin(\lambda^I t)\mathbf{v}^I - 2c_1^I \cos(\lambda^I t)\mathbf{v}^I - 2c_1^I \sin(\lambda^I t)\mathbf{v}^R\right) \\
&= 2e^{\lambda^R t}\Big( \left(c_1^R\mathbf{v}^R - c_1^I\mathbf{v}^I\right) \cos(\lambda^I t) - \left(c_1^R\mathbf{v}^I + c_1^I\mathbf{v}^R\right) \sin(\lambda^I t)\Big)
\end{aligned}$$

**NOTE:** The last relation indicates that the dynamics of $\mathbf{y}_{\mathrm{ho}}$ are "elliptical"! This is due to the trigonometry with either a converging (if $\lambda^R < 0$), or diverging (if $\lambda^R > 0$), or neutral (if $\lambda^R = 0$) character. Confer at this point the Linear Stability Analysis appendix of our lecture notes.

**Remark 2.** *In the case of a non-homogenous system of ODEs, e.g.* $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{R}$*, then in addition to the general solution* $\mathbf{y}_{ho}$ *of homogenous, we need to identify one special solution* $\mathbf{y}_{sp}$ *of the non-homogenous problem (just one suffices and we usually "guess"). Together, they provide the general solution of the non-homogenous:*

$$\mathbf{y} = \mathbf{y}_{ho} + \mathbf{y}_{sp}$$